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A KIRILLOV MODEL OF A PRINCIPAL SERIES REPRESENTATION OF $\mathrm{GL}_2(\mathcal{D})$

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0. Introduction

Let F be a non-Archimedean local field of arbitrary characteristic and $\mathcal{D}$ a central finite dimensional division algebra over F. Godement [1] constructed a model of an irreducible admissible representation (π, V) of $\mathrm{GL}_2(F)$, which is called the Kirillov model of (π, V) and is denoted by $\mathcal{K}(\pi)$. $\mathcal{K}(\pi)$ is realized as a certain space consisting of locally constant functions on F^* that vanish outside some compact subset of F. On $\mathcal{K}(\pi)$, upper triangular matrices act as

$$\pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) f(x) = \psi_F(d^{-1}xb)\omega_\pi(d) f(d^{-1}xa),$$

where ω_π is the central character of π and ψ_F is a non-trivial additive character of F. Godement obtained an irreducibility criterion of principal series representations by using the theory of Kirillov models, and then classified principal series representations of $\mathrm{GL}_2(F)$.

Prasad and Raghuram [2] developed the theory of Kirillov models for admissible representations of $\mathrm{GL}_2(\mathcal{D})$. Let (π, V) be an admissible representation of $\mathrm{GL}_2(\mathcal{D})$ and $V_{N,\Psi}$ the twisted Jacquet module of (π, V) with respect to a non-trivial additive character Ψ of $\mathcal{D}$. The Kirillov model of (π, V) is defined to be a certain space consisting of $V_{N,\Psi}$-valued locally constant functions on $\mathcal{D}^*$. If f is an element of the Kirillov model of (π, V), f vanishes outside some compact subset of $\mathcal{D}$ and upper triangular matrices act as

$$\pi\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) f(X) = \Psi(D^{-1}XB)\pi_{N,\Psi}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} f(D^{-1}XA).$$

In this paper we study a Kirillov model of a principal series representation $V(\pi_1, \pi_2)$ of $\mathrm{GL}_2(\mathcal{D})$ induced from an irreducible representation $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ of $\mathcal{D}^* \times \mathcal{D}^*$. Any element of $V(\pi_1, \pi_2)$ is a $V_1 \otimes V_2$-valued locally constant function on $\mathrm{GL}_2(\mathcal{D})$ and $\mathrm{GL}_2(\mathcal{D})$ acts on $V(\pi_1, \pi_2)$ by right translations. Even if $V(\pi_1, \pi_2)$ is not irreducible, we construct its Kirillov model as follows. The element ξ_φ of the Kirillov model of $V(\pi_1, \pi_2)$ corresponding to $\varphi \in V(\pi_1, \pi_2)$ is given as a distribution on $C_c^\infty(\mathcal{D})$ by the form

$$\xi_\varphi(X) = |X|^{1/2} 1 \otimes \pi_2(X) \sum_{n\in\mathbb{Z}} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\varphi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}\right) dY,$$

where $\mathfrak{v}$ denotes an additive valuation on $\mathcal{D}$. Raghuram [3] proved that the defining infinite series of ξ_φ converges. We give a proof of this fact by a different way from Raghuram in Lemma 2.2. As a consequence of the convergence of the series, we know that the Kirillov model is realized as a certain space of functions on $\mathcal{D}^*$. The asymptotic behavior of ξ_φ around 0 characterizes a principal series representation $V(\pi_1, \pi_2)$. Although Raghuram studied a behavior of $\hat{\phi}$ around 0, our statement in Theorem 2.3 is more precise than Raghuram's one.

Moreover, we give a condition under when the map $\phi \mapsto \hat{\phi}$ is injective in Proposition 2.4 and Theorem 2.6. From this theorem we get a sufficient condition for irreducibility of the principal series representations in Corollary 2.7. If the characteristic of F is 0, an irreducibility criterion of the principal series representations of $\mathrm{GL}_n(\mathcal{D})$ was given by Tadić [4] by using the theories of the Langlands classification and Hopf algebras. If we apply the results of Tadić to $\mathrm{GL}_2(\mathcal{D})$ case, the principal series representation $V(\pi_1, \pi_2)$ is reducible if and only if $\pi_2(X) = |X|^{\pm 1}\pi_1(X)$ for all $X \in \mathcal{D}^*$ when the characteristic of F is 0. As a consequence of this fact and Theorem 2.6 we know that if $\dim_F \mathcal{D} \neq 1$ and the characteristic of F is 0, there exists a reducible principal series representation $V(\pi_1, \pi_2)$ such that the maps from $V(\pi_1, \pi_2)$ to its Kirillov model and from $V(\pi_1, \pi_2)^\vee$ to its Kirillov model are injective. If $\dim_F \mathcal{D} = 1$, such representations do not exist.

1. Preliminaries

1.1. Notations. In this paper $\mathbb{Z}$ denotes the ring of integers and $\mathbb{C}$ the field of complex numbers as usual. Let F be a non-Archimedean local field of arbitrary characteristic, $\mathfrak{O}_F$ the integer ring of F, $\mathfrak{P}_F$ the unique maximal ideal of $\mathfrak{O}_F$, q the cardinality of $\mathfrak{O}_F/\mathfrak{P}_F$, and ϖ_F the prime element of F. The additive valuation $\mathfrak{v}_F$ and the multiplicative valuation $|\ |_F$ on F are normalized so that $|\varpi_F|_F = q^{-\mathfrak{v}_F(\varpi_F)} = q^{-1}$. We fix a nontrivial additive character ψ_F of F so chosen that the maximal fractional ideal in F on which ψ_F is trivial is $\mathfrak{O}_F$. Let $\mathcal{D}$ denote a central division algebra of dimension d^2 over F, $\mathfrak{O}$ the maximal order of $\mathcal{D}$, and $\mathfrak{P}$ the unique maximal ideal of $\mathfrak{O}$. Notice that the cardinality of $\mathfrak{O}/\mathfrak{P}$ is equal to q^d. There is a generator ϖ of $\mathfrak{P}$ as $\varpi^d = \varpi_F$. The additive valuation and the multiplicative valuation $|\ |$ on $\mathcal{D}$ are normalized so that $|\varpi| = q^{-\mathfrak{v}(\varpi)} = q^{-d}$. Let $T_{\mathcal{D}/F}$ be the reduced trace from $\mathcal{D}$ to F. Let Ψ be the additive character of $\mathcal{D}$ obtained by composing $T_{\mathcal{D}/F}$ and the character ψ_F. Let dX be the Haar measure on $\mathcal{D}$ normalized so that the volume of $\mathfrak{O}^*$ is $(1-q^{-d})^{-1}$.

Let $\mathrm{M}_2(\mathcal{D})$ be the matrix algebra of 2×2 matrices with entries in $\mathcal{D}$, $G =$

$GL_2(\mathcal{D}) = M_2(\mathcal{D})^*$ the unit group of $M_2(\mathcal{D})$, P the subgroup of upper triangular matrices in G and N the unipotent radical of P consisting of matrices with 1's on diagonal. The Shalika subgroup S is defined to be the subgroup of G consisting of the matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right)$ for $A \in \mathcal{D}^*$ and $B \in \mathcal{D}$. The subgroup of S consisting of the matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ for all $A \in \mathcal{D}^*$ is denoted by $\Delta(\mathcal{D}^*)$.

For a totally disconnected locally compact topological space X and an arbitrary vector space V, let $C^\infty(X, V)$ be the space consisting of V-valued locally constant functions on X and $C_c^\infty(X, V)$ be the subspace of $C^\infty(X, V)$ consisting of compactly supported functions. If V is one dimensional, we write simply $C^\infty(X)$ and $C_c^\infty(X)$ for $C^\infty(X, V)$ and $C_c^\infty(X, V)$, respectively.

Proposition 1.1. *Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then G is decomposed into the disjoint union of P and $PwP = PwN = NwP$.*

The subset PwP is called the big cell.

Proposition 1.2. *The additive character Ψ of $\mathcal{D}$ is a constant on $\mathfrak{P}^{1-d}$.*

For the proof, refer to [5, Chapter 10].

1.2. Admissible representations and Kirillov models. Let (π, V) be a representation of G. In this paper, the representation space V is always a vector space over $\mathbb{C}$. (π, V) is called admissible if the stabilizer subgroup of v in G is open for all $v \in V$ and the subspace which consists of all elements that are invariant under G' is finite dimensional for all open subgroup G' of G.

Let (π_1, V_1) and (π_2, V_2) be two irreducible representations of $\mathcal{D}^*$. We extend π_1, π_2 to a representation of P on which N acts trivially. Let $V(\pi_1, \pi_2)$ denote the representation of G induced from $\pi_1 \otimes \pi_2$ of P. Namely,

$$V(\pi_1,\pi_2) = \left\{ \varphi \in C^\infty(G, V_1 \otimes V_2) \;\middle|\; \begin{array}{l} \varphi\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} g\right) = \left|AD^{-1}\right|^{1/2} \times \pi_1(A) \otimes \pi_2(D)\varphi(g) \\ \left(\text{for all } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in P \text{ and } g \in G\right) \end{array} \right\}$$

and G acts on $V(\pi_1, \pi_2)$ by right translations. Then we obtain an admissible representation. Such a representation is called a principal series representation.

The following lemma is proved in the same way as [1, Theorem 5].

Lemma 1.3. *The contragredient representation of $V(\pi_1, \pi_2)$ is isomorphic to $V(\pi_1^\vee, \pi_2^\vee)$, where $\pi_i^\vee$ denote the contragredient representation of π_i.*

We study the Kirillov model in order to investigate when a principal series representation is irreducible. Let (π, V) be an admissible representation of G. Let $V(N, \Psi)$ be the subspace of V spanned by $\pi\left(\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\right)v - \Psi(X)v$ for all v in V and X in $\mathcal{D}$. The twisted Jacquet module $V_{N,\Psi}$ of V is defined as $V/V(N, \Psi)$. $V_{N,\Psi}$ is an S-module and the maximal quotient of V on which N acts via Ψ. It is known that if (π, V) is irreducible, $V_{N,\Psi}$ is finite dimensional. The next lemma was proved by Prasad and Raghuram in [2, Theorem 2.1].

Lemma 1.4. *The twisted Jacquet module $V(\pi_1, \pi_2)_{N,\Psi}$ of a principal series representation $V(\pi_1, \pi_2)$ is isomorphic with $V_1 \otimes V_2$ as $\Delta(\mathcal{D}^*)$-modules.*

DEFINITION 1.1. For any infinite dimensional admissible representation (π, V) of G, let L be the natural projection from V to $V_{N,\Psi}$. Let ξ_v be the function on $\mathcal{D}^*$ defined by $\xi_v(X) = L\left(\pi\left(\left(\begin{smallmatrix} X & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)v\right)$. Let $\mathcal{K}(\pi)$ denote the space consisting of functions ξ_v for all v in V. $\mathcal{K}(\pi)$ is called the Kirillov model of π.

The action of any element $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ of P on $\mathcal{K}(\pi)$ is easy to describe, which is

$$\pi\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right)\xi(X) = \Psi(D^{-1}XB)\pi_{N,\Psi}\left(\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\right)\xi(D^{-1}XA)$$

for all ξ in $\mathcal{K}(\pi)$ and X in $\mathcal{D}^*$. From this formula, each $V_{N,\Psi}$-valued function ξ of $\mathcal{K}(\pi)$ is locally constant on $\mathcal{D}^*$ and vanishes outside some compact subset of $\mathcal{D}$ because the stabilizer subgroup of ξ is open. The G-intertwining operator $v \mapsto \xi_v$ is injective if (π, V) is irreducible. Prasad and Raghuram proved the following lemma [2, Theorem 3.1].

Lemma 1.5. *For an admissible representation π, the Kirillov model $\mathcal{K}(\pi)$ contains the space $C_c^\infty(\mathcal{D}^*, V_{N,\Psi})$. Moreover, if π is a principal series representation, $C_c^\infty(\mathcal{D}^*, V_{N,\Psi})$ is a proper subspace of $\mathcal{K}(\pi)$.*

2. Main results

2.1. Asymptotic behavior of an element of a Kirillov model. In this section, we study the Kirillov model of a principal series representation of $\mathrm{GL}_2(\mathcal{D})$. Since $\mathcal{D}^*$ is not always commutative, the irreducible representation of $\mathcal{D}^*$ is not one-dimensional. However since $\mathcal{D}^*$ is compact modulo the center F^*, the irreducible representation is finite-dimensional. Let (π_1, V_1), (π_2, V_2) be two irreducible representations of $\mathcal{D}^*$.

The element ξ_φ in the Kirillov model of $V(\pi_1, \pi_2)$ corresponding to φ is defined as

$$\xi_\varphi(X) = |X|^{1/2} 1 \otimes \pi_2(X) \sum_{n\in\mathbb{Z}} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\varphi\left(w^{-1}\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}\right) dY.$$

This map $\varphi \mapsto \xi_\varphi$ is a G-intertwining operator, but not always injective.

We introduce the functions ϕ on $\mathcal{D}$ such that $\phi(X) = \varphi\left(w^{-1}\left(\begin{smallmatrix}1 & X\\ 0 & 1\end{smallmatrix}\right)\right)$. Let $\mathcal{F}(\pi_1, \pi_2)$ denote the space of such functions on $\mathcal{D}$. All functions ϕ of $\mathcal{F}(\pi_1, \pi_2)$ are locally constant on $\mathcal{D}$ and $|X|\pi_1(X)\otimes\pi_2(X^{-1})\phi(X)$ are constant vectors for $|X|$ large. We define $\hat{\phi}$ of ϕ as

$$\hat{\phi}(X) = \sum_{n\in\mathbb{Z}} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\phi(Y)\, dY. \tag{1}$$

$\hat{\phi}$ makes sense if this is regarded as a Fourier transform of ϕ in the sense of distribution on $C_c^\infty(\mathcal{D}^*)$.

Lemma 2.1. *The map $\varphi \mapsto \xi_\varphi$ is injective if and only if the map $\phi \mapsto \hat{\phi}$ is injective.*

Proof. The map $\varphi \mapsto \xi_\varphi$ is a composition of the maps $\varphi \mapsto \phi$, $\phi \mapsto \hat{\phi}$ and $\hat{\phi} \mapsto \xi_\phi$. The map $\hat{\phi} \mapsto \xi_\varphi$ is obviously isomorphic.

Since the big cell is dense in G, φ is completely determined on G by the corresponding ϕ. Hence the map $\varphi \mapsto \phi$ is an isomorphism from $V(\pi_1, \pi_2)$ to $\mathcal{F}(\pi_1, \pi_2)$. □

As a consequence of this lemma, it is important to consider the map $\phi \mapsto \hat{\phi}$. We start to consider of the convergence of the series of (1).

Lemma 2.2. *The series of* (1) *converges and the function vanishes outside some compact subset of $\mathcal{D}$.*

Proof. It is clear that $\mathcal{F}(\pi_1, \pi_2)$ is the direct sum of $C_c^\infty(\mathcal{D}, V_1\otimes V_2)$ and the subspace spanned by the functions

$$\phi_v(X) = \begin{cases} |X|^{-1}\pi_1(X^{-1})\otimes\pi_2(X)v & \text{if } |X| \geq 1 \\ 0 & \text{if } |X| < 1 \end{cases}$$

for all $v \in V_1 \otimes V_2$. If $\phi \in C_c^\infty(\mathcal{D}, V_1 \otimes V_2)$, $\phi \mapsto \hat{\phi}$ is a usual Fourier transform and therefore the series converges on every compact subset of $\mathcal{D}^*$.

Before considering ϕ_v, we give a filtration to $V_1 \otimes V_2$. We denote by f the minimal number such that $\pi_1(X)\otimes\pi_2(Y)v = v$ for all v in $V_1 \otimes V_2$ and X,Y in $1 + \mathfrak{P}^f$. Let

$$W'_f = V_1 \otimes V_2,$$
$$W'_{i-1} = \{v \in W'_i \mid \pi_1(X)\otimes\pi_2(Y)v = v \text{ (for all } X, Y \in 1 + \mathfrak{P}^{i-1})\} \quad \text{for } 2 \leq i \leq f,$$
$$W'_0 = \{v \in W'_1 \mid \pi_1(X)\otimes\pi_2(Y)v = v \text{ (for all } X, Y \in \mathfrak{O}^*)\}.$$

There exists an $\mathfrak{O}^* \times \mathfrak{O}^*$-invariant scalar product $\langle\ ,\ \rangle$ on $V_1 \otimes V_2$. Indeed, if we fix a scalar product $(\ ,\ )$ on $V_1 \otimes V_2$, then $\langle\ ,\ \rangle$ may be given by

$$\langle v, w\rangle = \int_{\mathfrak{O}^*}\int_{\mathfrak{O}^*} (\pi_1(X)\otimes\pi_2(Y)v, \pi_1(X)\otimes\pi_2(Y)w)\, d^*Y\, d^*X.$$

Let

$$W_i = \{v \in W_i' \mid \langle v, v'\rangle = 0 \text{ (for all } v' \in W_{i-1}')\},$$

for $1 \le i \le f$ and $W_0 = W_0'$. Then $V_1 \otimes V_2 = \bigoplus_{i=0}^{f} W_i$ and if $i \neq j$, $\langle v_i, v_j\rangle = 0$ for all $v_i \in W_i$ and $v_j \in W_j$. Notice that if W_0 is not $\{0\}$, $V_1 \otimes V_2$ is one-dimensional because all $\pi_1(X)\otimes\pi_2(Y)$, $X, Y \in \mathcal{D}^*$, are commutative with each other on W_0. If v_i is an element of W_i, then

$$\phi_{v_i}(X) = \begin{cases} |X|^{-1}\pi_1(X^{-1})\otimes\pi_2(X)v_i & \text{if } |X| \ge 1 \\ 0 & \text{if } |X| < 1, \end{cases}$$

and $\hat{\phi}_{v_i}$ is equal to

$$\sum_{n\le 0}\int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1})\otimes\pi_2(Y)v_i\, d^*Y.$$

If $i = 0$, then

$$\begin{aligned}
&\int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1})\otimes\pi_2(Y)v_0\, d^*Y \\
&= \int_{\mathfrak{O}^*} \overline{\Psi(X\varpi^n Y)}\pi_1(Y^{-1}\varpi^{-n})\otimes\pi_2(\varpi^n Y)v_0\, d^*Y \\
&= \pi_1(\varpi^{-n})\otimes\pi_2(\varpi^n)v_0 \int_{\mathfrak{O}^*} \overline{\Psi(X\varpi^n Y)}\, d^*Y \\
&= \pi_1(\varpi^{-n})\otimes\pi_2(\varpi^n)v_0 \int_{\mathfrak{O}} \left(\overline{\Psi(X\varpi^n Y)} - |\varpi|\overline{\Psi(X\varpi^{n+1}Y)}\right) dY.
\end{aligned}$$

Since Ψ is trivial on $\mathfrak{P}^{1-d}$, $\int_{\mathfrak{O}}\left(\overline{\Psi(X\varpi^n Y)} - |\varpi|\overline{\Psi(X\varpi^{n+1}Y)}\right)dY \neq 0$ is equivalent to $X\varpi^{n+1} \in \mathfrak{P}^{1-d}$. Hence $\hat{\phi}_{v_0}$ vanishes outside some compact subset of $\mathcal{D}$ and the series turns out to be a finite sum whenever $\mathfrak{v}(X)$ is fixed.

Let $i \neq 0$. Since $v_i \in W_i$,

$$\begin{aligned}
&\int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1})\otimes\pi_2(Y)v_i\, d^*Y \\
&= \int_{\mathfrak{O}^*/1+\mathfrak{P}^i}\int_{1+\mathfrak{P}^i} \overline{\Psi(X\varpi^n AB)}\pi_1(B^{-1}A^{-1}\varpi^{-n})\otimes\pi_2(\varpi^n AB)v_i\, d^*B\, d^*A
\end{aligned}$$

$$= \int_{\mathfrak{O}^*/1+\mathfrak{P}^i} \pi_1(A^{-1}\varpi^{-n}) \otimes \pi_2(\varpi^n A) \int_{1+\mathfrak{P}^i} \overline{\Psi(X\varpi^n AB)} v_i \, d^*B \, d^*A$$
$$= \int_{\mathfrak{O}^*/1+\mathfrak{P}^i} \overline{\Psi(X\varpi^n A)} \pi_1(A^{-1}\varpi^{-n}) \otimes \pi_2(\varpi^n A) v_i \, d^*A \int_{\mathfrak{P}^i} \overline{\Psi(X\varpi^n AB)} \, dB.$$

Since Ψ is trivial on $\mathfrak{P}^{1-d}$, $\int_{\mathfrak{P}^i} \overline{\Psi(X\varpi^n AB)} \, dB \neq 0$ is equivalent to $X\varpi^n A \in \mathfrak{P}^{1-d}$. Hence $\hat{\phi}_{v_i}$ vanishes outside some compact subset of $\mathcal{D}$ and the series turn out to be a finite sum whenever $\mathfrak{v}(X)$ is fixed.

This completes the proof since any function in $\mathcal{F}(\pi_1, \pi_2)$ can be written as a finite sum of the above functions. □

By this lemma the Kirillov model is realized as a certain space consisting of locally constant functions on $\mathcal{D}^*$.

REMARK 2.1. Raghuram also considered the convergence of the series (1) in [3] as follows. For $\mathfrak{v}(X)$ large, let

$$A(X) = \sum_{n \leq \mathfrak{v}(x)} \int_{\mathfrak{v}(T)=n} \overline{\Psi(T)} (\pi_1(T^{-1}) \otimes \pi_2(T)) \, d^*T.$$

$A(X)$ is an element of $\mathrm{End}(V_1 \otimes V_2)$. Then

$$\hat{\phi}_v(X) = \left(1 \otimes \pi_2(X)^{-1}\right) \cdot A(X) \cdot (\pi_1(X) \otimes 1) v$$

where the notations are the same as Lemma 2.2. He analyzed $A(X)$ and proved that the defining series of $A(X)$ is a finite sum.

Raghuram also calculated the asymptotic behavior of $\hat{\phi}$ around 0 and obtained

$$\hat{\phi}(X) = (1 \otimes \pi_2(X^{-1})) \cdot A(X) \cdot (\pi_1(X) \otimes 1) v_1 + v_2$$

for $|X|$ enough small. By the proof of Lemma (2.2), we can calculate $A(X)$ more precisely.

Let ω_i be the central characters of π_i for $i = 1, 2$ and $\omega = \omega_1 \cdot \omega_2^{-1}$.

Theorem 2.3. *For each $\phi \in \mathcal{F}(\pi_1, \pi_2)$, there exist four vectors $v_\alpha, v_\beta, v_\gamma, v_\delta$ in $V_1 \otimes V_2$ such that*

$$\hat{\phi}(X) = \left((1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1) + \sum_{t=0}^{[m/d]} \omega(\varpi^{td}) A_2 + A_3(m) \right) v_\alpha + \pi_1(X) \otimes \pi_2(X^{-1}) v_\beta + m v_\gamma + v_\delta \tag{2}$$

for $X \in \mathfrak{P}^m$, $X \notin \mathfrak{P}^{m+1}$ *with* m *large. Here*

$$A_1 = \sum_{1-d-f\le n\le 1-d} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(Y)}\pi_1(Y)\otimes\pi_2(Y^{-1})\,d^*Y,$$

$$A_2 = \sum_{1-d\le n\le 0} \int_{\mathfrak{v}(Y)=n} \pi_1(Y^{-1})\otimes\pi_2(Y)\,d^*Y,$$

$$A_3(m) = \sum_{1-d-m\le n\le -d-[m/d]d} \int_{\mathfrak{v}(Y)=n} \pi_1(Y^{-1})\otimes\pi_2(Y)\,d^*Y,$$

considered as elements of $\mathrm{End}(V_1\otimes V_2)$.

Proof. Similarly as in previous lemma, we start from the case ϕ is in $C_c^\infty(\mathcal{D}, V_1\otimes V_2)$. Since $\phi \mapsto \hat{\phi}$ is Fourier transform, in some neighborhood of 0, $\hat{\phi}(X)$ is a constant vector $\int_{\mathcal{D}}\phi(Y)\,dY$.

Let $m=\mathfrak{v}(X)$ be enough large. From the proof of the previous lemma, we have

$$\hat{\phi}_v(X) = \sum_{-d-f-m\le n\le 0} \int_{\mathfrak{v}(y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1})\otimes\pi_2(Y)v\,d^*Y$$

for v in $V_1\otimes V_2$. If v_0 is a non-zero element of W_0, π_1 and π_2 are characters. Then,

$$\begin{aligned}\hat{\phi}_{v_0}(X) &= \sum_{-d-m\le n\le 0} \int_{\mathfrak{v}(y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1})\pi_2(Y)v_0\,d^*Y\\ &= \sum_{-d-m\le n\le 0} \pi_1(\varpi^{-n})\pi_2(\varpi^n)v_0\int_{\mathfrak{O}^*}\overline{\Psi(X\varpi^nY)}\,d^*Y\\ &= \sum_{-d-m\le n\le 0} \pi_1(\varpi^{-n})\pi_2(\varpi^n)v_0\int_{\mathfrak{O}}\left(\overline{\Psi(X\varpi^nY)} - |\varpi|\overline{\Psi(X\varpi^{n+1}Y)}\right)dY.\end{aligned}$$

If we assume $\pi_1(\varpi)\pi_2(\varpi^{-1})\neq 1$, since Ψ is trivial on $\mathfrak{P}^{1-d}$,

$$\begin{aligned}\hat{\phi}_{v_0}(X) &= -|\varpi|\pi_1(\varpi^{d+m})\pi_2(\varpi^{-d-m})v_0 + (1-|\varpi|)\sum_{1-d-m\le n\le 0}\pi_1(\varpi^{-n})\pi_2(\varpi^n)v_0\\ &= -\pi_1(X)\otimes\pi_2(X^{-1})\\ &\quad\times\left((1-|\varpi|)\frac{\pi_1(\varpi^d)\otimes\pi_2(\varpi^{-d})}{1-\pi_1(\varpi)\otimes\pi_2(\varpi^{-1})} + |\varpi|\pi_1(\varpi^d)\otimes\pi_2(\varpi^{-d})\right)v_0\\ &\quad+\frac{1}{1-\pi_1(\varpi)\otimes\pi_2(\varpi^{-1})}v_0.\end{aligned}$$

The last is the behavior of $\hat{\phi}_{v_0}$ around 0 in this case.

If we assume $\pi_1(\varpi)\pi_2(\varpi^{-1}) = 1$,

$$
\begin{aligned}
\hat{\phi}_{v_0}(X) &= -|\varpi|\pi_1(\varpi^{d+m})\pi_2(\varpi^{-d-m})v_0 + (1-|\varpi|) \sum_{1-d-m\le n\le 0} \pi_1(\varpi^{-n})\pi_2(\varpi^{n})v_0 \\
&= -|\varpi|v_0 + (1-|\varpi|)(d+m)v_0 \\
&= m(1-|\varpi|)v_0 + ((1-|\varpi|)d - |\varpi|)v_0.
\end{aligned}
$$

The last is the behavior of $\hat{\phi}_{v_0}$ around 0 in this case.

Next, we assume v_i is an element of W_i for $i \neq 0$. Since Ψ is trivial on $\mathfrak{P}^{1-d}$,

$$
\begin{aligned}
\hat{\phi}_{v_i}(X) = & \sum_{1-d-f-m\le n\le -d-m} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(XY)}\pi_1(Y^{-1}) \otimes \pi_2(Y)v_i \, d^*Y \\
& + \sum_{1-d-m\le n\le 0} \int_{\mathfrak{v}(Y)=n} \pi_1(Y^{-1}) \otimes \pi_2(Y)v_i \, d^*Y \\
= & (1 \otimes \pi_2(X^{-1})) \\
& \times \left(\sum_{1-d-f\le n\le -d} \int_{\mathfrak{v}(Y)=n} \overline{\Psi(Y)}\pi_1(Y^{-1}) \otimes \pi_2(Y) \, d^*Y \right) (\pi_1(X) \otimes 1)v_i) \, d^*Y \\
& + \sum_{1-d-[m/d]d\le n\le 0} \int_{\mathfrak{v}(Y)=n} \pi_1(Y^{-1}) \otimes \pi_2(Y)v_i \, d^*Y \\
& + \sum_{1-d-m\le n\le -d-[m/d]d} \int_{\mathfrak{v}(Y)=n} \pi_1(Y^{-1}) \otimes \pi_2(Y)v_i \, d^*Y \\
= & (1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1)v_i \\
& + \sum_{t=0}^{[m/d]} \omega(\varpi^{td}) \left(\sum_{1-d\le n\le 0} \int_{v(Y)=n} \pi_1(Y^{-1}) \otimes \pi_2(Y)v_i \, d^*Y \right) + A_3(m)v_i.
\end{aligned}
$$

Then the asymptotic behavior around 0 is

$$
\hat{\phi}_{v_i}(X) = (1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1)v_i + \sum_{t=0}^{[m/d]} \omega(\varpi^{td})A_2 v_i + A_3(m)v_i
$$

in this case.

Any function in $\mathcal{F}(\pi_1, \pi_2)$ is a finite sum of above functions. Hence (2) is obtained. ☐

2.2. Injectivity of the map to a Kirillov model. Here we study the condition under when the map from $V(\pi_1, \pi_2)$ to its Kirillov model is injective. Since this map is G-intertwining, $V(\pi_1, \pi_2)$ is reducible if the map has non-zero kernel.

Proposition 2.4. *The mapping $\phi \mapsto \hat{\phi}$ is injective unless there exists a non-zero subspace of $V_1 \otimes V_2$ on which $\pi_1(X) \otimes \pi_2(X^{-1})$ acts as $|X|^{-1}$, in which case its kernel is the set of constant vector-valued functions in $\mathcal{F}(\pi_1, \pi_2)$.*

Proof. We fix a basis of n-dimensional vector space $V_1 \otimes V_2$. Then, $\hat{\phi}(X)$ is written as $(\hat{\phi}_1(X), \ldots, \hat{\phi}_n(X))$ and also $\phi(X)$ is $(\phi_1(X), \ldots, \phi_n(X))$, where each $\hat{\phi}_i$ is the Fourier transform of ϕ_i. If $\hat{\phi}_i = 0$ on $\mathcal{D}^*$, the measure $\hat{\phi}_i(X)\,dX$ is proportional to Dirac measure, which means ϕ_i is a constant on $\mathcal{D}$. Hence ϕ is a constant vector on $\mathcal{D}$. This happen if and only if there exists a non-zero subspace in $V_1 \otimes V_2$ on which $\pi_1(X) \otimes \pi_2(X^{-1})$ acts as $|X|^{-1}$. □

Proposition 2.5. *Let H be an arbitrary group, (π_1, V_1) and (π_2, V_2) finite dimensional irreducible representations of H, and χ a one dimensional representation of H. There exists a non-zero element v of $V_1 \otimes V_2$ such that $\pi_1(X) \otimes \pi_2(X^{-1})v = \chi(X)v$ for all $X \in H$ if and only if $\pi_1 = \chi \cdot \pi_2$ and* $\dim V_1 = \dim V_2 = 1$.

Proof. We assume there exists a non-zero element v of $V_1 \otimes V_2$ such that $\pi_1(X) \otimes \pi_2(X^{-1})v = \chi(X)v$ for all $X \in H$ and (π_1, V_1) and (π_2, V_2) are finite dimensional and irreducible. Notice that

$$\pi_1(X) \otimes 1v = \chi(X)(1 \otimes \pi_2(X))v.$$

Any element of $V_1 \otimes V_2$ is written as

$$\sum_i a_i(\pi_1(Y_i) \otimes 1)v,$$

where the sum is finite, $a_i \in \mathbb{C}^*$, and $Y_i \in H$. For any element X of H, one has

$$\begin{aligned}
&\pi_1(X) \otimes \pi_2(X^{-1}) \left(\sum_i a_i(\pi_1(Y_i) \otimes 1)v \right) \\
&= \sum_i a_i(1 \otimes \pi_2(X^{-1}))(\pi_1(XY_i) \otimes 1)v \\
&= \sum_i a_i(1 \otimes \pi_2(Y_i))(\pi_1(XY_i) \otimes \pi_2((XY_i)^{-1}))v \\
&= \sum_i a_i \chi(XY_i)(1 \otimes \pi_2(Y_i))v \\
&= \chi(X) \sum_i a_i(\pi_1(Y_i) \otimes 1)v.
\end{aligned}$$

Hence $\pi_1(X) \otimes \pi_2(X^{-1})$ acts on $V_1 \otimes V_2$ as $\chi(X)$. Next we consider the action of $\pi_1(XY)\otimes 1$ on $V_1\otimes V_2$ for all $X, Y\in H$. If w is any element of $V_1\otimes V_2$,

$$\begin{aligned}(\pi_1(XY)\otimes 1)w &= \chi(Y)(\pi_1(X)\otimes\pi_2(Y))w\\ &= \chi(Y)(1\otimes\pi_2(Y))(\pi_1(X)\otimes 1w)\\ &= (\pi_1(YX)\otimes 1)w.\end{aligned}$$

By Schur's lemma, $\dim V_1 = 1$. Similarly, $\dim V_2 = 1$.

The converse is obvious. □

These two propositions yield immediately the next theorem.

Theorem 2.6. *The map from an induced representation $V(\pi_1,\pi_2)$ to its Kirillov model is injective unless $\pi_1 = |\ |^{-1}\cdot\pi_2$ and $\dim V_1 = \dim V_2 = 1$.*

By this theorem we obtain a sufficient condition for the reducibility of a principal series representation.

Corollary 2.7. *If $\dim V_1 = \dim V_2 = 1$ and $\pi_1 = |\ |^{\pm 1}\cdot\pi_2$, $V(\pi_1,\pi_2)$ is reducible.*

Proof. Since the map $V(\pi_1,\pi_2)\ni\varphi\mapsto\xi_\varphi\in\mathcal{K}(\pi)$ is a G-intertwining operator, if this map is not injective, $V(\pi_1,\pi_2)$ is reducible. By Lemma 1.3, the map from $V(\pi_1,\pi_2)^\vee$ to its Kirillov model is not injective if $\pi_1 = |\ |\cdot\pi_2$ and $\dim V_1 = \dim V_2 = 1$. □

Tadić obtained the irreducibility criterion of principal series representations of $GL_n(\mathcal{D})$ when the characteristic of F is 0 by using theories of Langlands classification and Hopf algebras [4, Lemma 2.5 and 4.2]. The following theorem is a $GL_2(\mathcal{D})$ case of the results of Tadić.

Theorem 2.8 (Tadić). *When the characteristic of F is 0, the representation $V(\pi_1,\pi_2)$ is reducible if and only if $\pi_1 = |\ |^{\pm 1}\pi_2$.*

As a consequence of Corollary 2.7 and Theorem 2.8, if $d\geq 2$ and the characteristic of F is 0, there exists a reducible principal series representation $V(\pi_1,\pi_2)$ such that the maps from $V(\pi_1,\pi_2)$ to $\mathcal{K}(\pi)$ and from $V(\pi_1,\pi_2)^\vee$ to $\mathcal{K}(\pi)^\vee$ are injective. If $d = 1$, i.e. $\mathcal{D}$ is a commutative field, such representation $V(\pi_1,\pi_2)$ does not exist [1, Theorem 6].

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