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ON THE EQUILIBRIUM MEASURE OF RECURRENT MARKOV PROCESSES

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0. Introduction

In the case of planar Brownian motion, if we denote $h(x, y) = -\frac{1}{\pi} \log|x-y|$, the following results are well known (see [13], [16]). (i) If F is a non-polar compact set, then there exists a probability measure ξ_F on F such that $\int h(x, y)\xi_F(dy)$ equals a constant $R(F)$ on F except on a polar set. The measure ξ_F and the constant $R(F)$ are respectively called the equilibrium measure and Robin's constant of F . (ii) A compact set F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int h(x, y)\xi(dy)$ is locally bounded.

In this paper we shall be concerned with the similar problem for recurrent Hunt processes with strong Feller resolvent. In our case, in place of $h(x, y)$, we shall use a density $g(x, y)$ of a potential kernel $G(x, dy)$ of X relative to the invariant measure $\mu(dy)$. Unfortunately, our density $g(x, y)$ is not equal to $h(x, y)$ in the case of planar Brownian motion but equal to $h(x, y) + f(x) + g(y)$ with some locally bounded functions f and g (see §4).

Now we shall outline the contents of this paper. Let X be a recurrent Hunt process with strong Feller resolvent and μ an invariant measure of X . If we are given a certain finite non-negative continuous additive functional A of X then we can construct a potential kernel G of X by means of time change and killing based upon A ([4], [12]). In this paper we shall suppose, for simplicity, that $A_t = \int_0^t I_C(X_s) ds$ for an arbitrary fixed non-null compact set C but the similar argument can be applicable for a large class of functionals A .

In section 1, some preliminary results are established. Among others, a potential kernel K_A and an invariant measure ν_A of the time changed process by A are described. In section 2, for any other finite non-negative continuous additive functional B , a potential kernel K_B and an invariant measure ν_B of the time changed process by B are constructed by making use of K_A and ν_A . In section 3, let us introduce the duality hypothesis that there exists a dual process \hat{X} (of X relative to μ) satisfying those regularity conditions like X .

We shall then construct a kernel function $g(x, y)$ such that $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. finely] continuous, finite except on a polar set and $K_B(x, dy) = g(x, y)\nu_B(dy)$ [resp. $\hat{K}_B(dx, y) = g(x, y)\nu_B(dx)$] for all continuous additive functionals B , where \hat{K}_B is the potential kernel of the time changed process of \hat{X} by the dual functional \hat{B} of B . In particular, when $B_t = t$, we have $G(x, dy) = g(x, y)\mu(dy)$ and $\hat{G}(dx, y) = g(x, y)\mu(dx)$, where \hat{G} is the potential kernel of \hat{X} associated with $\hat{A}_t = \int_0^t I_C(\hat{X}_s) ds$ in the sense of section 2. In this sense, our function $g(x, y)$ may be called the potential kernel function associated with (G, \hat{G}) . In section 4, we introduce the notion of potential kernel function $h(x, y)$ in a more general sense and then establish a relation between $h(x, y)$ and $g(x, y)$. In section 5, we shall show the equilibrium principle. This means that, if $F = \text{supp}(B)$, then there is a probability measure ξ_F on $\text{supp}(\hat{B})$ such that $\int g(x, y)\xi_F(dy) = R(F)$ on F . In our case, the equilibrium measure ξ_F and Robin's constant $R(F)$ have intuitive probabilistic meanings. If X and \hat{X} are equivalent, the results of section 5 have simpler forms and the analogous potential principles to classical potential theory hold. This case is treated in section 6. There a characterization of the equilibrium measure by means of energy is also given.

1. Notations and preliminary results

Let E be a locally compact Hausdorff space with countable base, \mathcal{E} the Borel σ -field on E and \mathcal{E}^* the σ -field obtained by the universal completion of \mathcal{E} . If \mathcal{A} is a σ -field of subsets of E then the classes of all bounded \mathcal{A} -measurable functions, all bounded non-negative \mathcal{A} -measurable functions and all bounded \mathcal{A} -measurable functions with compact support are denoted by $b\mathcal{A}$, $b\mathcal{A}_+$ and $b\mathcal{A}_c$, respectively.

Throughout in this paper, let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a recurrent Hunt process on E with strong Feller resolvent, that is, a Hunt process satisfying

- (i) (Recurrence); For all $f \in b\mathcal{E}_+$, $G^0f(x) = E^x[\int_0^\infty f(X_t) dt] \equiv 0$ or $\equiv \infty$ on E .
- (ii) (Strong Feller property of resolvent); For all $p > 0$ and $f \in b\mathcal{E}$, $G^pf(x) = E^x[\int_0^\infty e^{-pt}f(X_t) dt]$ is bounded continuous.

In this case, it is well known that there exists a unique (except a constant multiple) invariant Radon measure μ of X , which is positive on every open sets (see [1], [2]). Let Φ be the family of all non-negative continuous additive functionals (abbreviated CAF) $A = (A_t)_{t \geq 0}$ of X such that $A_t < \infty$ a.s. for all $t < \infty$ and let Φ^+ be the subfamily of functionals $A \in \Phi$ which are not equivalent to the zero functional. If $A \in \Phi^+$ then $P^x(A_\infty = \infty) = 1$ for all x ([1]). For $A \in \Phi^+$ and $p \geq 0$ we define a kernel K_A^p by

$$(1.1) \quad K_A^p f(x) = E^x \left[\int_0^\infty e^{-pA_t} f(X_t) dA_t \right].$$

Note that $(K_A^p)_{p>0}$ is the resolvent of the time changed process of X by A .

Moreover, for $A, B \in \Phi^+$ and $p, q \geq 0$ we define two auxiliary kernels $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$ as follows:

$$(1.2) \quad U_{A,B}^{p,q} f(x) = E^x \left[\int_0^\infty e^{-pA_t - qB_t} f(X_t) dA_t \right],$$

$$(1.3) \quad V_{A,B}^{p,q} f(x) = E^x \left[\int_0^\infty e^{-pA_t - qB_t} f(X_t) dB_t \right].$$

Obviously, $U_{A,B}^{p,q} = V_{B,A}^{q,p}$. The family $(U_{A,B}^{p,q})_{p>0}$ is the resolvent of the time changed process by A_t of the e^{-qB_t} -subprocess of X . If $B_t = t$, we shall write $K_A^{p,q}$ for $U_{A,(t)}^{p,q}$ and $G_A^{p,q}$ for $V_{A,(t)}^{p,q}$, i.e.,

$$(1.4) \quad K_A^{p,q} f(x) = E^x \left[\int_0^\infty e^{-pA_t - qt} f(X_t) dA_t \right],$$

$$(1.5) \quad G_A^{p,q} f(x) = E^x \left[\int_0^\infty e^{-pA_t - qt} f(X_t) dt \right].$$

Note that $U_{A,B}^{p,0} = K_A^{p,0} = K_A^p$, $V_{A,B}^{0,q} = K_B^q$ and $G_A^{0,q} = G^q$ (the resolvent of X).

In the sequel, if there is no danger of confusion, the suffices A, B will be often omitted.

Lemma 1.1 (Nagasawa-Sato [10; theorem 2.1 and 2.2]). *Write $U^{p,q}$ and $V^{p,q}$ for $U_{A,B}^{p,q}$ and $V_{A,B}^{p,q}$. For all $p > 0, p' > 0, q \geq 0, q' \geq 0$ and $f \in b\mathcal{E}^*$,*

$$(1.6) \quad U^{p,q} f - U^{p',q'} f + (p-p') U^{p,q} U^{p',q'} f + (q-q') V^{p,q} U^{p',q'} f = 0,$$

$$(1.7) \quad V^{p,q} f - V^{p',q'} f + (p-p') V^{p,q} V^{p',q'} f + (q-q') U^{p,q} V^{p',q'} f = 0.$$

If, in particular, $U^{0,q_0} |f|$ [resp. $V^{q_0,0} |f|$] is bounded for some $q_0 \geq 0$ then $U^{0,q} |f|$ [resp. $V^{q,0} |f|$] is bounded for all $q > 0$ and (1.6) [resp. (1.7)] holds for all $p, p', q, q' \geq 0$ satisfying $p+q > 0$ and $p'+q' > 0$.

Lemma 1.2 ([12; lemma 2.2]). *There exists an increasing sequence $\{E_n\}_{n \geq 1}$ [resp. $\{F_n\}_{n \geq 1}$] of subsets in \mathcal{E}^* such that $\bigcup_{n \geq 1} E_n = E$ [resp. $\bigcup_{n \geq 1} F_n = E$] and $U^{0,1}(\cdot, E_n)$ [resp. $V^{1,0}(\cdot, F_n)$] is bounded for all $n \geq 1$.*

Lemma 1.3 (Blumenthal-Gettoor [3; III, section 5]). *If $A \in \Phi^+$ then $G_A^{p,0}(\cdot, F)$ is bounded for all compact set F and $p > 0$.*

A set C is said to be null if it is a set of potential zero relative to $(G^p)_{p>0}$. Let C be an arbitrary (but fixed) non-null compact subset of E and let us assume that μ is normalized on C as $\mu(C) = 1$.

In the remainder of this paper, unless otherwise stated, the CAF $A=(A_t)_{t \geq 0} \in \Phi^+$ always represents the CAF defined by

$$(1.8) \quad A_t = \int_0^t I_C(X_s) ds.$$

Then for every $B \in \Phi^+$,

$$U_{A,B}^{0,1} 1(x) = E^x \left[\int_0^\infty e^{-Bt} I_C(X_t) dt \right] = G_B^{1,0}(x, C)$$

is bounded by lemma 1.3. Moreover we have

Lemma 1.4. *For any $p, q > 0$ and $f \in b\mathcal{E}^*$, the functions $K_A^p f$ and $G_A^{p,q} f$ are bounded continuous. In case $f \in b\mathcal{E}_c^*$, $G_A^{p,0} f$ is bounded continuous for all $p > 0$.*

Proof. Drop the suffix A in the related kernels. For any $p > 0$ and $f \in b\mathcal{E}^*$ we have, from (1.7),

$$G^p f - G^{p,p} f - p K^{0,p} G^{p,p} f = 0.$$

Since $K^{0,p} g = G^p(I_C g)$ for any $g \in b\mathcal{E}^*$ the function

$$\begin{aligned} G^{p,p} f &= G^p f - p G^p(I_C G^{p,p} f) \\ &= G^p(f - p I_C G^{p,p} f) \end{aligned}$$

is bounded continuous by the strong Feller property of G^p . Therefore,

$$G^{p,q} f = G^{p,p}(f + (q-p) G^{p,p} G^{p,q} f)$$

is bounded continuous. If $f \in b\mathcal{E}_c^*$ then $G^{p,0} f$ is bounded by lemma 1.3, so that the above equality for $q=0$ shows that $G^{p,0} f$ is bounded continuous.

Since $(K_A^p)_{p>0}$ is a strong Feller resolvent by lemma 1.4, the mapping $x \rightarrow K_A^1(x, \cdot)$ of the compact set C into the space of measures over C is strongly continuous by a theorem of Mokobodzki (see Meyer [9]). Since, in addition, $K_A^1(x, \cdot)$ are equivalent for all $x \in E$, we have

$$\sup_{x, y \in C} \frac{1}{2} \|K_A^1(x, \cdot) - K_A^1(y, \cdot)\| \equiv a < 1.$$

Thus there exists a unique invariant probability measure ν_A of K^1 such that

$$(1.9) \quad \sup_{x \in E} \|(K_A^1)^{n+1}(x, \cdot) - \nu_A(\cdot)\| \leq 2a^n$$

for all $n \geq 0$ ([7; lemma 1.3]). Therefore the kernel

$$(1.10) \quad K_A(x, F) = \sum_{n=1}^\infty [(K_A^1)^n(x, F) - \nu_A(F)]$$

is well defined and satisfies

$$(I - K_A^1)K_A f = K_A^1 f - \langle \nu_A, f \rangle$$

for all $f \in b\mathcal{C}^*$.

Lemma 1.5. *The kernel K_A defined by (1.10) satisfies*

$$(1.11) \quad \limsup_{p \rightarrow 0} \sup_{x \in B} \|K_A^p(x, \cdot) - \frac{\nu_A(\cdot)}{p} - K_A(x, \cdot)\| = 0,$$

in particular,

$$(1.12) \quad \limsup_{p \rightarrow 0} \sup_{x \in B} \|pK_A^p(x, \cdot) - \nu_A(\cdot)\| = 0.$$

Proof. From the resolvent equation for (K_A^p) we have

$$\begin{aligned} K_A^p(x, \cdot) &= K_A^1 \sum_{n=0}^{\infty} (1-p)^n (K_A^1)^n(x, \cdot) \\ &= \sum_{n=1}^{\infty} (1-p)^{n-1} \{(K_A^1)^n(x, \cdot) - \nu_A(\cdot)\} + \frac{\nu_A(\cdot)}{p} \end{aligned}$$

for all $x \in E$ and $0 < p < 1$. Thus it follows that

$$\begin{aligned} &\|K_A^p(x, \cdot) - \frac{\nu_A(\cdot)}{p} - K_A(x, \cdot)\| \\ &\leq \sum_{n=1}^{\infty} \{1 - (1-p)^{n-1}\} \|(K_A^1)^n(x, \cdot) - \nu_A(\cdot)\| \\ &\leq \sum_{n=1}^{\infty} \{1 - (1-p)^{n-1}\} 2a^{n-1} = 2 \left\{ \frac{1}{1-a} - \frac{1}{1-a(1-p)} \right\}. \end{aligned}$$

Therefore the lemma follows.

2. An invariant measure and a potential kernel of (K_B^p)

Similarly to [4] and [12], for any $B \in \Phi^+$, an invariant measure ν_B and a potential kernel K_B of $(K_B^p)_{p>0}$ can be constructed by making use of ν_A and K_A defined in section 1. In [12], we have treated only the case of $B_i = t$ but the same arguments are valid for all $B \in \Phi^+$. We shall outline it in the form of our present use.

For any $B \in \Phi^+$ define the measure ν_B by

$$(2.1) \quad \nu_B = \nu_A V_{A,B}^{1,0}$$

Then ν_B charges no semipolar set and satisfies the following properties.

Lemma 2.1. *The measure ν_B is a σ -finite invariant measure of $(K_B^p)_{p>0}$. In particular, $\nu_{(t)} = \mu$.*

Proof. (cf. [12; theorem 2.7]) Since $V_{A,B}^{1,0}(\cdot, F_n)$ is bounded for all n by

lemma 1.2, ν_B is σ -finite. Integrating the equality

$$V_{A,B}^{1,0} - K_B^p + K_A^1 K_B^p - p V_{A,B}^{1,0} K_B^p = 0$$

by ν_A we have

$$\nu_B = p \nu_B K_B^p,$$

that is, ν_B is an invariant measure of (K_B^p) .

By the uniqueness of the invariant measure of X , $\nu_{(t)}$ is a constant multiple of μ , say,

$$\nu_{(t)} = \nu_A G_A^{1,0} = b \mu$$

for some constant b . Since $\mu(C)=1$ we have

$$b = \nu_A G_A^{1,0}(C) = \nu_A K^1(C) = \nu_A(C) = 1,$$

Hence $\nu_{(t)} = \mu$.

Lemma 2.2 (cf. [4; proposition 2]). *For any $B, B' \in \Phi^+$,*

$$(2.2) \quad \nu_{B'} = p \nu_B V_{B',B'}^{p,0}.$$

In particular, ν_B is the measure associated with B in the sense of Revuz ([14]). Moreover it holds that $\nu_A = \mu|_C$, where $\mu|_C$ is the restriction of μ to C .

Proof. Similarly to lemma 1.1, we can prove easily that

$$(2.3) \quad V_{A,B'}^{q,0} - V_{B',B'}^{p,0} + q K_A^q V_{B',B'}^{p,0} - p V_{A,B}^{q,0} V_{B',B'}^{p,0} = 0$$

for sufficiently many $f \in b\mathcal{E}^*$. Letting $q=1$ and integrating by ν_A , (2.2) follows. Set $B_i=t$ at (2.2) then $\nu_{B'} = p \nu_{(t)} V_{(t),B'}^{p,0} = p \mu K_B^{0,p}$ by lemma 2.1. Hence $\nu_{B'}$ is the measure associated with B' . In particular, when $B'=A$, it follows that $\langle \nu_A, f \rangle = p \langle \mu, K_A^{0,p} f \rangle = p \langle \mu, G^p I_C f \rangle = p \langle \mu G^p, I_C f \rangle = \langle \mu, I_C f \rangle = \langle \mu|_C, f \rangle$.

Define a kernel K_B by

$$(2.4) \quad K_B(x, \cdot) = K_A V_{A,B}^{1,0}(x, \cdot) + V_{A,B}^{1,0}(x, \cdot) - \nu_B(\cdot).$$

In case $B_i=t$ we shall denote K_B by G , which is the kernel we have constructed in [12]. Obviously, $K_B(x, \cdot)$ is a σ -finite signed measure on E and, for any $n \geq 1$, the total variation of $K_B(x, \cdot)$ on F_n are uniformly bounded for all $x \in E$ by (1.9) and lemma 1.2. Similarly, for any compact set F , the total variation of $G(x, \cdot)$ on F is uniformly bounded for all $x \in E$ by lemma 1.3. If we denote the total variation of a measure on F by $\|\cdot\|_F$, then the following theorem holds.

Theorem 2.3. *For all $n \geq 1$,*

$$(2.5) \quad \limsup_{p \rightarrow 0} \sup_{x \in E} \|V_{A,B}^{p,0}(x, \cdot) - \frac{\nu_B(\cdot)}{p} - K_B(x, \cdot)\|_{F_n} = 0,$$

and in particular,

$$(2.6) \quad \limsup_{p \rightarrow 0} \sup_{x \in E} \|pV_{A,B}^{p,0}(x, \cdot) - \nu_B(\cdot)\|_{F_n} = 0.$$

If $B_t = t$ then we can take arbitrary compact set in place of F_n .

Proof. Write $V^{p,0}$ for $V_{A,B}^{p,0}$. For any Borel subset D of F_n

$$V^{p,0}(x, D) - V^{1,0}(x, D) + pK_A^p V^{1,0}(x, D) - K_A^p V^{1,0}(x, D) = 0$$

from (1.7). This can be written, by noting (2.1),

$$\begin{aligned} \{V^{p,0}(x, D) - \frac{\nu_B(D)}{p}\} - V^{1,0}(x, D) + pK_A^p V^{1,0}(x, D) \\ - (K_A^p - \frac{1}{p}\nu_A)V^{1,0}(x, D) = 0. \end{aligned}$$

Thus we have

$$\begin{aligned} \|V^{p,0}(x, \cdot) - \frac{\nu_B(\cdot)}{p} - K_B(x, \cdot)\|_{F_n} \\ \leq \| \{pK_A^p(x, \cdot) - \nu_A(\cdot)\} V^{1,0} \|_{F_n} \\ + \| \{K_A^p(x, \cdot) - \frac{\nu_A(\cdot)}{p} - K_A(x, \cdot)\} V^{1,0} \|_{F_n}. \end{aligned}$$

This proves the theorem from lemma 1.5.

Corollary 1. *If $f \in b\mathcal{E}^*$ vanishes outside of some F_n , then*

$$(2.7) \quad (I - pK_B^p)K_B f = K_B^p f - U_{A,B}^{0,p} 1 \langle \nu_B, f \rangle$$

for all $p > 0$. If $V_{A,B}^{1,0} 1$ is bounded, then

$$(2.8) \quad K_B(I - pK_B^p)f = K_B^p f - \langle \nu_A, K_B^p f \rangle$$

for all $p > 0$ and $f \in b\mathcal{E}^*$.

Proof. Suppose that $V^{1,0} 1$ is bounded, then obviously (2.5) holds for E in place of F_n , so we have

$$\begin{aligned} K_B(I - pK_B^p)f(x) &= \lim_{q \rightarrow 0} (V^{q,0} - \frac{1}{q}\nu_B)(I - pK_B^p)f(x) \\ &= \lim_{q \rightarrow 0} V^{q,0}(I - pK_B^p)f(x) = \lim_{q \rightarrow 0} (K_B^p f - qK_A^q K_B^q f)(x) \\ &= K_B^p f(x) - \langle \nu_A, K_B^p f \rangle, \end{aligned}$$

from (1.13). The proof of (2.7) is similar.

Let us denote

$$(2.9) \quad N_B = \{f; f \in b\mathcal{E}^*, = 0 \text{ outside of some } F_n \text{ and } \langle \nu_B, f \rangle = 0\},$$

$$(2.10) \quad N = \{f; f \in b\mathcal{E}^*, \langle \mu, f \rangle = 0\} .$$

DEFINITION. If a kernel H on E satisfies the condition that (i) for any $f \in N_B$ [resp. $f \in N$], $Hf \in b\mathcal{E}^*$ and that (ii) for any $f \in N_B$ [resp. $f \in N$], $(I - pK_B^p)Hf = K_B^p f$ [resp. $(I - pG^p)Hf = G^p f$] for all $p > 0$, then we shall say that H is a potential kernel of $(K_B^p)_{p>0}$ [resp. X].

Corollary 2. *The kernels K_B and G are the potential kernels of $(K_B^p)_{p>0}$ and X , respectively.*

Corollary 3. *For every compact set F , the function $G(\cdot, F)$ is finely continuous.*

Proof. Set $B_t = t$ at (2.7)

$$G(x, F) = pG^p G(x, F) + G^p(x, F) - (K_A^{0,p} 1(x)) \mu(F) .$$

Since $K_A^{0,p} 1$ is p -excessive, the result is obvious.

3. Hypothesis of duality and the kernel function $g(x, y)$

In this section we shall assume that there exists a Hunt process \hat{X} with strong Feller resolvent \hat{G}^p such that X and \hat{X} are in duality relative to μ . It follows that \hat{X} is also recurrent and μ is the invariant measure of \hat{X} .

Let $\hat{\Phi}^+$ be the family of all non-zero non-negative finite continuous additive functionals of \hat{X} . For any $\hat{A}, \hat{B} \in \hat{\Phi}^+$, we define $\hat{U}_{A,B}^{p,q}$ etc. by

$$f \hat{U}_{A,B}^{p,q}(x) = \hat{E}^x \left[\int_0^\infty e^{-p\hat{A}_t - q\hat{B}_t} f(\hat{X}_t) d\hat{A}_t \right]$$

etc. (in general, a kernel with respect to the dual process \hat{X} is written such as $\hat{K}(D, x)$, so that \hat{K} operates to function from the right side and to measure from the left).

By Revuz [14; theorem VII. 1], for any $B \in \Phi^+$, there exists a polar set P_B and a CAF $\hat{B} \in \hat{\Phi}^+$ of \hat{X} restricted to $E - P_B$ such that $\nu_B = \hat{K}_B^{0,1} \mu$. Also, by [14; theorem VII. 2], there exists a jointly measurable kernel function $g_B^{p,q}(x, y)$ satisfying

- (i) $g_B^{p,q}(\cdot, y)$ [resp. $g_B^{p,q}(x, \cdot)$] is finely [resp. cofinely] continuous and q -excessive [resp. q -coexcessive] relative to the resolvent $(G_B^{p,q})_{q>0}$ [resp. $(\hat{G}_B^{p,q})_{q>0}$] for all $p > 0$ and $y \in E - P_B$ [resp. $x \in E$],
- (ii) For all $p, q > 0$ and $x \in E$, $K_B^{p,q}(x, dy) = g_B^{p,q}(x, y) \nu_B(dy)$, $G_B^{p,q}(x, dy) = g_B^{p,q}(x, y) \mu(dy)$ and for all $p, q > 0$ and $y \in E - P_B$, $\hat{K}_B^{p,q}(dx, y) = g_B^{p,q}(x, y) \nu_B(dx)$, $\hat{G}_B^{p,q}(dx, y) = g_B^{p,q}(x, y) \mu(dx)$.

As before, the set C with $\mu(C) = 1$ is fixed and A is given by (1.8). If $B = A$, P_B may be supposed to be empty and the dual CAF of A is given exactly by

$$(3.1) \quad \hat{A}_t = \int_0^t I_C(\hat{X}_s) ds.$$

In the following, unless otherwise stated, \hat{A} always represents this CAF and we shall drop the suffix A in $g_A^{1,q}$. Further, we shall denote $g^q(x, y)$ for $g_A^{0,q}(x, y)$, which is Kunita-Watanabe's potential kernel function. Note that $g_B^{0,q}(x, y) = g^q(x, y)$ for all $B \in \Phi^+$. Form the resolvent equation (1.6), for any $q > 0$,

$$(3.2) \quad \begin{aligned} g^{1,q}(x, y) &= g^q(x, y) - K_A^{1,q} g^q(x, y) \\ &= g^q(x, y) - g^q \hat{K}_A^{1,q}(x, y) \end{aligned}$$

on $\{(x, y); g^q(x, y) < \infty\}$. Hence for any $y \in E$, $K_A^{1,q} g^q(x, y) = g^q \hat{K}_A^{1,q}(x, y)$ a.a. $x(\mu)$. Since both sides of the equality are q -excessive, it holds for all $x, y \in E$ (cf. Gettoor [5; theorem 2.5]).

Lemma 3.1. For all $x \in E$ and $B \in \Phi^+$,

$$(3.3) \quad V_{A',B}^{1,0}(x, dy) = g^{1,0}(x, y) \nu_B(dy).$$

Proof. Set $A'_t = A_t + qt$. Replacing A' , $\{t\}$, B for A , B , B' in (2.3) we have

$$V_{A',B}^{1,0} f = V_{\{t\},B}^{q,0} f - K_{A'}^{1,q} V_{\{t\},B}^{q,0} f + q V_{A',\{t\}}^{1,0} V_{\{t\},B}^{q,0} f,$$

for sufficiently many functions f . Noting that $V_{\{t\},B}^{q,0} = K_B^{0,q}$, $K_{A'}^{1,q} = K_A^{1,q} + qG_A^{1,q}$ and $V_{A',\{t\}}^{1,0} = G_A^{1,q}$, it follows that

$$\begin{aligned} E^x \left[\int_0^\infty e^{-A_t - qt} f(X_t) dB_t \right] &= V_{A',B}^{1,0} f(x) \\ &= K_B^{0,q} f(x) - K_A^{1,q} K_B^{0,q} f(x) \\ &= \int \{g^q(x, y) - K_A^{1,q} g^q(x, y)\} f(y) \nu_B(dy) \\ &= \int \{g^q(x, y) - K_A^{1,q} g^q(x, y)\} f(y) \nu_B(dy) \\ &= \int \{g^{1,q}(x, y) f(y)\} \nu_B(dy) \end{aligned}$$

The last equality follows from (3.2) since ν_B has no mass on the polar set $\{y; g^q(x, y) = \infty\}$. Letting $q \rightarrow 0$ we have the result.

Dually, if \hat{B} is the dual CAF of B then

$$(3.4) \quad \hat{V}_{A,B}^{1,0}(dx, y) = g^{1,0}(x, y) \nu_B(dx) \quad \text{for all } y \in P_B.$$

Hence we have

Corollary. For all $f, g \in b(\mathcal{E}^*)^+$,

$$(3.5) \quad \int f(x)V_{A,B}^{1,0}g(x)\nu_A(dx) = \int f\hat{U}_{A,B}^{1,0}(y)g(y)\nu_B(dy)$$

$$(3.6) \quad \int f\hat{V}_{A,B}^{1,0}(y)g(y)\nu_A(dy) = \int f(x)U_{A,B}^{1,0}g(x)\nu_B(dx).$$

Since

$$K_A(x, dy) = K_B^1(x, dy) - \nu_A(dy) + K_A K_A^1(x, dy),$$

it is easy to show that, for each x , $K_A(x, \cdot)$ is absolutely continuous relative to ν_A and its density is given by

$$g^{1,0}(x, y) - 1 + K_{AB}g^{1,0}(x, y)$$

up to a set of ν_A -measure 0.

However, in order to solve the problem proposed in the introduction, we have to choose a more elaborated density $g(x, y)$. To do this, we need one more preliminary observation.

For all $x, y \in E$ and $n \geq 1$, set

$$(3.7) \quad f_n(x, y) = (\hat{K}_A^1)^{n-1}g^{1,0}(x, y) - 1 = g^{1,0}(K_A^1)^{n-1}(x, y) - 1,$$

then

$$(3.8) \quad \begin{aligned} f_n(x, y)\nu_A(dy) &= (K_A^1)^n(x, dy) - \nu_A(dy) \\ f_n(x, y)\nu_A(dx) &= (\hat{K}_A^1)^n(dx, y) - \nu_A(dx). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \int |f_n(x, y)|\nu_A(dy) = \sum_{n=1}^{\infty} \|(K_A^1)^n(x, \cdot) - \nu_A(\cdot)\| < \infty$$

for all $x \in E$ from (1.9) and (3.8), the series $\sum_{n=1}^{\infty} f_n(x, y)$ converges absolutely for $a.a.x(\nu_A)$. Similarly for all $y \in E$, $\sum_{n=1}^{\infty} f_n(x, y)$ converges absolutely for $a.a.x(\nu_A)$.

Also

$$\begin{aligned} \int_D \sum_{n=1}^{\infty} f_n(x, y)\nu_A(dy) &= \sum_{n=1}^{\infty} \int_D f_n(x, y)\nu_A(dy) \\ &= \sum_{n=1}^{\infty} \{(K_A^1)^n(x, D) - \nu_A(D)\} = K_A(x, D) \end{aligned}$$

for all $D \in \mathcal{E}$, that is, $\sum_{n=1}^{\infty} f_n(x, \cdot)$ is a density of $K_A(x, \cdot)$ relative to ν_A . Dually, $\sum_{n=1}^{\infty} f_n(\cdot, y)$ is a density of

$$(3.9) \quad \hat{K}_A(\cdot, y) = \sum_{n=1}^{\infty} \{(\hat{K}_A^1)^n(\cdot, y) - \nu_A(\cdot)\}$$

relative to ν_A . Here the proof of the strong convergence of (3.9) is similar to (1.9).

Lemma 3.2. *There exists a Borel subset Γ of $E \times E$ satisfying the following conditions.*

- (i) *Set $\Gamma_y = \{x; (x, y) \in \Gamma\}$ and $\hat{\Gamma}_x = \{y; (x, y) \in \Gamma\}$, then Γ_y^c and $\hat{\Gamma}_x^c$ are polar for all $x, y \in E$.*
- (ii) *For all $(x, y) \in \Gamma$, $\sum_{n=1}^{\infty} f_n(x, y)$ converges absolutely, $|K_A|g^{1,0}(x, y) < \infty$ and $g^{1,0}|\hat{K}_A|(x, y) < \infty$, where $|K_A|(x, \cdot)$ is the total variation measure of $K_A(x, \cdot)$.*
- (iii) *For all $(x, y) \in \Gamma$,*

$$(3.10) \quad \begin{aligned} \sum_{n=1}^{\infty} f_n(x, y) &= g^{1,0}(x, y) - 1 + K_A g^{1,0}(x, y) \\ &= g^{1,0}(x, y) - 1 + g^{1,0} \hat{K}_A(x, y). \end{aligned}$$

We define the kernel function $g(x, y)$ by

$$(3.11) \quad \begin{aligned} g(x, y) &= (3.10) && \text{if } (x, y) \in \Gamma \\ &= \infty && \text{if } (x, y) \notin \Gamma. \end{aligned}$$

By the lemma, it is easy to see that the function $g(\cdot, y)$ [resp. $g(x, \cdot)$] is finely [resp. cofinely] continuous on the fine [resp. cofine] open set Γ , [resp. $\hat{\Gamma}_x$] for all y [resp. all x] $\in E$.

Proof. Noting that,

$$\begin{aligned} |f_{n+1}(x, y)| &= |K_A^1 \{(K_A^1)^{n-1} g^{1,0} - 1\}(x, y)| \\ &= \left| \int K_A^1(x, dz) \left\{ \int (K_A^1)^{n-2}(z, du) g^{1,0}(u, v) g^{1,0}(v, y) \nu_A(dv) \right. \right. \\ &\quad \left. \left. - \int g^{1,0}(v, y) \nu_A(dv) \right\} \right| \\ &= \left| \int K_A^1(x, dz) \int \{(K_A^1)^{n-2} g^{1,0}(z, v) - 1\} g^{1,0}(v, y) \nu_A(dv) \right| \\ &= |K_A^1 f_{n-1} \hat{K}_A^1(x, y)| \leq K_A^1 |f_{n-1}| \hat{K}_A^1(x, y) \end{aligned}$$

for $n \geq 2$ and

$$\begin{aligned} &\int |(K_A^1)^n(x, dz) - \nu_A(dz)| g^{1,0}(z, y) \\ &= \int |(K_A^1)^{n-1} g^{1,0}(x, z) - 1| g^{1,0}(z, y) \nu_A(dz) \\ &= \int |K_A^1 \{(K_A^1)^{n-2} g^{1,0} - 1\}(x, z)| g^{1,0}(z, y) \nu_A(dz) \\ &\leq K_A^1 |f_{n-1}| \hat{K}_A^1(x, y) \end{aligned}$$

for $n \geq 2$, let us define the set Γ by

$$\Gamma = \{(x, y); |f_1|(x, y) + |f_2|(x, y) + \sum_{n=1}^{\infty} K_A^1 |f_n| \hat{K}_A^1(x, y) < \infty\} .$$

Then the proofs of (ii) and (iii) are obvious. For the proof of (i) set

$$\xi_x(dy) = \delta_x(dy) + K_A^1(x, dy) + \sum_{n=1}^{\infty} (K_A^1 |f_n|)(x, y) \nu_A(dy) .$$

Then

$$\xi_x(E) = 2 + \sum_{n=1}^{\infty} \int K_A^1(x, dy) |(K_A^1)^n(y, \cdot) - \nu_A(\cdot)| < \infty .$$

Moreover, it is easy to see that,

$$\Gamma = \{(x, y); \int \xi_x(dz) g^{1,0}(z, y) < \infty\} .$$

Hence $\hat{\Gamma}_x^c$ is polar if and only if $\int \xi_x(dz) g^{1,0}(z, y) < \infty$ except on a polar set. Since $g^{1,0}$ is the potential kernel function of the e^{-A_t} -subprocess (which is a transient Hunt process on E (13; III.3.16)) of X , the potential $\int \xi_x(dz) g^{1,0}(z, y)$ of the bounded measure ξ_x is finite except on a polar set if it is finite for $a.a.y(\mu)$ ([3; VI.2.3]). Since, for all $f \in b\mathcal{C}_c^+$,

$$\begin{aligned} & \iint \xi_x(dz) g^{1,0}(z, y) f(y) \mu(dy) \\ &= \int \xi_x(dz) G_A^{1,0} f(z) \leq \|G_A^{1,0} f\| \xi_x(E) < \infty , \end{aligned}$$

by lemma 1.3, $\int \xi_x(dz) g^{1,0}(z, y) < \infty a.a.y(\mu)$. Therefore $\hat{\Gamma}_x^c$ is polar. Similarly Γ_y^c is polar.

Suppose we are given a CAF $B \in \Phi^+$ and let \hat{B} be its dual CAF. Just as (2.4), define a kernel \hat{K}_B by

$$(3.12) \quad \hat{K}_B(dx, y) = \hat{V}_{A,B}^{1,0} \hat{K}_A(dx, y) + \hat{V}_{A,B}^{1,0}(dx, y) - \nu_B(dx) ,$$

for $y \in P_B$, where \hat{K}_A is the kernel defined by (3.9). In the case $B_t = t$ denote \hat{G}_B by \hat{G} . For these kernels, the dual results of section 2 are valid.

Theorem 3.3. For all $x \in E, y \in E$ and $z \in E - P_B$,

$$(3.13) \quad \begin{aligned} K_A(x, dy) &= g(x, y) \nu_A(dy), \quad \hat{K}_A(dx, y) = g(x, y) \nu_A(dx) \\ K_B(x, dy) &= g(x, y) \nu_B(dy) \quad \text{and} \quad \hat{K}_B(dx, z) = g(x, z) \nu_B(dx) . \end{aligned}$$

Proof. The first two equalities have been already proved. For the proof of the third equality, take a function $f \in b\mathcal{C}^+$ such that $V_{A,B}^{1,0} f$ is bounded. Then, since ν_B charges no polar set, it follows from lemma 3.1 and 3.2 that

$$\iint |g(x, y)| f(y) \nu_B(dy) \leq V_{A,B}^{1,0} f(x) + \langle \nu_B, f \rangle + |K_A| V_{A,B}^{1,0} f(x) < \infty .$$

Hence

$$\begin{aligned} & \int g(x, y) f(y) \nu_B(dy) \\ &= V_{A,B}^{1,0} f(x) - \langle \nu_B, f \rangle + K_A V_{A,B}^{1,0} f(x) = K_B f(x) . \end{aligned}$$

The last equality follows similarly.

Corollary. For all $x \in E$ and $y \in E$,

$$(3.14) \quad G(x, dy) = g(x, y) \mu(dy) \quad \text{and} \quad \hat{G}(dx, y) = g(x, y) \mu(dx) .$$

For a measure ξ on E , let us denote

$$(3.15) \quad G^{1,0} \xi(x) = \int g^{1,0}(x, y) \xi(dy) ,$$

$$(3.16) \quad G \xi(x) = \int g(x, y) \xi(dy) ,$$

if they are well defined.

Let X_A and \hat{X}_A be the subprocesses of X and \hat{X} by the multiplicative functionals $M_t = e^{-A_t}$ and $\hat{M}_t = e^{-\hat{A}_t}$, respectively. Then a set is polar if and only if it is polar relative to X_A or \hat{X}_A . Moreover, as we have seen at lemma 1.4, the resolvents $(G_A^{1,p})_{p>0}$ and $(\hat{G}_A^{1,p})_{p>0}$ of the processes X_A and \hat{X}_A are strong Feller, so that, it is well known that a compact set F is non-polar if and only if $G^{1,0} \xi$ is locally bounded for some non-zero finite measure ξ on F . Also, it is well known that if $G^{1,0} \xi$ is locally bounded then ξ charges no polar set (see [3; p. 285]). Hence we have the following theorem.

Theorem 3.4. *If F is a compact subset of E , then F is non-polar if and only if there exists a non-zero finite measure ξ on F such that $\int |g(x, y)| \xi(dy)$ is locally bounded.*

Proof. It is enough to prove that $G^{1,0} \xi$ is locally bounded if and only if $\int |g(x, y)| \xi(dy)$ is locally bounded.

If $G^{1,0} \xi$ is locally bounded for some non-zero finite measure ξ then ξ charges no polar set and hence, in particular, $\xi(\hat{\Gamma}_x^c) = 0$ for all $x \in E$. So, it follows that,

$$\int |g(x, y)| \xi(dy) \leq G^{1,0} \xi(x) + \xi(E) + |K_A| G^{1,0} \xi(x) .$$

In the right side of the inequality, since $G^{1,0} \xi$ is bounded on the compact set C , the last two terms are bounded. Therefore, $\int |g(x, y)| \xi(dy)$ is locally bounded.

Conversely, if $\int |g(x, y)| \xi(dy)$ is locally bounded then $\xi(\hat{\Gamma}_x^c) = 0$ from the definition of $g(x, y)$. Therefore, for any $x \in E$,

$$g(x, y) = g^{1,0}(x, y) - 1 + \int g^{1,0}(x, z) g(z, y) \nu_A(dz)$$

a.a. $y(\xi)$. Thus

$$\begin{aligned} G^{1,0}\xi(x) &\leq \int |g(x, y)| \xi(dy) + \xi(E) \\ &\quad + \int g^{1,0}(x, z) \left\{ \int |g(z, y)| \xi(dy) \right\} \nu_A(dz). \end{aligned}$$

Since $G^{1,0}\nu_A(x) = K_A^1 1(x) = 1$,

$$\begin{aligned} &\int g^{1,0}(x, z) \left\{ \int |g(z, y)| \xi(dy) \right\} \nu_A(dz) \\ &\leq \sup_{z \in \sigma} \int |g(z, y)| \xi(dy). \end{aligned}$$

Therefore the theorem is proved.

4. Potential kernel functions

By the corollary of theorem 3.3, we shall say that $g(x, y)$ is the *potential kernel function associated with* (G, \hat{G}) . Moreover the kernel function $g(x, y)$ satisfies several regularity conditions (corollaries 2 and 3 of theorem 2.3, lemma 3.2).

We now extend the notion of potential kernel functions.

DEFINITION. An $\mathcal{E}^* \times \mathcal{E}^*$ -measurable kernel function $h(x, y)$ is said to be a *potential kernel function* if the following conditions are satisfied.

- (i) Set $H(x, dy) = h(x, y) \mu(dy)$ and $\hat{H}(dx, y) = h(x, y) \mu(dx)$. Then H and \hat{H} are the potential kernels of X and \hat{X} such that Hf and $f\hat{H}$ are well defined and locally bounded for all $f \in b\mathcal{E}_c^*$. Moreover, the functions $H(\cdot, F)$ and $\hat{H}(F, \cdot)$ are finely and cofinely continuous for any compact set F , respectively.
- (ii) The sections $(\Gamma_h)_y^c$ and $(\hat{\Gamma}_h)_x^c$ (see §3) of the set $\Gamma_h^c = \{(x, y); |h(x, y)| = \infty\}$ are polar sets and the functions $h(\cdot, y)$ and $h(x, \cdot)$ are finely and cofinely continuous on the fine and cofine open sets $(\Gamma_h)_y$ and $(\hat{\Gamma}_h)_x$ for all $x, y \in E$, respectively.

We shall show how any potential kernel function $h(x, y)$ is related to $g(x, y)$. Recall that $\Gamma^c = \{(x, y); |g(x, y)| = \infty\}$.

Theorem 4.1. *If $h(x, y)$ is a potential kernel function of X , then*

$$(4.1) \quad g(x, y) = h(x, y) - H(x, C) - \hat{H}(C, y) + H(C, C),$$

for all $(x, y) \in \Gamma \cap \Gamma_h$, where $H(C, C) = \int_C H(x, C) \mu(dx)$.

Proof. If $f \in N$ then, by (i), $Gf - Hf$ is bounded and satisfies $(I - pG^p)(Gf - Hf) = 0$, so that, $Gf - Hf$ equals a constant on E . Particularly, set $f = I_F - \mu(F)I_C \in N$ for a relatively compact set $F \in \mathcal{E}^*$ then, since $G(\cdot, C) = 0$,

$$(4.2) \quad G(x, F) - H(x, F) + H(x, C) \mu(F) = a$$

for some constant a . Integrating both sides of (4.2) by $\nu_A = \mu|_C$ and noting that $\nu_A G = 0$, we have

$$-H(C, F) + H(C, C) \mu(F) = a.$$

Thus,

$$G(x, F) = H(x, F) - H(x, C) \mu(F) - H(C, F) + H(C, C) \mu(F).$$

Therefore, for all $x \in E$, (4.1) holds for a.a. $y(\mu)$. Since μ is equivalent to $\hat{G}^p(\cdot, y)$ for all $p > 0$ and $y \in E([1])$, μ charges all cofine open sets. Hence, for all $x \in E$, (4.1) holds for cofinely dense $y \in E$. Since both sides of (4.1) are cofinely continuous relative to y on the cofine open set $\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$, (4.1) holds for all $y \in \hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x$.

If $B \in \Phi^+$ then, since the associated measure ν_B of B has no mass on any semipolar set, we have

Corollary 1. *If $h(x, y)$ is a potential kernel function of X , then the kernels $H_B(x, dy) = h(x, y) \nu_B(dy)$ and $\hat{H}_B(dx, y) = h(x, y) \nu_B(dx)$ are potential kernels of (K_B^h) and (\hat{K}_B^h) , respectively.*

Corollary 2. *Let $h(x, y)$ be a potential kernel function such that $\Gamma_h \subset \Gamma$, then a compact subset F of E is non-polar if and only if $\int |h(x, y)| \xi(dy)$ is locally bounded for some non-zero finite measure ξ on F . In particular, if X and \hat{X} are equivalent, then F is non-polar iff $\int |h(x, y)| \xi(dy)$ is bounded on F for some ξ as above.*

Proof. It is enough to show that $\int |g(x, y)| \xi(dy)$ is locally bounded if and only if $\int |h(x, y)| \xi(dy)$ is locally bounded.

If $\int |g(x, y)| \xi(dy)$ is locally bounded, then ξ charges no polar set by theorem 3.4 and, in particular, $\xi(\{\hat{\Gamma}_x \cap (\hat{\Gamma}_h)_x\}^c) = 0$. Hence it follows from (4.1) that $\int |h(x, y)| \xi(dy)$ is locally bounded.

Conversely, if $\int |h(x, y)| \xi(dy)$ is locally bounded, then ξ has no mass on $(\hat{\Gamma}_h)_x^c$ for all $x \in E$, so that (4.1) holds a.a. $y(\xi)$ for all $x \in E$. Hence $\int |g(x, y)| \xi(dy)$ is locally bounded. If X and \hat{X} are equivalent, then all semipolar sets are polar.

Hence $\sup_{x \in E} G^{1,0}\xi(x) = \sup_{x \in F} G^{1,0}\xi(x)$ ([3]), so that the last part of the corollary is obvious from the proof of theorem 3.4.

REMARK. If $h(x, y)$ is a potential kernel function of X , then the kernel function $h'(x, y)$ defined by

$$(4.3) \quad \begin{aligned} h'(x, y) &= h(x, y) && \text{if } (x, y) \in \Gamma \cap \Gamma_h \\ &= \infty && \text{if } (x, y) \notin \Gamma \cap \Gamma_h \end{aligned}$$

is a potential kernel function of X . For this kernel function, the hypothesis $\Gamma_h' \subset \Gamma$ of the corollary 2 holds obviously.

REMARK. So far we have fixed a compact set C and assumed that $\mu(C) = 1$. If we delete such normalization condition, the only minor change is necessary; ν_A equals $[\mu(C)]^{-1}\mu|_C$ for $\mu|_C$ and $\nu_{(t)}$ equals $[\mu(C)]^{-1}\mu$. It then follows that $G(x, dy) = g(x, y)[\mu(C)]^{-1}\mu(dy)$.

For two compact sets C_1 and C_2 , let G_1, G_2 and g_1, g_2 be their associated potential kernels and kernel functions. Let μ be an arbitrary invariant measure (not necessarily normalized either on C_1 or C_2). By an argument similar to the proof of theorem 4.1, we have $G_1(x, F) - \frac{G_1(C_2, F)}{\mu(C_2)} = G_2(x, F) - \frac{\mu(F)}{\mu(C_1)}G_2(x, C_1)$. By the preceding remark, we obtain the following relation:

$$(4.4) \quad \frac{g_1(x, y)}{\mu(C_1)} - \frac{1}{\mu(C_2)}G_1(C_2, y) = \frac{g_2(x, y)}{\mu(C_2)} - \frac{1}{\mu(C_1)}G_2(x, C_1)$$

on $\Gamma g_1 \cap \Gamma g_2$.

5. Equilibrium measure

Let \mathbf{F} be the family of all non-empty relatively compact sets F which is the fine support of some CAF $B \in \Phi^+$. In this section we shall fix a set $F \in \mathbf{F}$ and the corresponding CAF B . Let $\{\hat{F}_n\}_{n \geq 1}$ be an increasing sequence satisfying that $\cup \hat{F}_n = E$ and $\hat{V}_{A, B}^{1,0}(\hat{F}_n, \cdot)$ are bounded for all n . The existence of such a sequence is the same as in lemma 1.2. Define the continuous additive functionals $B^n \in \Phi$ by $B_t^n = \int_0^t I_{F_n \cap \hat{F}_n}(X_s) dB_s$. Then the fine support of each B^n is relatively compact. The kernels defined by A and B^n are denoted by $U_n^{p,q}$ and $V_n^{p,q}$. By the definition of $B^n, V_n^{1,0}|f|$ and $|f| \hat{V}_n^{1,0}$ are bounded for all $f \in b\mathcal{E}^*$. Let ν_B be the measure associated with B as before and set $\nu_n(\cdot) = \nu_B(\cdot \cap F_n \cap \hat{F}_n)$. The fine support of ν_B is equal to F (see [14; remark II.2]) and ν_n is the measure associated with B^n . Write K_n for K_{B^n} . It follows that $K_n f$ is well defined and bounded for all $f \in b\mathcal{E}^*$.

Lemma 5.1. *If $B^n \neq 0$ then, for all $p > 0$,*

$$(5.1) \quad pK_n(U_n^{0,p}1) + U_n^{0,p}1 \equiv R_n(p)$$

is a finite constant on E .

Proof. If $B^n \neq 0$ then $B^n \in \Phi^+$, so that, by theorem 2.3, formulas (1.6) and (2.2), and lemma 1.5,

$$\begin{aligned} pK_n(U_n^{0,p}1)(x) &= p \lim_{q \rightarrow 0} \left\{ V_n^{q,0}(x, \cdot) - \frac{1}{q} \nu_n \right\} U_n^{0,p}1 \\ &= \lim_{q \rightarrow 0} \left\{ pV_n^{q,0}U_n^{0,p}1(x) - \frac{1}{q} \right\} \\ &= \lim_{q \rightarrow 0} \left\{ qK_n^q U_n^{0,p}1(x) - U_n^{0,p}1(x) \right\} \\ &= \nu_A U_n^{0,p}1 - U_n^{0,p}1(x). \end{aligned}$$

Therefore,

$$(5.2) \quad pK_n(U_n^{0,p}1)(x) + U_n^{0,p}1(x) = \nu_A U_n^{0,p}1 = R_n(p)$$

is a constant.

Let T_F be the hitting time of the set F , $\tau = \inf \{t; B_t > 0\}$ and $\tau^n = \inf \{t; B_t^n > 0\}$, where $\inf \phi = \infty$. Then, $T_F = \tau$ a.s. (see [3; proposition V.3.5]) and $\tau^n \downarrow \tau$ a.s. as $n \uparrow \infty$. Since $R_n(p) = E^{\nu_A}[\int_0^{\tau^n} e^{-pB_t^n} I_C(X_t) dt]$ decreases when n or p increases, the limit

$$(5.3) \quad R(F) = \lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} R_n(p) = \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} R_n(p)$$

exists and it is finite since $B \neq 0$ (see the proof of lemma 5.2 below).

DEFINITION. We shall call the constant $R(F)$ as Robin's constant of F (relative to the potential kernel function $g(x, y)$).

$$\textbf{Lemma 5.2.} \quad R(F) = E^{\nu_A}[\int_0^{T_F} I_C(X_t) dt].$$

Proof. Since $B \neq 0$, $U_n^{0,1}$ is bounded for all large n . Hence, for all $p \geq 1$ and large n ,

$$R_n(p) \leq R_n(1) = \nu_A U_n^{0,1} < \infty,$$

Therefore, by the Lebesgue theorem,

$$\begin{aligned} \lim_{p \rightarrow \infty} R_n(p) &= \lim_{p \rightarrow \infty} E^{\nu_A}[\int_0^{\tau^n} e^{-pB_t^n} I_C(X_t) dt] \\ &= E^{\nu_A}[\int_0^{\tau^n} (\lim_{p \rightarrow \infty} e^{-pB_t^n}) I_C(X_t) dt] \\ &= E^{\nu_A}[\int_0^{\tau^n} I_C(X_t) dt]. \end{aligned}$$

Letting $n \rightarrow \infty$ we have the result.

REMARK. From the lemma 5.2, Robin's constant $R(F)$ of F does not depend on the choice of B .

Lemma 5.3. *If $F \in \mathcal{F}$ then there exists a probability measure ξ_F on \bar{F} such that*

$$(5.4) \quad \lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} V_n^{1,0}(pU_n^{0,p}1)(x) = G^{1,0}\xi_F(x)$$

for a.a. $x(\mu)$. Moreover, $V_n^{1,0}(pU_n^{0,p}1)(x)$ are uniformly bounded for all $x \in E$, $p \geq 1$ and large n .

Proof. From (1.6), for all $p > 0$ and $n \geq 1$ such that $B^n \neq \emptyset$,

$$V_n^{1,0}(pU_n^{0,p}1)(x) = 1 - U_n^{0,p}1(x) + K_A^1 U_n^{0,p}1(x).$$

As in the proof of lemma 5.2, $U_n^{0,p}1(x)$ are uniformly bounded for all $x \in E$, $p \geq 1$ and $n \geq 1$ such that $B^n \neq \emptyset$, and

$$\lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} U_n^{0,p}1(x) = U^{0,\infty}1(x) \equiv E^x \left[\int_0^\tau I_C(X_t) dt \right].$$

Hence,

$$\lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} V_n^{1,0}(pU_n^{0,p}1)(x) = 1 - U^{0,\infty}1(x) + K_A^1 U^{0,\infty}1(x),$$

boundedly. Define a measure $\xi_{p,n}$ on the compact set \bar{F} by

$$\xi_{p,n}(dy) = pU_n^{0,p}1(y) \nu_n(dy) \text{ for } p > 0 \text{ and } n \geq 1 \text{ such that } B^n \neq \emptyset, \text{ then}$$

$$\xi_{p,n}(E) = \langle \nu_n, pU_n^{0,p}1 \rangle = \langle \nu_A, 1 \rangle = 1.$$

Thus there exists a sequence $p_k \rightarrow \infty$ such that $\{\xi_{p_k, n}\}_{k \geq 1}$ converges weakly to a probability measure ξ_n on \bar{F} as $k \rightarrow \infty$, for all n . Therefore, we can choose a subsequence $\{\xi_{n_m}\}$ of $\{\xi_n\}$ which converges weakly to a probability measure ξ_F on \bar{F} as $m \rightarrow \infty$. Taking an arbitrary function $f \in b\mathcal{C}_c^*$, we have

$$\begin{aligned} & \int f(x) \{1 - U^{0,\infty}1(x) + K_A^1 U^{0,\infty}1(x)\} \mu(dx) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int f(x) V_{n_m}^{1,0}(p_k U_{n_m}^{0,p_k}1)(x) \mu(dx) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int f(x) G^{1,0}\xi_{p_k, n_m}(x) \mu(dx) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int f \hat{G}_A^{1,0}(y) \xi_{p_k, n_m}(dy) \\ &= \int f \hat{G}_A^{1,0}(y) \xi_F(dy) = \int f(x) G^{1,0}\xi_F(x) \mu(dx), \end{aligned}$$

where we used the boundedness and continuity of $f\hat{G}_A^{1,0}$, which follows from the dual facts of lemmas 1.3 and 1.4. Therefore,

$$(5.5) \quad 1 - U^{0,\infty}1(x) + K_A^1 U^{0,\infty}1(x) = G^{1,0}\xi_F(x), \quad \text{for } a.a.x(\mu).$$

Let \hat{B} be the dual CAF of B as in section 3 and let \hat{F} be the cofine support of B . As before, \hat{F} is the cofine support of ν_B . Set $\hat{\tau} = \inf \{t; \hat{B}_t > 0\}$, then $\hat{\tau} = \hat{T}_F$ a.s. \hat{P}^x for all $x \in E - P_B$, where \hat{T}_F is the hitting time of \hat{F} relative to \hat{X} .

Lemma 5.4. For all $f \in b\mathcal{E}^*$,

$$(5.6) \quad \int f(y) \xi_F(dy) = \hat{E}^{\nu_A}[f(\hat{X}_{\hat{\tau}})].$$

In particular, ξ_F is a probability measure on \hat{F} which attains no mass on any polar set.

Proof. It is enough to show the equality (5.6) for $f \in \mathcal{C}_c$. If $f \in \mathcal{C}_c$, then by the corollary of lemma 3.1,

$$\begin{aligned} \int f(y) \xi_F(dy) &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int f(y) p_k U_{n_m}^{0,p_k} 1(y) \nu_{n_m}(dy) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int p_k f \hat{V}_{n_m}^{0,p_k}(y) \nu_A(dy) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \hat{E}^{\nu_A} \left[p_k \int_0^\infty e^{-p_k(\hat{B}_{n_m}^n)_t} f(\hat{X}_t) d(\hat{B}_{n_m}^n)_t \right] \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \hat{E}^{\nu_A} \left[\int_0^\infty e^{-u} f(\hat{X}_{\hat{\tau}_{n_m}(u/p_k)}) du \right], \end{aligned}$$

where $(\hat{B}^n)_t = \int_0^t I_{F_n \cap F_n^c}(\hat{X}_u) d\hat{B}_u$ is the dual CAF of B^n and $\hat{\tau}_n(s) = \inf \{u; (\hat{B}^n)_u > s\}$. Since, for all $n \geq 1$, $\hat{\tau}_n(s) \rightarrow \hat{\tau}_n \equiv \hat{\tau}_n(0)$ a.s. as $s \rightarrow 0$,

$$\begin{aligned} \int f(y) \xi_F(dy) &= \lim_{m \rightarrow \infty} \hat{E}^{\nu_A} \left[\int_0^\infty e^{-u} f(\hat{X}_{\hat{\tau}_{n_m}}) du \right] \\ &= \lim_{m \rightarrow \infty} \hat{E}^{\nu_A} [f(\hat{X}_{\hat{\tau}_{n_m}})]. \end{aligned}$$

Also, since $\hat{\tau}_{n_m} \rightarrow \hat{\tau}$ a.s. when $m \rightarrow \infty$, the lemma follows.

Theorem 5.5 (Equilibrium principle). Let $F \in \mathcal{E}^*$ be a relatively compact subset of E and suppose that there exists a CAF $B \in \Phi^+$ with fine support F . Then there exists a unique probability measure ξ_F on \hat{F} such that

$$(5.7) \quad G\xi_F(x) = \text{a constant on } F.$$

Here, \hat{F} is the cofine support of the dual CAF \hat{B} of B and the constant is equal to Robin's constant $R(F)$ of F . The measure ξ_F is given by (5.6) and called the equilibrium measure of F .

Proof. Let us show that the measure ξ_F in lemma 5.3 satisfies

$$(5.8) \quad G\xi_F(x) + U^{0,\infty}1(x) = R(F) \quad \text{everywhere on } E.$$

This proves (5.7) since $U^{0,\infty}1(x) = E^x[\int_0^T I_C(X_t) dt] = 0$ on F . From (2.2), (2.4) and lemma 5.1,

$$\begin{aligned} R_n(p) &= K_n(pU_n^{0,p}1)(x) + U_n^{0,p}1(x) \\ &= K_A V_n^{1,0}(pU_n^{1,p}1)(x) + V_n^{1,0}(pU_n^{0,p}1)(x) \\ &\quad - p\nu_n U_n^{0,p}1 + U_n^{0,p}1(x) \\ &= K_A V_n^{1,0}(pU_n^{0,p}1)(x) + V_n^{1,0}(pU_n^{0,p}1)(x) \\ &\quad - 1 + U_n^{0,p}1(x), \end{aligned}$$

since $p\nu_n U_n^{0,p}1 = \nu_A(C) = 1$. Let $p \rightarrow \infty$ and $n \rightarrow \infty$, then, as we have seen in lemmas 5.2 and 5.3, $R_n(p) \rightarrow R(F)$ and $V_n^{1,0}(pU_n^{0,p}1)(x) \rightarrow G^{1,0}\xi_F(x)$ a.a. $x(\mu)$, boundedly. Since $K_A(x, \cdot)$ is a bounded signed measure and which is absolutely continuous relative to μ , we have, for a.a. $x(\mu)$,

$$\begin{aligned} (5.9) \quad R(F) &= K_A G^{1,0}\xi_F(x) + G^{1,0}\xi_F(x) - 1 + U^{0,\infty}1(x) \\ &= \int \{K_A g^{1,0}(x, y) + g^{1,0}(x, y) - 1\} \xi_F(dy) + U^{0,\infty}1(x) \\ &= G\xi_F(x) + U^{0,\infty}1(x), \end{aligned}$$

from lemmas 3.2 and 5.4. Denote

$$\xi(dy) = \xi_F(dy) + \int \sum_{n=1}^{\infty} \{(K_A^1)^n - \nu_A\} g^{1,0}(dy, z) \xi_F(dz),$$

then ξ is a bounded signed measure on F and

$$G\xi_F(x) = G^{1,0}\xi(x) - 1.$$

Since $G^{1,0}\xi_F(x)$ is bounded, $G^{1,0}\xi(x)$ is the difference of two bounded excessive functions relative to $(G_A^{1,p})_{p>0}$. Therefore,

$$\lim_{p \rightarrow \infty} pG_A^{1,p}G^{1,0}\xi(x) = G^{1,0}\xi(x) \quad \text{for all } x \in E.$$

Moreover,

$$pG_A^{1,p}1(x) = E^x[\int_0^{\infty} \exp(-\int_0^{t/p} I_C(X_s) ds - t) dt] \rightarrow 1,$$

as $p \rightarrow \infty$ for all $x \in E$ and $G_A^{1,p}(x, \cdot)$ is absolutely continuous relative to μ , for all $x \in E$ and $p > 0$. Thus, operating $pG_A^{1,p}$ to both sides of (5.9) and letting $p \rightarrow \infty$, we have

$$R(F) = G\xi_F(x) + \lim_{p \rightarrow \infty} pG_A^{1,p}U^{0,\infty}1(x) \quad \text{for all } x \in E.$$

Therefore, it is enough to show that

$$(5.10) \quad \lim_p pG_A^{1,p} U^{0,\infty} 1(x) = U^{0,\infty} 1(x) \quad \text{for all } x \in E.$$

Let $p > 1$ then,

$$\begin{aligned} pG_A^{1,p} U^{0,\infty} 1(x) &= E^x \left[\int_0^\infty p \exp \left\{ - \int_0^t I_C(X_s) ds - pt \right\} E^{X_t} \left[\int_0^\tau I_C(X_u) du \right] dt \right] \\ &\leq E^x \left[\int_0^\infty p e^{-pt} \left\{ \int_t^{t+\tau\theta_t} I_C(X_u) du \right\} dt \right] \\ &\leq E^x \left[\int_0^\infty e^{-t} \left\{ \int_{t/p}^{(t/p)+\tau\theta_t/p} I_C(X_u) du \right\} dt \right] \\ &\leq E^x \left[\int_0^\infty e^{-t} \left\{ \int_t^{t+\tau\theta_t} I_C(X_u) du \right\} dt \right] + 1 \\ &\leq \|U^{0,\infty} 1\| + 1. \end{aligned}$$

Thus, noting that $\lim_{t \rightarrow 0} (t + \tau\theta_t) = \tau$ (see [3; p. 214]), by the Lebesgue theorem,

$$\begin{aligned} \lim_{p \rightarrow \infty} pG_A^{1,p} U^{0,\infty} 1(x) &= \lim_{p \rightarrow \infty} E^x \left[\int_0^\infty \exp \left\{ - \int_0^{t/p} I_C(X_s) ds - t \right\} dt \int_{t/p}^{(t/p)+\tau\theta_t/p} I_C(X_u) du \right] \\ &= \int_0^\infty e^{-t} E^x \left[\int_0^\tau I_C(X_u) du \right] dt = U^{0,\infty} 1(x). \end{aligned}$$

Now, it remains only the proof of the uniqueness. Let ξ be a bounded signed measure on \hat{F} satisfying $\xi(E) = 0$ and $G\xi(x) = a$, for some constant a , on F . For the proof of uniqueness we claim that $\xi = 0$. Integrating both sides of $G\xi(x) = a$ ($x \in F$) by $f(x)\nu_n(dx)$, we have

$$(5.11) \quad \int f \hat{K}_n(y) \xi(dy) = a \langle \nu_n, f \rangle$$

for all $f \in b\mathcal{E}^*$ and $n \geq 1$, where $\hat{K}_n(dx, y) = g(x, y)\nu_n(dx)$ as before. Set $f = g(I - p\hat{K}_{B^n}^p)$ for a bounded continuous function g . It follows, from the dual result of (2.8), that

$$f \hat{K}_n(y) = g(I - p\hat{K}_{B^n}^p) \hat{K}_n(y) = g \hat{K}_{B^n}^p(y) - \langle g \hat{K}_{B^n}^p, \nu_A \rangle,$$

for all $y \in P_{B^n}$, $n \geq 1$ and $p > 0$. Substituting this function into (5.11), we have

$$\int g \hat{K}_{B^n}^p(y) \xi(dy) = 0 \quad \text{for all } n \geq 1 \text{ and } p > 0,$$

because $\xi(E) = 0$, $\langle g(I - p\hat{K}_{B^n}^p), \nu_n \rangle = 0$ and ξ vanishes outside of $\hat{F} \subset E - P_B \subset E - P_{B^n}$. Therefore, similarly to the proof of lemma 5.4, we have

$$\int \hat{E}^y[g(\hat{X}_t)] \xi(dy) = 0.$$

This implies that $\int g(y) \xi(dy) = 0$, since $\hat{P}^y[\hat{r} = 0] = 1$ for all $y \in \hat{P}$.

6. Symmetric case

In this section we shall assume, in addition, that $g^p(x, y) = g^p(y, x)$ for all $p > 0$ and $x, y \in E$, that is, X and \hat{X} are equivalent. In this case, as is well known (see [3; proposition VI. 4. 10]), any semipolar set is polar. Hence, for every compact set F , the set $F - F'$ is polar, where F' is the set of all regular points of F (see [3; II. 3.3]). Therefore F is a projective set (see [3; V. 4.5]). Hence, by considering the projection of CAF $\{t\}$, there exists a CAF B such that

$$(6.1) \quad E^x[e^{-T_F}] = E^x\left[\int_0^\infty e^{-t} dB_t\right]$$

and $\text{supp}(B) = F'$ ([3; V. 4.6 and 4.7]), where $\text{supp}(B)$ is the fine support of B . Obviously, F is a polar set if and only if the corresponding CAF B is zero. Let $T = \inf\{t; B_t = \infty\}$. We have

$$1 \geq E^x[e^{-T_F}] \geq E^x\left[\int_0^T e^{-t} dB_t\right] \geq E^x[e^{-T} B_T].$$

This implies that $T = \infty$ a.s. P^x for all $x \in E$, that is, $B \in \Phi$. Let \hat{B} be the dual CAF of B then, under our present hypothesis, the corresponding polar set P_B may be supposed empty (see the proof of [14; VII. 1]) and the cofine support \hat{F} of \hat{B} is equal to $F' = F'$, since the fine and cofine topologies coincide, where F' is the set of all coregular points of F . Therefore, by theorem 5.5 we have

Theorem 6.1. *If F is a non-polar compact subset of E , then there exists a unique probability measure ξ_F on F' such that*

$$(6.2) \quad G\xi_F(x) = R(F) \quad \text{on } F'.$$

Here, the measure ξ_F and the constant $R(F)$ are given by

$$(6.3) \quad \xi_F(dy) = \hat{P}^y_A[\hat{X}_{\hat{r}_F} \in dy] \quad \text{and}$$

$$(6.4) \quad R(F) = E^y_A\left[\int_0^{T_F} I_C(X_t) dt\right],$$

respectively. The measure ξ_F is called the equilibrium measure of F (relative to the potential kernel function $g(x, y)$).

Corollary. *Under the hypothesis of theorem 6.1 there exists a unique probability measure ξ_F on F such that $G\xi_F$ is bounded on F and satisfies (6.2).*

Proof. It is enough to prove the uniqueness. Suppose that a measure ξ on F satisfies the conditions of the corollary. Then since $G\xi$ is bounded on F , ξ charges no polar set (see the proofs of theorem 3.4 and corollary 2 of theorem 4.1). Hence ξ is a measure on F^r , so that the corollary follows from theorem 6.1.

REMARK. By the proof of the corollary, the result of the corollary may be replaced by the following result. "There exists a unique probability measure on F which attains no mass on any polar set and satisfies (6.2)".

By using the relation (4.1) of $g(x, y)$ and an arbitrary potential kernel function $h(x, y)$, we would like to investigate the equilibrium principle relative to $h(x, y)$. At present, however, we can get only a partial result on this problem; we have to impose very strong conditions on $h(x, y)$ and we do not know even if the logarithmic potential kernel function of planar Brownian motion satisfies these conditions. Our conditions are the following.

(H1) For every compact set D ,

$$(6.5) \quad \lim_{p \rightarrow 0} \sup_{x, y \in D} |g^p(x, y) - \phi(p) - h(x, y)| = 0$$

for some function ϕ and a potential kernel function h .

(H2) For all $p > 0$ and bounded continuous function f , $K_B^p f$ is continuous, where B is a CAF with fine support F^r , as before.

To find the equilibrium measure ξ relative to h , we shall attempt a formal calculation. Suppose that a probability measure ξ on F satisfies $H\xi(x) = \int h(x, y)\xi(dy) = a$ on F^r for some constant a . Then, from (4.1), for all $x \in F^r$

$$\begin{aligned} G\xi(x) &= H\xi(x) - H(x, C) - \int \hat{H}(C, y)\xi(dy) + H(C, C) \\ &= -H(x, C) + a - \int \hat{H}(C, y)\xi(dy) + H(C, C). \end{aligned}$$

Operating $I - pK_B^p$ and integrating by $fd\nu_B$, it follows that

$$(6.6) \quad \langle f, (I - pK_B^p)G\xi \rangle_{\nu_B} = -\langle f, (I - pK_B^p)H(\cdot, C) \rangle_{\nu_B}.$$

The left side of (6.6) becomes

$$\begin{aligned} \langle f, (I - pK_B^p)G\xi \rangle_{\nu_B} &= \langle f(I - p\hat{K}_B^p)\hat{K}_B, \xi \rangle \\ &= \langle f\hat{K}_B^p, \xi \rangle - \langle f\hat{K}_B^p, \nu_A \rangle \end{aligned}$$

from the dual formula of (2.8).

On the other hand, the right side of (6.6) becomes

$$\begin{aligned}
-\langle f, (I-pK_B^p)H(\cdot, C) \rangle_{\nu_B} &= -\lim_{q \rightarrow 0} \langle f, (I-pK_B^p)(G^q(\cdot, C) - \phi(q)) \rangle_{\nu_B} \\
&= -\lim_{\tau \rightarrow 0} \langle f, (I-pK_B^p)G^q(\cdot, C) \rangle_{\nu_B} = \lim_{\tau \rightarrow 0} \langle f, -G_B^{p,0}(\cdot, C) + qG_B^{p,0}G^q(\cdot, C) \rangle_{\nu_B} \\
&= -\langle f\hat{K}_B^p, \nu_A \rangle + \lim_{\tau \rightarrow 0} \langle f, qG_B^{p,0}G^q(\cdot, C) \rangle_{\nu_B}.
\end{aligned}$$

Hence (6.6) becomes

$$\langle f\hat{K}_B^p, \xi \rangle = \lim_{\tau \rightarrow 0} \langle f, qG_B^{p,0}G^q(\cdot, C) \rangle_{\nu_B}.$$

Multiplying p and letting $p \rightarrow \infty$ we have

$$(6.7) \quad \langle f, \xi \rangle = \lim_{p \rightarrow \infty} \lim_{q \rightarrow 0} pq \langle f, G_B^{p,0}G^q(\cdot, C) \rangle_{\nu_B}.$$

Theorem 6.2. *Let F be a non-polar compact subset of E . Under the hypothesis (H1) and (H2), there exists a unique probability measure ξ on F^r such that*

$$(6.8) \quad H\xi = \text{a constant on } F^r.$$

Proof. Since

$$\langle 1, pqG_B^{p,0}G^q(\cdot, C) \rangle_{\nu_B} = pq \langle 1\hat{K}_B^p\hat{G}^q, \nu_A \rangle = 1,$$

the measure $pqG_B^{p,0}G^q(x, C)\nu_B(dx)$ is a probability measure on F for any $p, q > 0$. Hence, for all $p > 0$, we can choose a sequence $q_n \rightarrow 0$ and a probability measure ξ_p on F such that $pq_n G_B^{p,0}G^{q_n}(x, C)\nu_B(dx) \rightarrow \xi_p(dx)$, weakly. Similarly, there exists a sequence $p_m \rightarrow \infty$ and a probability measure ξ on F such that $\xi_{p_m} \rightarrow \xi$, weakly. From the hypothesis (H2), for all bounded continuous function f ,

$$\begin{aligned}
\langle p_k f\hat{K}_B^k, \xi \rangle &= \lim_m \lim_n \langle p_k f\hat{K}_B^k, p_m q_n G_B^{p_m,0}G^{q_n}(\cdot, C) \rangle_{\nu_B} \\
&= \lim_m \lim_n \langle p_k p_m q_n f\hat{K}_B^k \hat{K}_B^{p_m} \hat{G}^{q_n}, \nu_A \rangle \\
&= \lim_m \lim_n \frac{1}{p_m - p_k} \langle p_k p_m q_n f(\hat{K}_B^k - \hat{K}_B^{p_m}) \hat{G}^{q_n}, \nu_A \rangle \\
&= \lim_m \lim_n \frac{1}{p_m - p_k} p_k p_m q_n \langle f\hat{K}_B^k \hat{G}^{q_n}, \nu_A \rangle \\
&= \lim_n p_k q_n \langle f\hat{K}_B^k \hat{G}^{q_n}, \nu_A \rangle = \langle f, \xi_k \rangle.
\end{aligned}$$

Hence, letting $k \rightarrow \infty$, it follows that

$$\lim_k \langle p_k f\hat{K}_B^k, \xi \rangle = \langle f, \xi \rangle,$$

that is, $\xi = \hat{E}^\xi[f(\hat{X}_\tau)] = E^\xi[f(X_{T_F})]$. So that ξ is a measure on F^r .

To prove (6.8), let f be a bounded continuous function with compact support. By restricting the CAT B as in section 5, we may suppose that $V_{A,B}^{1,0}1$ is bounded. Then, since $f\hat{G}$ is bounded and continuous from (2.7), we have

$$\begin{aligned}
\langle f, G\xi \rangle_\mu &= \langle f\hat{G}, \xi \rangle = \lim_m \lim_n \langle f\hat{G}, p_m q_n G_B^{\hat{p}m, 0} G^{q_n}(\cdot, C) \rangle_{\nu_B} \\
&= \lim_m \lim_n p_m \langle f\hat{G}, G_B^{\hat{p}m, 0}(\cdot, C) - G^{q_n}(\cdot, C) + p_m K_B^{\hat{p}m} G^{q_n}(\cdot, C) \rangle_{\nu_B} \\
&= \lim_m \lim_n p_m \langle f, K_B G_B^{\hat{p}m, 0}(\cdot, C) - K_B(I - p_m K_B^{\hat{p}m}) G^{q_n}(\cdot, C) \rangle_\mu \\
&= \lim_m \lim_n p_m \langle f, K_B G_B^{\hat{p}m, 0}(\cdot, C) - K_B^{\hat{p}m} G^{q_n}(\cdot, C) + \nu_A K_B^{\hat{p}m} G^{q_n}(C) \rangle_\mu
\end{aligned}$$

from (2.8). By the definition of K_B and H ,

$$\begin{aligned}
\lim_m p_m K_B G_B^{\hat{p}m, 0}(x, C) &= \lim_m \lim_{q \rightarrow 0} p_m \left\{ V_{A, B}^{q, 0} - \frac{\nu_B}{q} \right\} U_{A, B}^{0, \hat{p}m} 1(x) \\
&= \lim_m \lim_{q \rightarrow 0} \left\{ p_m V_{A, B}^{q, 0} U_{A, B}^{0, \hat{p}m} 1(x) - \frac{1}{q} \right\} \\
&= \lim_m \lim_{q \rightarrow 0} \left\{ K_A^q 1 - U_{A, B}^{0, \hat{p}m} 1 + q K_A^q U_{A, B}^{0, \hat{p}m} 1 - \frac{1}{q} \right\}(x) \\
&= \lim_m \{ -U_{A, B}^{0, \hat{p}m} 1 + \nu_A U_{A, B}^{0, \hat{p}m} 1 \}(x) \\
&= -E^x \left[\int_0^{T_F} I_C(X_s) ds \right] + E^{\nu_A} \left[\int_0^{T_F} I_C(X_s) ds \right], \\
\lim_m \lim_n p_m \{ K_B^{\hat{p}m} G^{q_n}(x, C) - \nu_A K_B^{\hat{p}m} G^{q_n}(C) \} \\
&= \lim_m \lim_n p_m [K_B^{\hat{p}m} \{ G^{q_n}(x, C) - \phi(q_n) \} - \nu_A K_B^{\hat{p}m} \{ G^{q_n}(\cdot, C) - \phi(q_n) \}] \\
&= \lim_m p_m \{ K_B^{\hat{p}m} H(x, C) - \nu_A K_B^{\hat{p}m} H(C) \} \\
&= E^x [H(X_{T_F}, C)] - E^{\nu_A} [H(X_{T_F}, C)].
\end{aligned}$$

Hence

$$\begin{aligned}
\langle f, G\xi \rangle_\mu &= \langle f, -E^x \left[\int_0^{T_F} I_C(X_s) ds \right] - E^{\nu_A} [E(X_{T_F}, C)] \rangle_\mu \\
&\quad + \langle f, 1 \rangle_\mu E^{\nu_A} \left[\int_0^{T_F} I_C(X_s) ds + H(X_{T_F}, C) \right].
\end{aligned}$$

Therefore

$$G\xi(x) = -E^x \left[\int_0^{T_F} I_C(X_s) ds + H(X_{T_F}, C) \right] + E^{\nu_A} \left[\int_0^{T_F} I_C(X_s) ds + H(X_{T_F}, C) \right]$$

for $a.a.x(\mu)$. In particular,

$$G\xi(x) = -H(x, C) + E^{\nu_A} \left[\int_0^{T_F} I_C(X_s) ds + H(X_{T_F}, C) \right]$$

for $a.a.x \in F^r(\mu)$, and hence for all $x \in F^r$. Hence, by (4.1), (6.8) holds. If ξ_1 and ξ_2 are measures on F^r satisfying (6.8), then $G(\xi_1 - \xi_2)$ equals to a constant on F^r . Hence $\xi_1 = \xi_2$ by the proof of theorem 5.5.

In the classical case, the equilibrium measure is characterized as the measure which minimize the energy. In our case, the analogous result holds. Denote \mathcal{M} the family of all bounded signed measures ξ on E with compact support

such that $\int |g(x, y)| |\xi|(dy)$ is bounded, $\mathcal{N}^+ = \{\xi \geq 0; \xi \in \mathcal{N}\}$ and $\mathcal{N}^0 = \{\xi \in \mathcal{N}; \xi(E) = 0\}$. For $\xi, \zeta \in \mathcal{N}$, define the mutual energy of ξ and ζ by

$$(6.9) \quad (\xi, \zeta) = \iint g(x, y) \xi(dx) \zeta(dy).$$

Denote (ξ, ξ) by $I(\xi)$ and call it the energy of ξ .

Lemma 6.3. *If $\xi \in \mathcal{N}^0$, then $I(\xi)$ is non-negative. Moreover, $I(\xi) = 0$ if and only if $\xi = 0$.*

Proof. Suppose that $\xi \in \mathcal{N}^0$. Since $G^{1,0}|\xi|(x)$ is bounded,

$$\begin{aligned} & \int \sum_{n=1}^{\infty} |(K_A^1)^n - \nu_A|(x, dz) G^{1,0}|\xi|(z) \\ & \leq \|G^{1,0}|\xi|\| \left\| \sum_{n=1}^{\infty} \|(K_A^1)(x, \cdot) - \nu_A\| \right\| \end{aligned}$$

converges uniformly in x . Hence for any $\varepsilon > 0$ there exists a number N such that

$$\left| \sum_{n=N}^{\infty} \int \{(K_A^1)^n - \nu_A\}(x, dz) G^{1,0}\xi(z) \right| < \varepsilon$$

for all $x \in E$. From our definition of $g(x, y)$, for $(x, y) \in \Gamma$,

$$\begin{aligned} g(x, y) &= g^{1,0}(x, y) - 1 + \sum_{n=1}^{N-1} \{(K_A^1)^n - \nu_A\} g^{1,0}(x, y) \\ &\quad + \varepsilon(x, y, N), \end{aligned}$$

where $\varepsilon(x, y, N) = \sum_{n=N}^{\infty} \{(K_A^1)^n - \nu_A\} g^{1,0}(x, y)$. Since $\xi(\hat{\Gamma}_x) = 0$, for all $x \in E$,

$$\begin{aligned} I(\xi) &= \iint g^{1,0}(x, y) \xi(dx) \xi(dy) + \sum_{n=1}^{N-1} \iint (K_A^1)^n g^{1,0}(x, y) \xi(dx) \xi(dy) \\ &\quad + \iint \varepsilon(x, y, N) \xi(dx) \xi(dy). \end{aligned}$$

From the resolvent equation (1.7), we have

$$g^{2,0}(x, y) - g^{1,0}(x, y) + K_A^2 g^{1,0}(x, y) = 0.$$

This combined with $g^{1,0}(x, y) \geq g^{2,0}(x, y)$, we have

$$K_A^2 g^{2,0}(x, y) \leq K_A^2 g^{1,0}(x, y) \leq g^{1,0}(x, y),$$

so that

$$\int g^{2,0}(x, z) g^{2,0}(z, y) \nu_A(dz) \leq g^{1,0}(x, y).$$

Hence we have, from the symmetry of $g^2(x, y)$

$$\iint g^{1,0}(x, y) \xi(dx) \xi(dy) \geq \iint \{g^{2,0}(x, y) \xi(dy)\}^2 \nu_A(dx) \geq 0.$$

Similarly it follows that

$$\sum_{n=1}^{N-1} \iint (K_A^1)^n g^{1,0}(x, y) \xi(dx) \xi(dy) \geq 0.$$

Therefore $I(\xi) \geq -\varepsilon$ and hence $I(\xi) \geq 0$.

Suppose that $I(\xi) = 0$. By a routine argument, we have $|(\xi, \zeta)|^2 \leq I(\xi)I(\zeta)$ for all $\zeta \in \mathcal{M}^0$. Hence $(\xi, \zeta) = 0$ for all $\zeta \in \mathcal{M}^0$. This implies that $G\xi$ equals to a constant on E . Integrating by ν_A , we can see that the constant is equals to 0. Hence $\xi = 0$.

Theorem 6.4. *The equilibrium measure ξ_F of a compact set F is the unique measure which attains the*

$$(6.10) \quad \min \{I(\xi); \xi \in \mathcal{M}^+, \xi(E) = 1, \text{ support of } \xi \subseteq F\},$$

and Robin's constant $R(F)$ equals the minimum value of (6.10).

Proof. The proof is similar to the classical case [16]. If a measure ξ satisfies the conditions of (6.10), then, since $G\xi_F = R(F)$ on F except a polar subset of F and ξ charges no polar set,

$$\begin{aligned} I(\xi) &= I(\xi - \xi_F) - I(\xi_F) + 2(\xi, \xi_F) \\ &= I(\xi - \xi_F) + R(F). \end{aligned}$$

Since $\xi - \xi_F \in \mathcal{M}^0$, this implies the result by lemma 6.2.

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