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Osaka University
A KIRILLOV MODEL OF A PRINCIPAL SERIES REPRESENTATION OF GL\(_2(D)\)

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0. Introduction

Let \(F\) be a non-Archimedean local field of arbitrary characteristic and \(D\) a central finite dimensional division algebra over \(F\). Godement [1] constructed a model of an irreducible admissible representation \((\pi, V)\) of \(GL_2(F)\), which is called the Kirillov model of \((\pi, V)\) and is denoted by \(\mathcal{K}(\pi)\). \(\mathcal{K}(\pi)\) is realized as a certain space consisting of locally constant functions on \(F^\ast\) that vanish outside some compact subset of \(F\). On \(\mathcal{K}(\pi)\), upper triangular matrices act as

\[
\pi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) f(x) = \psi_F(d^{-1}xb)\omega_\pi(d)f(d^{-1}xa),
\]

where \(\omega_\pi\) is the central character of \(\pi\) and \(\psi_F\) is a non-trivial additive character of \(F\). Godement obtained an irreducibility criterion of principal series representations by using the theory of Kirillov models, and then classified principal series representations of \(GL_2(F)\).

Prasad and Raghuram [2] developed the theory of Kirillov models for admissible representations of \(GL_2(D)\). Let \((\pi, V)\) be an admissible representation of \(GL_2(D)\) and \(V_{N_\psi}\) the twisted Jacquet module of \((\pi, V)\) with respect to a non-trivial additive character \(\Psi\) of \(D\). The Kirillov model of \((\pi, V)\) is defined to be a certain space consisting of \(V_{N_\psi}\)-valued locally constant functions on \(D^\ast\). If \(f\) is an element of the Kirillov model of \((\pi, V)\), \(f\) vanishes outside some compact subset of \(D\) and upper triangular matrices act as

\[
\pi\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) f(X) = \Psi(D^{-1}XB)\pi_{N_\psi}\left(\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\right) f(D^{-1}XA).
\]

In this paper we study a Kirillov model of a principal series representation \(V(\pi_1, \pi_2)\) of \(GL_2(D)\) induced from an irreducible representation \((\pi_1 \otimes \pi_2, V_1 \otimes V_2)\) of \(D^\ast \times D^\ast\). Any element of \(V(\pi_1, \pi_2)\) is a \(V_1 \otimes V_2\)-valued locally constant function on \(GL_2(D)\) and \(GL_2(D)\) acts on \(V(\pi_1, \pi_2)\) by right translations. Even if \(V(\pi_1, \pi_2)\) is not irreducible, we construct its Kirillov model as follows. The element \(\xi_\varphi\) of the Kirillov model of \(V(\pi_1, \pi_2)\) corresponding to \(\varphi \in V(\pi_1, \pi_2)\) is given as a distri-
bution on \( C_c^\infty(D) \) by the form
\[
\xi_\varphi(X) = |X|^{1/2} \otimes \pi_2(X) \sum_{n \in \mathbb{Z}} \int_{v(Y) = n} \Psi(XY) \varphi \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \right) dY,
\]
where \( v \) denotes an additive valuation on \( D \). Raghuram [3] proved that the defining infinite series of \( \xi_\varphi \) converges. We give a proof of this fact by a different way from Raghuram in Lemma 2.2. As a consequence of the convergence of the series, we know that the Kirillov model is realized as a certain space of functions on \( D^* \). The asymptotic behavior of \( \xi_\varphi \) around 0 characterizes a principal series representation \( V(\pi_1, \pi_2) \). Although Raghuram studied a behavior of \( \hat{\phi} \) around 0, our statement in Theorem 2.3 is more precise than Raghuram’s one.

Moreover, we give a condition under when the map \( \phi \mapsto \hat{\phi} \) is injective in Proposition 2.4 and Theorem 2.6. From this theorem we get a sufficient condition for irreducibility of the principal series representations in Corollary 2.7. If the characteristic of \( F \) is 0, an irreducibility criterion of the principal series representations of \( \text{GL}_n(D) \) was given by Tadić [4] by using the theories of the Langlands classification and Hopf algebras. If we apply the results of Tadić to \( \text{GL}_2(D) \) case, the principal series representation \( V(\pi_1, \pi_2) \) is reducible if and only if \( \pi_2(X) = |X|^{\pm 1} \pi_1(\hat{X}) \) for all \( X \in D^* \) when the characteristic of \( F \) is 0. As a consequence of this fact and Theorem 2.6 we know that if \( \dim_F D \neq 1 \) and the characteristic of \( F \) is 0, there exists a reducible principal series representation \( V(\pi_1, \pi_2) \) such that the maps from \( V(\pi_1, \pi_2) \) to its Kirillov model and from \( V(\pi_1, \pi_2) \) to its Kirillov model are injective. If \( \dim_F D = 1 \), such representations do not exist.

1. Preliminaries

1.1. Notations. In this paper \( \mathbb{Z} \) denotes the ring of integers and \( \mathbb{C} \) the field of complex numbers as usual. Let \( F \) be a non-Archimedean local field of arbitrary characteristic, \( O_F \) the integer ring of \( F \), \( \mathfrak{P}_F \) the unique maximal ideal of \( O_F \), \( q \) the cardinality of \( O_F/\mathfrak{P}_F \), and \( \varpi_F \) the prime element of \( F \). The additive valuation \( v_F \) and the multiplicative valuation \( | \cdot |_F \) on \( F \) are normalized so that \( |\varpi_F|_F = q^{-v_F(\varpi_F)} = q^{-1} \). We fix a nontrivial additive character \( \psi_F \) of \( F \) so chosen that the maximal fractional ideal in \( F \) on which \( \psi_F \) is trivial is \( O_F \). Let \( \mathcal{D} \) denote a central division algebra of dimension \( d^2 \) over \( F \), \( \mathfrak{D} \) the maximal order of \( \mathcal{D} \), and \( \mathfrak{P} \) the unique maximal ideal of \( \mathfrak{D} \). Notice that the cardinality of \( \mathfrak{D}/\mathfrak{P} \) is equal to \( q^d \). There is a generator \( \varpi \) of \( \mathfrak{P} \) as \( \varpi^d = \varpi_F \). The additive valuation and the multiplicative valuation \( | \cdot | \) on \( \mathcal{D} \) are normalized so that \( |\varpi| = q^{-v(\varpi)} = q^{-d} \). Let \( T_{\mathcal{D}/F} \) be the reduced trace from \( \mathcal{D} \) to \( F \). Let \( \Psi \) be the additive character of \( \mathfrak{D} \) obtained by composing \( T_{\mathcal{D}/F} \) and the character \( \psi_F \). Let \( dX \) be the Haar measure on \( \mathcal{D} \) normalized so that the volume of \( O_F^* \) is \( (1 - q^{-d})^{-1} \).

Let \( M_2(D) \) be the matrix algebra of \( 2 \times 2 \) matrices with entries in \( D \), \( G = \)}
The Kirillov model of representation of $GL_2(D)$

Let $GL_2(D) = M_2(D)^* \text{ the unit group of } M_2(D)$, $P$ the subgroup of upper triangular matrices in $G$ and $N$ the unipotent radical of $P$ consisting of matrices with 1’s on diagonal. The Shalika subgroup $S$ is defined to be the subgroup of $G$ consisting of the matrices of the form $(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix})$ for $A \in D^*$ and $B \in D$. The subgroup of $S$ consisting of the matrices of the form $(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix})$ for all $A \in D^*$ is denoted by $\Delta(D^*)$.

For a totally disconnected locally compact topological space $X$ and an arbitrary vector space $V$, let $C^\infty(X, V)$ be the space consisting of $V$-valued locally constant functions on $X$ and $C^\infty_c(X, V)$ be the subspace of $C^\infty(X, V)$ consisting of compactly supported functions. If $V$ is one dimensional, we write simply $C^\infty(X)$ and $C^\infty_c(X)$ for $C^\infty(X, V)$ and $C^\infty_c(X, V)$, respectively.

**Proposition 1.1.** Let $w = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Then $G$ is decomposed into the disjoint union of $P$ and $PwP = PwN = NwP$.

The subset $PwP$ is called the big cell.

**Proposition 1.2.** The additive character $\Psi$ of $D$ is a constant on $\mathfrak{p}^{1-d}$.

For the proof, refer to [5, Chapter 10].

1.2. Admissible representations and Kirillov models. Let $(\pi, V)$ be a representation of $G$. In this paper, the representation space $V$ is always a vector space over $\mathbb{C}$. $(\pi, V)$ is called admissible if the stabilizer subgroup of $v$ in $G$ is open for all $v \in V$ and the subspace which consists of all elements that are invariant under $G'$ is finite dimensional for all open subgroup $G'$ of $G$.

Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be two irreducible representations of $D^*$. We extend $\pi_1, \pi_2$ to a representation of $P$ on which $N$ acts trivially. Let $V(\pi_1, \pi_2)$ denote the representation of $G$ induced from $\pi_1 \otimes \pi_2$ of $P$. Namely,

$$V(\pi_1, \pi_2) = \left\{ \varphi \in C^\infty(G, V_1 \otimes V_2) \middle| \varphi \left( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} g \right) = |AD^{-1}|^{1/2} \pi_1(A) \otimes \pi_2(D) \varphi(g) \right\}$$

and $G$ acts on $V(\pi_1, \pi_2)$ by right translations. Then we obtain an admissible representation. Such a representation is called a principal series representation.

The following lemma is proved in the same way as [1, Theorem 5].

**Lemma 1.3.** The contragredient representation of $V(\pi_1, \pi_2)$ is isomorphic to $V(\pi_1^\vee, \pi_2^\vee)$, where $\pi_i^\vee$ denote the contragredient representation of $\pi_i$.

We study the Kirillov model in order to investigate when a principal series repre-
sentation is irreducible. Let \((\pi, V)\) be an admissible representation of \(G\). Let \(V(N, \Psi)\) be the subspace of \(V\) spanned by \(\pi((\frac{1}{2}, \frac{1}{2}))v - \Psi(X)v\) for all \(v\) in \(V\) and \(X\) in \(\mathcal{D}\). The twisted Jacquet module \(V_{N, \Psi}\) of \(V\) is defined as \(V/V(N, \Psi)\). \(V_{N, \Psi}\) is an \(S\)-module and the maximal quotient of \(V\) on which \(N\) acts via \(\Psi\). It is known that if \((\pi, V)\) is irreducible, \(V_{N, \Psi}\) is finite dimensional. The next lemma was proved by Prasad and Raghuram in [2, Theorem 2.1].

**Lemma 1.4.** The twisted Jacquet module \(V(\pi_1, \pi_2)_{N, \Psi}\) of a principal series representation \(V(\pi_1, \pi_2)\) is isomorphic with \(V_1 \otimes V_2\) as \(\Delta(\mathcal{D}^*)\)-modules.

**Definition 1.1.** For any infinite dimensional admissible representation \((\pi, V)\) of \(G\), let \(L\) be the natural projection from \(V\) to \(V_{N, \Psi}\). Let \(\xi_v\) be the function on \(\mathcal{D}^*\) defined by \(\xi_v(X) = L(\pi((\frac{X}{0}))v)\). Let \(K(\pi)\) denote the space consisting of functions \(\xi_v\) for all \(v\) in \(V\). \(K(\pi)\) is called the Kirillov model of \(\pi\).

The action of any element \(\left(\begin{array}{cc} A & B \\ 0 & D \end{array}\right)\) of \(P\) on \(K(\pi)\) is easy to describe, which is

\[
\pi(\left(\begin{array}{cc} A & B \\ 0 & D \end{array}\right)) \xi(X) = \Psi(D^{-1}XB)\pi_{N, \Psi}(\left(\begin{array}{cc} D & 0 \\ 0 & D \end{array}\right)) \xi(D^{-1}XA)
\]

for all \(\xi\) in \(K(\pi)\) and \(X\) in \(\mathcal{D}^*\). From this formula, each \(V_{N, \Psi}\)-valued function \(\xi\) of \(K(\pi)\) is locally constant on \(\mathcal{D}^*\) and vanishes outside some compact subset of \(\mathcal{D}\) because the stabilizer subgroup of \(\xi\) is open. The \(G\)-intertwining operator \(v \mapsto \xi_v\) is injective if \((\pi, V)\) is irreducible. Prasad and Raghuram proved the following lemma [2, Theorem 3.1].

**Lemma 1.5.** For an admissible representation \(\pi\), the Kirillov model \(K(\pi)\) contains the space \(C_\infty(\mathcal{D}^*, V_{N, \Psi})\). Moreover, if \(\pi\) is a principal series representation, \(C_\infty(\mathcal{D}^*, V_{N, \Psi})\) is a proper subspace of \(K(\pi)\).

2. **Main results**

2.1. **Asymptotic behavior of an element of a Kirillov model.** In this section, we study the Kirillov model of a principal series representation of \(\text{GL}_2(\mathcal{D})\). Since \(\mathcal{D}^*\) is not always commutative, the irreducible representation of \(\mathcal{D}^*\) is not one-dimensional. However since \(\mathcal{D}^*\) is compact modulo the center \(F^*\), the irreducible representation is finite-dimensional. Let \((\pi_1, V_1), (\pi_2, V_2)\) be two irreducible representations of \(\mathcal{D}^*\).

The element \(\xi_\varphi\) in the Kirillov model of \(V(\pi_1, \pi_2)\) corresponding to \(\varphi\) is defined as

\[
\xi_\varphi(X) = |X|^{1/2} \otimes \pi_2(X) \sum_{n \in \mathbb{Z}} \int_{\mathcal{D}} \frac{\Psi(XY)\varphi \left(w^{-1} \left(\begin{array}{cc} 1 & Y \\ 0 & 1 \end{array}\right)\right)}{v(Y)} dY,
\]
This map \( \varphi \mapsto \xi_\varphi \) is a \( G \)-intertwining operator, but not always injective.

We introduce the functions \( \phi \) on \( D \) such that \( \phi(X) = \varphi \left( w^{-1} \left( \begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix} \right) \right) \). Let \( \mathcal{F}(\pi_1, \pi_2) \) denote the space of such functions on \( D \). All functions \( \phi \) of \( \mathcal{F}(\pi_1, \pi_2) \) are locally constant on \( D \) and \( |X| \pi_1(X) \otimes \pi_2(X^{-1}) \phi(X) \) are constant vectors for \( |X| \) large. We define \( \hat{\phi} \) of \( \phi \) as

\[
\hat{\phi}(X) = \sum_{n \in \mathbb{Z}} \int_{\nu(Y)=n} \Psi(XY) \phi(Y) \, dY.
\]

\( \hat{\phi} \) makes sense if this is regarded as a Fourier transform of \( \phi \) in the sense of distribution on \( C_c^\infty(D^*) \).

**Lemma 2.1.** The map \( \varphi \mapsto \xi_\varphi \) is injective if and only if the map \( \phi \mapsto \hat{\phi} \) is injective.

**Proof.** The map \( \varphi \mapsto \xi_\varphi \) is a composition of the maps \( \varphi \mapsto \phi \), \( \phi \mapsto \hat{\phi} \) and \( \hat{\phi} \mapsto \xi_\phi \). The map \( \hat{\phi} \mapsto \xi_\phi \) is obviously isomorphic.

Since the big cell is dense in \( G \), \( \varphi \) is completely determined on \( G \) by the corresponding \( \phi \). Hence the map \( \varphi \mapsto \phi \) is an isomorphism from \( V(\pi_1, \pi_2) \) to \( \mathcal{F}(\pi_1, \pi_2) \).

As a consequence of this lemma, it is important to consider the map \( \phi \mapsto \hat{\phi} \). We start to consider of the convergence of the series of (1).

**Lemma 2.2.** The series of (1) converges and the function vanishes outside some compact subset of \( D \).

**Proof.** It is clear that \( \mathcal{F}(\pi_1, \pi_2) \) is the direct sum of \( C_c^\infty(D, V_1 \otimes V_2) \) and the subspace spanned by the functions

\[
\phi_v(X) = \begin{cases} |X|^{-1} \pi_1(X^{-1}) \otimes \pi_2(X)v & \text{if } |X| \geq 1 \\ 0 & \text{if } |X| < 1 \end{cases}
\]

for all \( v \in V_1 \otimes V_2 \). If \( \phi \in C_c^\infty(D, V_1 \otimes V_2) \), \( \phi \mapsto \hat{\phi} \) is a usual Fourier transform and therefore the series converges on every compact subset of \( D^* \).

Before considering \( \hat{\phi}_v \), we give a filtration to \( V_1 \otimes V_2 \). We denote by \( f \) the minimal number such that \( \pi_1(X) \otimes \pi_2(Y)v = v \) for all \( v \in V_1 \otimes V_2 \) and \( X, Y \in 1 + \mathfrak{F}^f \).

Let

\[
W_0 = \{ v \in W'_1 \mid \pi_1(X) \otimes \pi_2(Y)v = v \ \text{for all } X, Y \in \mathfrak{F}^0 \},
\]

\[
W_i = \{ v \in W'_i \mid \pi_1(X) \otimes \pi_2(Y)v = v \ \text{for all } X, Y \in \mathfrak{F}^i \} \quad \text{for } 2 \leq i \leq f,
\]

\[
W'_0 = \{ v \in W'_1 \mid \pi_1(X) \otimes \pi_2(Y)v = v \ \text{for all } X, Y \in \mathfrak{F}^* \},
\]

\[
W'_f = V_1 \otimes V_2.
\]
There exists an $\mathcal{D}^* \times \mathcal{D}^*$-invariant scalar product $\langle \ , \rangle$ on $V_1 \otimes V_2$. Indeed, if we fix a scalar product $\langle \ , \rangle$ on $V_1 \otimes V_2$, then $\langle \ , \rangle$ may be given by

$$\langle v, w \rangle = \int_{\mathcal{D}^*} \int_{\mathcal{D}^*} (\pi_1(X) \otimes \pi_2(Y))v, \pi_1(X) \otimes \pi_2(Y)w \, d^*Y \, d^*X.$$ 

Let

$$W_i = \{ v \in W_i' \mid \langle v, v' \rangle = 0 \ (\text{for all } v' \in W_i'_{-1}) \},$$

for $1 \leq i \leq f$ and $W_0 = W_0'$. Then $V_1 \otimes V_2 = \bigoplus_{i=0}^f W_i$ and if $i \neq j$, $\langle v_i, v_j \rangle = 0$ for all $v_i \in W_i$ and $v_j \in W_j$. Notice that if $W_0$ is not $\{0\}$, $V_1 \otimes V_2$ is one-dimensional because all $\pi_1(X) \otimes \pi_2(Y)$, $X, Y \in \mathcal{D}^*$, are commutative with each other on $W_0$. If $v_i$ is an element of $W_i$, then

$$\phi_{v_i}(X) = \begin{cases} |X|^{-1} \pi_1(X^{-1}) \otimes \pi_2(X) v_i & \text{if } |X| \geq 1 \\ 0 & \text{if } |X| < 1, \end{cases}$$

and $\hat{\phi}_{v_i}$ is equal to

$$\sum_{n \leq 0} \int_{v(Y) = n} \Psi(XY)\pi_1(Y^{-1}) \otimes \pi_2(Y) v_i \, d^*Y.$$

If $i = 0$, then

$$\int_{v(Y) = n} \Psi(XY)\pi_1(Y^{-1}) \otimes \pi_2(Y) v_0 \, d^*Y$$

$$= \int_{\mathcal{D}^*} \Psi(X\varpi^n)\pi_1(Y^{-1}\varpi^{-n}) \otimes \pi_2(\varpi^n) v_0 \, d^*Y$$

$$= \pi_1(\varpi^{-n}) \otimes \pi_2(\varpi^n) v_0 \int_{\mathcal{D}^*} \Psi(X\varpi^n) \, d^*Y$$

$$= \pi_1(\varpi^{-n}) \otimes \pi_2(\varpi^n) v_0 \int_{\mathcal{D}^*} (\Psi(X\varpi^n) - |\varpi|\Psi(X\varpi^{n+1})) \, dY.$$

Since $\Psi$ is trivial on $\mathcal{P}^{1-d}$, $\int_{\mathcal{D}} (\Psi(X\varpi^n) - |\varpi|\Psi(X\varpi^{n+1})) \, dY \neq 0$ is equivalent to $X\varpi^{n+1} \in \mathcal{P}^{1-d}$. Hence $\hat{\phi}_{v_0}$ vanishes outside some compact subset of $\mathcal{D}$ and the series turns out to be a finite sum whenever $v(X)$ is fixed.

Let $i \neq 0$. Since $v_i \in W_i$,

$$\int_{v(Y) = n} \Psi(XY)\pi_1(Y^{-1}) \otimes \pi_2(Y) v_i \, d^*Y$$

$$= \int_{\mathcal{D}^*/1+\mathcal{P}^d} \int_{1+\mathcal{P}^d} \Psi(X\varpi^nAB)\pi_1(B^{-1}A^{-1}\varpi^{-n}) \otimes \pi_2(\varpi^nAB) v_i \, d^*B \, d^*A.$$


\[ = \int_{\mathcal{D}^* \cdot 1 + \mathfrak{p}^i} \pi_1(A^{-1} \omega^{-n}) \otimes \pi_2(\omega^n A) \int_{1 + \mathfrak{p}^i} \overline{\Psi(\mathcal{X} \omega^n A B)} \psi d^* B d^* A \]
\[ = \int_{\mathcal{D}^* \cdot 1 + \mathfrak{p}^i} \overline{\Psi(\mathcal{X} \omega^n A)} \pi_1(A^{-1} \omega^{-n}) \otimes \pi_2(\omega^n A) \psi d^* A \int_{\mathfrak{p}^i} \overline{\Psi(\mathcal{X} \omega^n A B)} dB. \]

Since \( \Psi \) is trivial on \( \mathfrak{p}^{1-d} \), \( \int_{\mathfrak{p}^i} \overline{\Psi(\mathcal{X} \omega^n A B)} dB \neq 0 \) is equivalent to \( X \omega^n A \in \mathfrak{p}^{1-d} \). Hence \( \hat{\phi}_v \) vanishes outside some compact subset of \( \mathcal{D} \) and the series turn out to be a finite sum whenever \( v(X) \) is fixed.

This completes the proof since any function in \( \mathcal{F}(\pi_1, \pi_2) \) can be written as a finite sum of the above functions. \( \square \)

By this lemma the Kirillov model is realized as a certain space consisting of locally constant functions on \( \mathcal{D}^* \).

**Remark 2.1.** Raghuram also considered the convergence of the series (1) in [3] as follows. For \( v(X) \) large, let
\[
A(X) = \sum_{n \leq v(X)} \int_{v(T) = n} \overline{\Psi(T)}(\pi_1(T^{-1}) \otimes \pi_2(T)) d^* T.
\]

\( A(X) \) is an element of \( \text{End}(V_1 \otimes V_2) \). Then
\[
\hat{\phi}_v(X) = (1 \otimes \pi_2(X)^{-1}) \cdot A(X) \cdot (\pi_1(X) \otimes 1) v
\]
where the notations are the same as Lemma 2.2. He analyzed \( A(X) \) and proved that the defining series of \( A(X) \) is a finite sum.

Raghuram also calculated the asymptotic behavior of \( \hat{\phi} \) around 0 and obtained
\[
\hat{\phi}(X) = (1 \otimes \pi_2(X^{-1})) \cdot A(X) \cdot (\pi_1(X) \otimes 1) v_1 + v_2
\]
for \( |X| \) enough small. By the proof of Lemma (2.2), we can calculate \( A(X) \) more precisely.

Let \( \omega_i \) be the central characters of \( \pi_i \) for \( i = 1, 2 \) and \( \omega = \omega_1 \cdot \omega_2^{-1} \).

**Theorem 2.3.** For each \( \phi \in \mathcal{F}(\pi_1, \pi_2) \), there exist four vectors \( v_{\alpha}, v_{\beta}, v_{\gamma}, v_5 \) in \( V_1 \otimes V_2 \) such that
\[
\hat{\phi}(X) = \left( (1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1) + \sum_{r=0}^{\lfloor m/d \rfloor} \omega(\omega^{ld})A_2 + A_3(m) \right) v_{\alpha}
\]
\[
+ \pi_1(X) \otimes \pi_2(X^{-1}) v_{\beta} + m v_{\gamma} + v_5
\]
for $X \in \mathcal{P}^n$, $X \notin \mathcal{P}^{n+1}$ with $m$ large. Here

$$A_1 = \sum_{1-d-f \leq n \leq 1-d} \int_{v(Y) = n} \overline{\psi(Y)} \pi_1(Y) \otimes \pi_2(Y^{-1}) d^*Y,$$

$$A_2 = \sum_{1-d \leq n \leq 0} \int_{v(Y) = n} \pi_1(Y^{-1}) \otimes \pi_2(Y) d^*Y,$$

$$A_3(m) = \sum_{1-d-m \leq n \leq -d - \lfloor m/d \rfloor} \int_{v(Y) = n} \pi_1(Y^{-1}) \otimes \pi_2(Y) d^*Y,$$

considered as elements of $\text{End}(V_1 \otimes V_2)$.

Proof. Similarly as in previous lemma, we start from the case $\phi$ is in $C^\infty_c(D, V_1 \otimes V_2)$. Since $\phi \mapsto \hat{\phi}$ is Fourier transform, in some neighborhood of 0, $\hat{\phi}(X)$ is a constant vector $\int_D \hat{\phi}(Y) dY$.

Let $m = v(X)$ be enough large. From the proof of the previous lemma, we have

$$\hat{\phi}_v(X) = \sum_{-d-f \leq m \leq n \leq 0} \int_{v(Y) = n} \overline{\psi(XY)} \pi_1(Y^{-1}) \otimes \pi_2(Y) v d^*Y$$

for $v$ in $V_1 \otimes V_2$. If $v_0$ is a non-zero element of $W_0$, $\pi_1$ and $\pi_2$ are characters. Then,

$$\hat{\phi}_{v_0}(X) = \sum_{-d-m \leq n \leq 0} \int_{v(Y) = n} \overline{\psi(XY)} \pi_1(Y^{-1}) \pi_2(Y) v_0 d^*Y$$

$$= \sum_{-d-m \leq n \leq 0} \pi_1(\varpi^{-n}) \pi_2(\varpi^n) v_0 \int_{\Delta^*} \overline{\psi(X \varpi^n Y)} d^*Y$$

$$= \sum_{-d-m \leq n \leq 0} \pi_1(\varpi^{-n}) \pi_2(\varpi^n) v_0 \int_{\Delta} \left( \overline{\psi(X \varpi^n Y)} - |\varpi| \overline{\psi(X \varpi^{n+1} Y)} \right) dY. $$

If we assume $\pi_1(\varpi) \pi_2(\varpi^{-1}) \neq 0$, since $\Psi$ is trivial on $\mathcal{P}^{1-d}$,

$$\hat{\phi}_{v_0}(X) = -|\varpi| \pi_1(\varpi^{d+m}) \pi_2(\varpi^{-d-m}) v_0 + (1 - |\varpi|) \sum_{1-d-m \leq n \leq 0} \pi_1(\varpi^{-n}) \pi_2(\varpi^n) v_0$$

$$= -\pi_1(X) \otimes \pi_2(X^{-1})$$

$$\times \left( 1 - |\varpi| \right) \frac{\pi_1(\varpi^d) \otimes \pi_2(\varpi^{-d})}{1 - \pi_1(\varpi) \otimes \pi_2(\varpi^{-1})} + |\varpi| \pi_1(\varpi^d) \otimes \pi_2(\varpi^{-d}) \right) v_0$$

$$+ \frac{1}{1 - \pi_1(\varpi) \otimes \pi_2(\varpi^{-1})} v_0.$$

The last is the behavior of $\hat{\phi}_{v_0}$ around 0 in this case.
If we assume \( \pi_1(\varpi)\pi_2(\varpi^{-1}) = 1 \),

\[
\hat{\phi}_{\mu}(X) = -|\varpi|\pi_1(\varpi^{d+m})\pi_2(\varpi^{-d-m})\psi_0 + (1 - |\varpi|) \sum_{1-d-m \leq n \leq 0} \pi_1(\varpi^{-n})\pi_2(\varpi^n)\psi_0
\]

\[
= -|\varpi|\psi_0 + (1 - |\varpi|)(d + m)\psi_0
\]

\[
= m(1 - |\varpi|)\psi_0 + ((1 - |\varpi|)d - |\varpi|)\psi_0.
\]

The last is the behavior of \( \hat{\phi}_{\mu} \) around 0 in this case.

Next, we assume \( \psi_i \) is an element of \( W_i \) for \( i \neq 0 \). Since \( \Psi \) is trivial on \( \mathfrak{p}^{1-d} \),

\[
\hat{\phi}_{\psi}(X) = \sum_{1-d-f-m \leq n \leq -d-m} \int_{v(Y) = n} \Psi(XY)\pi_1(Y^{-1}) \otimes \pi_2(Y)\psi_i d^*Y
\]

\[
+ \sum_{1-d-m \leq n \leq 0} \int_{v(Y) = n} \pi_1(Y^{-1}) \otimes \pi_2(Y)\psi_i d^*Y
\]

\[
= (1 \otimes \pi_2(X^{-1})) \times \left( \sum_{1-d-f-m \leq n \leq -d} \int_{v(Y) = n} \Psi(YY)\pi_1(Y^{-1}) \otimes \pi_2(Y) d^*Y \right) \left( \pi_1(X) \otimes 1 \right)\psi_i d^*Y
\]

\[
+ \sum_{1-d-m \leq n \leq -d-m/d} \int_{v(Y) = n} \pi_1(Y^{-1}) \otimes \pi_2(Y)\psi_i d^*Y
\]

\[
= (1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1)\psi_i
\]

\[
+ \sum_{\lfloor n/d \rfloor} \omega(\varpi^{d_0}) \left( \sum_{1-d-m \leq n \leq 0} \int_{v(Y) = n} \pi_1(Y^{-1}) \otimes \pi_2(Y)\psi_i d^*Y \right) + A_3(m)\psi_i
\]

Then the asymptotic behavior around 0 is

\[
\hat{\phi}_{\psi}(X) = (1 \otimes \pi_2(X^{-1})) \cdot A_1 \cdot (\pi_1(X) \otimes 1)\psi_i + \sum_{\lfloor n/d \rfloor} \omega(\varpi^{d_0})A_2\psi_i + A_3(m)\psi_i
\]

in this case.

Any function in \( \mathcal{F}(\pi_1, \pi_2) \) is a finite sum of above functions. Hence (2) is obtained.

\[\square\]

2.2. Injectivity of the map to a Kirillov model. Here we study the condition under when the map from \( V(\pi_1, \pi_2) \) to its Kirillov model is injective. Since this map is \( G \)-intertwining, \( V(\pi_1, \pi_2) \) is reducible if the map has non-zero kernel.
Proposition 2.4. The mapping $\phi \mapsto \hat{\phi}$ is injective unless there exists a non-zero subspace of $V_1 \otimes V_2$ on which $\pi_1(X) \otimes \pi_2(X^{-1})$ acts as $|X|^{-1}$, in which case its kernel is the set of constant vector-valued functions in $\mathcal{F}(\pi_1, \pi_2)$.

Proof. We fix a basis of $n$-dimensional vector space $V_1 \otimes V_2$. Then, $\hat{\phi}(X)$ is written as $(\hat{\phi}_1(X), \ldots, \hat{\phi}_n(X))$ and also $\phi(X)$ is $(\phi_1(X), \ldots, \phi_n(X))$, where each $\hat{\phi}_i$ is the Fourier transform of $\phi_i$. If $\hat{\phi}_i = 0$ on $\mathcal{D}^*$, the measure $\hat{\phi}_i(X)\,dX$ is proportional to Dirac measure, which means $\hat{\phi}_i$ is a constant on $\mathcal{D}$. Hence $\phi$ is a constant vector on $\mathcal{D}$. This happen if and only if there exists a non-zero subspace in $V_1 \otimes V_2$ on which $\pi_1(X) \otimes \pi_2(X^{-1})$ acts as $|X|^{-1}$. \hfill \Box

Proposition 2.5. Let $H$ be an arbitrary group, $(\pi_1, V_1)$ and $(\pi_2, V_2)$ finite dimensional irreducible representations of $H$, and $\chi$ a one dimensional representation of $H$. There exists a non-zero element $\nu$ of $V_1 \otimes V_2$ such that $\pi_1(X) \otimes \pi_2(X^{-1})\nu = \chi(X)\nu$ for all $X \in H$ if and only if $\pi_1 = \chi \cdot \pi_2$ and $\dim V_1 = \dim V_2 = 1$.

Proof. We assume there exists a non-zero element $\nu$ of $V_1 \otimes V_2$ such that $\pi_1(X) \otimes \pi_2(X^{-1})\nu = \chi(X)\nu$ for all $X \in H$ and $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are finite dimensional and irreducible. Notice that

$$\pi_1(X) \otimes 1\nu = \chi(X)(1 \otimes \pi_2(X))\nu.$$ 

Any element of $V_1 \otimes V_2$ is written as

$$\sum_i a_i (\pi_1(Y_i) \otimes 1)\nu,$$

where the sum is finite, $a_i \in \mathbb{C}^*$, and $Y_i \in H$. For any element $X$ of $H$, one has

$$\pi_1(X) \otimes \pi_2(X^{-1}) \left( \sum_i a_i (\pi_1(Y_i) \otimes 1)\nu \right)$$

$$= \sum_i a_i (1 \otimes \pi_2(X^{-1}))(\pi_1(XY_i) \otimes 1)\nu$$

$$= \sum_i a_i (1 \otimes \pi_2(Y_i))(\pi_1(XY_i) \otimes \pi_2((XY_i)^{-1}))\nu$$

$$= \sum_i a_i \chi(XY_i)(1 \otimes \pi_2(Y_i))\nu$$

$$= \chi(X) \sum_i a_i (\pi_1(Y_i) \otimes 1)\nu.$$ 

Hence $\pi_1(X) \otimes \pi_2(X^{-1})$ acts on $V_1 \otimes V_2$ as $\chi(X)$. Next we consider the action
of \( \pi_1(XY) \otimes 1 \) on \( V_1 \otimes V_2 \) for all \( X, Y \in H \). If \( w \) is any element of \( V_1 \otimes V_2 \),

\[
(\pi_1(XY) \otimes 1)w = \chi(Y)(\pi_1(X) \otimes \pi_2(Y))w = \chi(Y)(1 \otimes \pi_2(Y))(\pi_1(X) \otimes 1)w = (\pi_1(YX) \otimes 1)w.
\]

By Schur’s lemma, \( \dim V_1 = 1 \). Similarly, \( \dim V_2 = 1 \).

The converse is obvious. \( \square \)

These two propositions yield immediately the next theorem.

**Theorem 2.6.** The map from an induced representation \( V(\pi_1, \pi_2) \) to its Kirillov model is injective unless \( \pi_1 = | -1 \cdot \pi_2 \) and \( \dim V_1 = \dim V_2 = 1 \).

By this theorem we obtain a sufficient condition for the reducibility of a principal series representation.

**Corollary 2.7.** If \( \dim V_1 = \dim V_2 = 1 \) and \( \pi_1 = | \pm 1 \cdot \pi_2 \), \( V(\pi_1, \pi_2) \) is reducible.

Proof. Since the map \( V(\pi_1, \pi_2) \ni \varphi \mapsto \xi_\varphi \in \mathcal{K}(\pi) \) is a \( G \)-intertwining operator, if this map is not injective, \( V(\pi_1, \pi_2) \) is reducible. By Lemma 1.3, the map from \( V(\pi_1, \pi_2)^\vee \) to its Kirillov model is not injective if \( \pi_1 = | \cdot \pi_2 \) and \( \dim V_1 = \dim V_2 = 1 \).

Tadić obtained the irreducibility criterion of principal series representations of \( \text{GL}_n(D) \) when the characteristic of \( F \) is 0 by using theories of Langlands classification and Hopf algebras [4, Lemma 2.5 and 4.2]. The following theorem is a \( \text{GL}_2(D) \) case of the results of Tadić.

**Theorem 2.8** (Tadić). When the characteristic of \( F \) is 0, the representation \( V(\pi_1, \pi_2) \) is reducible if and only if \( \pi_1 = | \pm 1 \cdot \pi_2 \).

As a consequence of Corollary 2.7 and Theorem 2.8, if \( d \geq 2 \) and the characteristic of \( F \) is 0, there exists a reducible principal series representation \( V(\pi_1, \pi_2) \) such that the maps from \( V(\pi_1, \pi_2) \) to \( \mathcal{K}(\pi) \) and from \( V(\pi_1, \pi_2)^\vee \) to \( \mathcal{K}(\pi)^\vee \) are injective. If \( d = 1 \), i.e. \( D \) is a commutative field, such representation \( V(\pi_1, \pi_2) \) does not exist [1, Theorem 6].

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