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# AN ENERGY-THEORETIC APPROACH TO THE HITCHIN-KOBAYASHI CORRESPONDENCE FOR MANIFOLDS, II

Dedicated to Professor Eugenio Calabi on his eightieth birthday

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## Abstract

Recently, Donaldson proved asymptotic stability for a polarized algebraic manifold  $M$  with polarization class admitting a Kähler metric of constant scalar curvature, essentially when the linear algebraic part  $H$  of  $\text{Aut}^0(M)$  is semisimple. The purpose of this paper is to give a generalization of Donaldson's result to the case where the polarization class admits an extremal Kähler metric, even when  $H$  is not semisimple.

## 0. Introduction

For a connected polarized algebraic manifold  $(M, L)$  with an extremal Kähler metric in the polarization class  $c_1(L)_{\mathbb{R}}$ , we consider the Kodaira embedding

$$\Phi_m = \Phi_{|L^{\otimes m}|} : M \hookrightarrow \mathbb{P}^*(V_m), \quad m \gg 1,$$

where  $\mathbb{P}^*(V_m)$  denotes the set of all hyperplanes in  $V_m := H^0(M, \mathcal{O}(L^m))$  through the origin. For the identity component  $\text{Aut}^0(M)$  of the group of holomorphic automorphisms of  $M$ , let  $H$  denote its maximal connected linear algebraic subgroup. Replacing the ample holomorphic line bundle  $L$  by some positive integral multiple of  $L$  if necessary, we may fix an  $H$ -linearization of  $L$ , i.e., a lift to  $L$  of the  $H$ -action on  $M$  such that  $H$  acts on  $L$  as bundle isomorphisms covering the  $H$ -action on  $M$ , and may further assume that the natural  $H$ -equivariant maps

$$\text{pr}_m : \bigotimes^m V_1 \rightarrow V_m, \quad m = 1, 2, \dots,$$

are surjective (cf. [18], Theorem 3). In this paper, applying a method in [15], we shall generalize a result in Donaldson [3] about stability to extremal Kähler cases:

**Main Theorem.** *For a polarized algebraic manifold  $(M, L)$  as above with an extremal Kähler metric in the polarization class, there exists an algebraic torus  $T$  in  $H$  such that the image  $\Phi_m(M)$  in  $\mathbb{P}^*(V_m)$  is stable relative to  $T$  (cf. Section 2 and [14]) for  $m \gg 1$ .*

In particular in [16], by an argument as in [3], an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold  $M$  will be shown to be unique<sup>1</sup> up to the action of the group  $H$ .

Fix once for all an extremal Kähler metric  $\omega_0$  in the polarization class in Main Theorem. By a result of Calabi [1], the identity component  $K$  of the group of isometries of  $(M, \omega_0)$  is a maximal compact connected subgroup of  $H$ . For the identity component  $Z$  of the center of  $K$ , we consider the complexification  $Z^\mathbb{C}$  of  $Z$  in  $H$ . Then we shall see that Main Theorem is true for  $T = Z^\mathbb{C}$  (cf. Section 1).

One may ask why relative stability in place of ordinary stability has to be considered in our study. The reason why we choose relative stability is because, in general, the obstruction in [13] to asymptotic semistability does not vanish (cf. [17]). Thus, as to the group action on  $V_m$  related to stability, we must replace the full special linear group  $\mathrm{SL}(V_m)$  of  $V_m$  by its subgroup  $G_m(T)$  (see (1.3)), where the algebraic torus  $T$  in  $Z^\mathbb{C}$  is chosen in such a way that the obstruction vanishes when restricted to  $G_m(T)$ , i.e.,  $G'_m(T)$  fixes  $\hat{M}_m$  (cf. Section 1). Note also that  $G_m(T)$  is a direct product of special linear groups. To see why we choose such a group  $G_m(T)$  in place of  $\mathrm{SL}(V_m)$ , we may compare our stability with that of holomorphic vector bundles. Recall that a holomorphic vector bundle splitting into a direct sum of stable vector bundles often appears in the boundary of a compactified moduli space of stable vector bundles. Similarly for our stability of manifolds, a splitting phenomenon occurs for  $V_m$  in (1.2). Roughly speaking, we consider the moduli space of all  $M$ 's with fixed decomposition data (1.2), where same type of construction of moduli spaces occurs typically for the Hodge decomposition in the variation of Hodge structures.

We now explain the difficulty which we encounter in applying the method of [15]. Such a difficulty comes up when we use the estimate of Phong and Sturm [21]. By applying a stability criterion in [15] of Hilbert-Mumford's type, we write the vector space  $\mathfrak{p}_m$  as an orthogonal direct sum

$$\mathfrak{p}_m = \mathfrak{p}'_m \oplus \mathfrak{p}''_m \quad (\text{cf. Section 3}),$$

and then check the stability of  $\hat{M}_m$  along the orbits of the one-parameter subgroups in  $G_m(T)$  generated by elements of  $\mathfrak{p}''_m$ . Though  $\mathfrak{p}$  and  $\mathfrak{p}''_m$  are transversal by the equality  $\mathfrak{p}'_m = \mathfrak{p}_m \cap \mathfrak{p}$ , we further need the orthogonality of  $\mathfrak{p}$  and  $\mathfrak{p}''_m$  in order to apply directly the estimate in [21]. Since such an orthogonality does not generally hold, we are in

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<sup>1</sup>For uniqueness of extremal Kähler metrics, Chen and Tian recently obtain a more general result without any projectivity condition.

trouble, but still the situation is not so bad (see (3.17), (3.18)), and this overcomes the difficulty.

### 1. Reduction of Main Theorem

In this section, by introducing necessary notation, we reduce the proof of Main Theorem to showing Theorems A and B below. Throughout this paper, we fix once for all a pair  $(M, L)$  of a connected projective algebraic manifold  $M$  and an ample holomorphic line bundle  $L$  over  $M$  as in the introduction. For  $V_m$  in the introduction, we put  $N_m := \dim_{\mathbb{C}} V_m - 1$ , where the positive integer  $m$  is such that  $L^m$  is very ample. Let  $n$  and  $d$  be respectively the dimension of  $M$  and the degree of the image  $M_m := \Phi_m(M)$  in the projective space  $\mathbb{P}^*(V_m)$ . Fixing an  $H$ -linearization of  $L$  as in the introduction, we consider the associated representation:  $H \rightarrow \mathrm{PGL}(V_m)$ . Pulling it back by the finite unramified cover:  $\mathrm{SL}(V_m) \rightarrow \mathrm{PGL}(V_m)$ , we obtain an isogeny

$$(1.1) \quad \iota: \tilde{H} \rightarrow H,$$

where  $\tilde{H}$  is an algebraic subgroup of  $\mathrm{SL}(V_m)$ . On the other hand, for an algebraic torus  $T$  in  $H$ , the  $H$ -linearization of  $L$  naturally induces a faithful representation

$$H \rightarrow \mathrm{GL}(V_m),$$

and this gives a  $T$ -action on  $V_m$  for each  $m$ . Then we have a finite subset  $\Gamma_m = \{\chi_1, \chi_2, \dots, \chi_{v_m}\}$  of the free Abelian group  $\mathrm{Hom}(T, \mathbb{C}^*)$  of all characters of  $T$  such that the vector space  $V_m = H^0(M, \mathcal{O}(L^m))$  is uniquely written as a direct sum

$$(1.2) \quad V_m = \bigoplus_{k=1}^{v_m} V_T(\chi_k),$$

where for each  $\chi \in \mathrm{Hom}(T, \mathbb{C}^*)$ , we set  $V_T(\chi) := \{s \in V_m; t \cdot s = \chi(t)s \text{ for all } t \in T\}$ . Define an algebraic subgroup  $G_m = G_m(T)$  of  $\mathrm{SL}(V_m)$  by

$$(1.3) \quad G_m := \prod_{k=1}^{v_m} \mathrm{SL}(V_T(\chi_k)),$$

and the associated Lie subalgebra of  $\mathrm{sl}(V_m)$  will be denoted by  $\mathfrak{g}_m$ . Here,  $G_m$  and  $\mathfrak{g}_m$  possibly depend on the choice of the algebraic torus  $T$ , and if necessary, we denote these by  $G_m(T)$  and  $\mathfrak{g}_m(T)$ , respectively. The  $T$ -action on  $V_m$  is, more precisely, a right action, while the  $G_m$ -action on  $V_m$  is a left action. Since  $T$  is Abelian, this  $T$ -action on  $V_m$  can be regarded also as a left action. Note that the group  $G_m$  acts diagonally on  $V_m$  in such a way that, for each  $k$ , the  $k$ -th factor  $\mathrm{SL}(V_T(\chi_k))$  of  $G_m$  acts just on the  $k$ -th factor  $V_T(\chi_k)$  of  $V_m$ . We now put

$$W_m := \{S^d(V_m)\}^{\otimes n+1},$$

where  $S^d(V_m)$  denotes the  $d$ -th symmetric tensor product of  $V_m$ . To the image  $M_m$  of  $M$ , we can associate a nonzero element  $\hat{M}_m$  in  $W_m^*$  such that the corresponding element  $[\hat{M}_m]$  in  $\mathbb{P}^*(W_m)$  is the Chow point of the irreducible reduced algebraic cycle  $M_m$  on  $\mathbb{P}^*(V_m)$ . Note that the  $G_m$ -action on  $V_m$  naturally induces a  $G_m$ -action on  $W_m$  and also on  $W_m^*$ . As in [14], the subvariety  $M_m$  of  $\mathbb{P}^*(V_m)$  is said to be *stable relative to  $T$*  or *semistable relative to  $T$* , according as the orbit  $G_m \cdot \hat{M}_m$  is closed in  $W_m^*$  or the closure of  $G_m \cdot \hat{M}_m$  in  $W_m^*$  does not contain the origin of  $W_m^*$ .

Let  $\Delta$  be the set of all algebraic subtori  $T$  of  $Z^\mathbb{C}$ . Take a Hermitian metric  $h_0$  for  $L$  such that  $c_1(L; h_0)$  is the extremal Kähler metric  $\omega_0$  in the Main Theorem. Let  $E$  be the extremal Kähler vector field for  $(M, \omega_0)$ , and let  $\mathfrak{k}$  be the Lie algebra of  $K$ . Let  $K^\mathbb{C}$  be the complexification of  $K$  in  $H$ . For  $\omega_0$  above, we further define  $\Delta_{\min}$  as the set of all  $T \in \Delta$  for which the statement of Theorem B in [14] is valid. Then, as the procedure in Section 6 of [14] shows, there exists a unique minimal element<sup>2</sup>, denoted by  $T_0$ , of  $\Delta_{\min}$  such that

$$\Delta_{\min} = \{T \in \Delta; T_0 \subset T\}.$$

For each  $T \in \Delta_{\min}$ , we put  $\tilde{T} := \iota^{-1}(T)$  and  $\tilde{Z}^\mathbb{C} := \iota^{-1}(Z^\mathbb{C})$ , and let  $G'_m(T)$  and  $Z'_m(T)$  be the identity components of  $G_m(T) \cap \tilde{H}$  and  $G_m(T) \cap \tilde{Z}^\mathbb{C}$ , respectively.

**DEFINITION 1.4.** For an algebraic torus  $T$  in  $\Delta_{\min}$ , we say that  $T$  is *irredundant*, if for all sufficiently large positive integers  $m$ ,  $\dim_{\mathbb{C}} K^\mathbb{C} = \dim_{\mathbb{C}} G'_m(T) + \dim_{\mathbb{C}} T$  (or equivalently  $\dim_{\mathbb{C}} Z^\mathbb{C} = \dim_{\mathbb{C}} Z'_m(T) + \dim_{\mathbb{C}} T$ ).

For instance,  $Z'_m(T) = \{1\}$  if  $T = Z^\mathbb{C}$ . In particular,  $Z^\mathbb{C}$  is irredundant. We now define subsets  $\Delta_0$  and  $\Delta_1$  of  $\Delta_{\min}$  by

$$\begin{aligned} \Delta_0 &: \text{the set of all irredundant elements in } \Delta_{\min}, \\ \Delta_1 &:= \{T \in \Delta_{\min}; G'_m(T) \cdot \hat{M}_m = \hat{M}_m \text{ for all } m \gg 1\}. \end{aligned}$$

**DEFINITION 1.5.** Let  $\Delta_L$  denote the set of all algebraic subtori  $T$  of  $Z^\mathbb{C}$  for which the statement of Main Theorem is valid.

Note that, if  $T'$  and  $T''$  are algebraic subtori of  $Z^\mathbb{C}$  with  $T' \subset T''$  and  $T' \in \Delta_L$ , then the stability criterion of Hilbert-Mumford type (cf. [14], Theorem 3.2) shows that  $T''$  also belongs to  $\Delta_L$ . We now pose the following:

**Theorem A.** *The algebraic torus  $Z^\mathbb{C}$  belongs to  $\Delta_1$ .*

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<sup>2</sup>The algebraic torus  $T_0$  is actually the closure in  $Z^\mathbb{C}$  of the complex Lie subgroup generated by the vector fields  $E, F_k$ ,  $k = 1, 2, \dots$  which appear in the asymptotic approximation (cf. (3.1) below) of the weighted analogues (cf. [14], 2.6) of balanced metrics.

**Theorem B.**  $\Delta_L \cap \Delta_0 = \Delta_0 \cap \Delta_1$ .

Once these theorems are proved, then by Theorem A, we have  $Z^\mathbb{C} \in \Delta_0 \cap \Delta_1$ . This together with Theorem B implies that  $Z^\mathbb{C} \in \Delta_L$ , completing the proof of Main Theorem.

If the extremal Kähler metric  $\omega_0$  above has a constant scalar curvature, and if the obstruction as in [13] vanishes, then we have both  $\Delta_1 = \Delta_{\min}$  and  $\{1\} \in \Delta_0$ . Hence in this case, Theorem B shows that  $\{1\} \in \Delta_L$ . This then proves the main theorem in [15].

It is very likely that the set  $\Delta_L$  has a natural minimal element closely related to the algebraic torus  $T_0$ . To see this, let us consider the case where  $M$  is an extremal Kähler toric Fano surface polarized by  $L = K_M^{-1}$ . Then  $M$  is possibly a complex projective plane blown up at  $r$  points with  $r \leq 3$ . If  $r = 0$  or  $3$ , then  $M$  admits a Kähler-Einstein metric, and  $\Delta_L$  has the unique minimal element  $\{1\}$  ( $= T_0$ ). On the other hand, if  $r = 1$ , then  $T_0$  coincides with  $Z^\mathbb{C}$ , and is the one-dimensional algebraic torus generated by the extremal Kähler vector field. Hence, in this case,  $\Delta_L$  has the unique minimal element  $T_0$ . Finally for  $r = 2$ , the involutive holomorphic symmetry of  $M$  switching the blown-up points allows us to regard  $T_0$  as the one-dimensional algebraic torus generated by the extremal Kähler vector field. It then follows that  $T_0$  again has to be a minimal element of  $\Delta_L$ .

## 2. Proof of Theorem A

In this section, we first prove Theorem A, and then make several definitions with a lemma added. Put  $\tilde{K}^\mathbb{C} := \iota^{-1}(K^\mathbb{C})$ .

Proof of Theorem A. In this proof, let  $T = Z^\mathbb{C}$ , and consider the associated set  $\Gamma_m = \{\chi_1, \chi_2, \dots, \chi_{v_m}\}$  of characters for  $m \gg 1$ . Since  $K^\mathbb{C}$  commutes with  $Z^\mathbb{C}$ , we have

$$(2.1) \quad \tilde{K}^\mathbb{C} \subset \mathrm{SL}(V_m) \cap \prod_{k=1}^{v_m} \mathrm{GL}(V_{Z^\mathbb{C}}(\chi_k)).$$

Recall that the extremal Kähler vector field  $E$  belongs to the Lie algebra of  $T_0$ . Hence, a theorem of Calabi [1] shows that  $G'_m(Z^\mathbb{C}) \subset G'_m(T_0) \subset \tilde{K}^\mathbb{C}$ . Hence,

$$(2.2) \quad G'_m(Z^\mathbb{C}) \cdot \tilde{Z}^\mathbb{C} \subset \tilde{K}^\mathbb{C}.$$

To complete the proof of Theorem A, we compare two groups  $[\tilde{K}^\mathbb{C}, \tilde{K}^\mathbb{C}]$  and  $G'_m(Z^\mathbb{C})$ . By (2.1), we obviously have  $[\tilde{K}^\mathbb{C}, \tilde{K}^\mathbb{C}] \subset G'_m(Z^\mathbb{C})$ . On the other hand,

$$\dim_{\mathbb{C}} [\tilde{K}^\mathbb{C}, \tilde{K}^\mathbb{C}] = \dim_{\mathbb{C}} \tilde{K}^\mathbb{C} - \dim_{\mathbb{C}} \tilde{Z}^\mathbb{C} \geq \dim_{\mathbb{C}} G'_m(Z^\mathbb{C}),$$

where the last inequality follows from (2.2) in view of the fact that the intersection of  $G'_m(Z^\mathbb{C})$  and  $\tilde{Z}^\mathbb{C}$  is a finite group. It now follows that  $G'_m(Z^\mathbb{C})$  coincides with  $[\tilde{K}^\mathbb{C}, \tilde{K}^\mathbb{C}]$ . Hence  $G'_m(Z^\mathbb{C}) \cdot \hat{M}_m = \hat{M}_m$ . Then by  $T_0 \subset Z^\mathbb{C}$ , we now obtain  $Z^\mathbb{C} \in \Delta_1$ , as required.  $\square$

Let  $h$  be a Hermitian metric for  $L$  such that  $\omega := c_1(L; h)$  is a  $K$ -invariant Kähler metric on  $M$ . Define a Hermitian metric  $\rho_h$  on  $V_m$  by

$$(2.3) \quad \rho_h(s, s') := \int_M (s, s')_{h^m} \omega^n, \quad s, s' \in V_m,$$

where  $(s, s')_{h^m}$  denotes the function on  $M$  obtained as the pointwise inner product of  $s, s'$  by  $h^m$ . Let  $\mathcal{S} := \{s_0, s_1, \dots, s_{N_m}\}$  be an orthonormal basis for  $V_m$  satisfying

$$\rho_h(s_i, s_j) = \delta_{ij}.$$

Let  $T \in \Delta_0 \cap \Delta_1$ . Then we say that  $\mathcal{S}$  is  $T$ -admissible, if each  $V_T(\chi_k)$ ,  $k = 1, 2, \dots$ , admits a basis  $\{s_{k,i}; i = 1, 2, \dots, n_k\}$  such that

$$(2.4) \quad s_{l(k,i)} = s_{k,i}, \quad i = 1, 2, \dots, n_k; \quad k = 1, 2, \dots, v_m,$$

where  $n_k := \dim_{\mathbb{C}} V_T(\chi_k)$ , and  $l(k, i) := (i-1) + \sum_{k'=1}^{k-1} n_{k'}$  for all  $k$  and  $i$  (cf. [14]). Let  $\mathfrak{t}_c := \text{Lie}(T_c)$  denote the Lie algebra of the maximal compact subgroup  $T_c$  of  $T$ . Put  $q := 1/m$  and  $\mathfrak{t}_{\mathbb{R}} := \sqrt{-1}\mathfrak{t}_c$ . For each  $F \in \mathfrak{t}_{\mathbb{R}}$ , we define

$$(2.5) \quad B_q(\omega, F) := \frac{n!}{m^n} \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} e^{-d\chi_k(F)} |s_{k,i}|_{h^m}^2,$$

where  $|s|_{h^m}^2 := (s, s)_{h^m}$  for all  $s \in V_m$ , and  $d\chi_k: \mathfrak{t}_{\mathbb{R}} \rightarrow \mathbb{R}$  denotes the restriction to  $\mathfrak{t}_{\mathbb{R}}$  of the differential at  $t = 1$  for the character  $\chi_k \in \text{Hom}(T, \mathbb{C}^*)$ .

As a final remark in this section, we give an upper bound for degrees of the characters in  $\Gamma_m$ . Let  $T$  be an algebraic torus sitting in  $Z^{\mathbb{C}}$ . By setting  $r := \dim_{\mathbb{C}} T$ , we identify  $T$  with the multiplicative group  $(\mathbb{C}^*)^r := \{t = (t_1, t_2, \dots, t_r); t_j \in \mathbb{C}^* \text{ for all } j\}$ . Since each  $\chi_k$  in (1.2) may depend on  $m$ , the character  $\chi_k$  will be rewritten as  $\chi_{m;k}$  until the end of this section. Then for each  $k \in \{1, 2, \dots, v_m\}$ ,

$$\chi_{m;k}(t) = \prod_{i=1}^r t_i^{\alpha(m, k, i)}, \quad t = (t_1, t_2, \dots, t_r) \in T,$$

for some integers  $\alpha(m, k, i)$  independent of the choice of  $t$ . Define a nonnegative integer  $\alpha_m$  by  $\alpha_m := \sup_{k=1}^{v_m} \sum_{i=1}^r |\alpha(m, k, i)|$ . Then we have the following upper bound for  $\alpha_m$ :

**Lemma 2.6.** *For all positive integers  $m$ , the inequality  $\alpha_m \leq m\alpha_1$  holds.*

Proof. Put  $S := \text{Ker } \text{pr}_m$ . Since the subspace  $S$  of  $\bigotimes^m V_1$  is preserved by the  $T$ -action, we have a  $T$ -invariant subspace, denoted by  $S^{\perp}$ , of  $\bigotimes^m V_1$  such that the

vector space  $\bigotimes^m V_1$  is written as a direct sum

$$\bigotimes^m V_1 = S \oplus S^\perp.$$

Then the restriction of  $\text{pr}_m$  to  $S^\perp$  defines a  $T$ -equivariant isomorphism  $S^\perp \cong V_m$ . On the other hand, the characters of  $T$  appearing in the  $T$ -action on  $\bigotimes^m V_1$  are

$$\chi_{\vec{k}}(t) := t_1^{\sum_{j=1}^m \alpha(1, k_j, 1)} t_2^{\sum_{j=1}^m \alpha(1, k_j, 2)} \cdots t_r^{\sum_{j=1}^m \alpha(1, k_j, r)}, \quad \vec{k} = (k_1, k_2, \dots, k_m) \in I^m,$$

where  $I^m$  is the Cartesian product of  $m$ -pieces of  $I := \{1, 2, \dots, v_m\}$ . Since  $S^\perp$  ( $\cong V_m$ ) is a subspace of  $\bigotimes^m V_1$ , we now obtain

$$\alpha_m \leq \max_{\vec{k} \in I^m} \sum_{i=1}^r \left| \sum_{j=1}^m \alpha(1, k_j, i) \right| \leq \max_{\vec{k} \in I^m} \sum_{j=1}^m \sum_{i=1}^r |\alpha(1, k_j, i)| \leq m\alpha_1,$$

as required.  $\square$

### 3. Proof of Theorem B

Fix an arbitrary element  $T$  of  $\Delta_0 \cap \Delta_1$ . Let  $m \gg 1$ . Then by [14], Theorem B, there exist  $F_k \in \mathfrak{t}_{\mathbb{R}}$ , real numbers  $\alpha_k \in \mathbb{R}$ , and smooth real-valued  $K$ -invariant functions  $\varphi_k$ ,  $k = 1, 2, \dots$ , on  $M$  such that, for each  $\ell \in \mathbb{Z}_{\geq 0}$ , we have

$$(3.1) \quad B_q(\omega(\ell), F(\ell)) = C_{q,\ell} + O(q^{\ell+2}), \quad m \gg 1,$$

where  $F(\ell) := (\sqrt{-1}E/2)q^2 + \sum_{j=0}^{\ell} q^{j+2}F_j$ ,  $h(\ell) := h_0 \exp(-\sum_{k=1}^{\ell} q^j \varphi_j)$ ,  $C_{q,\ell} := 1 + \sum_{j=0}^{\ell} \alpha_j q^{j+1}$ , and  $\omega(\ell) := c_1(L; h(\ell))$ . Let us now fix an arbitrary positive integer  $\ell$ . To each  $T$ -admissible orthonormal basis  $\mathcal{S} := \{s_0, s_1, \dots, s_{N_m}\}$  for  $(V_m; \rho_{h(\ell)})$ , we associate a basis  $\tilde{\mathcal{S}} := \{\tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{N_m}\}$  for  $V_m$  by

$$(3.2) \quad \tilde{s}_{k,i} = e^{-d\chi_k(F(\ell))/2} s_{k,i}, \quad i = 1, 2, \dots, n_k; \quad k = 1, 2, \dots, v_m,$$

where we put  $s_{l(k,i)} = s_{k,i}$  and  $\tilde{s}_{l(k,i)} = \tilde{s}_{k,i}$  by using the notation in (2.4).

**REMARK.** Lemma 2.6 above implies that  $|d\chi_{m;k}(F(\ell))| \leq C\alpha_1 q$  for some positive real constant  $C$  independent of the choice of  $m$  and  $k$ , where  $\chi_{m;k}$  is as in the last section. Hence in (3.1) above, for each fixed nonnegative integer  $\ell$ , there exists a positive constant  $C'$  independent of  $m$  and  $k$  such that

$$|e^{-d\chi_k(F(\ell))} - 1| \leq C'q, \quad k = 1, 2, \dots, v_m.$$

In particular, in (3.2) above, the integral  $\int_M \|\tilde{s}_{k,i}\|_{h^m}^2 \omega(\ell)^n$  ( $= e^{-d\chi_k(F(\ell))}$ ) converges to 1, uniformly in  $k$ , as  $m \rightarrow \infty$ .

We now consider the Kodaira embedding  $\Phi_m: M \rightarrow \mathbb{P}^*(V_m)$  defined by

$$\Phi_m(x) := (\tilde{s}_0(x) : \tilde{s}_1(x) : \cdots : \tilde{s}_{N_m}(x)), \quad x \in M,$$

where  $\mathbb{P}^*(V_m)$  is identified with  $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \cdots : z_{N_m})\}$  by the basis  $\tilde{\mathcal{S}}$ . Put  $M_m := \Phi_m(M)$ . Since  $\Delta_L \cap \Delta_0$  is a subset of  $\Delta_1 \cap \Delta_0$  (cf. [13], Section 3), the proof of Theorem B (and Main Theorem also) is reduced to showing the following assertion:

**Assertion.** *The orbit  $G_m(T) \cdot \hat{M}_m$  is closed in  $W_m^*$ .*

In the Hermitian vector space  $(V_m; \rho_{h(\ell)})$ , the subspaces  $V_T(\chi_k)$ ,  $k = 1, 2, \dots, v_m$ , are mutually orthogonal. Put

$$K_m := \prod_{k=1}^{v_m} \mathrm{SU}(V_T(\chi_k); \rho_{h(\ell)}), \quad \mathfrak{k}_m := \bigoplus_{k=1}^{v_m} \mathfrak{su}(V_T(\chi_k); \rho_{h(\ell)}).$$

Since  $T$  belongs to  $\Delta_1$ , the group  $G'_m = G'_m(T)$  coincides with the isotropy subgroup of  $G_m$  at  $\hat{M}_m \in W_m^*$ . Consider the Lie algebra  $\mathfrak{g}'_m := \mathrm{Lie}(G'_m)$  of  $G'_m$ . Put  $\mathfrak{h} := \mathrm{Lie}(H) = \mathrm{Lie}(\tilde{H})$ . Then by  $G'_m \subset \tilde{H} \subset \mathrm{SL}(V_m)$ , we have the inclusions

$$\mathfrak{g}'_m \hookrightarrow \mathfrak{h} \hookrightarrow \mathfrak{sl}(V_m).$$

Put  $\mathfrak{k}'_m := \mathrm{Lie}(K'_m)$ , where  $K'_m$  is the isotropy subgroup of  $K_m$  at the point  $\hat{M}_m \in W_m^*$ . Then  $\mathfrak{g}_m$  and  $\mathfrak{g}'_m$  are the complexifications of  $\mathfrak{k}_m$  and  $\mathfrak{k}'_m$ , respectively (cf. [1]). Put  $\mathfrak{p}_m := \sqrt{-1}\mathfrak{k}_m$  and  $\mathfrak{p}'_m := \sqrt{-1}\mathfrak{k}'_m$ . We further define

$$\mathfrak{K}_m := \left\{ \bigoplus_{k=1}^{v_m} \mathfrak{u}(V_T(\chi_k); \rho_{h(\ell)}) \right\} \cap \mathfrak{su}(V_m; \rho_{h(\ell)}), \quad \mathfrak{P}_m := \sqrt{-1}\mathfrak{K}_m.$$

By the above inclusions of Lie algebras (see also (2.1)), we can regard  $\mathfrak{p} := \sqrt{-1}\mathfrak{k}$  as a Lie subalgebra of  $\mathfrak{P}_m$ . Let  $\omega_{\mathrm{FS}}$  be the Fubini-Study metric on  $\mathbb{P}^*(V_m)$  defined by

$$\omega_{\mathrm{FS}} := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_{\alpha=0}^{N_m} |z_\alpha|^2 \right).$$

For each  $Q \in \mathfrak{P}_m$ , let  $\mathcal{Q}$  be the associated holomorphic vector field on  $\mathbb{P}^*(V_m)$ . By the notation for  $t = 0$  in Step 1 later in the proof of Assertion, we obtain a vector field  $\mathcal{Q}_{TM_m}$  on  $M_m$  via the orthogonal projection of  $\mathcal{Q}$  along  $M_m$  to tangential directions. Then we have a unique real-valued function  $\varphi_Q$  on  $\mathbb{P}^*(V_m)$  satisfying both  $\int_{\mathbb{P}^*(V_m)} \varphi_Q \omega_{\mathrm{FS}}^{N_m} = 0$  and

$$i_{\mathcal{Q}} \left( \frac{\omega_{\mathrm{FS}}}{m} \right) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \varphi_Q.$$

Let  $\square_{M, \text{FS}} := -\bar{\partial}^* \bar{\partial}$  denote the Laplacian on functions on the Kähler manifold  $(M, \Phi_m^* \omega_{\text{FS}})$ . Define a positive semidefinite  $K'_m$ -invariant inner product  $(\cdot, \cdot)$  on  $\mathfrak{P}_m$  by setting

$$\begin{aligned} (\mathcal{Q}_1, \mathcal{Q}_2) &:= \frac{1}{m^2} \int_{M_m} ((\mathcal{Q}_1)_{TM_m}, (\mathcal{Q}_2)_{TM_m})_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \\ &= \frac{\sqrt{-1}}{2\pi} \int_{M_m} \partial \varphi_{\mathcal{Q}_2} \wedge \bar{\partial} \varphi_{\mathcal{Q}_1} \wedge n \omega_{\text{FS}}^{n-1} = \int_{M_m} (\bar{\partial} \varphi_{\mathcal{Q}_1}, \bar{\partial} \varphi_{\mathcal{Q}_2})_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \\ &= - \int_M \varphi_{\mathcal{Q}_1} (\square_{M, \text{FS}} \varphi_{\mathcal{Q}_2}) \Phi_m^* \omega_{\text{FS}}^n \in \mathbb{R} \end{aligned}$$

for all  $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathfrak{P}_m$ . Restrict this inner product to  $\mathfrak{p}_m$ . Then the inner product  $(\cdot, \cdot)$  on  $\mathfrak{p}_m$  is positive definite on  $\mathfrak{p}$  and hence on  $\mathfrak{p}'_m$ . As vector spaces,  $\mathfrak{P}_m$  and  $\mathfrak{p}_m$  are written respectively as orthogonal direct sums

$$\mathfrak{P}_m = \mathfrak{p} \oplus \mathfrak{p}^\perp, \quad \mathfrak{p}_m = \mathfrak{p}'_m \oplus \mathfrak{p}''_m,$$

where  $\mathfrak{p}^\perp$  is the orthogonal complement of  $\mathfrak{p}$  in  $\mathfrak{P}_m$ , and moreover  $\mathfrak{p}''_m$  is the orthogonal complements of  $\mathfrak{p}'_m$  in  $\mathfrak{p}_m$  (cf. [15]). Hence if  $\mathcal{Q} \in \mathfrak{p}^\perp$ , then for any holomorphic vector field  $\mathcal{W}$  on  $M_m$ , we have

$$\begin{aligned} &\frac{1}{m^2} \int_{M_m} (\mathcal{Q}_{TM_m}, \mathcal{W}_{TM_m})_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \\ &= \int_{M_m} (\bar{\partial} \varphi_{\mathcal{Q}}, \bar{\partial} \theta_0 + \bar{\partial}(\varphi_{W^1} + \sqrt{-1}\varphi_{W^2}))_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \\ &= \frac{\sqrt{-1}}{2\pi} \int_{M_m} \{\theta_0 \wedge \bar{\partial} \varphi_{\mathcal{Q}} + \partial(\varphi_{W^1} - \sqrt{-1}\varphi_{W^2}) \wedge \bar{\partial} \varphi_{\mathcal{Q}}\} \wedge n \omega_{\text{FS}}^{n-1} = 0, \end{aligned}$$

where  $i_{\mathcal{W}}(\omega_{\text{FS}}/m)$  on  $M_m$  is known to be expressible as  $\bar{\partial} \theta_0 + \bar{\partial}(\varphi_{W^1} + \sqrt{-1}\varphi_{W^2})$  for some holomorphic 1-form  $\theta_0$  on  $M_m$  and elements  $W^1, W^2$  in  $\mathfrak{p}$ . We consider the open neighbourhood (cf. [15])

$$U_m := \{X \in \mathfrak{p}''_m; \zeta(\text{ad } X)\mathfrak{p}'_m \cap \mathfrak{p}''_m = \{0\}\}$$

of the origin in  $\mathfrak{p}''_m$ , where  $\zeta: \mathbb{R} \rightarrow \mathbb{R}$  is a real analytic function defined by  $\zeta(x) := x(e^x + e^{-x})/(e^x - e^{-x})$ ,  $x \neq 0$ , and  $\zeta(0) = 0$ . By operating  $(\sqrt{-1}/2\pi) \partial \bar{\partial} \log$  on both sides of (3.1), we obtain

$$(3.3) \quad \Phi_m^* \omega_{\text{FS}} \equiv m \omega(\ell), \quad \text{mod } q^{\ell+2}.$$

For an element  $X$  of  $\mathfrak{P}_m$  (later we further assume  $X \in \mathfrak{p}''_m$ ), there exists a  $T$ -admissible orthonormal basis  $\mathcal{T} := \{\tau_0, \tau_1, \dots, \tau_{N_m}\}$  for  $(V_m, \rho_{h(\ell)})$  such that the infinitesimal action of  $X$  on  $V_m$  can be diagonalized in the form

$$X \cdot \tau_\alpha = \gamma_\alpha(X) \tau_\alpha$$

for some real constants  $\gamma_\alpha = \gamma_\alpha(X)$ ,  $\alpha = 0, 1, \dots, N_m$ , satisfying  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X) = 0$ . As in (3.2), we consider the associated basis  $\tilde{\mathcal{T}} = \{\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{N_m}\}$  for  $V_m$ , where  $\tilde{\tau}_{k,i} := e^{-d\chi_k(F(\ell))/2} \tau_{k,i}$ . By setting

$$\lambda_X(e^t) := \exp(tX), \quad t \in \mathbb{R},$$

we consider the one-parameter group  $\lambda_X: \mathbb{R}_+ \rightarrow \{\prod_{k=1}^{N_m} \text{GL}(V_T(\chi_k))\} \cap \text{SL}(V_m)$  associated to  $X$ . Then  $\lambda_X(e^t) \cdot \tau_\alpha = e^{t\gamma_\alpha} \tau_\alpha$  for all  $\alpha$  and all  $t \in \mathbb{R}$ . Moreover,

$$(3.4) \quad \Phi_m^* \varphi_X = \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(X) |\tilde{\tau}_\alpha|^2}{m \sum_{\alpha=0}^{N_m} |\tilde{\tau}_\alpha|^2}, \quad X \in \mathfrak{P}_m.$$

Let  $\eta_m$  be the Kähler form on  $M$  defined by  $\eta_m := (1/m) \Phi_m^* \omega_{\text{FS}}$ . To each  $X \in \mathfrak{P}_m$ , we can associate a real constant  $c_X$  such that  $\phi_X := c_X + \Phi_m^* \varphi_X$  on  $M$  satisfies

$$\int_M \phi_X \eta_m^n = 0.$$

**Proof of Assertion.** Fix an arbitrary element  $0 \neq X$  of  $\mathfrak{p}_m''$ , and define a real-valued function  $f_{X,m}(t)$  on  $\mathbb{R}$  by

$$f_{X,m}(t) := \log \|\lambda_X(e^t) \cdot \hat{M}_m\|_{\text{CH}(\rho_{\mathbb{R}(\ell)})}.$$

For this  $X$ , we consider the associated  $\gamma_\alpha(X)$ ,  $\alpha = 0, 1, \dots, N_m$ , defined in the above. From now on,  $X$  regarded as a holomorphic vector field on  $\mathbb{P}^*(V_m)$  will be denoted by  $\mathcal{X}$ . By [26] (see also [14], 4.5), we have  $\ddot{f}_{X,m}(t) \geq 0$  for all  $t$ . Then by [15], Lemma 3.4, it suffices to show the existence of a real number  $t_X^{(m)}$  such that

$$(3.5) \quad \dot{f}_{X,m}(t_X^{(m)}) = 0 < \ddot{f}_{X,m}(t_X^{(m)}) \quad \text{and} \quad t_X^{(m)} \cdot X \in U_m.$$

In the below, real numbers  $C_i$ ,  $i = 1, 2, \dots$ , always mean positive real constants independent of the choice of  $m$  and  $X$ . Moreover by abuse of terminology, we write  $m \gg 1$ , if  $m$  satisfies  $m \geq m_0$  for a sufficiently large  $m_0$  independent of the choice of  $X$ . Then the proof of Assertion will be divided into the following eight steps:

**STEP 1.** Put  $\lambda_t := \lambda_X(e^t)$  and  $M_{m,t} := \lambda_t(M_m)$  for each  $t \in \mathbb{R}$ . Metrically, we identify the normal bundle of  $M_{m,t}$  in  $\mathbb{P}^*(V_m)$  with the subbundle  $TM_{m,t}^\perp$  of  $T\mathbb{P}^*(V_m)|_{M_{m,t}}$  obtained as the orthogonal complement of  $TM_{m,t}$  in  $T\mathbb{P}^*(V_m)|_{M_{m,t}}$ . Hence,  $T\mathbb{P}^*(V_m)|_{M_{m,t}}$  is differentiably written as the direct sum  $TM_{m,t} \oplus TM_{m,t}^\perp$ . Associated to this, the restriction  $\mathcal{X}|_{M_{m,t}}$  of  $\mathcal{X}$  to  $M_{m,t}$  is written as

$$\mathcal{X}|_{M_{m,t}} = \mathcal{X}_{TM_{m,t}} \oplus \mathcal{X}_{TM_{m,t}^\perp}$$

for some smooth sections  $\mathcal{X}_{TM_{m,t}}$  and  $\mathcal{X}_{TM_{m,t}^\perp}$  of  $TM_{m,t}$  and  $TM_{m,t}^\perp$ , respectively. Then the second derivative  $\ddot{f}_{X,m}(t)$  is (see for instance [14], [21]) given by

$$(3.6) \quad \ddot{f}_{X,m}(t) = \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n \geq 0.$$

Since the Kodaira embedding  $\Phi^{\tilde{\mathcal{T}}}: M \rightarrow \mathbb{P}^{N_m}(\mathbb{C})$  defined by

$$\Phi^{\tilde{\mathcal{T}}}(p) := (\tilde{\tau}_0(p) : \tilde{\tau}_1(p) : \dots : \tilde{\tau}_{N_m}(p))$$

coincides with  $\Phi_m$  above up to an isometry of  $(V_m, \rho_{h(\ell)})$ , we may assume without loss of generality that  $\Phi^{\tilde{\mathcal{T}}}$  is chosen as  $\Phi_m$ .

STEP 2. In view of the orthogonal decomposition  $\mathfrak{P}_m = \mathfrak{p}^\perp \oplus \mathfrak{p}$ , we can express  $X$  as an orthogonal sum

$$X = X' + X''$$

for some  $X' \in \mathfrak{p}$  and  $X'' \in \mathfrak{p}^\perp$ . Since  $\omega(\ell)$  is  $K$ -invariant (cf. [14]), the group  $K$  acts isometrically on  $(V_m, \rho_{h(\ell)})$ . Now, there exists a  $T$ -admissible orthonormal basis  $\mathcal{B} := \{\beta_0, \beta_1, \dots, \beta_{N_m}\}$  for  $V_m$  such that the infinitesimal action of  $X''$  on  $V_m$  is written as

$$X'' \cdot \beta_\alpha = \gamma_\alpha(X'') \beta_\alpha, \quad \alpha = 0, 1, \dots, N_m,$$

for some real constants  $\gamma_\alpha(X'')$ ,  $\alpha = 0, 1, \dots, N_m$ , satisfying  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X'') = 0$ . By the notation as in (3.2), we consider the associated basis  $\tilde{\mathcal{B}} := \{\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_{N_m}\}$  for  $V_m$ . Then

$$(3.7) \quad \phi_{X''} = \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'') |\tilde{\beta}_\alpha|^2}{m \sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2},$$

where  $\hat{\gamma}_\alpha(X'') := \gamma_\alpha(X'') + mc_{X''}$ . Now,  $X'$  and  $X''$  regarded as holomorphic vector fields on  $\mathbb{P}^*(V_m)$  will be denoted by  $\mathcal{X}'$  and  $\mathcal{X}''$ , respectively. Associated to the expression  $T\mathbb{P}^*(V_m)|_{M_{m,t}} = TM_{m,t} \oplus TM_{m,t}^\perp$  as differentiable vector bundles, the restrictions  $\mathcal{X}'|_{M_{m,t}}$ ,  $\mathcal{X}''|_{M_{m,t}}$  of  $\mathcal{X}'$  and  $\mathcal{X}''$  to  $M_{m,t}$  are respectively written as

$$\mathcal{X}'|_{M_{m,t}} = \mathcal{X}'_{TM_{m,t}} \oplus \mathcal{X}'_{TM_{m,t}^\perp}$$

and

$$\mathcal{X}''|_{M_{m,t}} = \mathcal{X}''_{TM_{m,t}} \oplus \mathcal{X}''_{TM_{m,t}^\perp},$$

where  $\mathcal{X}'_{TM_{m,t}}$ ,  $\mathcal{X}''_{TM_{m,t}}$  are smooth sections of  $TM_{m,t}$ , and  $\mathcal{X}'_{TM_{m,t}^\perp}$ ,  $\mathcal{X}''_{TM_{m,t}^\perp}$  are smooth sections of  $TM_{m,t}^\perp$ . Then by  $X' \in \mathfrak{p}$ , we have

$$(3.8) \quad \mathcal{X}'_{TM_{m,t}^\perp} = 0, \quad \text{i.e.,} \quad \mathcal{X}_{TM_{m,t}^\perp} = \mathcal{X}''_{TM_{m,t}^\perp}.$$

STEP 3. Since  $T$  is irredundant, we have  $\mathfrak{g}'_m(T) + \mathfrak{t} = \mathfrak{k}^{\mathbb{C}}$ , i.e.,  $\mathfrak{p}'_m + \sqrt{-1}\mathfrak{t}_c = \mathfrak{p}$ , where these are equalities as Lie subalgebras of  $\mathfrak{h}$ . From now on until the end of this step, as in the preceding steps, we regard both  $\mathfrak{p}'_m$  and  $\mathfrak{k}^{\mathbb{C}}$  as Lie subalgebras of  $\mathfrak{sl}(V_m)$ . Hence, as Lie subalgebras of  $\mathfrak{sl}(V_m)$ , we have

$$\mathfrak{p} = \mathfrak{p}'_m + \sqrt{-1}\tilde{\mathfrak{t}}_c,$$

where we put  $\tilde{\mathfrak{t}}_c := \text{Lie}(\tilde{T}_c)$  for the maximal compact subgroup  $\tilde{T}_c$  of  $\tilde{T} := \iota^{-1}(T)$ . Then we can write  $X' \in \mathfrak{p}$  as a sum

$$X' = Y + W$$

for some  $Y \in \mathfrak{p}'_m$  and some  $W \in \sqrt{-1}\tilde{\mathfrak{t}}_c$ . Note that the holomorphic vector fields  $\mathcal{Y}$  and  $\mathcal{W}$  on  $\mathbb{P}^*(V_m)$  induced by  $Y$  and  $W$ , respectively, are tangent to  $M$ . By  $[Y, W] = 0$ , there exists a  $T$ -admissible orthonormal basis  $\{\sigma_0, \sigma_1, \dots, \sigma_{N_m}\}$  for  $V_m$  such that

$$\begin{cases} Y \cdot \sigma_{\alpha} = \gamma_{\alpha}(Y)\sigma_{\alpha}, & \alpha = 0, 1, \dots, N_m; \\ W \cdot \sigma_{k,i} = b_k \sigma_{k,i}, & k = 1, 2, \dots, v_m, \end{cases}$$

for some real constants  $\gamma_{\alpha}(Y)$  and  $b_k$ , where in the last equality, we put  $\sigma_{l(k,i)} := \sigma_{l(k,i)}$  by using the same notation  $l(k, i)$  as in (2.4). By setting  $\tilde{\sigma}_{k,i} = e^{-d\chi_k(F(\ell))/2}\sigma_{k,i}$ , we later consider the basis  $\{\tilde{\sigma}_0, \tilde{\sigma}_1, \dots, \tilde{\sigma}_{N_m}\}$  for  $V_m$ . Note that  $\sum_{\alpha=0}^{N_m} \gamma_{\alpha}(Y) = \sum_{k=1}^{v_m} n_k b_k = 0$ . Since both  $X$  and  $Y$  belong to  $\mathfrak{p}_m$ , it follows from  $X = X' + X'' = Y + W + X''$  that

$$\sum_{i=1}^{n_k} \gamma_{k,i}(X) = \sum_{i=1}^{n_k} \gamma_{k,i}(Y) = 0 \quad \text{and} \quad n_k b_k = - \sum_{i=1}^{n_k} \gamma_{k,i}(X'') \quad \text{for all } k,$$

where  $\gamma_{k,i} := \gamma_{l(k,i)}$  with  $l(k, i)$  as in (2.4). Note that  $X \in \mathfrak{p}_m''$  and  $X'' \in \mathfrak{p}^{\perp}$ , where by  $\mathfrak{p}'_m + \sqrt{-1}\mathfrak{t} = \mathfrak{p}$ , the space  $\mathfrak{p}^{\perp}$  is perpendicular to  $\mathfrak{p}'_m$ . Then by  $Y \in \mathfrak{p}'_m$  and  $X = Y + W + X''$ , we have

$$(Y, Y) + (Y, W) = (Y, X) = 0$$

in terms of the inner product  $(\ , \ )$  on  $\mathfrak{P}_m$ . Hence  $(Y, Y) = -(Y, W) \leq \sqrt{(Y, Y)(W, W)}$ . It now follows that

$$(3.9) \quad \int_{M_m} |\mathcal{Y}|_{M_m}|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n = m^2(Y, Y) \leq m^2(W, W) = \int_{M_m} |\mathcal{W}|_{M_m}|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n.$$

The integral on the right-hand side is, for  $m \gg 1$ ,

$$\begin{aligned} & \int_M \frac{\left( \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} |\tilde{\sigma}_{k,i}|_{h(\ell)}^2 \right) \left( \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} b_k^2 |\tilde{\sigma}_{k,i}|_{h(\ell)}^2 \right) - \left( \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} b_k |\tilde{\sigma}_{k,i}|_{h(\ell)}^2 \right)^2}{\left( \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} |\tilde{\sigma}_{k,i}|_{h(\ell)}^2 \right)^2} m^n \eta_m^n \\ & \leq m^n \int_M \frac{\sum_{k=1}^{v_m} \sum_{i=1}^{n_k} b_k^2 |\tilde{\sigma}_{k,i}|_{h(\ell)}^2}{\sum_{k=1}^{v_m} \sum_{i=1}^{n_k} |\tilde{\sigma}_{k,i}|_{h(\ell)}^2} \eta_m^n \\ & \leq \frac{n!}{2} \int_M \sum_{k=1}^{v_m} \sum_{i=1}^{n_k} b_k^2 |\tilde{\sigma}_{k,i}|_{h(\ell)}^2 \eta_m^n \leq C_1 \sum_{k=1}^{v_m} n_k b_k^2 \end{aligned}$$

for some  $C_1$ , where in the last two inequalities, we used the Remark right after (3.2). Hence, by setting  $\hat{\gamma}_\alpha(Y) := \gamma_\alpha(Y) + mc_Y$ , we see from (3.9) that

$$\begin{aligned} (3.10) \quad & \int_M \frac{\left( \sum_{\alpha=0}^{N_m} |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right) \left( \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2 |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right) - \left( \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y) |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right)^2}{\left( \sum_{\alpha=0}^{N_m} |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right)^2} \eta_m^n \\ & \leq q^n C_1 \sum_{k=1}^{v_m} n_k b_k^2. \end{aligned}$$

Define real numbers  $f_1$  and  $f_2$  by

$$\begin{cases} f_1 := \int_M \left( \sum_{\alpha=0}^{N_m} |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right)^{-1} \left( \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2 |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right) \eta_m^n, \\ f_2 := \int_M \left\{ \left( \sum_{\alpha=0}^{N_m} |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right)^{-1} \left( \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y) |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right) \right\}^2 \eta_m^n. \end{cases}$$

If  $f_1 \geq 2f_2$ , then by (3.10) and the Remark right after (3.2), we have

$$\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2 \leq C_2 \sum_{k=1}^{v_m} n_k b_k^2, \quad \text{if } m \gg 1,$$

for some  $C_2$ . Next, assume  $f_1 < 2f_2$ . Then for

$$\phi_Y := \left( m \sum_{\alpha=0}^{N_m} |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right)^{-1} \left( \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y) |\tilde{\sigma}_\alpha|_{h(\ell)}^2 \right),$$

the left-hand side of (3.10) divided by  $m \gg 1$  is written as

$$\int_{M_m} |\mathcal{Y}|_{M_m}|_{(\omega_{\text{FS}}/m)}^2 \left( \frac{\omega_{\text{FS}}}{m} \right)^n = \int_M |\bar{\partial} \phi_Y|_{\eta_m}^2 \eta_m^n \geq C_3 \int_M \phi_Y^2 \eta_m^n,$$

for some  $C_3$ , because the Kähler manifolds  $(M, \eta_m)$ ,  $m \gg 1$ , have bounded geometry (see also the Remark right after (3.2)). Hence, by  $f_1 < 2f_2$  and (3.10), we see that, for  $m \gg 1$ ,

$$\begin{aligned} q^n C_1 \sum_{k=1}^{v_m} n_k b_k^2 &\geq m C_3 \int_M \phi_Y^2 \eta_m^n = C_3 q f_2 \\ &> \frac{C_3 q f_1}{2} \geq C_4 q^{n+1} \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2 \end{aligned}$$

for some  $C_4$ , where in the last inequality, we used the Remark right after (3.2). By  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(Y) = 0$ , we here observe that  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 \leq \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(Y)^2$ . Hence, whether  $f_1 \geq 2f_2$  or not, there always exists  $C_5$  such that, for  $m \gg 1$ ,

$$(3.11) \quad q \sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 \leq C_5 \sum_{k=1}^{v_m} n_k b_k^2.$$

STEP 4. Put  $P := W + X''$ . In view of [26], Theorem 1.6, a weighted version of (3.4.2) in [26] is true (cf. [14], [19]). Hence by  $T \in \Delta_1$ , we obtain

$$(3.12) \quad \dot{f}_{X,m}(0) = \dot{f}_{P,m}(0) = (n+1) \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|_{h(\ell)}^2} \Phi_m^* \omega_{\text{FS}}^n,$$

where  $\gamma_{k,i}(P) := b_k + \gamma_{k,i}(X'')$  ( $= \gamma_{l(k,i)}(P)$ ). Let  $C_{q,\ell} = 1 + \sum_{j=0}^{\ell} \alpha_j q^{j+1}$  be as in (3.1). Then by (3.1) and (3.3), there exist a function  $u_{m,\ell}$  and a 2-form  $\theta_{m,\ell}$  on  $M$  such that

$$(3.13) \quad \begin{cases} B_q(\omega(\ell), F(\ell)) = \frac{n!}{m^n} \sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|_{h(\ell)}^2 = C_{q,\ell} + u_{m,\ell} q^{\ell+2}; \\ \eta_m = \frac{1}{m} \Phi_m^* \omega_{\text{FS}} = \omega(\ell) + \theta_{m,\ell} q^{\ell+2}, \end{cases}$$

where we have the inequalities  $\|u_{m,\ell}\|_{C^0(M)} \leq C_6$  and  $\|\theta_{m,\ell}\|_{C^0(M, \omega_0)} \leq C_7$  for some  $C_6$  and  $C_7$  (cf. Remark 2.11; see also [25], [14]). Hence, if  $m \gg 1$ ,

$$\begin{aligned} &\frac{|\dot{f}_{X,m}(0)|}{(n+1)!} \\ &\leq \int_M \frac{\left( \sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2 \right) \left\{ 1 + \sum_{i=1}^{\infty} (-u_{m,\ell} q^{\ell+2} / C_{q,\ell})^i \right\} \{ \omega(\ell) + \theta_{m,\ell} q^{\ell+2} \}^n}{C_{q,\ell}}. \end{aligned}$$

Here,  $\left\{ 1 + \sum_{i=1}^{\infty} (-u_{m,\ell} q^{\ell+2} / C_{q,\ell})^i \right\} \{ \omega(\ell) + \theta_{m,\ell} q^{\ell+2} \}^n$  is written as  $(1 + w_{m,\ell}) \omega(\ell)^n$  for some function  $w_{m,\ell}$  on  $M$  such that the inequality  $\|w_{m,\ell}\|_{C^0(M)} \leq C_8$  holds for some

$C_8$ . Then by  $\int_M \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2 \right\} \omega(\ell)^n = \sum_{k=1}^{v_m} \left\{ e^{d\chi_k(F(\ell))} \sum_{i=1}^{n_k} (b_k + \gamma_{k,i}(X'')) \right\} = 0$ , we have

$$\begin{aligned} |\dot{f}_{X,m}(0)| &\leq (n+1)! q^{\ell+2} \int_M \left| \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2}{C_{q,\ell}} \right| \cdot |w_{m,\ell}| \omega(\ell)^n \\ &= (n+1)! q^{\ell+2-n} \int_M \left( 1 + \frac{u_{m,\ell} q^{\ell+2}}{C_{q,\ell}} \right) \left| \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|_{h(\ell)}^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|_{h(\ell)}^2} \right| \cdot \left| \frac{w_{m,\ell}}{n!} \right| \omega(\ell)^n. \end{aligned}$$

In view of (3.4), by setting  $\hat{\phi} := (m \sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2)^{-1} (\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|^2)$ , there exist  $C_9$  and  $C_{10}$  such that, for  $m \gg 1$ ,

$$(3.14) \quad |\dot{f}_{X,m}(0)| \leq q^{\ell+1-n} C_9 \|\hat{\phi}\|_{L^1(M, \omega(\ell))} \leq q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M, \omega(\ell))}.$$

STEP 5. Note that  $0 \leq n_k^2 b_k^2 = \left\{ \sum_{i=1}^{n_k} \gamma_{k,i}(X'') \right\}^2 \leq n_k \sum_{i=1}^{n_k} \gamma_{k,i}(X'')^2$  holds for all  $k$  by the Cauchy-Schwarz inequality. Hence

$$(3.15) \quad \sum_{k=1}^{v_m} n_k b_k^2 \leq \sum_{\alpha=0}^{N_m} \gamma_\alpha(X'')^2.$$

From  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X'') = 0$  and  $\hat{\gamma}_\alpha(X'') = \gamma_\alpha(X'') + mc_{X''}$ , it follows that  $\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 = (N_m + 1)(mc_{X''})^2 + \sum_{\alpha=0}^{N_m} \gamma_\alpha(X'')^2$ . In particular,

$$(3.16) \quad \sum_{\alpha=0}^{N_m} \gamma_\alpha(X'')^2 \leq \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2.$$

Since  $\gamma_{k,i}(P) = b_k + \gamma_{k,i}(X'')$ , (3.15) and (3.16) above imply that

$$(3.17) \quad \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \leq 2 \left\{ \sum_{k=1}^{v_m} n_k b_k^2 + \sum_{\alpha=0}^{N_m} \gamma_\alpha(X'')^2 \right\} \leq 4 \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2.$$

By  $X = Y + P$ , we have  $\sum_{\alpha=0}^{N_m} \gamma_\alpha(X)^2 \leq 2 \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 + \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\}$ , because  $\gamma_\alpha(X) = \gamma_\alpha(Y) + \gamma_\alpha(P)$ . Hence, by (3.11), (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} (3.18) \quad q \sum_{\alpha=0}^{N_m} \gamma_\alpha(X)^2 &\leq 2q \sum_{\alpha=0}^{N_m} \gamma_\alpha(Y)^2 + 2q \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \\ &\leq 2C_5 \sum_{k=1}^{v_m} n_k b_k^2 + \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \leq C_{11} \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \end{aligned}$$

for  $m \gg 1$ , where we put  $C_{11} := 4 + 2C_5$ . Fix a positive real number  $\ell_0$  independent of the choice of  $m$  and  $X$ . Put  $\delta_0 := q^{1/2+\ell_0} / \sqrt{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2}$ . Then by (3.18), we have

$0 < \delta_0 < \sqrt{C_{11}}/\bar{\gamma}$ , where  $\bar{\gamma} := \max\{|\gamma_\alpha(X)|; \alpha = 0, 1, \dots, N_m\}$ . In view of Step 1 of [15], Section 4, by assuming  $|t| \leq \delta_0$ , we see that the family of Kähler manifolds  $(M, q\Phi_m^*\lambda_t^*\omega_{\text{FS}})$  have bounded geometry.

STEP 6. At the beginning of this step, we shall show the inequality (3.19) below as an analogue of [21], (5.9), by proving that an argument of Phong and Sturm [21] for  $\dim H = 0$  is valid also for  $\dim H > 0$ . To see this, we consider the following exact sequence of holomorphic vector bundles

$$0 \rightarrow TM_{m,t} \rightarrow T\mathbb{P}^*(V_m)|_{M_{m,t}} \rightarrow TM_{m,t}^\perp \rightarrow 0,$$

where  $TM_{m,t}^\perp$  is regarded as the normal bundle of  $M_{m,t}$  in  $\mathbb{P}^*(V)$ . The pointwise estimate (cf. [21], (5.16)) of the second fundamental form for this exact sequence has nothing to do with  $\dim H$ , and as in [21], (5.15), it gives the inequality

$$\int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \geq C_{12} \int_{M_{m,t}} |\bar{\partial} \mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} \omega_{\text{FS}}^n,$$

for some  $C_{12}$ . Let  $\mathcal{A}^{0,p}(T_M)$ ,  $p = 0, 1$ , denote the sheaf of germs of smooth  $(0, p)$ -forms on  $M$  with values in the holomorphic tangent bundle  $TM$  of  $M$ , and endow  $M$  with the Kähler metric  $(1/m)\Phi_m^*\lambda_t^*\omega_{\text{FS}}$ . We then consider the operator  $\square_{TM} := -\bar{\partial}^\# \bar{\partial}$  on  $\mathcal{A}^{0,0}(T_M)$ , where  $\bar{\partial}^\# : \mathcal{A}^{0,1}(T_M) \rightarrow \mathcal{A}^{0,0}(T_M)$  is the formal adjoint of  $\bar{\partial} : \mathcal{A}^{0,0}(T_M) \rightarrow \mathcal{A}^{0,1}(T_M)$ . Since by Step 1, the Kähler metrics  $q\Phi_m^*\lambda_t^*\omega_{\text{FS}}$  has bounded geometry, the first positive eigenvalue of the operator  $-\square_{TM}$  on  $\mathcal{A}^{0,0}(T_M)$  is bounded from below by  $C_{13}$ . Hence, by  $X'' \in \mathfrak{p}^\perp$ ,

$$\int_{M_{m,t}} |\bar{\partial} \mathcal{X}_{TM_{m,t}^\perp}''|^2_{(\omega_{\text{FS}}/m)} \left(\frac{\omega_{\text{FS}}}{m}\right)^n \geq C_{13} \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{(\omega_{\text{FS}}/m)} \left(\frac{\omega_{\text{FS}}}{m}\right)^n.$$

Since  $\bar{\partial} \mathcal{X}_{TM_{m,t}^\perp}'' = -\bar{\partial} \mathcal{X}_{TM_{m,t}}''$ , it now follows that

$$\begin{aligned} \ddot{f}_{X,m}(t) &= \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} \omega_{\text{FS}}^n \\ (3.19) \quad &\geq C_{12} C_{13} q \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} \omega_{\text{FS}}^n. \end{aligned}$$

In view of the equality  $|\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} + |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} = |\mathcal{X}_{TM_{m,t}}''|^2_{\omega_{\text{FS}}}$ , by adding the integral  $C_{12} C_{13} q \int_{M_{m,t}} |\mathcal{X}_{TM_{m,t}^\perp}''|^2_{\omega_{\text{FS}}} \omega_{\text{FS}}^n$  to both sides of (3.19) and by dividing the resulting

inequality by  $(1 + C_{12}C_{13}q)$ , we see that, for some  $C_{14}$  and  $C_{15}$ ,

$$\begin{aligned}
 \ddot{f}_{X,m}(t) &= \int_M \Phi_m^* \lambda_t^* \left( |\mathcal{X}_{TM_{m,t}^\perp}''|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n \right) \\
 (3.20) \quad &\geq C_{14}q \int_M \Phi_m^* \lambda_t^* \left( |\mathcal{X}_{|M_{m,t}}''|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n \right) \\
 &\geq C_{15}q \int_{M_m} |\mathcal{X}_{|M_m}''|_{\omega_{\text{FS}}}^2 \omega_{\text{FS}}^n \geq C_{15}q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n,
 \end{aligned}$$

where  $\Theta := (\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2)^{-2} \{ (\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2) (\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|^2) - (\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'') |\tilde{\beta}_\alpha|^2)^2 \}$  is nonnegative everywhere on  $M$ . Then by (3.14) and (3.20),

$$\begin{cases} \dot{f}_{X,m}(\delta_0) \geq \dot{f}_{X,m}(0) + C_{15}\delta_0 q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n \\ \geq -q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M, \omega(\ell))} + C_{15}\delta_0 q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n, \\ \dot{f}_{X,m}(-\delta_0) \leq \dot{f}_{X,m}(0) - C_{15}\delta_0 q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n \\ \leq q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M, \omega(\ell))} - C_{15}\delta_0 q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n. \end{cases} \quad (3.21)$$

By (3.20) and [15], Lemma 3.4, the proof of Main Theorem is reduced to showing the following three conditions for all  $m \gg 1$ :

$$\text{i) } \dot{f}_{X,m}(\delta_0) > 0 > \dot{f}_{X,m}(-\delta_0), \quad \text{ii) } \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n > 0, \quad \text{iii) } t_X^{(m)} \cdot X \in U_m.$$

Since iii) follows from Remark 3.31 below, we have only to prove i) and ii). Then by (3.21), it suffices to show the following for all  $m \gg 1$ :

$$(3.22) \quad C_{15}\delta_0 q \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n - C_{10}q^{\ell+1-n} \|\hat{\phi}\|_{L^2(M, \omega(\ell))} > 0.$$

Let us define real numbers  $\hat{e}_1, \hat{e}_2, e_1, e_2$  by setting

$$\begin{aligned} \hat{e}_1 &:= \int_M \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \omega(\ell)^n, \quad \hat{e}_2 := \int_M \left( \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'') |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \right)^2 \omega(\ell)^n, \\ e_1 &:= \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \omega(\ell)^n, \quad e_2 := \int_M \left( \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P) |\tilde{\beta}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{\beta}_\alpha|^2} \right)^2 \omega(\ell)^n. \end{aligned}$$

By the Cauchy-Schwarz inequality, we always have  $\hat{e}_1 \geq \hat{e}_2$  and  $e_1 \geq e_2$ . Now, the following cases are possible:

CASE 1:  $\hat{e}_1 > 2\hat{e}_2$ .

CASE 2:  $\hat{e}_1 \leq 2\hat{e}_2$ .

In view of the identities in (3.13), we can write

$$\begin{aligned}\hat{e}_1 &= q^n n! \int_M \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 |\tilde{\beta}_\alpha|_{h(\ell)}^2}{1 + \sum_{\alpha=0}^{\ell} \alpha_k q^{k+1} + u_{m,\ell} q^{\ell+2}} \omega(\ell)^n, \\ e_1 &= q^n n! \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 |\tilde{\beta}_\alpha|_{h(\ell)}^2}{1 + \sum_{\alpha=0}^{\ell} \alpha_k q^{k+1} + u_{m,\ell} q^{\ell+2}} \omega(\ell)^n, \\ \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n &= m^n \int_M \Theta \{\omega(\ell) + \theta_{m,\ell} q^{\ell+2}\}^n,\end{aligned}$$

and hence, given a positive real number  $0 < \varepsilon \ll 1$ , both  $\hat{e}_1$  and  $\int_M \Theta \Phi_m^* \omega_{\text{FS}}^n$  above are estimated, for all  $m \gg 1$ , by

$$(3.23) \quad (1 - \varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\} \leq \frac{\hat{e}_1}{n!} \leq (1 + \varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\},$$

$$(3.24) \quad (1 - \varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\} \leq \frac{e_1}{n!} \leq (1 + \varepsilon) q^n \left\{ \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\},$$

$$(3.25) \quad (1 - \varepsilon) q^{-n} \int_M \Theta \omega(\ell)^n \leq \int_M \Theta \Phi_m^* \omega_{\text{FS}}^n \leq (1 + \varepsilon) q^{-n} \int_M \Theta \omega(\ell)^n,$$

where we used the Remark right after (3.2). Moreover, we can write  $e_2$  in the form

$$(3.26) \quad q^{-2} \|\hat{\phi}\|_{L^2(M, \omega(\ell))}^2 = e_2 \leq e_1.$$

STEP 7. We first consider Case 1. Then from (3.17), (3.23), (3.24), (3.25), (3.26),  $\hat{e}_2 < \hat{e}_1/2$  and the definition of  $\delta_0$ , it follows that

$$\begin{aligned}\text{L.H.S. of (3.22)} &\geq (1 - \varepsilon) C_{15} \delta_0 q^{1-n} \int_M \Theta \omega(\ell)^n - q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M, \omega(\ell))} \\ &\geq (1 - \varepsilon) C_{15} q^{1-n} \delta_0 (\hat{e}_1 - \hat{e}_2) - q^{\ell+2-n} C_{10} \sqrt{e_1} \\ &\geq \frac{(1 - \varepsilon) C_{15} \delta_0 q^{1-n} \hat{e}_1}{2} - (1 + \varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \left\{ n! \sum_{\alpha=0}^{N_m} \gamma_\alpha(P)^2 \right\}^{1/2} \\ &\geq \frac{(1 - \varepsilon)^2 C_{15} \delta_0 q \left\{ \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\} n!}{2} - 2(1 + \varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \left\{ n! \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\}^{1/2} \\ &\geq \left\{ n! \sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha(X'')^2 \right\}^{1/2} \left\{ \frac{\sqrt{n!} (1 - \varepsilon)^2 C_{15} q^{\ell_0+3/2}}{2} - 2(1 + \varepsilon)^{1/2} C_{10} q^{\ell+2-n/2} \right\},\end{aligned}$$

for  $m \gg 1$ . Now we see that, if  $\ell > (n-1)/2 + \ell_0$ , then  $q^{\ell+2-n/2}/q^{\ell_0+3/2}$  converges to

0 as  $m \rightarrow \infty$ . Thus, if  $m \gg 1$ , then by choosing  $\ell$  such that  $\ell > (n-1)/2 + \ell_0$ , we now see from the computation above that L.H.S. of (3.22) is positive, as required.

STEP 8. Let us finally consider Case 2. For each fixed  $\ell$ , the Kähler form  $\eta_m$  converges to  $\omega_0$ , as  $m \rightarrow \infty$ , in  $C^j(M)$ -norm for all positive integers  $j$  (cf. [25], [14]; see also the Remark right after (3.2)). Note that  $X'' \in \mathfrak{p}^\perp$ . In view of  $\int_M \phi_{X''} \eta_m^n = 0$ , we see that

$$(3.27) \quad \|\phi_{X''}\|_{L^2(M, \eta_m)}^2 \leq C_{16} \|\bar{\partial} \phi_{X''}\|_{L^2(M, \eta_m)}^2 = C_{16} \|\Phi_m^* \mathcal{X}_{TM_m}''\|_{L^2(M, \eta_m)}^2,$$

for some  $C_{16}$ , where by abuse of terminology, the differential  $(\Phi_m^{-1})_*: TM_m \rightarrow TM$  is denoted by  $\Phi_m^*$ . Moreover, by (3.19) applied to  $t = 0$ , we obtain

$$(3.28) \quad \|\Phi_m^* \mathcal{X}_{TM_m}''\|_{L^2(M, \eta_m)}^2 \leq (C_{12} C_{13})^{-1} q^{-1} \|\Phi_m^* \mathcal{X}_{TM_m^\perp}''\|_{L^2(M, \eta_m)}^2.$$

From now on until the end of this proof, we assume that  $m \gg 1$ . By (3.27) together with (3.28) and (3.7), there exist  $C_{17}$  and  $C_{18}$  such that

$$(3.29) \quad \begin{aligned} \|\Phi_m^* \mathcal{X}_{TM_m^\perp}''\|_{L^2(M, \eta_m)} &\geq C_{17} q^{1/2} \|\phi_{X''}\|_{L^2(M, \eta_m)} \\ &\geq C_{18} q^{1/2} \|\phi_{X''}\|_{L^2(M, \omega(\ell))} = C_{18} q^{3/2} \sqrt{\hat{e}_2}. \end{aligned}$$

We now observe the pointwise estimate  $q^{1/2} |\mathcal{X}_{|M_m}''|_{\omega_{\text{FS}}} = |\mathcal{X}_{|M_m}''|_{\eta_m} \geq |\mathcal{X}_{TM_m^\perp}''|_{\eta_m}$ . Hence by (3.20) and (3.29), we obtain

$$(3.30) \quad \begin{aligned} \ddot{f}_{X,m}(t) &\geq C_{15} \int_{M_m} (q^{1/2} |\mathcal{X}_{|M_m}''|_{\omega_{\text{FS}}})^2 \omega_{\text{FS}}^n \\ &\geq C_{15} q^{-n} \|\Phi_m^* \mathcal{X}_{TM_m^\perp}''\|_{L^2(M, \eta_m)}^2 \geq C_{19} q^{3-n} \hat{e}_2, \end{aligned}$$

for some  $C_{19}$ . As in deducing (3.21) from (3.14) and (3.20), we obtain by (3.14) and (3.30) the inequalities

$$\dot{f}_{X,m}(\delta_0) \geq R$$

and

$$\dot{f}_{X,m}(-\delta_0) \leq -R,$$

where  $R := -q^{\ell+1-n} C_{10} \|\hat{\phi}\|_{L^2(M, \omega(\ell))} + C_{19} \delta_0 q^{3-n} \hat{e}_2$ . Hence, it suffices to show that  $R > 0$ . In view of the definition of  $\hat{\phi}$  and  $e_2$ , we see from  $\hat{e}_1 \leq 2\hat{e}_2$  and (3.26) that

$$R = C_{19} \delta_0 q^{3-n} \hat{e}_2 - C_{10} q^{\ell+2-n} \sqrt{e_2} \geq \frac{C_{19} \delta_0 q^{3-n} \hat{e}_1}{2} - C_{10} q^{\ell+2-n} \sqrt{e_1}.$$

Here by (3.23) and (3.24), we obtain

$$\begin{aligned} \frac{\delta_0 q^{3-n} \hat{e}_1}{q^{\ell+2-n} \sqrt{e_1}} &= q^{3/2+\ell_0-\ell} \sqrt{\frac{\hat{e}_1}{e_1}} \sqrt{\frac{\hat{e}_1}{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha (X'')^2}} \\ &\geq C_{20} q^{(3+n)/2+\ell_0-\ell} \sqrt{\frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_\alpha (X'')^2}{\sum_{\alpha=0}^{N_m} \gamma_\alpha (P)^2}} \geq \frac{C_{20}}{2} q^{(3+n)/2+\ell_0-\ell} \end{aligned}$$

for some  $C_{20}$ , where the last inequality follows from (3.17). Therefore, by choosing  $\ell$  such that  $\ell > (3+n)/2 + \ell_0$ , we now conclude that  $R > 0$  for  $m \gg 1$ , as required.  $\square$

**REMARK 3.31.** In the above proof, it is easy to check the condition iii) in Step 6 as follows: In view of  $|t_X^{(m)}| < \delta_0$ , it suffices to show that, if  $m \gg 1$ , then

$$(3.32) \quad t \cdot X \in U_m, \quad \text{for all } (t, X) \in \mathbb{R} \times \mathfrak{p}_m'' \quad \text{with } |t| < \delta_0.$$

For each  $Q \in \mathfrak{p}$ , let  $u_Q \in C^\infty(M)_\mathbb{R}$  denote the Hamiltonian function for the holomorphic vector field  $Q$  on the Kähler manifold  $(M, \omega_0)$  characterized by the equalities

$$i_Q \omega_0 = \frac{\sqrt{-1}}{2\pi} \bar{\partial} u_Q$$

and

$$\int_M u_Q \omega_0^n = 0.$$

Define compact subsets  $\Sigma(\mathfrak{p}'_m)$ ,  $\Sigma(\mathfrak{p})$  of  $\mathfrak{p}$  by setting

$$\begin{cases} \Sigma(\mathfrak{p}'_m) := \{Q \in \mathfrak{p}'_m; \|\bar{\partial} u_Q\|_{L^2(M, \omega_0)} = 1\}, \\ \Sigma(\mathfrak{p}) := \{Q \in \mathfrak{p}; \|\bar{\partial} u_Q\|_{L^2(M, \omega_0)} = 1\}. \end{cases}$$

Choose an orthonormal basis  $\mathcal{S} := \{s_0, s_1, \dots, s_{N_m}\}$  for the Hermitian vector space  $(V_m, \rho_{h(\ell)})$ . For the space  $\mathcal{H}_m$  of all Hermitian matrices of order  $N_m+1$ , define a norm

$$\mathcal{H}_m \rightarrow \mathbb{R}_{\geq 0}, \quad A = (a_{\alpha\beta}) \mapsto \|A\|_m := \sqrt{\text{tr } A^* A} = \sqrt{\sum_{\alpha, \beta} |a_{\alpha\beta}|^2}$$

on  $\mathcal{H}_m$ . Let  $m \gg 1$ . The infinitesimal action of  $\mathfrak{p}_m$  on  $V_m$  is given by

$$Q \cdot s_\beta = \sum_{\alpha=0}^{N_m} s_\alpha \gamma_{\alpha\beta}(Q), \quad Q \in \mathfrak{p}_m,$$

where  $\gamma_Q = (\gamma_{\alpha\beta}(Q)) \in \mathcal{H}_m$  denotes the representation matrix of  $Q$  on  $V_m$  with respect to  $\mathcal{S}$ . Let  $X \in \mathfrak{p}_m''$ , and let  $\delta_0$  be as in Step 5 above. For  $t \in \mathbb{R}$  with  $|t| < \delta_0$ , we put  $\tilde{X} := tX$ . In order to prove (3.32) above, it suffices to show

$$(3.33) \quad \zeta(\text{ad } \tilde{X})Q \notin \mathfrak{p}_m'' \quad \text{for all } Q \in \Sigma(\mathfrak{p}_m').$$

Let  $Q \in \Sigma(\mathfrak{p}_m')$ . For a suitable choice of a basis  $\mathcal{S}$  as above, we may assume that the representation matrix  $\gamma_Q$  of  $Q$  is a real diagonal matrix. Note also that  $\text{tr } \gamma_Q = 0$ . Let  $\Phi_m: M \rightarrow \mathbb{P}^{N_m}(\mathbb{C})$  be the Kodaira embedding of  $M$  defined by (cf. (3.2))

$$\Phi_m(p) := (\tilde{s}_0(p) : \tilde{s}_1(p) : \cdots : \tilde{s}_{N_m}(p)).$$

In view of the definition  $\eta_m := \Phi_m^* \omega_{FS}/m$  of  $\eta_m$ , the Hamiltonian function  $\phi_Q$  on  $(M, \eta_m)$  associated to the holomorphic vector field  $Q$  is expressed in the form

$$\phi_Q = \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q) |\tilde{s}_\alpha|^2}{m \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|^2}.$$

We define  $\hat{\gamma}_Q := (\hat{\gamma}_{\alpha\beta}(Q)) \in \mathcal{H}_m$  by setting  $\hat{\gamma}_{\alpha\beta}(Q) := \{\gamma_{\alpha\alpha}(Q) + mc_Q\} \delta_{\alpha\beta}$  for Kronecker's delta  $\delta_{\alpha\beta}$ . As in deducing (3.16) from  $\hat{\gamma}_\alpha(X'') = \gamma_\alpha(X'') + mc_{X''}$ , we easily see that

$$\|\gamma_Q\|_m^2 \leq \|\hat{\gamma}_Q\|_m^2.$$

Recall that  $\eta_m$  is expressible as  $\omega_0 + (\sqrt{-1}/2\pi)q \partial\bar{\partial}\xi_m$  for some real-valued smooth function  $\xi_m$  on  $M$  such that

$$(3.34) \quad \|\xi_m\|_{C^3(M)} \leq C_{21},$$

where all  $C_j$ 's in this remark are positive constants independent of the choice of  $m$ ,  $X$  and  $Q$ . We now observe that

$$(3.35) \quad \phi_Q = u_Q + q(Q\xi_m).$$

Note that  $Q \in \Sigma(\mathfrak{p}_m') \subset \Sigma(\mathfrak{p})$ . Since  $Q$  sits in the compact set  $\Sigma(\mathfrak{p})$ , and since  $\Sigma(\mathfrak{p})$  is independent of the choice of  $m$ , there exist  $C_{22}$  and  $C_{23}$  such that

$$0 < C_{22} \leq \int_M u_Q^2 \omega_0^n \left( = \int_M \phi_Q^2 \eta_m^n \right) \leq C_{23}.$$

Note that both  $\eta_m$  and  $\omega(\ell)$  converge to  $\omega_0$  as  $m \rightarrow \infty$  (see (3.3) and the statement at the beginning of Step 8). Note also that, by the Remark right after (3.2), the function  $(n!/m^n) \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|_{h(\ell)}^2$  on  $M$  converges uniformly to 1, as  $m \rightarrow \infty$ . Again by the

Remark right after (3.2), it now follows from the Cauchy-Schwarz inequality that, for  $m \gg 1$ ,

$$(3.36) \quad \begin{aligned} \|\gamma_Q\|_m^2 &= \sum_{\alpha=0}^{N_m} \gamma_{\alpha\alpha}(Q)^2 \geq C_{24} m^n \int_M \frac{\sum_{\alpha=0}^{N_m} \gamma_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|^2}{\sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|^2} \omega(\ell)^n \\ &\geq C_{24} m^{n+2} \int_M (\Phi_m^* \varphi_Q)^2 \omega(\ell)^n \geq C_{25} m^{n+2} \int_M (\Phi_m^* \varphi_Q)^2 \eta_m^n \end{aligned}$$

for some  $C_{24}$  and  $C_{25}$ , where  $\Phi_m^* \varphi_Q$  is as in (3.4). Then for  $m \gg 1$ ,

$$\begin{aligned} C_{26} &= \max_{J \in \Sigma(\mathbf{P})} \int_M |J|_{\omega_0}^2 \omega_0^n \geq \int_M |Q|_{\omega_0}^2 \omega_0^n \geq C_{27} \int_M |Q|_{\eta_m}^2 \eta_m^n \\ &= C_{27} m \int_M \left\{ \frac{\sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2}{m^2 \sum_{\alpha=0}^{N_m} |\tilde{s}_\alpha|_{h(\ell)}^2} - \phi_Q^2 \right\} \eta_m^n \\ &\geq \frac{C_{28}}{m^{n+1}} \left\{ \int_M \sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2 \eta_m^n \right\} - C_{29} m \\ &\geq \frac{C_{30}}{m^{n+1}} \left\{ \int_M \sum_{\alpha=0}^{N_m} \hat{\gamma}_{\alpha\alpha}(Q)^2 |\tilde{s}_\alpha|_{h(\ell)}^2 \omega(\ell)^n \right\} - C_{29} m \\ &\geq C_{31} \frac{\|\hat{\gamma}_Q\|_m^2}{m^{n+1}} - C_{29} m, \end{aligned}$$

for some  $C_{26}$ ,  $C_{27}$ ,  $C_{28}$ ,  $C_{29}$ ,  $C_{30}$  and  $C_{31}$ . Hence, if  $m \gg 1$ , then

$$(3.37) \quad \|\gamma_Q\|_m^2 \leq \|\hat{\gamma}_Q\|_m^2 \leq C_{32} m^{n+2}.$$

for some  $C_{32}$ . Now by  $\zeta(0) = 1$ , we define a real analytic function  $\tilde{\zeta} = \tilde{\zeta}(x)$  on  $\mathbb{R}$  satisfying  $\tilde{\zeta}(0) = 0$  by

$$\tilde{\zeta}(x) := \zeta(x) - 1.$$

For  $X \in \mathfrak{p}_m''$  above, by choosing an orthonormal basis for  $(V_m, \rho_{h(\ell)})$  possibly distinct from the original one, we may assume that the representation matrix  $\gamma_X$  of  $X$  is a real diagonal matrix. Recall that  $\tilde{X} = tX$ , where  $|t| < \delta_0 := q^{1/2+\ell_0}/\|\hat{\gamma}_{X''}\|_m$ . Put  $\tilde{X}'' := tX''$ . Then by (3.18),

$$\sqrt{\frac{q}{C_{11}}} \|\gamma_{\tilde{X}}\|_m \leq \|\hat{\gamma}_{\tilde{X}''}\|_m = |t| \cdot \|\hat{\gamma}_{X''}\|_m \leq q^{1/2+\ell_0},$$

i.e.,  $\|\gamma_{\tilde{X}}\|_m \leq \sqrt{C_{11}} q^{\ell_0}$ . Hence, if  $m \gg 1$ ,

$$(3.38) \quad \|\gamma_{\tilde{\zeta}(\text{ad } \tilde{X})Q}\|_m \leq C_{33} q^{\ell_0} \|\gamma_Q\|_m,$$

for some  $C_{33}$ . Now by the same argument as in (3.36), we see that, for some  $C_{34}$ ,

$$(3.39) \quad \|\gamma_{\zeta(\text{ad } \tilde{X})Q}\|_m^2 \geq C_{34}m^{n+2} \int_M \Phi_m^* \varphi_{\zeta(\text{ad } \tilde{X})Q}^2 \eta_m^n, \quad \text{if } m \gg 1.$$

Put  $a_m := \sqrt{\int_M \Phi_m^* \varphi_{\zeta(\text{ad } \tilde{X})Q}^2 \eta_m^n}$ . Then for  $m \gg 1$ , by (3.37), (3.38) and (3.39),

$$(3.40) \quad a_m \leq C_{35}q^{\ell_0}$$

for some  $C_{35}$ . Consider the Laplacians  $\square_{\eta_m}$  and  $\square_{\omega_0}$  on functions for the Kähler manifolds  $(M, \eta_m)$  and  $(M, \omega_0)$ , respectively. Note that  $\zeta(\text{ad } \tilde{X})Q = Q + \tilde{\zeta}(\text{ad } \tilde{X})Q$ . Then for  $m \gg 1$ , by (3.35), we obtain

$$(3.41) \quad \begin{aligned} & \left| \int_M (\square_{\eta_m} \phi_Q) \phi_{\zeta(\text{ad } \tilde{X})Q} \eta_m^n \right| \\ &= \left| \int_M (\square_{\eta_m} \phi_Q) (\phi_Q + \Phi_m^* \varphi_{\zeta(\text{ad } \tilde{X})Q}) \eta_m^n \right| \\ &\geq \|\bar{\partial} \phi_Q\|_{L^2(M, \eta_m)}^2 - \left| \int_M (\square_{\eta_m} \phi_Q) (\Phi_m^* \varphi_{\zeta(\text{ad } \tilde{X})Q}) \eta_m^n \right| \\ &\geq \|\bar{\partial} \{u_Q + q(Q\xi_m)\}\|_{L^2(M, \eta_m)}^2 - a_m \|\square_{\eta_m} \{u_Q + q(Q\xi_m)\}\|_{L^2(M, \eta_m)} \\ &\geq (1 - \epsilon)R_m - (1 + \epsilon)a_m S_m, \end{aligned}$$

where we put  $R_m := \|\bar{\partial} \{u_Q + q(Q\xi_m)\}\|_{L^2(M, \omega_0)}^2$  and  $S_m := \|\square_{\omega_0} \{u_Q + q(Q\xi_m)\}\|_{L^2(M, \omega_0)}$ , and  $\epsilon \ll 1$  is a positive constant independent of the choice of  $m$ ,  $X$  and  $Q$ . Since  $Q$  belongs to the compact set  $\Sigma(\mathfrak{p})$ , by (3.34) and the equality  $\|\bar{\partial} u_Q\|_{L^2(M, \omega_0)} = 1$ , we obtain constants  $C_{36}$  and  $C_{37}$  such that

$$(3.42) \quad \begin{cases} R_m \geq 1 - 2q \|\bar{\partial} (Q\xi_m)\|_{L^2(M, \omega_0)} \geq 1 - C_{36}q, \\ S_m \leq \|\square_{\omega_0} u_Q\|_{L^2(M, \omega_0)} + q \|\square_{\omega_0} (Q\xi_m)\|_{L^2(M, \omega_0)} \leq C_{37}. \end{cases}$$

Then for  $m \gg 1$ , by (3.40), (3.41) and (3.42), we finally obtain

$$\left| \int_M (\square_{\eta_m} \phi_Q) \phi_{\zeta(\text{ad } \tilde{X})Q} \eta_m^n \right| \geq (1 - \epsilon)(1 - C_{36}q) - (1 + \epsilon)C_{35}C_{37}q^{\ell_0} > 0,$$

which implies (3.33), as required.

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