Title	Kaluza-Klein Modes in Braneworld Cosmology
Author(s)	南辻, 真人
Citation	大阪大学, 2006, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/46474
rights	
Note	

理中 10898

OSAKA UNIVERSITY   JAPAN

DEPARTMENT OF EARTH AND SPACE SCIENCE,
GRADUATE SCHOOL OF SCIENCE

D-Thesis

Kaluza-Klein Modes
in Braneworld Cosmology

(ブレーン宇宙論における
カルツァー・クラインモード)

Masato Minamitsuji

(南辻　真人)

March 2006

Abstract

In this thesis, we discuss the dynamics and effects of Kaluza-Klein (KK) modes in braneworld cosmology in the context of the second Randall-Sundrum (RS) model. KK modes are waves propagating in the extra-dimension, and may affect cosmology and gravity on the brane non-trivially.

First, we discuss the backreaction of KK graviton modes qualitatively. For gravitons which are produced by high energy particle interactions on the brane, we assume the KK gravitons as ingoing null dust flux which is emitted from the brane radially. We discuss the bulk geometry, the brane dynamics in the bulk and show that a strong, visible, naked singularity can be formed in the bulk in a paticular situation. Then, for gravitons of a single KK mode, which is produced quantum mechanically in the whole bulk during brane inflation, we derive the effective stress-energy tensor, adopting the averaging procedure where the existence of the brane is taken into account. We show that a (massive) KK mode behaves effectively as cosmic dust with negative energy density on the brane. The negativity of the energy density can be explained phyically in terms of the energy conservation law in the bulk, which is satisfied in the five-dimensional spacetime with a maximally symmetric three-space.

In reality, however, what we observe is the sum of an infinite number of KK modes. Thus for KK modes produced quantum mechanically, we have to determine its amplitude in terms of quantum field theory. As is well-known, however, there is a significant pathology in attempting to quantify the quantized KK backreaction. It is the divergence of the sum as one approaches the brane from the bulk, even after a conventional ultra-violet (UV) regularization. We show that a finite brane thickness can regularize this divergence and the size of quantum backreaction can be naturally reduced to below that of the background stress-energy tensor.

Finally, as a more general extension of the RS model, we discuss the linearized effective gravity on the brane in the Einstein Gauss-Bonnet (EGB) theory. We show that in the EGB theory the effective gravity on a cosmological (de Sitter) brane is four-dimensional on all distance scales, from short distances to large distances. We also show that on high energy expanding branes as well as on low energy expanding ones, effective gravity becomes four-dimensional.

PACS numbers: 04.50. ⊦ h, 98.80.Cq

Acknowledgments

Contents

1

Introduction

The idea that there are extra-dimensions other than the usual four-dimensions (i.e., three spatial dimensions and one time dimension) has been discussed for a long time, since the proposals for unification of fundamental interactions by Kaluza [1] and Klein [2]. Recent progress in string theory revives the idea of using extra-dimensions for unification and suggests the novel possibility that our universe is, in reality, a four-dimensional submanifold, called a *brane*, embedded into a higher-dimensional spacetime, called a *bulk* [3, 4]. This gives new paradigm for cosmology and gravity, which is called *braneworld*. In braneworld, interactions other than gravity are trapped on the brane, whereas only gravity can propagate into the bulk. This is a quite different picture from one in the KK theory. Several scenarios which realize braneworld have been discussed [5, 6, 7, 8, 9]. Expecially among them, the scenarios which were proposed by Randall and Sundrum (RS) have been attracted much attention, because it succeeds in the localization of gravity through a new mechanism, the warping of the extra-dimension [9]. This model has been given phenomenological grounds from various aspects of higher-dimensional theories of gravity.[1]

In this Chapter, first we briefly give a historical review of the braneworld model proposed by RS, cosmology realized on it and various extensions of the RS model. And then, we state the purpose of this thesis and introduce the outline.

1.1   Randall-Sundrum (RS) braneworld

In this thesis, we assume the five-dimensional bulk (i.e., codimension one). We consider the five-dimensional Einstein theory with a negative cosmological constant ($\Lambda_5 < 0$) and a brane

$$S_{\text{RS}} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left(\overset{(5)}{R} - 2\Lambda_5 \right) + \int d^4x \sqrt{-q}\, \sigma \,, \tag{1.1}$$

---

[1] For more reviews about braneworlds, see e.g., [10, 11, 12, 13]

where $q_{\mu\nu}$ is the induced metric on the brane. κ_5^2 is the five-dimensional gravitational constant. The brane has a positive tension $\sigma > 0$. We assume the $\mathbb{Z}_2$ (mirror) symmetry with respect to the brane. This means that we first cut a five-dimensional spacetime, then make a copy of the piece of spacetime and finally glue them.

Without any dynamical matter other than the metric, the bulk solution becomes Anti- de Sitter(AdS) spacetime,

$$ds^2 = dy^2 + e^{-2|y|/\ell}\eta_{\mu\nu}dx^\mu dx^\nu \,, \tag{1.2}$$

bounded by a brane located at $y = 0$ where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric and

$$\ell = \sqrt{-\frac{6}{\Lambda_5}} \,, \tag{1.3}$$

is the curvature radius of the AdS spacetime. In order to realize the Minkowski spacetime on the brane, the brane tension is tuned, for the effective four-dimensional cosmological constant to vanish as

$$\Lambda_4 := \frac{1}{12}\kappa_5^4\sigma^2 + \frac{1}{2}\Lambda_5 = 0 \,. \tag{1.4}$$

Though the extra-dimension is infinitely extended, due to the exponential warping of it the effective volume is still finite. [2] The effective Planck scale on the brane is given by the integration of the gravitational action over the extra-dimension;

$$M_{\rm pl}^2 = 2M_5^3 \int_0^\infty dy\, e^{-2y/\ell} = M_5^3\ell, \tag{1.5}$$

where $M_5^3 = \kappa_5^{-2}$ is the five-dimensional Planck scale.

In the above solution, the brane geometry becomes the four-dimensinal Minkowski spacetime. When the tuning condition Eq.(1.4) is broken as $\Lambda_4 > 0$, the brane becomes a de Sitter (dS) spacetime. The bulk geometry is also AdS and the metric in the Gaussian-normal coordinate is given by

$$ds^2 = dy^2 + (H\ell)^2 \sinh^2(y/\ell)\,\gamma_{\mu\nu}dx^\mu dx^\nu \,, \tag{1.6}$$

where $\gamma_{\mu\nu}$ is a four-dimensional dS metric with scalar curvature

$$R(\gamma) = 12H^2 \,, \quad H^2 = \frac{1}{3}\Lambda_4 \,, \tag{1.7}$$

and in this case the brane locates at $y = y_0$ as $H = 1/(\ell\sinh(H\ell))$. One side of the bulk region is restricted to $0 < y < y_0$. The dS brane solution is useful

---

[2] The extra-dimension can be compactified by puttting another brane with a negatve tension of the same size. This model has been discussed as a possible resolution of the hierarchy problem in particle physics, though in this thesis we mainly focus on single brane models [8].

in discussing cosmological inflation on the brane. Note that the size of the extra-dimension itself is not inflating. Inflation is described as a motion of the brane in the warped bulk.

In the case that there is no usual matter and only (time-independent) tension on the brane, the four-dimensional part of bulk metric in a Gaussian-normal coordinate can be written in a separable form as Eqs. (1.2) and (1.6). However, as discussed later, in the case that there is time dependent matter component on the brane, the bulk metric can not be written in a separable form in general. The brane trajectory in the bulk is determined through the junction condition across the brane [14].

1.2 Linearized gravity in RS braneworld and Kaluza-Klein (KK) modes

Then, in the RS (single) brane model, we consider the bulk metric perturbations and the linearized effective gravity realized on the brane. In the bulk, the metric Eq.(1.2) is perturbed as

$$\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}(y, x^\mu) \,, \tag{1.8}$$

where for convenience we take the gauge condition $h_{a5} = 0$ and $h^\mu{}_\mu = h^\mu{}_{\nu,\mu} = 0$, which is called RS gauge [9, 15], whereas the brane position is fluctuated as $y = 0 \rightarrow \varphi(x^\mu)$ in general. The bulk metric perturbations and the fluctuation of the brane position can be analyzed by using the bulk Einstein equations and the junction conditions.

For simplicity, we focus on the bulk metric perturbations. The perturbations are separable as $h_{\mu\nu}^{(m)} = f_m(y)\tilde{h}_{\mu\nu}^{(m)}(x^\mu)$ where m^2 is the eigenvalue which correponds to mass of gravitons for a four-dimensional observer,

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\right) \tilde{h}_{\mu\nu}^{(m)}(x^\mu) = 0 \,. \tag{1.9}$$

The mass spectrum is determined by

$$\left(-\frac{d^2}{dz^2} + V(z)\right) \hat{f}_m(z) = m^2 \hat{f}_m(z) \,,$$

$$V(z) = \frac{15}{4(|z| + \ell)^2} - \frac{3}{\ell}\delta(z) \,, \tag{1.10}$$

where $z = \int dy e^{-y/\ell}$ and $\hat{f}_m$ is the rescaled function of f_m multiplied by powers of the warp factor. $V(z)$ is interpretted as the potential for perturbations. The delta function part generates a bound state $m^2 = 0$, which is called the *zero mode*. This zero mode can realize four-dimensional gravity on the brane. On the other hands, there is continuum spectrum $m^2 > 0$ of excited modes, which are called *Kaluza-Klein (KK) modes*. They are the main subject of this thesis. They correspond to waves propagating in the

bulk and are observed as the infinite number of massive modes on the brane. The general solution for the metric perturbation is written as;

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + \int_0^\infty dm\, h_{\mu\nu}^{(m)} \,. \tag{1.11}$$

As an example, for putting a static, point-like source of mass M on the brane, the gravitational potential on the brane at large distances $r \gg \ell$ becomes

$$h_{00} = \frac{2G_4 M}{r} \left(1 + \mathcal{O}\left(\frac{\ell^2}{r^2}\right)\right) \,. \tag{1.12}$$

Thus the gravity on the brane is four-dimensional and the small correction comes from the contribution of KK modes [15]. From the experimental tests, Newton's law is confirmed up to sub-millimeter scales. So, we obtain the bound on the size of the extra-dimension as $\ell < 0.1$mm. Thus the size of the extra-dimension may be much larger than the Planck size. The non-linear extension has been done by using the long wavelength (low energy expansion) approximation [16].

For a dS brane, a potential for perturbations like Eq.(1.10) is obtained. The only differenece is the existence of the mass gap between the zero mode ($m^2 = 0$) and KK modes ($m^2 > 9H^2/4$). This is an important nature for a dS brane. During brane inflation, KK graviton modes may be produced but they decay rapidly due to the presence of the mass gap [17]. Non-linear realization of zero mode (and KK modes) has been discussed, see, e.g., [18].

In brane cosmology, these KK modes, especially these graviton modes, are *naturally* produced at the early stage of the brane universe. These modes may be excited mainly by the following two mechanisms: The first mechanism is that they are generated by high energy particle interactions on the brane. Non-standard particle interactions on the brane at the early stage may produce gravitons. From the assumptions in braneworld, gravitons can escape into the bulk. [3] The second one is that they are produced by quantum flutuations in the whole bulk. In the cases that the brane univserse undergoes inflation or an inflationary brane universe is created via quantum tunneling in five-dimensions, KK gravitational modes may be produced quantum mechanically or exist from the beginning [18]. In the former case, the amount of KK modes is determined by the initial condition on the brane. In the latter case, it is determined by the normalization condition for each KK mode in the bulk in terms of quantum field theory.

In the case that the background bulk metric is not separable in the Gaussian normal coordinate, e.g., for general cosmological branes other than Minkoswski and dS ones, KK modes are not well-defined in general. The

---

[3]One might worry about the production of low energy KK gravitons in the particle interactions in the context of the standard model. But, the spectrum of KK modes is vanishing for these smaller mass scales as $m \to 0$. So, KK gravitons are not excited in the conventional particle interactions on the earth.

bulk geometry around a brane with spatial flatness and cosmic scale factor $a(t)$ (and located at $y = 0$), is given by the metric [19]

$$ds^2 = -N^2(t,y)dt^2 + Q^2(t,y)a^2(t)dx^2 + dy^2 , \qquad (1.13)$$

where

$$Q(t,y) = \cosh(y/\ell) - \eta \sinh(|y|/\ell)$$
$$N(t,y) = \cosh(y/\ell) - \left(\eta + \frac{\dot{\eta}}{H}\right)\sinh(|y|/\ell), \qquad (1.14)$$

with

$$\eta = \sqrt{H^2\ell^2 + 1} = \sqrt{\left(\frac{\dot{a}}{a}\right)^2 \ell^2 + 1}. \qquad (1.15)$$

Although the bulk geometry is still AdS, clearly, this metric is non-separable. However, in the low energy limit characterized by $H\ell \ll 1$ and $\dot{H}\ell^2 \ll 1$, we have $\eta \simeq 1$ and $\dot{\eta}/H \ll 1$ so that the metric can be approximated by

$$ds^2 = dy^2 + e^{-2|y|/\ell}\left(-dt^2 + a^2(t)dx^2\right), \qquad (1.16)$$

which is now separable. In these cases, they are approximately well-defined.

1.3 Brane Cosmology

In the case that there is time-dependent matter on the brane, the brane geometry deviates from both Minkowski and dS, even in assuming a cosmological symmetry. Here, we take a slightly different approach. In the previous sections, we set a Gaussian normal coordinate with respect to the brane. In this sense, it is somewhat *brane-based*. Now, we set a static coordinate in the bulk. In this *bulk-based* picture, the brane describes a trajectory in the bulk.

We assume that the bulk geometry is AdS-Schwarzshild spacetime

$$ds^2 = -\left(K + \frac{r^2}{\ell^2} - \frac{M_0}{r^2}\right)dt^2 + \left(K + \frac{r^2}{\ell^2} - \frac{M_0}{r^2}\right)^{-1}dr^2 + r^2 d\Sigma_{(K,3)}^2, \qquad (1.17)$$

rather than purely AdS, where M_0 is the mass of a black hole sitting in the bulk and $K = 1, 0, -1$. The case $M_0 = 0$ is just the AdS solution. The line element $d\Sigma_{(K,3)}^2$ correspond to the three-sphere ($K = +1$), the three-dimensional flat space ($K = 0$) and the three-hyperboloid ($K = -1$), respectively. Then, from the junction condition the cosmological evolution of the brane (i.e., its trajectory in the bulk) is given by [20, 21]

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\Lambda_4 + \frac{\kappa_4^2}{3}\rho + \frac{1}{\ell^2}\left(\frac{\rho}{\sigma}\right)^2 + \frac{M_0}{a^4}, \qquad (1.18)$$

where $a(\tau)$ is the scale factor of the brane expansion as a function of the proper time on the brane τ. The brane locates at the coordinate radius $r = a(\tau)$. The four-dimensional cosmological constant is given by

$$\Lambda_4 = \frac{\kappa_5^4}{12}\sigma^2 - \frac{3}{\ell^2} = \frac{\kappa_5^4}{12}\sigma^2 + \frac{1}{2}\Lambda_5 \,. \tag{1.19}$$

If the condition Eq. (1.4) is satisfied, then the effective cosmological constant vanishes. This is just a generalization of Friedmann equation in the standard cosmology to braneworld cosmology.

In the effective Friedmann Eq. (1.18), the M_0 term is just coming from the tidal effect of the bulk black hole and behaves as usual radiation on the brane. Thus, this term is called *dark radiation* [19, 12]. Dark radiation can be seen as an additional relativistic degree of freedom other than those in the standard model of particle physics. Such additional degrees of freedom increase the expansion rate of the Universe and affect the Big Bang nucleosynthesis (BBN) significantly, especially the abundance of ^{4}He, because in the case of a faster expansion rate there is less time for neutrinos to dacay between the time of the weak interaction freezeout and the onset of BBN, and the ratio of neutrons to protons becomes larger [22, 23]. Thus, in order to realize the successful nucleosynthesis, observational constraints on the dark radiation have been discussed, see e.g., Ref. [23].

The other main difference from the conventional Friedmann equation is the presence of the squared density term. When the brane universe is high energy, i.e., $\rho \gg \sigma$, this term dominates the cosmological evolution of the brane significantly [19, 12]. For radiation dominated universe, $\rho \propto a^{-4}$, the scale factor increases as $a \propto \tau^{1/4}$, whereas in standard cosmology $a \propto \tau^{1/2}$. The squared density term does not affect BBN and CMB significantly because this term dominates at much earlier time than the epoch of BBN and recombination.

1.4 Extensions of the RS brane model

In previous sections, we have assumed that there is no dynamical degree of freedom other than the spacetime metric in the bulk. More realistically, e.g., from the stringy point of view, it would be natural that there are dynamical degrees of freedom other than the metric. We also have assumed that the bulk gravitational theory is the Einstein gravity. But, in higher-dimensions we may add curvature corrections in the gravitational theory in the bulk. We may also add an induced gravity term into the boundary action.

From these considerations, several extensions of the braneworld from the original RS model have been proposed. In this thesis, we also discuss KK modes in these extended models.

1.4.1 Bulk scalar fields and bulk inflaton models

We may add some dynamical degrees of freedom other than the spacetime metric, e.g., a scalar field, in the bulk theory. These bulk fields appear as a result of dimensional reductions of higher-dimensional theories, e.g., moduli fields, dilatons and so on. As a simple model, we can consider brane models with a bulk scalar field,

$$
\begin{aligned}
S &= \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left(\overset{(5)}{R} + \kappa_5^2 \left(-(\partial\phi)^2 - 2V(\phi) \right) \right) \\
&+ \int d^4x \sqrt{-q} \left(-\sigma(\phi) \right) .
\end{aligned}
\tag{1.20}
$$

In the context of the braneworld, the models including a bulk scalar field has firstly been discussed from phenomenology, see e.g., [24, 25, 26, 27]. In paticular situations, this type of scalar field behaves as an inflaton, namely the dynamics of the field induces a cosmological inflation on the brane, see e.g., [28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. In these cases, this type of scalar field is often called *bulk inflaton*. The effective scalar field potential induced on the brane is given by [26]

$$
\Lambda_4 = \frac{1}{2}\kappa_5^2 \left(V(\phi) + \frac{1}{6}\kappa_5^2 \left(\sigma(\phi) \right)^2 - \frac{1}{8} \left(\frac{d\sigma(\phi)}{d\phi} \right)^2 \right) .
\tag{1.21}
$$

The static (time-independent) bulk fields are also used for supporting *thick braneworlds*, i.e., classical domain walls in the bulk, see e.g.,[38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. We will discuss quantum fluctuations on a thick brane model in Chapter 4.

1.4.2 Einstein Gauss-Bonnet (EGB) braneworld

We also may add higher-curvature terms into the bulk gravitational theory. In the four dimensions, the tensor gravitational theory is uniquely given by the Einstein theory, but in higher dimensions, higher order curvature corrections may also be added. For instance, the gravitational theory including the Gauss-Bonnet (GB) term;

$$
\begin{aligned}
S_{\text{EGB}} &= \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \\
&\times \left[\overset{(5)}{R} + \alpha \left(\overset{(5)}{R}{}^2 - 4 \overset{(5)}{R}_{cd} \overset{(5)}{R}{}^{cd} + \overset{(5)}{R}_{cdef} \overset{(5)}{R}{}^{cdef} \right) - 2\Lambda_5 \right] \\
&+ \int d^4x \sqrt{-q} \left(-\sigma \right) ,
\end{aligned}
\tag{1.22}
$$

gives the most natural tensor gravity in five dimensions [4]. α is coupling parameter of the GB term of dimensions $(length)^2$. Note that the GB term

---

[4]For general apects of the gravitatinal theory with the GB term, including a boundary brane, see e.g., [50, 51, 52]

is just a topological quantity in four-dimensional spacetime, namely just gives a total derivative term for the variation of the gravitational action, but it becomes dynamical in higher-dimensions. This type of correction also appears as a low energy correction in the perturbative approach to string theory. Braneworld models in the *Einstein-Gauss-Bonnet (EGB)* theory have predicted new, interesting phenomena. A part of them will be discussed in this thesis. Here we introduce some basic cosmological features of this model.

The AdS solution whose metric is given by, for instance, Eq. (1.2) or Eq. (1.6), has two possible branches,

$$\frac{1}{\ell^2} = \frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda_5}{3}}\right).$$ (1.23)

The $(-)$ branch is reduced to the solution in the Einstein gravity for $\alpha \to 0$, whereas the $(+)$ branch gives the completely new sequence of solution in the EGB theory, which is known to be unstable for perturbations.

The modified Friedmann equation in this model, neglecting the dark radiation term (i.e., without a black hole in the bulk), is given by

$$H^2 = \frac{1}{4\alpha}\left[\left(1 + \frac{4}{3}\alpha\Lambda_5\right)^{1/2}\cosh\left(\frac{2}{3}x\right) - 1\right],$$ (1.24)

$$\rho + \sigma = \left[\frac{2}{\alpha\kappa_5^4}(1 + \frac{4}{3}\alpha\Lambda_5)^{3/2}\right]^{1/2}\sinh x.$$

For the high energy regime, $x \gg 1$, we obtain $H^2 \propto \rho^{2/3}$. Cosmologies in the EGB braneworld have been studied, see e.g., [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 64, 66, 67, 68]. Also higher-dimensional black holes in the EGB theory have been studied, see e.g., [69, 70, 71, 72, 73, 74, 75, 76, 77, 78].

1.4.3   Dvali-Gabadazze-Porrati (DGP) braneworld

There are other possible extensions of RS model, although we will not discuss them explicitly in this thesis. The most familiar (and interesting) one is adding an induced gravity term (i.e., four-dimensional scalar curvature) into the boundary action;

$$S_{\mathrm{DGP}} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(\overset{(5)}{R} - 2\Lambda_5\right) + \int d^4x\sqrt{-q}\left(\frac{1}{2\kappa_4^2}\overset{(4)}{R} - \sigma\right).$$ (1.25)

This type of model was originally discussed by Dvali, Gabadazze and Porrati [79] and is called *DGP* model. The scalar curvature term on the brane is assumed to be induced by quantum effects in the bulk.

In this model, again neglecting the dark radiation term, the modified Friedmann equation on the brane is obtained as [80]

$$H^2 + \frac{K}{a^2} = \frac{\kappa_4^2}{3}\left[\rho + \rho_0\left\{1 + \epsilon\left(1 - \frac{2\eta\Lambda_4}{\kappa_4^2\rho_0} + \frac{2\eta\rho}{\rho_0}\right)^{1/2}\right\}\right],$$ (1.26)

where $\epsilon = \pm 1$ denotes the possible branches of the bulk solutions and

$$\eta := \frac{6\kappa_4^2}{\kappa_5^4}, \quad \rho_0 := \sigma + \frac{6\kappa_4^2}{\kappa_5^4}. \tag{1.27}$$

Λ_4 is the effective cosmological constant given by Eq. (1.19). Assuming a flat bulk ($\Lambda_5 = 0$) and zero tension brane ($\sigma = 0$), i.e., $\Lambda_4 = 0$, if we choose $\epsilon = +1$ branch, at a later time, neglecting ρ terms, we obtain

$$H^2 = 4\frac{\kappa_4^4}{\kappa_5^4} := r_c^{-2} \tag{1.28}$$

(We also choose $K = 0$). Although we assume a vanishing bulk cosmological constant and a vanishing brane tension, we obtain an accelerating universe with the expansion rate given by $r_c = \kappa_5^2/(2\kappa_4^2) = M_{\rm pl}^2/(2M_5^3)$ [81]. This late-time *self-acceleration* is an interesting feature in this type of model. As we readily see, five dimensional effects dominate at larger distance scales $r > r_c$. In this sense, the self-acceleration may result from the contributions of KK modes. The self-accelerating universe has been discussed as one of the possibilities for the present acceleration of the expansion of our Universe, so-called dark energy, from the theoretical and phenomenological aspects, see, e.g., [82, 83, 84, 85, 86] and from the observational aspects, see, e.g., [87, 88, 89, 90].

1.5 The purpose of this thesis

In this thesis, we focus on the Kaluza-Klein (KK) modes in braneworld cosmology in the context of the second RS model including its extensions. The name "KK modes" just comes from Kaluza-Klein theory [1, 2] and their behavior in extra-dimensions is quite similar both in KK theory and in RS braneworld. As we will see in this thesis, from four-dimensional observers in the braneworld context the behavior of KK modes is quite different from those in the original KK theory.

Braneworlds predict novel, interesting phenomena and hopefully give possible resolutions of difficuluties in the standard four-dimensional cosmology. For instance, as discussed in the previous section, the solution of self-accelerating universe in DGP braneworld gives a possibility for the present acceleration of the expansion of the Universe, without introducing any explicit dark energy source. Then, five-dimensional effects, i.e., KK modes, play the role of the dark energy effectively. As in this case, KK modes in braneworld cosmology may have further, intriguing effects, which should be investigated. In the near future, the detection of signals from extra-dimensions might also be realized in high-energy accelerators and/or cosmological observations. From this aspect, we need to clarify the signals from extra-dimensions in braneworld context.

On the other hand, recent observations are highly in agreement with the predictions of four-dimensional general relativity. Cosmological observations are also consistent with standard model of cosmology. So, in the present situation, in order for braneworld cosmology to be viable, it is required that predictions of it should be at least consistent with these observations. In this sense, the amount of the KK modes on the brane which appear as corrections to four-dimensional theory should not be so large. In the case that it is large, some of the detailed assumptions of the model should be modified, from realistic point of views. We need to evaluate the amount of the KK modes carefully. Any confliction with observations and experiments may tell us more realistic extentions of braneworld.

Keeping the things stated above in mind, we have planed to investigate the behavior of KK modes in RS braneworld cosmology, especially their dynamics and effects, from various aspects, in order to clarify their qualitative nature on a cosmological brane, to cure their pathological nature and to find new dynamics of them in more general context of brane cosmology. In this thesis, we report our results of these investigations at the present.

1.6   Outline of this thesis

This thesis is organized as follows: In *Chapter 2*, we first introduce the effective gravitational equations on the brane by using the geometrical projection method. This formalism gives an useful tool to analyze non-linear aspects of the effective theory on the brane. Furthermore, for a cosmological brane with a maximally symmetric three-space, we show that there are local conservation laws as a natural extension of those in a four-dimensional spherically symmetric spacetime. For such a brane model, we derive a closed set of equations to describe the motion of the brane in the bulk in terms of these locally conserved quantities.

In *Chapter 3*, we discuss the backreaction of KK gravitons produced in the early stage of brane cosmology, in the bulk and on the brane. For KK gravitons which are produced by high energy particle interactions on the brane, we assume them as null dust flux emitted radially from the brane. We discuss the bulk geometry and brane cosmology. We also discuss the possibility of forming a naked singularity in the bulk. In terms of the locally conserved quantities in the bulk with a maximally symmetric three-space, which are derived in the previous Chapter, we also derive a set of equations which dominate the cosmological evolution of the brane in the bulk. On the other hand, we derive the effective stress energy tensor for gravitons of a KK mode, which are produced quantum mechanically in the whole bulk and discuss its qualitative nature on the brane and in the bulk for a single KK mode.

In connection with the second case in the previous Chapter, what we should note is that what we observe on the brane is the sum of an infinite

number of KK modes. Also, we need to treat KK modes in terms of quantum field theory because they are produced quantum mechanically. As is well-known, however, the sum suffers divergence as one approaches the brane from the bulk, even after a conventional UV regularization. In *Chapter 4*, we propose a new regularization scheme for the sum of all KK modes on the brane. We consider a thick brane model and show that a finite brane thickness can regularize the KK modes on the brane. Then, we demonstrate that the size of the quantum backreaction can be reduced below that of the background thick brane. From thes discussions, we give a theoretical bound on the brane thickness parameter.

Finally, we extend our attention to braneworld model with the Gauss-Bonnet curvature correction. In *Chapter 5*, we discuss the linearized effective gravity on a cosmological (de Sitter) brane in the five-dimensional Einstein Gauss-Bonnet theory. We show that there are quite novel features in the linearized effective theory on the brane, that the effective gravity on the brane is four-dimensional on all scales, from short distances to large distances. KK modes, to sum up, play the role of a scalar field degree of freedom in the effective four-dimensional theory on short distance scales. We also show that on high energy expanding branes as well as on low energy ones, effective gravity becomes also four-dimensional.

In *Chapter 6*, we shall summarize this thesis and mention issues related to our work.

2

Non-linear effective equations in the bulk and on the brane

Based on [91] (See also [34] for the case with bulk dynamical degree of freedom other than the spacetime metric), we first derive the effective gravitational equations on the brane by using the geometrical projection method. Then, especially for the brane which has cosmological symmetry, i.e., homogeneity and isotropy, we derive a set of equations to describe the motion of the brane in the bulk in terms of these locally conserved quantities defined in the bulk [92].

2.1 Effective gravitational theory on the brane

We set Gaussian normal coordinate around a brane

$$ds^2 \;=\; g_{ab}dx^a dx^b = \left(n_a n_b + q_{ab}\right) dx^a dx^b = dy^2 + q_{ab}dx^a dx^b \qquad (2.1)$$

where we define the bulk coordinate $dy = n_a dx^a$. n^a and q_{ab} are unit vector field normal to time-like hypersurfaces and induced meric on the hypersurface, respectively. We assume that the brane is located at $y = 0$. The Latin indices $\{a, b\}$ denote tensors defined in the bulk whereas the Greek indices $\{\mu, \nu\}$ those defined on the brane in this thesis.

We start from the five-dimensional Einstein theory

$$\overset{(5)}{G}_{ab} + \Lambda_5 g_{ab} = \kappa_5^2 T_{ab} + \kappa_5^2 S_{ab}\delta(y) \,, \qquad (2.2)$$

and the surface stress-tensor is given by the summation of the tension and localized matter

$$S_{ab} = -\sigma q_{ab} + \tau_{ab} \,, \qquad (2.3)$$

where σ denotes the brane tension and τ_{ab} does brae localized matter. In this thesis, we denote curvature tensors defined in the bulk as $"\overset{(5)}{A}"$ whereas them defined on the brane as simply $"A"$.

Using the geometrical identities, we obtain

$$
\begin{aligned}
G_{ab} &= -\frac{1}{2}\Lambda_5 q_{ab} + \frac{2}{3}\kappa_5^2 \left(T_{cd}q_a^c q_b^d + \left(T_{cd}n^c n^d - \frac{1}{4}T^c{}_c \right) q_{ab} \right) \\
&+ KK_{ab} - K_{ac}K^c{}_b - \frac{1}{2}q_{ab}\left(K^2 - K^{cd}K_{cd} \right) - E_{ab}\,,
\end{aligned} \tag{2.4}
$$

where $K_{ab} = q_a^c q_b^d \nabla_c n_d$ is the extrinsic curvature on the hypersurface whose metric is given by q_{ab} (∇_a is the covariant derivative with respect to the bulk metric g_{ab}). And

$$
E_{ab} = \overset{(5)}{C}{}_{cdef} n^c n^e q_a^d q_b^f\,, \tag{2.5}
$$

where $\overset{(5)}{C}{}_{abcd}$ is five-dimensional Weyl tensor.

Then, we consider the junction (maching) condition across the brane,

$$
[q_{\mu\nu}] = 0
$$

$$
[K_{\mu\nu}] = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3}q_{\mu\nu}S \right)\,. \tag{2.6}
$$

Furthermore, taking the assumption of the $\mathbb{Z}_2$-symmetry with respect to the brane into account,

$$
K_{\mu\nu}\Big|_{y\to 0+} = -\frac{\kappa_5^2}{2}\left(S_{\mu\nu} - \frac{1}{3}q_{\mu\nu}S \right)\,. \tag{2.7}
$$

Substituting Eq. (2.7) into Eq. (2.4), we obtain the desired effective gravitational equation on the brane,

$$
G_{\mu\nu} + \Lambda_4 q_{\mu\nu} = \kappa_4^2 \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} + \kappa_5^2 T_{\mu\nu}^{(b)} - E_{\mu\nu}\,, \tag{2.8}
$$

where

$$
T_{\mu\nu}^{(b)} = \frac{2}{3}\left(T_{cd}q_\mu^c q_\nu^d + \left(T_{cd}n^c n^d - \frac{1}{4}T^c{}_c \right) q_{\mu\nu} \right)\,,
$$

$$
\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau_\nu^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}q_{\mu\nu}\tau^2\,. \tag{2.9}
$$

Here we define

$$
\kappa_4^2 = \frac{1}{6}\kappa_5^4\sigma^2\,,
$$

$$
\Lambda_4 = \frac{1}{2}\Lambda_5 + \frac{1}{12}\kappa_5^4\sigma^2\,, \tag{2.10}
$$

which are read as four-dimensional gravtational constant and cosmological constant, respectively. If there is only the first term in the right-handed-size of Eq. (2.8), the conventional four-dimensional Einstein gravity is recovered.

The $\pi_{\mu\nu}$ term is quadratic order of stress-energy tensor of brane-localized matter. This term gives the term proportional to ρ^2 in the modified Friedmann equation Eq. (1.18) for a cosmological brane. $T_{\mu\nu}^{(b)}$ term is just coming from the projection of the stress-energy tensor of the bulk degrees of freedom. $E_{\mu\nu}$ is the contribution of the bulk gravitational field, which is one of the most important quantities in brane cosmology.

Stricktly speaking, the set of effective equations on the brane is not closed even if the evolutions of the bulk metric and other bulk dynamical degrees of freedom are solved in the bulk. In general, we need to derive the equations of motion of $E_{\mu\nu}$ [93, 34]. However, for a cosmological brane with a maximally symmetric three-space, the form of $E_{\mu\nu}$ can be specified because of the high degree of symmetry, apart from the depencence on the initial hypersurface.

2.2 Local conservation laws and brane cosmology

In this section, we discuss the general property of a dynamical bulk spacetime with maximally symmetric three-space, and consider cosmology on the brane. First, we derive local conservation laws in the bulk, as generalization of the local energy conservation law in a spherically symmetric spacetime in four-dimensions [94, 95, 96].

Next, we introduce the brane as a boundary of the dynamical spacetime. The effective Friedmann equation is determined via the junction condition and it is shown that the local mass corresponds to the generalized dark radiation. Finally, we show that the projected Weyl tensor on the brane is uniquely related to the local mass.

2.2.1 Local conservation law

We assume that the bulk allows slicing by a maximally symmetric three-space. Then, the bulk metric can written in the double-null form

$$ds^2 = \frac{4r_{,u}r_{,v}}{\Omega}dudv + r(u,v)^2 d\Sigma_{(K,3)}^2, \tag{2.11}$$

where we refer to v and u as the advanced and retarded time coordinates, respectively. In Appendix A. 1, the explicit components of the connection and curvature in an $(n+2)$-dimensional spacetime with maximally symmetric n-space are listed.

First, we consider the Einstein equations in the bulk. They are given by

$$3\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,v}}{\Omega}\right|\right)_{,u} = \kappa_5^2 T_{uu}, \quad 3\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}}{\Omega}\right|\right)_{,v} = \kappa_5^2 T_{vv},$$

$$6\frac{r_{,u}r_{,v}}{r^2}\left(1 - \frac{K}{\Omega}\right) + 3\frac{r_{,uv}}{r} = \kappa_5^2 T_{uv} - \frac{2r_{,u}r_{,v}}{\Omega}\Lambda_5,$$

$$\left\{\frac{r^2\Omega}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv} + 4\frac{r_{,uv}}{r}\right] - \left(K - \Omega\right)\right\}\gamma_{ij} = \kappa_5^2 T_{ij}$$

$$-r^2\gamma_{ij}\Lambda_5, \tag{2.12}$$

where γ_{ij} is the intrinsic metric of the maximally symmetric three-space.

Now, we derive the local conservation law. We introduce a vector field in five-dimensional spacetime as

$$\xi^a = \frac{1}{2}\Omega\left(-\frac{1}{r_{,v}}\frac{\partial}{\partial v} + \frac{1}{r_{,u}}\frac{\partial}{\partial u}\right)^a. \tag{2.13}$$

From the form of the metric (2.11), we can readily see that ξ^a is conserved:

$$\sqrt{-g}\,\xi^a{}_{;a} = \left(\sqrt{-g}\,\xi^a\right)_{,a} = 2\sqrt{\gamma}\left((r^3 r_{,u})_{,v} - (r^3 r_{,v})_{,u}\right) = 0, \tag{2.14}$$

where $\gamma = \det\gamma_{ij}$. Note that, for an asymptotically constant curvature spacetime, the vector field ξ^a becomes asymptotically the timelike Killing vector field $-(\partial/\partial t)^a$.

With this vector field ξ^a, we define a new vector field,

$$\tilde{S}^a = \xi^b\tilde{T}_b{}^a, \tag{2.15}$$

where

$$\tilde{T}_{ab} = T_{ab} - \frac{1}{\kappa_5^2}\Lambda_5 g_{ab}. \tag{2.16}$$

Using the Einstein equations, the components of the vector field $\tilde{S}^a$ are given by

$$\kappa_5^2\sqrt{-g}\tilde{S}^v = \frac{3}{2}\left[r^2\left(K - \Omega\right)\right]_{,u}\sqrt{\gamma},$$

$$\kappa_5^2\sqrt{-g}\tilde{S}^u = -\frac{3}{2}\left[r^2\left(K - \Omega\right)\right]_{,v}\sqrt{\gamma}. \tag{2.17}$$

Then, we have the local conservation law as

$$\tilde{S}^a{}_{;a} = 0. \tag{2.18}$$

Since ξ^a is conserved separately, the conservation of $\tilde{S}^a$ implies that we have another conserved current S^a defined by

$$S^a := \xi^b T_b{}^a \left(= \tilde{S}^a + \frac{1}{\kappa_5^2}\Lambda_5\xi^a\right). \tag{2.19}$$

Thus we have the local conservation law for the energy-momentum tensor in the bulk.

From Eqs. (2.17), we readily see the local mass corresponding to $\tilde{S}^a$ is given by [94]

$$\tilde{M} := \left(K - \Omega\right)r^2, \tag{2.20}$$

where the factor $3/2$ in the original expression for $\tilde{S}^a$ is eliminated for later convenience. Alternatively, corresponding to S^a, we have another local mass that excludes the contribution of the bulk cosmological constant,

$$M := \tilde{M} - \frac{1}{6}\Lambda_5 r^4 = \left(K - \Omega\right)r^2 - \frac{1}{6}\Lambda_5 r^4. \tag{2.21}$$

In what follows, we focus on the matter part M, rather than on the whole mass $\tilde{M}$. It may be noted, however, that this decomposition of $\tilde{M}$ to the cosmological constant part and the matter part is rather arbitrary, as in the case of a bulk scalar field. Here we adopt this decomposition just for convenience. For example, this decomposition is more useful when we consider small perturbations on the static AdS-Schwarzschild bulk.

We note that, in the case of a spherically symmetric asymptotic flat space-time in four-dimensions (hence $K = +1$ and with no cosmological constant), this function M agrees with the Arnowitt-Deser-Misner (ADM) energy or the Bondi energy in the appropriate limits.

2.2.2 Local mass and a charge associated with Weyl tensor

From the five-dimensional Einstein equations (2.12), we can write down the local conservation equation for M in terms of the bulk energy-momentum tensor explicitly as

$$M_{,v} = \frac{2}{3}\kappa_5^2 r^3 \left(T^u{}_v r_{,u} - T^v{}_v r_{,v}\right),$$

$$M_{,u} = \frac{2}{3}\kappa_5^2 r^3 \left(T^v{}_u r_{,v} - T^v{}_v r_{,u}\right), \tag{2.22}$$

or in a bit more concise form,

$$dM = \frac{2}{3}\kappa_5^2 r^3 \left(T^u{}_v r_{,u} dv + T^v{}_u r_{,v} du - T^v{}_v dr\right). \tag{2.23}$$

Using the above, we can immediately write down two integral expressions for M given in terms of flux crossing the $u = $ constant hypersurfaces from v_1 to v_2, and flux crossing the $v = $ constant hypersurfaces from u_1 to u_2, respectively, as

$$M(v_2, u) - M(v_1, u) = \frac{2}{3}\kappa_5^2 \int_{v_1}^{v_2} dv\, r^3 \left(T^u{}_v r_{,u} - T^v{}_v r_{,v}\right)\Big|_{u=\text{const.}},$$

$$M(v, u_2) - M(v, u_1) = \frac{2}{3}\kappa_5^2 \int_{u_1}^{u_2} du\, r^3 \left(T^v{}_u r_{,v} - T^v{}_v r_{,u}\right)\Big|_{v=\text{const.}} \tag{2.24}$$

Finally, let us consider the Weyl tensor in the bulk. In the present case of a five-dimensional spacetime with maximally symmetric three-space, there exists only one non-trivial component of the Weyl tensor, say $\overset{(5)}{C}{}_{vu}{}^{vu}$. The explicit expressions for the components of the Weyl tensor are given in Appendix A. 1, Eqs. (A.7). Using the Bianchi identities and the Einstein equations, we have [97]

$$\overset{(n+2)}{C}{}_{abcd}{}^{;d} = J_{abc},\qquad(2.25)$$

where

$$J_{abc} = \frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]} + \frac{1}{(n+1)}g_{c[b}T_{;a]}\right).\qquad(2.26)$$

From this, we can show that there exists a conserved current,

$$Q^a = r\,\ell_b n_c J^{bca}\,;\quad Q^a{}_{;a} = 0\,,\qquad(2.27)$$

where ℓ_a and n_a are a set of two hypersurface orthogonal null vectors,

$$\begin{aligned}
\ell_a &= \sqrt{\frac{2}{\Omega}}\,(-r_{,v}\,dv)_a\,, & \ell^a &= -\sqrt{\frac{1}{2}\Omega}\,\frac{1}{r_{,u}}\left(\frac{\partial}{\partial u}\right)^a,\\
n_a &= \sqrt{\frac{2}{\Omega}}\,(r_{,u}\,du)_a\,, & n^a &= \sqrt{\frac{1}{2}\Omega}\,\frac{1}{r_{,v}}\left(\frac{\partial}{\partial v}\right)^a.
\end{aligned}\qquad(2.28)$$

The non-zero components are written explicitly as

$$Q^u = -rJ^{vu}{}_v\,,\quad Q^v = -rJ^{vu}{}_u\,,\qquad(2.29)$$

and we have

$$\left(r^4\,\overset{(5)}{C}{}_{vu}{}^{vu}\right)_{,v} = r^4 J^v{}_{vv}\,,\qquad(2.30)$$

$$\left(r^4\,\overset{(5)}{C}{}_{vu}{}^{vu}\right)_{,u} = r^4 J^u{}_{uu}\,.$$

These are very similar to Eqs. (2.17). It is clear that $r^4\,\overset{(5)}{C}{}_{vu}{}^{vu}$ defines a local charge associated with this conserved current.

Using the Einstein equations, we then find that this charge can be expressed in terms of M and the energy-momentum tensor as

$$r^4\,\overset{(5)}{C}{}_{vu}{}^{vu} = 3\tilde{M} + \frac{r^4}{6}\left(6\,\overset{(5)}{G}{}^v{}_v - \overset{(5)}{G}{}^i{}_i\right) = 3M + \frac{\kappa_5^2}{6}r^4\left(6T^v{}_v - T^i{}_i\right).\qquad(2.31)$$

This is one of the most important results in this paper. As we shall see below, the Weyl component $\overset{(5)}{C}{}_{vu}{}^{vu}$ is directly related to the projected Weyl tensor $E_{\mu\nu}$, and hence this relation gives explicitly how the local mass M and the local value of the energy-momentum tensor affects the brane dynamics.

2.2.3 Apparent horizons

As in the conventional four-dimensional gravity, the gravitational dynamics may lead to the formation of a black hole in the bulk. Rigorously speaking, the black hole formation can be discussed only by analyzing the global causal structure of a spacetime. Nevertheless, we discuss the black hole formation by studying the formation of an apparent horizon.

In four-dimensions, an apparent horizon is defined as a closed two-sphere on which the expansion of an outgoing (or ingoing) null geodesic congruence vanishes. Here, we extend the definition to our case and define an apparent horizon as a three-surface on which the expansion of a radial null geodesic congruence vanishes. Note that 'radial' here means simply those congruences that have only the (v, u) components, hence an apparent horizon will not be a closed surface if $K = 0$.

The expansions of the congruence of null geodesics forming the $u =$ constant and $v =$ constant hypersurfaces, respectively, are given by [94]

$$\rho_u = -\frac{1}{2}u^{;a}{}_{;a} = -\frac{1}{2r}\frac{\Omega}{r_{,u}}, \quad \rho_v = -\frac{1}{2}v^{;a}{}_{;a} = -\frac{1}{2r}\frac{\Omega}{r_{,v}}. \tag{2.32}$$

Naively, if $\Omega = 0$, one might think that both ρ_u and ρ_v vanish. However, from the regularity condition of the metric (2.11), we have

$$-4\frac{r_{,u}r_{,v}}{\Omega} > 0. \tag{2.33}$$

Hence, it must be that $r_{,u} = 0$ or $r_{,v} = 0$, if $\Omega = 0$. If $\Omega = r_{,v} = 0$, we have $\rho_u = 0$ and an apparent horizon for the outgoing null geodesics is formed, whereas if $\Omega = r_{,u} = 0$, we have $\rho_v = 0$ and an apparent horizon for the ingoing null geodesics is formed.

2.2.4 Brane cosmology

We now consider the dynamics of a brane in a dynamical bulk with maximally symmetric three-space [21]. The brane trajectory is parameterized as $(v, u) = \big(v(\tau), u(\tau)\big)$. Taking τ to be the proper time on the brane, we have

$$4\frac{r_{,u}r_{,v}}{\Omega}\dot{u}\dot{v} = -1, \tag{2.34}$$

on the brane, where $\dot{u} = du/d\tau$ and so on. The unit vector tangent to the brane (i.e., the five-velocity of the brane) is given by

$$v^a = \left(\dot{v}\frac{\partial}{\partial v} + \dot{u}\frac{\partial}{\partial u}\right)^a, \quad v_a = \frac{2r_{,u}r_{,v}}{\Omega}\left(\dot{u}\,dv + \dot{v}\,du\right)_a, \tag{2.35}$$

and the unit normal to the brane is given by

$$n^a = \left(-\dot{v}\frac{\partial}{\partial v} + \dot{u}\frac{\partial}{\partial u}\right)^a, \quad n_a = \frac{2r_{,u}r_{,v}}{\Omega}\left(\dot{u}\,dv - \dot{v}\,du\right)_a. \tag{2.36}$$

The components of the induced metric on the brane are calculated as

$$q_{\mu\nu} = \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} g_{ab}, \tag{2.37}$$

where μ, ν run from 0 to 3 and y^μ are the intrinsic coordinates on the brane with $y^0 = \tau$ and $y^i = x^i$ $(i = 1, 2, 3)$. Then the induced metric on the brane is given by

$$ds^2_{(4)} = -d\tau^2 + r(\tau)^2 d\Sigma^2_{(K,3)}, \tag{2.38}$$

The trajectory of the brane is determined by the junction condition under the $\mathbb{Z}_2$ symmetry with respect to the brane. The extrinsic curvature on the brane is determined as

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2} \left(S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu} \right), \tag{2.39}$$

where $S_{\mu\nu}$ is assumed to take the form

$$S^\mu_{\ \nu} = \text{diag.} \left(-\rho, p, p, p \right) - \sigma \delta^\mu_{\ \nu}, \tag{2.40}$$

with σ and ρ being the tension and energy density of the matter on the brane, respectively, as introduced previously, and p being the isotropic pressure of the matter on the brane. Substituting the induced metric (2.38) in Eq. (2.39), we obtain

$$r_{,u} \dot{u} = -\frac{r}{2} \left[\frac{\kappa_5^2}{6} \left(\rho + \sigma \right) - H \right], \tag{2.41}$$

$$r_{,v} \dot{v} = \frac{r}{2} \left[\frac{\kappa_5^2}{6} \left(\rho + \sigma \right) + H \right], \tag{2.42}$$

where $H = \dot{r}/r$. Multiplying the above two equations and using the normalization condition (2.34), we then obtain the effective Friedmann equation on the brane:

$$H^2 + \frac{K}{r^2} = \left(\frac{\kappa_5^4}{36} \sigma^2 - \frac{1}{\ell^2} \right) + \frac{\kappa_5^4}{18} \left(2\sigma\rho + \rho^2 \right) + \frac{M}{r^4}. \tag{2.43}$$

We see that M is a natural generalization of the dark radiation in the AdS-Schwarzschild case to a dynamical bulk from the comparison with the Eq. (1.18).

For a dynamical bulk, M varies in time. The evolution of M is determined by Eq. (2.23), and on the brane it gives

$$\begin{aligned}
\dot{M} &= M_{,v} \dot{v} + M_{,u} \dot{u} \\
&= \frac{2}{3} \kappa_5^2 r^4 \left[T_{vv} \left(\frac{1}{6} \kappa_5^2 (\rho + \sigma) - H \right) \dot{v}^2 - T_{uu} \left(\frac{1}{6} \kappa_5^2 (\rho + \sigma) + H \right) \dot{u}^2 \right] \\
&\quad - \frac{2}{3} \kappa_5^2 r^4 H T^v_{\ v}.
\end{aligned} \tag{2.44}$$

From the Codacci equation on the brane [91],

$$D_\nu K^\nu{}_\mu - D_\mu K^\nu{}_\nu = \kappa_5^2 T_{ab} n^b q^a{}_\mu \,, \tag{2.45}$$

where D_μ is the covariant derivative with respect to $q_{\mu\nu}$ and $K_{\mu\nu}$ is the extrinsic curvature of the brane, we obtain the equation for the energy transfer of the matter on the brane to the bulk,

$$\dot\rho + 3H(\rho + p) = 2\Big(-T_{vv}\dot v^2 + T_{uu}\dot u^2\Big). \tag{2.46}$$

Equations (2.43), (2.44) and (2.46) determine the cosmological evolution on the brane, once the bulk geometry is solved.

Now we relate the above result with the geometrical approach developed in the previous section, in particular with the projected Weyl tensor $E_{\mu\nu}$ on the brane, defined in Eq. (2.5). It has only one non-trivial component as

$$E_{\tau\tau} = \overset{(5)}{C}{}_{abcd} n^a n^c v^b v^d = 4 \overset{(5)}{C}{}_{uvuv} \dot u^2 \dot v^2 = - \overset{(5)}{C}{}_{vu}{}^{vu}. \tag{2.47}$$

Using Eq. (2.31), this can be uniquely decomposed into the part proportional to M and the part due to the projection of the bulk energy-momentum tensor on the brane. We find

$$E_{\tau\tau} = -\frac{3\tilde M}{r^4} + \frac16\Big(\overset{(5)}{G}{}^i{}_i - 6 \overset{(5)}{G}{}^v{}_v\Big) = -\frac{3M}{r^4} + \frac{\kappa_5^2}{6}\Big(T^i{}_i - 6T^v{}_v\Big). \tag{2.48}$$

If we eliminate the M/r^4 term from Eq. (2.43) by using this equation, we recover the effective Friedmann equation on the brane in the geometrical approach [91],

$$H^2 + \frac{K}{r^2} = \Big(\frac{\kappa_5^4}{36}\sigma^2 - \frac{1}{\ell^2}\Big) + \frac{\kappa_5^4}{18}\Big(2\sigma\rho + \rho^2\Big) + \kappa_5^2 T^{(b)}_{\tau\tau} - \frac{E_{\tau\tau}}{3}\,, \tag{2.49}$$

where $T^{(b)}_{\tau\tau}$ comes from the projection of the bulk energy-momentum tensor on the brane and is given in the present case by

$$T^{(b)}_{\tau\tau} = \frac16 T^i{}_i - T^v{}_v\,. \tag{2.50}$$

3

Backreaction of Kaluza-Klein gravitons

In this Chapter, we discuss the backreaction of KK gravitons, i.e., the bulk metric perturbations, in the bulk and on a cosmological brane. As we mentioned in the Chapter 1, it is considered that these KK gravitons are *naturally* produced at an initial high-energy stage of the brane universe. There are mainly two possibilities of the productions of KK gravitons, namely, via high energy particle interactions on the brane and quantum fluctuations in the bulk. Here, we show two different analyses corresponding to these two cases. In the first analysis, we are interested in the KK gravitons which are produced by non-standard particle interactions on the brane. We assume KK gravitons as null dust flux which is emitted from the brane radially [92]. First, we discuss the bulk geometry. We also derive a set of equations which describe the cosmological evolution of the brane, i.e., its trajectory into the bulk, by using the the locally conserved quanties defined in the previous Chapter. Then, we discuss the possibility of formation of a naked singularity in the bulk.

In the second analysis, we discuss the backreaction of KK gravitons which are produced quantum mechanically in the whole bulk, during brane inflation (or exist from the beginning). We derive their effective stress tensor by computing the curvature tensors up to the second order of perturbations and averaging them, taking the existence of the infinitely thin brane into account [98]. Taking out a single graviton KK mode, we derive its effective energy density and pressure, in the bulk and on the brane. We show that a KK mode, if sufficiently massive, behaves as cosmic dust but the energy density becomes negatve. Then, we discuss the physical reason of the negativity of the energy density in terms of the local conservation laws in the bulk, discussed in the previous Chapter.

3.1 Emission of radial Kaluza-Klein gravitons

In this section, by using the local mass derived in the previous Chapter, we discuss the backreaction of the bulk geometry and brane cosmology in the context of an ingoing null dust model [99]. Here, we are interested in KK gravitons which are produced by matter exicitations on the brane.

3.1.1 Set-up

The energy-momentum tensor of a null dust fluid takes the form,

$$T_{ab} = \mu_1 \ell_a \ell_b + \mu_2 n_a n_b, \tag{3.1}$$

where ℓ_a and n_a are the ingoing and outgoing null vectors, respectively, introduced in Eqs. (2.28). If we require that the energy-momentum conservation law is satisfied for the ingoing and outgoing null dust independently, we have

$$\mu_1 = \frac{\Omega}{(r_{,v})^2 r^3} \frac{f(v)}{2}, \qquad \mu_2 = \frac{\Omega}{(r_{,u})^2 r^3} \frac{g(u)}{2}, \tag{3.2}$$

where $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively, and have the dimension $(G_5 \times \text{mass})^{-1}$. We assume the positive energy density, i.e., $f(v) \geq 0$ and $g(u) \geq 0$. Thus, the non-trivial components of the energy-momentum tensor are

$$T_{vv} = \frac{f(v)}{r^3}, \qquad T_{uu} = \frac{g(u)}{r^3}. \tag{3.3}$$

To satisfy the local conservation law in an infinitesimal interval $(u, u+du)$ and $(v, v+dv)$, we find that the intensity functions $f(v)$ and $g(u)$ have to satisfy the relation,

$$f(v)\left(\frac{\Omega}{r_{,v}}\right)_{,u} = g(u)\left(\frac{\Omega}{r_{,u}}\right)_{,v}. \tag{3.4}$$

In general, if both $f(v)$ and $g(u)$ are non-zero, it seems almost impossible to find an analytic solution that satisfies Eq. (3.4). Hence we choose to set either $f(v) = 0$ or $g(u) = 0$. In the following discussion, we focus on the case that $g(u) = 0$, that is, the ingoing null dust.

For $g(u) = 0$, Eqs. (2.23) give

$$M_{,v} = \frac{1}{3}\kappa_5^2 \frac{\Omega}{r_{,v}} f(v), \qquad M_{,u} = 0. \tag{3.5}$$

The second equation implies $M = M(v)$. Substituting Eq. (3.3) into the Einstein equations (2.12), we find

$$\frac{\Omega}{r_{,v}} = e^{F(v)}, \tag{3.6}$$

where the function $F(v)$ describes the freedom in the rescaling off the null coordinate v. This equation is consistent with Eq. (3.4). Thus, we obtain the solution as

$$\Omega = r_{,v}e^{F(v)} = K + \frac{r^2}{\ell^2} - \frac{M(v)}{r^2} \; ; \quad M(v) = \frac{1}{3}\kappa_5^2 \int_{v_0}^{v} dv\, e^{F(v)}\, f(v) + M_0 ,(3.7)$$

where we have assumed that $f(v) = 0$ for $v < v_0$, that is, v_0 is the epoch at which the ingoing flux is turned on. For definiteness, we assume that the bulk is pure AdS at $v < v_0$ and set $M_0 = 0$ in what follows.

Transforming the double-null coordinates (v, u) to the half-null coordinates (v, r) as

$$r_{,u}du = dr - r_{,v}dv,   \tag{3.8}$$

the solution is expressed as

$$ds^2 = -4\Omega(r, v)e^{-2F(v)}dv^2 + 4e^{-F(v)}dv\, dr + r^2 d\Sigma_{(K,3)}^2,   \tag{3.9}$$

where Ω is given by the first of Eqs. (3.7). This is an ingoing Vaidya solution with a negative cosmological constant [99, 100]. For an arbitrary intensity function $f(v)$, this is an exact solution for the bulk geometry. Note that if we re-scale v as $dv \to d\bar{v} = e^{-F}dv$, $f(v)$ scales as $f(v) \to \bar{f}(v) = e^{-2F}f(v)$, which manifestly shows the invariance of the solution under this rescaling.

An apparent horizon for the outgoing radial null congruence is located on the three-space satisfying

$$\Omega = r_{,v} = 0, \quad \text{while} \quad r_{,u} = \text{finite}.   \tag{3.10}$$

This gives

$$r^2 = \frac{\ell^2}{2}\left(\sqrt{K^2 + 4\frac{M(v)}{\ell^2}} - K\right).   \tag{3.11}$$

The direction of the trajectory of the apparent horizon is given by

$$\frac{dr}{dv} = \frac{M_{,v}\ell^2 r}{2(r^4 + M\ell^2)} = \frac{\kappa_5^2 f(v)e^{F(v)}\ell^2 r}{6(r^4 + M\ell^2)}.   \tag{3.12}$$

Thus, for $f(v) > 0$, dr/dv is positive, which implies that the trajectory of the apparent horizon is spacelike.

For the case of $K = +1$ or $K = 0$, the apparent horizon originates from $r = 0$, while it originates from $r = \ell$ for $K = -1$. A schematic view of the null dust collapse is shown in Fig. 3.1. We assume that the the brane emits the ingoing flux during a finite interval (bounded by the dashed lines in the figures). For all the cases, the causal structures after the onset of emission are very similar. The spacelike singularity is formed at $r = 0$, but it is hidden inside the apparent horizon.
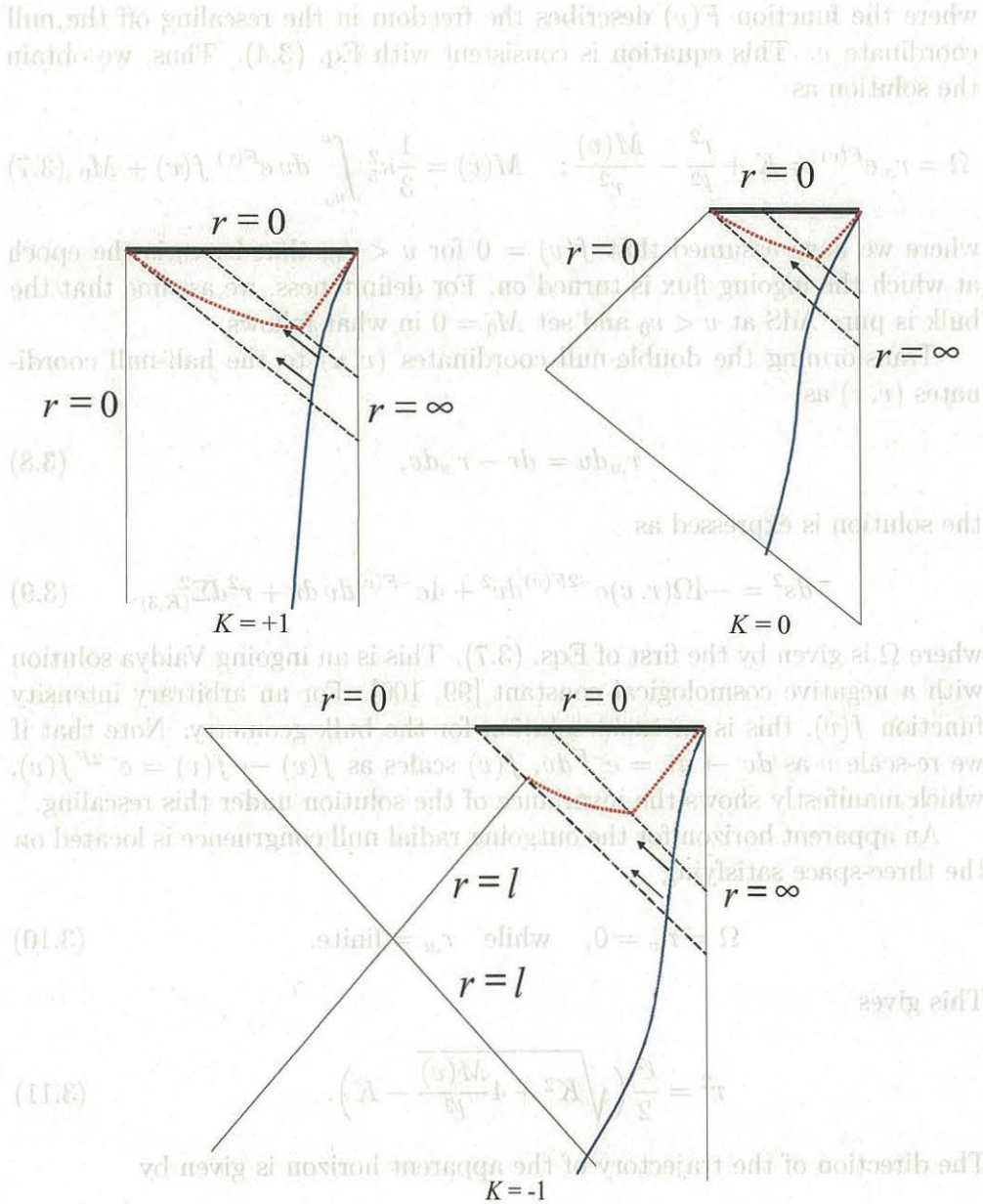
Figure 3.1: Causal structure of a spacetime with ingoing null dust for the cases of $K = +1$, 0 and -1. In each figure, The (almost vertical) wavy curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. The thick horizontal line at $r = 0$ represents the spacelike curvature singularity formed there. The ingoing flux is assumed to be emitted during a finite interval bounded by the dashed lines.

3.1.2 Brane trajectory in the bulk

In the null dust model, using Eq. (2.34), the proper time on the brane is related to the advanced time in the bulk as

$$\dot{v}_\pm = e^{F(v)} \frac{\dot{r} \pm \sqrt{\dot{r}^2 + \Omega}}{2\Omega} .$$

(3.13)

To determine the appropriate sign in the above, we require that the brane trajectory is timelike, hence $\dot{v} > 0$, and examine the signs of $\dot{v}_\pm$ for all possible cases:

(1)	$\dot{r} > 0,\ \Omega > 0$	$\rightarrow$	$\dot{v}_+ > 0,\ \dot{v}_- < 0.$
(2)	$\dot{r} > 0,\ \Omega < 0$	$\rightarrow$	$\dot{v}_+ < 0,\ \dot{v}_- < 0.$
(3)	$\dot{r} < 0,\ \Omega > 0$	$\rightarrow$	$\dot{v}_+ > 0,\ \dot{v}_- < 0.$
(4)	$\dot{r} < 0,\ \Omega < 0$	$\rightarrow$	$\dot{v}_+ > 0,\ \dot{v}_- > 0.$

From these, we can conclude the following. For an expanding brane, $\dot{r} > 0$, the brane exists always outside the horizon, $\Omega > 0$, and $\dot{v}$ is given by $\dot{v}_+$. On the other hand, a contracting brane, $\dot{r} < 0$, can exist either outside or inside of the horizon. Thus, if the brane is expanding initially, the trajectory is given by $\dot{v} = \dot{v}_+$, and it stays outside the horizon until it starts to recollapse, if ever. If the brane universe starts to recollapse, which is possible only in the case $K = +1$, by continuity, the trajectory is still given by $\dot{v} = \dot{v}_+$, and the brane universe is eventually swallowed into the black hole.

From the above result, we find

$$r_{,u}\dot{u} = \dot{r} - r_{,v}\dot{v} = \frac{\dot{r} - \sqrt{\dot{r}^2 + \Omega}}{2} < 0.$$

(3.14)

Using Eq. (2.41), this gives an upper bound of the Hubble parameter on the brane as

$$H < \frac{1}{6}\kappa_5^2\left(\rho + \sigma\right).$$

(3.15)

Let us now turn to the effective Friedmann equation on the brane. For simplicity, we tune the brane tension to the Randall-Sundrum value, $\kappa_5^2\sigma = 6/\ell$. The effective Friedmann equation on the brane is

$$H^2 + \frac{K}{r^2} = \frac{1}{18}\kappa_5^4\rho\sigma + \frac{1}{36}\kappa_5^4\rho^2 + \frac{M(\tau)}{r^4} ,$$

(3.16)

where $M(\tau) = M(v(\tau))$ for notational simplicity. From Eq. (2.46), the energy equation on the brane is given by

$$\dot{\rho} + 3\frac{\dot{r}}{r}\left(\rho + p\right) = -2\frac{f(\tau)}{r^3}\dot{v}^2,$$

(3.17)

where $f(\tau) = f(v(\tau))$. From Eq. (2.44), the time derivative of M is given by

$$\dot{M} = \frac{2}{3} r \kappa_5^2 \left[\frac{1}{6} \kappa_5^2 (\rho + \sigma) - H \right] f(\tau) \dot{v}^2. \tag{3.18}$$

Thus, from Eq. (3.15), M continues to increase on the brane.

The advanced time in the bulk is related to the proper time on the brane by $\dot{v}_+$ in Eq. (3.13). Specifically, using the equality,

$$\Omega = K + \frac{r^2}{\ell^2} - \frac{M}{r^2} = r^2 \left(\frac{\kappa_5^4}{36} (\rho + \sigma)^2 - H^2 \right), \tag{3.19}$$

on the brane, we have

$$\dot{v} = \frac{e^{F(v)}}{2r} \left(\frac{\kappa_5^2}{6} (\rho + \sigma) - H \right)^{-1}. \tag{3.20}$$

Note that the product $f \dot{v}^2$ is invariant under the rescaling of v. Once $f(\tau)$ is given, we can solve the system of equations (3.16), (3.17) and (3.18) self-consistently for a given initial condition, and determine the bulk geometry and the brane dynamics at the same time [1].

3.1.3 Formation of a naked singularity

In the previous subsections, we assumed that there is no naked singularity in the bulk. However, it has been shown that a naked singularity can be formed in the null dust collapse [2]. For instance, a naked singularity exists in a Vaidya spacetime when the flux of radiation rises from zero sufficiently slowly. We expect the same is true in the present case.

Without loss of generality, we set $e^{F(v)} = 2$. We consider the following situation. For $v < 0$, the bulk geometry is purely AdS. The radiative emission from the brane begins at $v = 0$. We choose the intensity function as

$$f(v) = \frac{2\lambda}{\kappa_5^2} v, \tag{3.21}$$

where λ is a positive constant. This corresponds to the self-similar Vaidya spacetime if the cosmological constant were absent [105]. The brane ceases to emit radiation at $v = v_0$ and the bulk becomes a static AdS-Schwarzschild for $v > v_0$. Thus the local mass is given by

$$M(v) = \begin{cases} 0 & (v < 0) \\ \frac{2}{3} \lambda v^2 & (0 \leq v \leq v_0) \\ \frac{2}{3} \lambda v_0^2 & (v_0 < v). \end{cases} \tag{3.22}$$

---

[1] Similar studies have been done, see e.g.,[100, 101, 102, 103, 104]

[2] Naked singularity formation in the Vaidya solution in four-dimensions has been discussed, see, e.g.,[105, 106, 107, 108, 109, 110]

The singularity is formed at $(r, v) = (0, 0)$, and it is naked if there exists a future-directed radial null geodesic emanating from it. The null geodesics then form a Cauchy horizon. The trajectory of a radial null geodesic is determined by the equation,

$$\frac{dr}{dv} = \frac{1}{2}\left(K + \frac{r^2(v)}{\ell^2} - \frac{M(v)}{r^2(v)}\right). \tag{3.23}$$

Let us analyze the above equation in the vicinity of $v = 0$. A future-directed radial null geodesic exists if $x := \lim_{v \to 0} dr/dv$ is positive. Using L'Hôpital's theorem, we obtain

$$x = \lim_{v \to 0}\frac{r(v)}{v} = \lim_{v \to 0}\frac{dr}{dv} = \frac{1}{2}\left(K - \frac{2\lambda}{3x^2}\right). \tag{3.24}$$

It is clear that the above equation has no solution when $K = 0$ or $K = -1$. Hence no naked singularity is formed for $K = 0$ or $K = -1$. Therefore, we consider the case $K = 1$. We introduce a function,

$$Q(x) = 3x^3 - \frac{3}{2}x^2 + \lambda. \tag{3.25}$$

Then, the condition for the naked singularity formation is that $Q(x) = 0$ has a solution for a positive x. The function $Q(x)$ has a minimal point at $x = 1/3$. Therefore, the singularity is naked if

$$Q(1/3) = -\frac{1}{18} + \lambda \leq 0, \tag{3.26}$$

that is,

$$0 < \lambda \leq \frac{1}{18}. \tag{3.27}$$

Thus, the bulk has a naked singularity for small values of λ, i.e., for the flux of radiation which rises slowly enough.

Our next interest is whether the naked singularity is local or global. If it is globally naked, it may be visible on the brane. To examine this, we integrate Eq. (3.23). In the vicinity of $v = 0$, we find

$$r_{\mathrm{null}}(v) = x_0\, v\left(1 + b\,\frac{v^2}{\ell^2} + \cdots\right) \tag{3.28}$$

where x_0 is the largest positive root of $Q(x) = 0$;

$$x_0 = \frac{1}{6}\left(1 + \left(1 - 36\lambda + i\,6\sqrt{2\lambda(1 - 18\lambda)}\right)^{1/3}\right.$$
$$\left. + \left(1 - 36\lambda - i\,6\sqrt{2\lambda(1 - 18\lambda)}\right)^{1/3}\right) \tag{3.29}$$

and

$$b = \frac{x_0^2}{2(5x_0 - 1)} \, . \tag{3.30}$$

From the form of $Q(x)$, we readily see that x_0 monotonically decreases from $1/2$ to $1/3$ as λ increases from 0 to $1/18$, and hence b is positive definite. We compare this trajectory with the trajectory of the apparent horizon. It is given by Eq. (3.11) with $K = +1$ In the vicinity of $v = 0$, it gives

$$r_{\text{app}}(v) = \sqrt{\frac{2\lambda}{3}} \, v \left(1 - \frac{\lambda}{8} \frac{v^2}{\ell^2} + \cdots \right) \, . \tag{3.31}$$

Since $x_0 > \sqrt{2\lambda/3}$ for all the values of λ in the range $0 < \lambda \leq 1/18$, and dr_{app}/dv is a decreasing function of v while dr_{null}/dv is an increasing function of v, it follows that the null geodesic lies in the exterior of the apparent horizon and the difference in the radius at the same v increases as v increases, at least when v is small. This suggests that the singularity is globally naked.

In Fig. 3.2, we plot the loci of the null geodesic and the apparent horizon. The result is clear. The null geodesic always stays outside of the apparent horizon, thus outside of the final event horizon at $v = v_0$. Mathematically, this is due to the cosmological constant term in Eq. (3.23), which strongly drives the null geodesic trajectory to larger values of r. Thus, we conclude that the naked singularity is global and visible on the brane. The causal structure in this case is illustrated in Fig. 3.3. Investigations on the effect of the visible singularity on the brane are necessary, but they are left for future work.

Finally, let us mention the strength of the naked singularity as we approach it along a radial null geodesic. Let w be an affine parameter of the geodesic, $w = 0$ be the singularity, and the tangent vector be denoted by $k^a = dx^a/dw$. We examine $\overset{(5)}{R}{}_{ab}k^a k^b$ and $\overset{(5)}{C}{}_{vu}{}^{vu}$. From Eq. (3.3) and the Einstein equations, we have

$$\overset{(5)}{R}{}_{ab} \, k^a k^b = \frac{\kappa_5^2 f(v)}{r^3} \left(\frac{dv}{dw} \right)^2 = \frac{2\lambda v}{r^3} \left(\frac{dv}{dw} \right)^2 \xrightarrow[w \to 0]{} \frac{2\lambda}{x_0(1 - x_0)^2} w^{-2} \, . \tag{3.32}$$

Also, from Eq. (2.31), we have

$$\overset{(5)}{C}{}_{vu}{}^{vu} = \frac{3M}{r^4} = \frac{2\lambda v^2}{r^4} \xrightarrow[w \to 0]{} \frac{2\lambda}{x_0^4} v^{-2} \propto w^{-\frac{2x_0}{1-x_0}} \, . \tag{3.33}$$

Thus the Ricci tensor and the Weyl tensor diverge as w^{-2} and $w^{-\frac{2x_0}{1-x_0}}$, respectively, These facts means that this singularity is a strong curvature singularity.
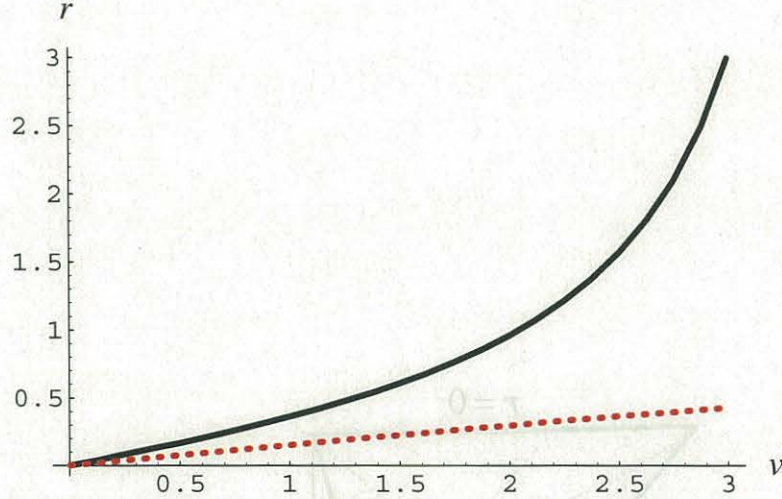
Figure 3.2: The loci of the null geodesic (the solid curve) and the apparent horizon (the dotted curve) on the (v, r)-plane, scaled in units of the AdS radius ℓ, in the critical case $\lambda = 1/18$. Their behaviors are qualitatively the same for all the other values of λ in the range $0 < \lambda < 1/18$.

3.2   Backreaction of KK gravitons in the bulk and on the brane

For KK modes which are produced in the whole bulk quantum mechanically, we take a different approach. We derive the effective stress energy tensor for KK gravitons by computing the curvature tensors up to the second order of perturbations. In this way, we can discuss the backreaction of non-radially emitted KK gravitons in a correct way. There is a problem about how to average the second order curvature tensors. We take an averaging procedure where the presence of the brane as an infinitely thin object is taken into consideration. The procedure is discussed in Appendix A. 2.

A similar analysis for KK modes of the massless, minimally coupled scalar field is discussed in Appendix B. Our discussion in this section is basically along this line, apart from the existence of the brane intrinsic contributions. We adopt a more general perspective by considering a $(d-1)$-brane embedded in a $(d+1)$-dimensional bulk spacetime, although we remain primarily interested by the case $d = 4$.
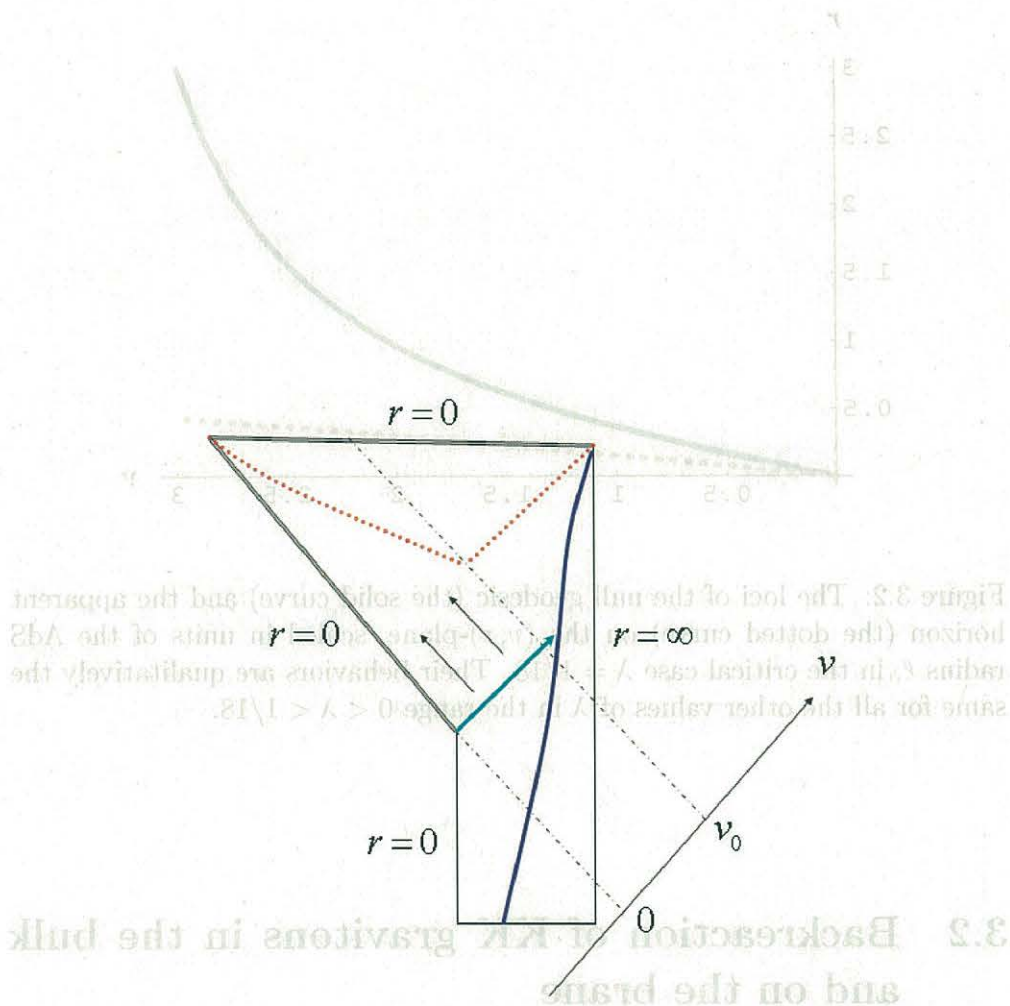
Figure 3.3: Causal structure of a spacetime with ingoing null dust when a naked singularity is formed. The wavy and almost vertical curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. A naked singularity is formed at $r = 0$ along the $v = 0$ null line. A radial, future directed null geodesic originating from the naked singularity (the right-pointed thick line) stays outside of the apparent horizon and reaches the brane.

3.2.1 Effective theory in the bulk

We now consider only pure gravity in the bulk. The action of the system is given by

$$S[g] = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x\sqrt{-g}\left(\overset{(d+1)}{R} - 2\Lambda_{d+1}\right) - \int d^dx\sqrt{-q}\,\sigma\,, \qquad (3.34)$$

where Λ_{d+1} is the bulk cosmological constant and σ is the brane tension. We mainly consider a dS brane background in this subsection and assume that its tension is larger than that of the corresponding RS value $2(d-1)/(\kappa_{d+1}^2\ell)$, where $\ell = (-d(d-1)/(2\Lambda_{d+1}))^{1/2}$ is the bulk AdS curvature radius.

We start from an unperturbed metric $\overset{[0]}{g}$, which is a solution of Einstein's equations and thus satisfies

$$\frac{\delta S}{\delta g}[g]\Big|_{\overset{[0]}{g}} = 0\,, \qquad (3.35)$$

where and in what follows the notation, $Q[a+g]\Big|_f$, means that a functional $Q[a+g]$ of g is evaluated for a function f, i.e.,

$$Q[a+g]\Big|_f = Q[a+f]\,. \qquad (3.36)$$

We then consider (small) linear perturbations of this metric, which we write $\epsilon\,\overset{[1]}{g}$ and such that its *average* vanishes i.e.,

$$\langle\overset{[1]}{g}\rangle = 0. \qquad (3.37)$$

Here we should specify our definition of averaging. We assume that the perturbation $\overset{[1]}{g}$ has a typical wavelength λ which is much smaller than the characteristic curvature radius L of the background $\overset{[0]}{g}$, $\lambda \ll L$. Then we take the average over a length scale much larger than λ but much smaller than L. In our case, we can take this average in the spacetime dimensions parallel to the brane. However, the situation is dramatically different in the direction of the extra spatial dimension because the brane is infinitesimally thin, which implies that the curvature radius along the extra dimension is infinitely small. Therefore one cannot take an average in that direction at or around the brane. Thus our averaging will include only the average over the $1 + (d-1)$ spacetime dimensions. (For spatially homogeneous perturbations, we take only the time average.)

What we are interested in is the correction to the original metric due to the backreaction of the metric perturbations. The total metric we consider can thus be written as

$$g_{\text{tot}} = \overset{[0]}{g} + \epsilon\,\overset{[1]}{g} + \epsilon^2\,\overset{[2]}{g}\,, \qquad (3.38)$$

where the quantity $\overset{[2]}{g}$ represents the backreaction due to the metric perturbations, so that the effective background (homogeneous) metric, after averaging, is given by

$$\bar{g} = \overset{[0]}{g} + \epsilon^2 \overset{[2]}{g} \, . \tag{3.39}$$

For convenience, the parameter ϵ is introduced as an expansion parameter, which is to be set to unity at the end of the calculation.

If we expand the action with respect to $\overset{[1]}{g}$, we have

$$S\left[\bar{g} + \epsilon \overset{[1]}{g}\right] = S\left[\bar{g}\right] + \frac{\delta S}{\delta g}[g]\Big|_{\bar{g}} \left(\epsilon \overset{[1]}{g}\right) + \frac{1}{2}\frac{\delta^2 S}{\delta g^2}[g]\Big|_{\bar{g}} \left(\epsilon \overset{[1]}{g}\right)^2 + \mathcal{O}(\epsilon^3) \, . \tag{3.40}$$

Hence the variation of the above expression with respect to $\overset{[1]}{g}$ yields

$$\epsilon\frac{\delta S}{\delta g}[\bar{g} + g]\Big|_{\epsilon\overset{[1]}{g}} = \epsilon\frac{\delta S}{\delta g}[g]\Big|_{\bar{g}} + \mathcal{O}(\epsilon^2) = \epsilon\frac{\delta S}{\delta g}[g]\Big|_{\overset{[0]}{g}} + \mathcal{O}(\epsilon^2) = \mathcal{O}(\epsilon^2) \, , \tag{3.41}$$

where we have used Eq. (3.35) in the final equality. This implies that, up to $\mathcal{O}(\epsilon)$, the equation of motion for the perturbation $\overset{[1]}{g}$ is given by

$$\frac{\delta S}{\delta g}[\overset{[0]}{g} + g]\Big|_{\epsilon\overset{[1]}{g}} = 0 \, . \tag{3.42}$$

On the other hand, the variation of the action with respect to $g_{\rm tot}$ gives

$$\begin{aligned}
0 = \frac{\delta S}{\delta g}[g]\Big|_{g_{\rm tot}} &= \frac{\delta S}{\delta g}[g]\Big|_{\bar{g}+\epsilon\overset{[1]}{g}} \\
&= \frac{\delta S}{\delta g}[g]\Big|_{\bar{g}} + \frac{\delta^2 S}{\delta g^2}[g]\Big|_{\bar{g}} \left(\epsilon \overset{[1]}{g}\right) + \frac{1}{2}\frac{\delta^3 S}{\delta g^3}[g]\Big|_{\bar{g}} \left(\epsilon \overset{[1]}{g}\right)^2 + \mathcal{O}(\epsilon^3) \\
&= \frac{\delta S}{\delta g}[g]\Big|_{\bar{g}} + \frac{\delta^2 S}{\delta g^2}[g]\Big|_{\bar{g}} \left(\epsilon \overset{[1]}{g}\right) + \frac{1}{2}\frac{\delta^3 S}{\delta g^3}[g]\Big|_{\overset{[0]}{g}} \left(\epsilon \overset{[1]}{g}\right)^2 \\
&\quad + \mathcal{O}(\epsilon^3) \, ,
\end{aligned} \tag{3.43}$$

where, to get the last expression, the argument of the coefficient of the third term, $\bar{g}$, has been replaced by $\overset{[0]}{g}$, which is justified within the accuracy of $\mathcal{O}(\epsilon^2)$. If one averages the above expression, the second term on the right-hand side vanishes and we obtain the equation that determines the backreaction-corrected background metric $\bar{g}$, in the form

$$\frac{\delta S}{\delta g}[g]\Big|_{\bar{g}} = -\frac{1}{2}\epsilon^2 \left\langle \overset{[1]}{g} \frac{\delta^3 S}{\delta g^3}[g]\Big|_{\overset{[0]}{g}} \overset{[1]}{g} \right\rangle \, . \tag{3.44}$$

Substituting the explicit form for the braneworld action, we find that Eq. (3.44) yields

$$\overset{(d+1)}{G}{}^{a}{}_{b} + \Lambda_{d+1}\delta^{a}{}_{b} = \kappa^2_{d+1}T^{a}{}_{b} + \bar{t}_{\rm (brane)}{}^{a}{}_{b} + \delta t_{\rm (brane)}{}^{a}{}_{b} \, , \tag{3.45}$$

where $\overset{(d+1)}{\bar{G}}$ is the background bulk Einstein tensor including the backreaction effects, i.e., for the metric $\bar{g}$. And the stress-energy tensor due to the backreaction in the bulk is given by

$$\kappa_{d+1}^2 T^a{}_b = -\left\langle \overset{(d+1)}{G}{}^{[2]}{}^a{}_b \right\rangle,$$ (3.46)

where $\overset{(d+1)}{G}{}^{[2]}{}^a{}_b$ is the bulk Einstein tensor at quadratic order. Here it may be worth noting that averaging is necessary for this effective stress-energy tensor to be physically meaningful, since there exists no locally covariant gravitational energy-momentum tensor due to the equivalence principle. The tensor $\bar{t}_{(\mathrm{brane})}{}^a{}_b$ corresponds to the brane energy-momentum tensor in the background configuration defined by the metric $\bar{g}$ and thus comes from the variation of the brane action in the left-hand side of (3.44). Finally, $\delta t_{(\mathrm{brane})}{}^a{}_b$, which comes from the brane-dependent part in the right-hand side of (3.44), denotes the backreaction due to the brane fluctuations and will be discussed later. The existence of this term is the most important difference when compared to the case of the scalar field, in which case the backreaction originates purely from the bulk.

Hereafter, we write $\overset{(d+1)}{\bar{G}}{}^a{}_b$ as $\overset{(d+1)}{G}{}^a{}_b$ for simplicity. For the moment, we concentrate on the effective theory in the bulk,

$$\overset{(d+1)}{G}{}^a{}_b + \Lambda_{d+1}\delta^a{}_b = \kappa_{d+1}^2 T^a{}_b.$$ (3.47)

Our first task is to evaluate the effective bulk energy-momentum tensor $T^a{}_b$, which is quadratic in the metric perturbations. Then we will take the limit to the brane.

We now identify the background metric $g^{[0]}$ with the separable metric of AdS_{d+1} bulk-dS brane spacetime and $g^{[1]}$ as the linear perturbation of this system. Namely,

$$ds^2 = dy^2 + b^2(y)\big(\gamma_{\mu\nu} + h_{\mu\nu}\big)dx^\mu dx^\nu, \quad h^\alpha{}_\alpha = h_\alpha{}^\beta{}_{|\beta} = 0,$$ (3.48)

where $b(y)$ is the warp factor defined in Eq. (B.12) and $\gamma_{\mu\nu}$ is the metric of a d-dimensional dS spacetime which is an extension of Eq. (B.13). Note that we have adopted the so-called RS gauge for the perturbations [9, 111]. The equation of motion for the perturbations in the bulk reads

$$\left[\frac{1}{b^d}\partial_y\big(b^d\partial_y\big) + \frac{1}{b^2}\big(\Box_d - 2H^2\big)\right]h_{\alpha\beta} = 0.$$ (3.49)

This equation is separable and one considers solutions of the form $h_{\alpha\beta} = f(y)\varphi_{\alpha\beta}(x^\mu)$, where $f(y)$ is the generalization of the solution of Eq. (B.15) to the case of a d-dimensional brane with boundary condition $\partial_y f(y) = 0$ at $y = y_0$ because $\partial_y h_{\alpha\beta} = 0$ on the brane. Similarly to the scalar case, the

separation constant m represents the effective mass of a KK graviton mode and satisfies $m > (d-1)H/2$. The d-dimensional part $\varphi^\alpha{}_\beta$ satisfies

$$\left[\Box_d - 2H^2\right]\varphi^\alpha{}_\beta = m^2\varphi^\alpha{}_\beta. \tag{3.50}$$

We focus on a KK mode with $m^2 \gg H^2$. Furthermore, for simplicity, we focus on perturbations of the tensor-type with respect to the spatial $(d-1)$-geometry, namely on those with $h^t{}_t = h^t{}_i = h^i{}_t = 0$. Taking the slicing of the de Sitter space with the flat spatial $(d-1)$-geometry, they will have the form,

$$h^i{}_j = \frac{f_m}{a^{(d-1)/2}} \cos(mt)\, Q^i{}_j, \tag{3.51}$$

where f_m is the amplitude of the KK mode and $Q^i{}_j$ is the polarization tensor on the flat $(d-1)$-space. The amplitude f_m can be determined, for instance, by the normalization condition if one considers a quantized perturbation theory.

As mentioned earlier, in order to obtain the stress-energy tensor that embodies the backreaction due to the metric perturbations, one needs to "average" the Einstein tensor at quadratic order, according to Eq. (3.46). The components of the bulk curvature tensors, up to quadratic order in the perturbations are listed in Appendix A. 2. As explained after Eq. (3.37), we take the spacetime average in the $1 + (d-1)$ dimensions parallel to the brane, but not along the extra dimension. In particular, because of the cosmological symmetry, we can take the average in the $(d-1)$ dimensions over the complete space. The derivatives along the extra dimension are replaced by using the field equation (3.49) and the boundary conditions on the brane. Our procedure is detailed in Appendix A. 2.

Using Eq. (A.24) of Appendix A. 2 and the computational rules detailed in Appendix A. 2, we obtain in the limit $y \to +0$ the expressions

$$\begin{aligned}
\left\langle {}^{(d+1)}\overset{[2]}{G}{}^y{}_y \right\rangle &= -\frac{1}{8}\left\langle h^{\rho\sigma}\Box_d h_{\rho\sigma} \right\rangle, \\
\left\langle {}^{(d+1)}\overset{[2]}{G}{}^\alpha{}_\beta \right\rangle &= -\frac{1}{2}\left\langle h^{\alpha\rho}\Box_d h_{\rho\beta} \right\rangle - \frac{d-3}{8d}\delta^\alpha{}_\beta\left\langle h^{\rho\sigma}\Box_d h_{\rho\sigma} \right\rangle \\
&\quad - \frac{1}{4}\left\langle h^{\rho\sigma|\alpha}h_{\rho\sigma|\beta} \right\rangle. \tag{3.52}
\end{aligned}$$

A priori, the effective energy-momentum tensor includes an anisotropic stress, to which each mode will contribute with a factor $O(m^2)$. However, if the perturbations are described by a random field which is statistically homogeneous and isotropic, the average over all modes of the anisotropic part must cancel. What remains is thus to justify the randomness of the perturbations. In this respect, the quantum fluctuations are indeed expected to have this property. Also, $\left\langle {}^{(d+1)}\overset{[2]}{G}{}^y{}_\nu \right\rangle$ vanishes on the brane by using the boundary conditions $\partial_y h_{\alpha\beta} = 0$ on it.

3.2.2 Backreaction on the brane

Let us now discuss the effect of the backreaction onto the brane. The projected gravitational equation on the brane reads

$$G^\alpha{}_\beta = -\Lambda_{\text{eff}} \, \delta^\alpha{}_\beta + \kappa_d^2 \tau^\alpha{}_\beta[h, h] + \kappa_{d+1}^2 \overset{(b)}{T}{}^\alpha{}_\beta - E^\alpha{}_\beta, \tag{3.53}$$

where

$$\Lambda_{\text{eff}} = \frac{d-2}{d} \Lambda_{d+1} + \frac{d-2}{8(d-1)} \kappa_{d+1}^4 \sigma^2 \,, \tag{3.54}$$

is the effective cosmological constant on the brane, and

$$\begin{aligned}
\kappa_{d+1}^2 \overset{(b)}{T}{}^\alpha{}_\beta &= \frac{d-2}{d-1} \kappa_{d+1}^2 \left[T^\alpha{}_\beta + \delta^\alpha{}_\beta \left(T^y_y - \frac{1}{d} T^\alpha{}_a \right) \right] \\
&= -\frac{d-2}{d-1} \left[\left\langle \overset{(d+1)}{G}{}^{[2]\,\alpha}{}_\beta \right\rangle + \delta^\alpha{}_\beta \left(\left\langle \overset{(d+1)}{G}{}^{[2]\,y}{}_y \right\rangle - \frac{1}{d} \left\langle \overset{(d+1)}{G}{}^{[2]\,a}{}_a \right\rangle \right) \right] \\
&= \frac{d-2}{2(d-1)} \left\langle h^{\alpha\rho} \Box_d \, h_{\rho\beta} \right\rangle + \frac{(d-2)(d-3)}{8d(d-1)} \delta^\alpha{}_\beta \left\langle h^{\rho\sigma} \Box_d \, h_{\rho\sigma} \right\rangle \\
&\quad + \frac{d-2}{4(d-1)} \left\langle h^{\rho\sigma|\alpha} h_{\rho\sigma|\beta} \right\rangle , \tag{3.55}
\end{aligned}$$

is the projection of the effective energy-momentum tensor of the bulk gravitons. The tensor $\tau^\alpha{}_\beta$, corresponding to $\delta t_{\text{(brane)}}{}^a{}_b$ of the previous subsection, describes the brane perturbation induced by the bulk perturbation. We will show in the next subsection that, for our purposes, this term can be neglected. We now concentrate on the effect of the effective energy-momentum of the bulk gravitons projected on the brane, i.e., the terms $\overset{(b)}{T}{}^\alpha{}_\beta$ and $E^\alpha{}_\beta$.

Let us first consider $\overset{(b)}{T}{}^\alpha{}_\beta$. Because of the assumed symmetries, i.e., the spatial homogeneity and isotropy, this gives in the brane an effective perfect fluid with some energy density and pressure. Decomposing the metric perturbations into KK modes, one finds that the contribution of a sufficiently massive mode to the energy density and pressure is given by

$$\kappa_{d+1}^2 \overset{(b)}{T}{}^t{}_t = -\frac{(d+3)(d-2)}{16d(d-1)} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left\langle Q^{k\ell} Q^*_{k\ell} \right\rangle ,$$

$$\kappa_{d+1}^2 \overset{(b)}{T}{}^i{}_j = \frac{(d^2+3)(d-2)}{16d(d-1)^2} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left\langle Q^{k\ell} Q^*_{k\ell} \right\rangle \delta^i{}_j . \tag{3.56}$$

We must also take into account the projection of the Weyl tensor on the brane, $E^\alpha{}_\beta$. Although this term is not included in the "bulk energy-monemtum" tensor because it is a part of the bulk Weyl tensor, it contributes nevertheless to the projected gravitational equations as an "energy-momentum" tensor. Although its direct evaluation is rather delicate, this term can be computed by resorting once more to the cosmological symmetry.

From Eq. (3.53), the contracted Bianchi identities $D^{\alpha(d)}G_\alpha{}^\beta = 0$, together with the conservation of $\tau_\alpha{}^\beta$, give

$$D^\mu E_{\mu\nu} = \kappa_{d+1}^2 D^\mu \overset{(b)}{T}{}_{\mu\nu}. \tag{3.57}$$

Because of the cosmological symmetry, the only non-trivial component of the above equation is the time component, which reads

$$\partial_t E^t{}_t + d\frac{\dot{a}}{a}E^t{}_t = \kappa_{d+1}^2\left(\partial_t \overset{(b)}{T}{}^t{}_t + (d-1)\frac{\dot{a}}{a}\overset{(b)}{T}{}^t{}_t - \frac{\dot{a}}{a}\overset{(b)}{T}{}^i{}_i\right), \tag{3.58}$$

where, on the left-hand side, we have used the property that $E_{\mu\nu}$ is traceless and thus $E^i{}_i = -E^t{}_t$. The integration then yields

$$E^t{}_t = \frac{\kappa_{d+1}^2}{a^d}\int_{t_0}^t dt'\, a^d\left(\partial_t \overset{(b)}{T}{}^t{}_t + (d-1)\frac{\dot{a}}{a}\overset{(b)}{T}{}^t{}_t - \frac{\dot{a}}{a}\overset{(b)}{T}{}^i{}_i\right). \tag{3.59}$$

As before, we neglect the contribution from the initial condition, which is valid at late times.

Substituting a KK graviton mode given by Eq. (3.51) into the integrand on the right-hand side of Eq. (3.59), and taking the time average, one finds

$$\begin{aligned}
&\kappa_{d+1}^2\left(\partial_t \overset{(b)}{T}{}^t{}_t + (d-1)\frac{\dot{a}}{a}\overset{(b)}{T}{}^t{}_t - \frac{\dot{a}}{a}\overset{(b)}{T}{}^i{}_i\right) \\
&= -\frac{(d^2+3)(d-2)}{16d(d-1)}\frac{H}{a^{d-1}}|f_m|^2 m^2\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle.
\end{aligned} \tag{3.60}$$

This gives, at late times,

$$E^t{}_t = -\frac{(d^2+3)(d-2)}{16d(d-1)}\frac{1}{a^{d-1}}m^2|f_m|^2\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle. \tag{3.61}$$

Because of the traceless nature of this tensor, we then obtain $E^i{}_j = -(1/(d-1))E^t{}_t\delta^i_j$.

The total contribution of the two tensors is therefore

$$\kappa_{d+1}^2\overset{(b)}{T}{}^t{}_t - E^t{}_t = \frac{d-2}{16}\frac{1}{a^{d-1}}m^2|f_m|^2\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle. \tag{3.62}$$

for the temporal part and

$$\kappa_{d+1}^2\overset{(b)}{T}{}^i{}_i - E^i{}_i = 0, \tag{3.63}$$

for the spatial part. This means that the contributions of a KK mode to the total effective energy density and pressure are respectively given by

$$\kappa_d^2\rho_{(\text{eff})} = -\frac{d-2}{16}\frac{1}{a^{d-1}}m^2|f_m|^2\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle,$$

$$\kappa_d^2 p_{(\text{eff})} = 0. \tag{3.64}$$

For instance, for $d = 4$, we obtain

$$\kappa_4^2 \rho_{(\text{eff})} = -\frac{1}{8 a^3} m^2 |f_m|^2 \left\langle Q^{k\ell} Q_{k\ell}^* \right\rangle,$$

$$\kappa_4^2 p_{(\text{eff})} = 0. \tag{3.65}$$

The effective isotropic pressure vanishes and the effective energy density is negative. This is the same as in the case of the scalar field discussed in the previous subsection.

We note that the bulk energy density of a KK mode on the brane remains positive

$$\kappa_{d+1}^2 \rho_{(\text{bulk})} := -\kappa_{d+1}^2 T^t{}_t = \frac{d+3}{16d} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left\langle Q^{k\ell} Q_{k\ell}^* \right\rangle > 0, \tag{3.66}$$

as in the scalar case, Eq. (B.27). It shows again that there is no singular effect in the bulk. The negativity of the effective energy density on the brane originates from the projected Weyl tensor $E_{\mu\nu}$.

3.2.3 Brane intrinsic contributions

We now consider the brane intrinsic contributions. In order to discuss the gravitational perturbations in the brane world, it is not sufficient to consider the contribution from the bulk. The brane perturbations must be taken into account as well. We take an approach in which we derive the second order boundary action and regard it as the action for an effective matter on the brane.

The brane is treated as a thin wall. In the thin wall approximation, the second order action on the boundary has been derived in the Appendix of [111]. When there is no ordinary matter on the brane and thus no brane bending mode, the second order boundary action is given by [111]

$$\delta^2 S = \frac{1}{2\kappa_{d+1}^2} \int_{\partial M} d^d x \sqrt{-\tilde{q}} \left[-\Delta k^{\rho\sigma} \tilde{h}_{\rho\sigma} + \frac{\sigma \kappa_{d+1}^2}{2(d-1)} \tilde{h}^{\rho\sigma} \tilde{h}_{\rho\sigma} \right], \tag{3.67}$$

where $\tilde{h}_{\mu\nu} = b^2 h_{\mu\nu}$, $k_{\rho\sigma} = \partial_y \tilde{h}_{\rho\sigma}/2$ and $\Delta Q = Q^{(+)} - Q^{(-)}$. For an AdS-bulk configuration and with the assumption of $\mathbb{Z}_2$ symmetry about the brane, this reduces to

$$\delta^2 S = \frac{3}{4(d-1)} \sigma \int_{\partial M} d^d x \sqrt{-\tilde{q}} \, h_{\rho\sigma} h^{\rho\sigma}. \tag{3.68}$$

The second order action can be regarded as an action for some effective matter induced on the brane

$$\int_{\partial M} d^d x \sqrt{-\tilde{q}} \, \mathcal{L}_m, \quad \mathcal{L}_m := \frac{3}{2(d-1)} \sigma h_{\rho\sigma} h^{\rho\sigma}. \tag{3.69}$$

Its variation with respect to the background metric $\tilde{q}_{\mu\nu}$ yields the induced matter energy-momentum tensor on the brane

$$\tau_{\alpha\beta} = \frac{2}{\sqrt{-\tilde{q}}}\frac{\delta}{\delta\tilde{q}^{\alpha\beta}}\left(\sqrt{-\tilde{q}}\,\mathcal{L}_m\right) = \frac{6}{d-1}\sigma\left(h_{\alpha\rho}h^{\rho}{}_{\beta} - \frac{1}{4}\tilde{q}_{\alpha\beta}h^{\rho\sigma}h_{\rho\sigma}\right)(3.70)$$

Note that, strictly speaking, $\tilde{q}_{\mu\nu}$ does not include the backreaction. However, as discussed previously, for perturbations with small amplitude, the linear perturbation equations are identical to those for the background metric in which the backreaction is taken into account. Thus we can add this term as a part of the (effective) matter contribution in the effective equation on the brane.

We can readily calculate the effective energy density and pressure of this contribution. One finds

$$\kappa_d^2\rho_{(\text{brane})} = \frac{3(d-2)}{16(d-1)^2}\kappa_{d+1}^4\sigma^2|f_m|^2\frac{1}{a^{d-1}}\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle,$$

$$\kappa_d^2 p_{(\text{brane})} = \frac{3(d-2)(-d+5)}{16(d-1)^3}\kappa_{d+1}^4\sigma^2|f_m|^2\frac{1}{a^{d-1}}\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle. \quad (3.71)$$

However, its contribution to $G^{\alpha}{}_{\beta}$ is of order

$$\kappa_{d+1}^4\sigma\tau^{\alpha}{}_{\beta} \sim \frac{1}{\ell^2}\left(1 + \left(H\ell\right)^2\right)h^{\rho\sigma}h_{\rho\sigma}, \quad (3.72)$$

where we use [111]

$$\sigma = \frac{2(d-1)}{\kappa_{d+1}^2\ell}\left(1 + \left(H\ell\right)^2\right)^{1/2}. \quad (3.73)$$

For the cases $H\ell \ll 1$ and $H\ell \gg 1$, the right-hand side of Eq. (3.72) is $\mathcal{O}(\ell^{-2}h^2)$ and $\mathcal{O}(H^2h^2)$, respectively. Thus as long as we consider sufficiently massive KK modes, with $m \gg \max\{\ell^{-1}, H\}$, the brane perturbations can be safely neglected and only the projected bulk contributions are relevant for the effective theory on the brane.

3.3 Brane dynamics in the bulk and the evolution of dark radiation

Intuitively, the "negative energy density" of a KK mode is rather puzzling. However, we can understand its cause by regarding the KK modes as a part of the dark component, such as the dark radiation. The energy density of the dark component evolves as

$$\dot{\rho}_{(D)} + 4H\rho_{(D)} = -2\left(1 + \frac{\rho}{\sigma}\right)T_{ab}u^an^b - 2(H\ell)T_{ab}n^an^b, \quad (3.74)$$

where u^a and n^a are the tangent and normal vectors to the brane, respectively. $\rho_{(D)}$ is related to the bulk local mass defined in the preceding Chapter as

$$M = \frac{\kappa_5^2}{3\ell}a^4\rho_{(D)}. \quad (3.75)$$

Since there is no matter on the brane and no brane-bulk energy exchange, the first term on the right-hand side of Eq. (3.74) vanishes and only the second one, related to the pressure transverse to the brane, survives. In terms of the energy conservation law in the bulk discussed in previous Chapter, this has the simple interpretation that the work done by the pressure on the brane to move it outward in the direction of the AdS infinity reduces the energy in the bulk. As a result, the dark energy density decreases, since the dark energy density on the brane is proportional to the total mass (energy) M.

We consider the case of a massless scalar field discussed in Appendix B, but the case of KK gravitons is essentially very similar. For a massless scalar field,

$$T_{ab}n^a n^b = \frac{1}{2}\dot{\phi}^2 > 0 \,. \tag{3.76}$$

Thus, the dark component decays faster than ordinary radiation. For the KK modes, after time averaging, we have

$$T_{ab}n^a n^b = \frac{1}{4\,a^3}\,|f_m|^2 m^2 \,. \tag{3.77}$$

The formal solution of Eq. (3.74) is

$$\rho_{(D)} = -\frac{2}{a^4}\int_{t_0}^{t} dt\, a^4 (H\ell)T_{ab}n^a n^b + \frac{C}{a^4}, \tag{3.78}$$

where C denotes the initial mass in the bulk. For the KK modes,

$$\int_{t_0}^{t} dt\, a^4 (H\ell)T_{ab}n^a n^b = \frac{\ell}{4}|f_m|^2 m^2 \int_{t_0}^{t} dt\, \dot{a} = \frac{\ell}{4}\Big(a(t) - a(t_0)\Big)|f_m|^2 m^2 \tag{3.79}$$

Hence

$$\rho_{(D)} = -\frac{\ell}{2a^3}|f_m|^2 m^2 + \frac{C}{a^4}\,, \tag{3.80}$$

where we have redefined the mass parameter C by absorbing into it the initial data dependent term of the integral (3.79).

Anyway, in the case of a dS brane (or a cosmological brane which slightly deviates from the dS geometry), the effective cosmological constant dominates the cosmological evolution and the KK effect does not have a significant impact on the brane. For a low energy brane, especially for a radiation-dominated brane, naively one might worry that this result would imply the appearance of a negative energy density within a finite time. However from Eq. (3.74), the bulk pressure term is proportional to H. Hence if $\dot{H} < 0$ at $H = 0$, the energy density will remain positive at the expense of rendering the universe to recollapse.

For simplicity, we consider the case where the cosmological evolution of the brane is determined solely by the dark component. Note that this discussion can be generalized when one considers ordinary dust or radiation

in addition to the dark component. The Hubble parameter on the brane is obtained from

$$H^2 = \frac{\kappa_5^2}{3\ell}\rho_{(D)} = -\frac{\kappa_5^2}{6}\frac{1}{a^3}|f_m|^2 m^2 + \frac{\kappa_5^2}{3\ell}\frac{C}{a^4} = -\frac{2\kappa_5^2}{3}T_{ab}n^a n^b + \frac{\kappa_5^2}{3\ell}\frac{C}{a^4}. \quad (3.81)$$

Taking the time derivative of this equation, we obtain

$$\dot{H} = \frac{\kappa_5^2}{4}\frac{1}{a^3}|f_m|^2 m^2 - \frac{2\kappa_5^2}{3\ell}\frac{C}{a^4} = \kappa_5^2 T_{ab}n^a n^b - \frac{2\kappa_5^2}{3\ell}\frac{C}{a^4}. \quad (3.82)$$

Therefore, at $H = 0$, we have

$$\dot{H} = -\frac{1}{3}T_{ab}n^a n^b < 0, \quad (3.83)$$

and the universe begins to collapse. Thus, the backreaction of the KK modes leads to a collapsing universe.

The situation is the same for the case of KK gravitons as long as the brane fluctuations are negligible, because we have

$$\kappa_5^2 T_{ab}n^a n^b = -\left\langle \overset{(2)}{G}{}^v{}_y \right\rangle = \frac{1}{8}m^2\left\langle h^{\rho\sigma}h_{\rho\sigma}\right\rangle$$

$$= \frac{1}{16}|f_m|^2 m^2 \frac{1}{a^3}\left\langle Q^{k\ell}Q^*_{k\ell}\right\rangle > 0. \quad (3.84)$$

Thus, provided that the brane fluctuations can be neglected, the brane universe will start to collapse within a finite time. For more realistic situations in cosmology, our result suggests that for a low energy brane the brane universe will eventually collapse unless the contribution of the true (normal) dust matter is larger than that of KK modes.

3.4   Summary and issues

In the latter half of this Chapter, we investigated the effect of a Kaluza-Klein (KK) graviton mode on brane cosmology by deriving the effective stress-energy tensor in the bulk and on the brane.

The KK gravitons, which are just the metric perturbations in the bulk, are produced during a de Sitter (dS) brane inflation phase via vacuum fluctuations. From the four-dimensional point of view they are effectively equivalent to massive gravitons with masses $m > 3H/2$, where H represents the dS expansion rate of the brane. The theory of linear perturbations reveals that the squared amplitude of a KK mode decays as a^{-3} and its contribution rapidly becomes negligible during the brane inflation. However, after brane inflation in the radiation-dominated Friedmann-Lemaître-Robertson-Walker (FLRW) era the background radiation energy density decays as a^{-4}, which implies that the contribution of KK gravitons may have a significant impact on brane cosmology at late times.

We first derived the effective energy-monemtum tensor of a KK mode in the bulk, and investigated the effect of the KK mode on the brane cosmology by projecting the effective energy-monemtum tensor on the brane. We have found that a massive KK mode behaves as negative energy density dust. In this Chapter, we discussed only the case of a dS brane background. However, in Appendix B, we confirmed that a KK mode of a bulk scalar field also gives a negative energy density on the brane for a low energy cosmological brane. Thus, the feature is rather generic for any separable background.

The negative energy density of a KK mode may sound rather puzzling. But, from the bulk point of view, we have shown that this result can be regarded as a natural consequence of the energy conservation law in the bulk. Here the essence is to recall that the dark radiation term, which behaves like radiation on the brane, describes the total mass in the bulk. Then, a very massive KK mode corresponds to a particle with a high momentum in the direction of the extra dimension, which exerts a pressure on the brane and pushes it outward in the direction of the AdS infinity. As a result, the energy in the bulk decreases, leading to the decrease of the dark energy term. Thus, a massive KK mode gives a negative contribution to the dark radiation term. This is why a KK mode behaves like negative energy density dust.

Note that the negative energy of a KK mode emerges only from the effective four-dimensional point of view on the brane. The bulk energy density for a KK mode still remains positive and thus there is no singular effect in the bulk.

However, for a general cosmological brane, one cannot define a KK mode since its very definition depends on the separability of the equations in the bulk. Nevertheless, considering the discussion from the bulk point of view given in the previous paragraph, it seems reasonable to expect that this back-reaction effect of the bulk metric perturbations persists for a general cosmological brane. Thus we conclude that the effect of very massive KK modes is to reduce the energy density on the brane and hence the expansion rate, and for a low energy brane the universe will recollapse unless the contribution of normal (true) dust matter is larger than that of the KK modes.

To quantify this effect in realistic cosmological models, there are some additional issues that remain to be resolved. We have considered only a single KK mode and calculated the effective energy density and pressure. In reality, one should integrate over all the KK modes that contribute to the cosmology of the brane. This requires first the knowledge of the whole spectrum of the KK modes, which will presumably be determined by vacuum fluctuations in the bulk. However, knowing the whole spectrum may not be enough, because a naive integral of the KK spectrum is expected to diverge. One would then need an appropriate regularization scheme. In connection with this, it may be important to take into account the thickness of a brane. In the next Chapter, we discuss the effect of a finite brane thickness on the KK spectrum.

4

A new regularizaton scheme for Kaluza-Klein modes on the brane

In this Chapter, we give a quantitatve study about the amplitude and back-reaction of the Kaluza-Klein (KK) modes of a massless scalar field, produced in the whole bulk quantum mechanically. In the previous Chapter, we have treated the KK gravitons classically, though they are considered to be produced quantumly mechanically.

As is well-known, the sum of the KK modes suffers from divergence as one approaches the brane from the bulk. It prevents us from evaluating fluctuations and backreactions exactly on the brane. This implies that one may take "structures" of the brane into account, to obtain successfully regularized KK contribution. In this Chapter, we propose a new regularization scheme for this type of divergence by a finite brane thickness.

As a demonstration of this scheme, we show that a finite brane thickness can regularize the quantum fluctuation on the brane in an explicit thick brane model. We consider KK modes of a massless scalar field evolving on the thick brane background. The reason we choose a massless scalar field is that especially for the minimally coupled case, the equation of motion of the scalar field is the same as that of the gravitons and also for technical simplicity. First, we consider the quantum fluctuations. We show that a finite brane thickness can regularize the quantum fluctuations on the brane [112]. We also calculate the amount of the scalar field backreacition for the minimally coupled case and show that the the amount of the backreaction can be reduced to below that of the background stress-energy [113]. We finally mention the case of KK gravitons.

4.1   A thick de Sitter brane model

We consider the Einstein theory coupled to a bulk scalar field,

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left(\overset{(d+1)}{R} - (\partial \chi)^2 - 2V(\chi) \right), \tag{4.1}$$

where the potential of the scalar field is given by the axion like form [48, 49],

$$V(\chi) = V_0 \left(\cos \left[\frac{\chi}{\chi_0} \right] \right)^{2(1-\sigma)} . \tag{4.2}$$

Note that we once shall set $\kappa_{d+1}^2 = 1$ in this Chapter and will back them if needed.

We shall assume a static configuration, namely χ depends on only the bulk coordinate and make the following metric ansatz

$$ds^2 = b(z)^2 (dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu) , \tag{4.3}$$

where $\gamma_{\mu\nu}$ denotes the metric of d-dimensional de Sitter (dS) spacetime. Following the above ansatz, we obtain the Einstein equations

$$\frac{d(d-1)}{2} \left(\frac{b'}{b} \right)^2 - \frac{d(d-1)}{2} H^2 = \frac{1}{2} \chi'^2 - b^2 V ,$$

$$(d-1) \left[\frac{b''}{b} + \frac{1}{2}(d-4) \left(\frac{b'}{b} \right)^2 - \frac{1}{2}(d-2)H^2 \right] = -\frac{1}{2} \chi'^2 - b^2 V \tag{4.4}$$

and the field equation for the scalar field is

$$\chi'' + (d-1) \frac{b'}{b} \chi' - b^2 \frac{\partial V}{\partial \chi} = 0 , \tag{4.5}$$

where the prime $'$ denotes the derivative with respect to z. Note that only two of these three equations are independent. For this potential, we find the solutions

$$b(z) = \left(\cosh \left(\frac{Hz}{\sigma} \right) \right)^{-\sigma} , \qquad \chi(z) = \chi_0 \sin^{-1} \left(\tanh \left(\frac{Hz}{\sigma} \right) \right) , \tag{4.6}$$

where

$$H^2 = \frac{2\sigma V_0}{(d-1)[1+(d-1)\sigma]} , \qquad \chi_0 = \sqrt{(d-1)\sigma(1-\sigma)} . \tag{4.7}$$

This solution represents a dS domain wall whose energy is localized at $z = 0$, i.e., the center of the wall. The parameter σ has the meaning of the thickness of the wall (brane) from the physical point of view. In order to keep the positivity in the square root, we should restrict the range to [48]

$$0 < \sigma < 1 . \tag{4.8}$$

The classical stabilities of the thick brane model against the tensor and scalar perturbations are discussed in Appendix C.1 and C.2, respectively.

4.2 Quantized scalar field perturbations

Our purpose is to discuss the quantized scalar field perturbations on a thick, inflating brane model. We achieve this by introducing another scalar field ϕ, which is coupled to the domain wall configuration and its fluctuations. Hence, we add the action of the scalar field ϕ to the original action Eq. (4.1), i.e.,

$$
\begin{aligned}
S = & \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left(\overset{(d+1)}{R} - (\partial\phi)^2 - 2V(\phi) \right) \\
& + \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left(-(\partial\phi)^2 - \xi \overset{(d+1)}{R} \phi^2 \right),
\end{aligned} \tag{4.9}
$$

where ξ is the scalar curvature coupling.

As discussed in [114, 115] the coupling of the field ϕ to ϕ can be ignored because its backreaction to the domain wall geometry is only important at higher order, $O(\phi^2)$.[1] This assumption allows us to treat the ϕ-field contribution perturbatively. The minimally coupled case, $\xi = 0$, will be of particular interest, because the perturbation equations are very similar to those for tensor perturbations of the metric (see Appendix C.1).

4.2.1 Dimensional reduction approach

We shall evaluate the amplitude of the quantum field ϕ, based on a dimensional reduction of the higher dimensional canonically quantized field. This method has been already discussed in [117] and we refer the reader to this reference for more details.

In this method, the action of ϕ is rewritten as

$$
S_\phi = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \, \phi \left(\Box_{d+1} - \xi \overset{(d+1)}{R} \right) \phi, \tag{4.10}
$$

where we set a regulator boundary at $z = L$ in order to obtain a *well-posed* quantum field theory on the dimensionally reduced spacetime. Then, the bulk modes become discrete and the solution is written as

$$
\phi(z, x^\mu) = \sum_n F_n(z) \varphi_n(x^\mu) H^{1/2}, \tag{4.11}
$$

where φ_n has the dimension of a scalar field in the d-dimensional dS spacetime. Due to the maximal symmetry of dS spacetime, we can integrate out the dependence on the transverse directions, x^μ, assuming that the vacuum respects the dS invariance. Hence, we shall drop it in the amplitude.

Integrating the action with respect to z, it is reduced to the summation of theories of a d-dimensional massive scalar field with mass m_n:

$$
S_\phi = \frac{1}{2} \sum_n \int d^d x \sqrt{-\gamma} \, \varphi_n(x^\mu) \left(\Box_d - m_n^2 \right) \varphi_n(x^\mu), \tag{4.12}
$$

---

[1]These works use the methods developed in [116].
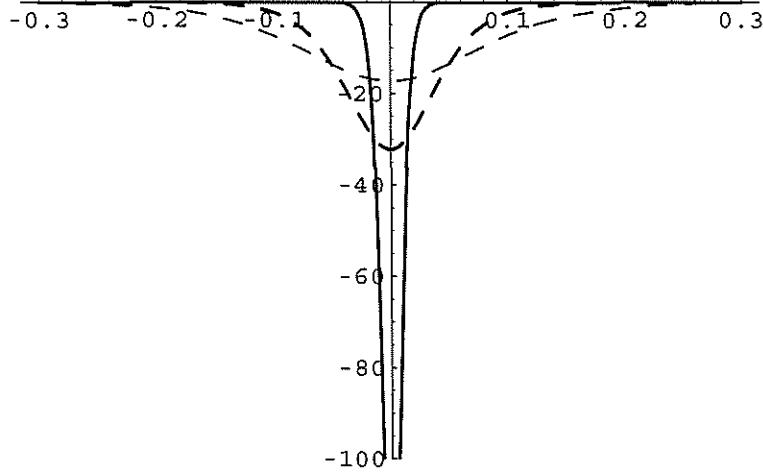
Figure 4.1: The potential for a minimally coupled bulk scalar field perturbation is shown as a function of Hz for a four-dimensional dS wall. The thick, thick-dashed and dashed curves correspond to the cases of $\sigma = 0.01, 0.05, 0.1$, respectively.

where we employed the normalization condition

$$2 \int_0^L (H\,dz)\, b^{d-1}(z) F_{q_n}(z) F_{q'_n}(z) = \delta_{nn'} . \tag{4.13}$$

Note that the multiplying factor of two is due to the $\mathbb{Z}_2$-symmetry. The mass-squared m_n^2 is given by

$$m_n^2 = q_n^2 H^2 + \frac{(d-1)^2}{4} H^2 . \tag{4.14}$$

We introduce a new function $f_{q_n}(z) := b^{(d-1)/2}(z) F_{q_n}(z)$, which obeys the Schrödinger like equation

$$- f''_{q_n}(z) + \tilde{V}(z) H^2 f_{q_n} = q_n^2 H^2 f_{q_n}(z) , \tag{4.15}$$

where

$$\tilde{V}(z) := -\left(\xi_c - \xi\right)\left\{ d(d-1) + \frac{2d}{\sigma} \right\} \frac{1}{\cosh^2(Hz/\sigma)} . \tag{4.16}$$

For the minimally coupled case, $\xi = 0$, this potential reduces to the one for the tensor perturbations, Eq. (C.3). In Fig. 4.1, we plot the potential for the $d = 4$ case explicitly for $\xi = 0$. It is evident that the potential becomes deeper for smaller values of σ.
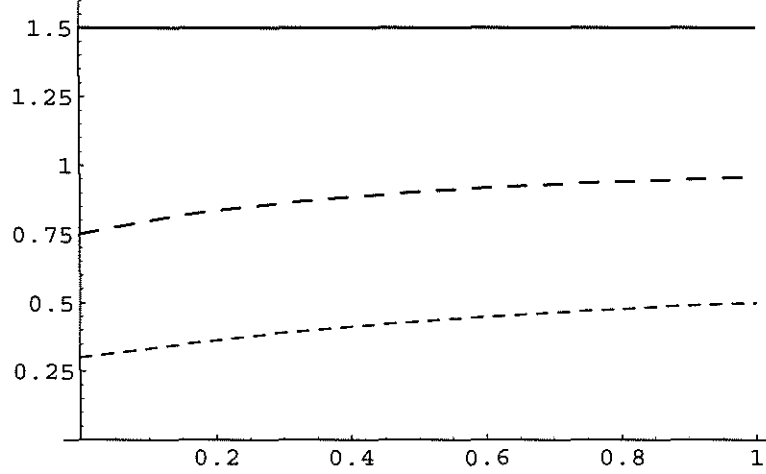
Figure 4.2: ν/σ is plotted as a function of thickness, σ. The thick, thick-dashed and dashed curves correspond to the cases for $\xi = 0, 3/32, 3/20$, respectively.

The solution of the KK modes is

$$f_{q_n}(z) = C_1 P_\nu^{i\sigma q_n}(x) + C_2 P_\nu^{-i\sigma q_n}(x), \tag{4.17}$$

where $P_\nu^\mu(x)$ denotes the Legendre functions of the first kind, $x :=$ $\tanh(Hz/\sigma)$ and

$$\nu := \frac{\sqrt{1 + 4(\xi_c - \xi)(d(d-1)\sigma^2 + 2d\sigma)} - 1}{2}. \tag{4.18}$$

The coupling

$$\xi_c = \frac{d-1}{4d} \tag{4.19}$$

denotes the conformal coupling strength, e.g., for the $d = 2$ case $\xi_c = 1/8$ and for the $d = 4$ case $\xi_c = 3/16$. In this article, we restrict the coupling to the range $0 \leq \xi \leq \xi_c$.

The mass of the bound state mode is given by

$$q_0 = \frac{i\nu}{\sigma}, \tag{4.20}$$

which has a maximal value of $(d-1)/2$ at $\xi = 0$. For $\xi < 0$, the bound state becomes tachyonic and non-normalizable. For $\xi = 0$, it is the zero mode and for $\xi = \xi_c$ it becomes the lowest mass KK mode, irrespective of the choice of σ. In Fig. 4.2, we plot ν/σ as a function of σ for several choice of ξ. We find that this ratio is almost independent of σ. Note that the $\xi = 0$ case is equivalent to the bulk tensor perturbations.

4.2.2 The zeta function method

Given the functions $f_{q_n}(z)$, the vacuum expectation value is defined by

$$\langle \phi^2(z) \rangle = \frac{2H}{b^{d-1}(z)} \sum_n f_{q_n}^2(z) \langle \varphi_{q_n}^2(x) \rangle , \qquad (4.21)$$

where the factor of two is due to the $\mathbb{Z}_2$-symmetry. From now on, we shall discuss the quantized field theory in Euclidean space, i.e., the metric is

$$ds^2 = b^2(z)\big(dz^2 + d\Sigma_d^2\big) , \qquad (4.22)$$

where $d\Sigma_d$ is the line element of S^d with unit radius, whose volume is given by

$$V_{S^d} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} . \qquad (4.23)$$

Thus, in order to consider the quantum fluctuations of a d-dimensional field, we assume that the vacuum is given by the Euclidean vacuum, which corresponds to the dS invariant, Bunch-Davis vacuum in the original Lorentzian spacetime.

For the d-sphere, S^d, any local quantities are related to global ones by simply dividing by the volume of the sphere (a property of maximally symmetric spaces; see [117]). Thus, we are particularly interested in the local vacuum expectation value as only a function of z (one non-trivial dimension), implying

$$K_n(t) = \int d^d x \sqrt{\gamma} K_n(x, x; t) , \qquad (4.24)$$

where K_n is the dS heat kernel for each mode n, see [117]. Thus, due to the maximal symmetry of dS space, the global heat kernel is simply related to the local one by

$$K_n(t) = \frac{V_{S^d}}{H^d} K_n(x, x; t) . \qquad (4.25)$$

At this stage it is convenient to rescale the amplitude as

$$\langle \tilde{\phi}^2(z) \rangle = \frac{b(z)^{d-1} V_{S^d}}{H^{d+1}} \langle \phi^2(z) \rangle , \qquad (4.26)$$

where overall factors can be restored at the end of the calculation. Now we may sum up all the KK modes in Eq. (4.21); however, as is well known, a naive summation over all the KK modes gives rise to unwanted divergences.

To deal with such a problem, we construct the *local* zeta function, $\zeta(z, s)$, along the lines of reference [117], where the parameter s is initially assumed

to be $\text{Re}(s) > (d+1)/2$ in $(d+1)$-dimensions. Once we obtain such a zeta function, after analytic continuation to $s \to 1$, we end up with

$$\langle \tilde{\phi}^2(z) \rangle = \lim_{s \to 1} \tilde{\zeta}(z, s), \tag{4.27}$$

where

$$\tilde{\zeta}(z, s) := \frac{b(z)^{d-1} V_{S^d}}{H^{d+1}} \zeta(z, s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} K(z, t). \tag{4.28}$$

$K(z, t)$ is the local heat kernel defined as

$$K(z, t) = 2 \sum_{n=1}^\infty f_{q_n}^2(z) K_n(t), \tag{4.29}$$

and

$$K_n(t) = \sum_{j=0}^\infty d_j \, e^{-[q_n^2 + (j+1/2)^2] H^2 t}, \tag{4.30}$$

where d_j is the degeneracy for each mode j given by

$$d_j = (2j + d - 1) \frac{(j + d - 2)!}{j!(d-1)!}, \tag{4.31}$$

is the global heat kernel for each KK mode. Note that the dimension of $\tilde{\zeta}$ is slightly different to the case discussed in [117], because of a difference in dimension of the warp factor.

4.2.3 Contour integral representation of the local zeta function

First, as a resolution to the *subtle* nature of the continuous modes, we introduce *another boundary* at $z = L$. This then enables us to evaluate the zeta function using the residue theorem, based on certain assumptions relating to the zeros of the function in the contour. Then, after constructing such a zeta function we show that we can take the one brane limit $L \to \infty$ in a well defined manner.[2]

The solution for the scalar field perturbations in general dimensions is given by

$$f_q(z) = N_q \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right), \tag{4.32}$$

where for convenience, we choose the second solution $R_\nu^{iq\sigma}(x)$ to satisfy

$$P_\nu^{iq\sigma}(x) R_\nu^{iq\sigma\prime}(x) - R_\nu^{iq\sigma}(x) P_\nu^{iq\sigma\prime}(x) = \frac{1}{1 - x^2}, \tag{4.33}$$

---

[2]The same approach cannot be used for the one-loop effective action because it is a global quantity, e.g., see the discussion in [118].

where $x = \tanh(Hz/\sigma)$. There are several candidates for $R_\nu^{iq\sigma}(x)$ such as

$$\frac{\Gamma(-iq\sigma + \nu + 1)}{\Gamma(iq\sigma + \nu + 1)} Q_\nu^{iq\sigma}(x) \,, \quad -\frac{\pi}{2i\sinh(\pi q\sigma)} P_\nu^{-iq\sigma}(x) \,, \tag{4.34}$$

and so on. For now we do not need to specify the explicit form of the second solution $R_\nu^{iq\sigma}(x)$, but only use the property of the Wronskian in Eq. (4.33).

To be specific, let us consider the case of Neumann boundary conditions. The boundary conditions at the center of the thick brane and the second boundary are respectively

$$f_q'(z)|_{z=0} = 0 \,, \quad f_q'(z)|_{z=L} = 0 \,. \tag{4.35}$$

Note, the thick brane is not a boundary, we just fix the z derivative of the field at a point to obtain a well-posed eigenvalue equation. From these boundary conditions, we get an equation which determines the KK mass spectrum as

$$P_\nu^{iq\sigma\prime}(0) R_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(x_L) R_\nu^{iq\sigma\prime}(0) = 0 \,. \tag{4.36}$$

We denote the solutions for the eigen-equation as q_n $(n = 1, 2, 3, \cdots)$ whose eigenfunctions are

$$f_{q_n}(z) = N_{q_n}\left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x)\right) \,, \tag{4.37}$$

where

$$\frac{\alpha_{q_n}}{\beta_{q_n}} = \frac{R_\nu^{iq_n\sigma\prime}(0)}{P_\nu^{iq_n\sigma\prime}(0)} = \frac{R_\nu^{iq_n\sigma\prime}(x_L)}{P_\nu^{iq_n\sigma\prime}(x_L)} \,. \tag{4.38}$$

We assume $q_1 < q_2 < q_3 < \cdots$, respectively. Note that the final equality is satisfied only for $q = q_n$. Without loss of generality, we can choose $\alpha_q = R_\nu^{iq\sigma\prime}(0)$ and $\beta_q = P_\nu^{iq\sigma\prime}(0)$. We shall also require the normalization constant for n-th mode which is found to be

$$\begin{aligned}
N_{q_n}^{-2} &= 2\int_0^L (H\,dz)\left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x)\right)^2 \\
&= -\frac{1}{\sigma q_n}\frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\partial_q\left(R_\nu^{iq\sigma\prime}(0) P_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(0) R_\nu^{iq\sigma\prime}(x_L)\right)_{q=q_n} \tag{4.39}
\end{aligned}$$

See, Appendix D.1.

Now we have all the necessary tools to calculate the zeta function by applying the residue theorem as follows: from the equations given above, the normalized mode functions can be written as

$$f_{q_n}^2(z) = \frac{\sigma q_n G(q_n, z)}{\partial_q F(q)|_{q=q_n}} \,, \tag{4.40}$$

where,

$$F(q) = -\left(R_\nu^{iq\sigma\prime}(0) P_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(0) R_\nu^{iq\sigma\prime}(x_L)\right) \tag{4.41}$$

and

$$G(q, z) = \left(R_\nu^{iq\sigma'}(0) P_\nu^{iq\sigma}(x) P_\nu^{iq\sigma'}(0) R_\nu^{iq\sigma}(x) \right) \left(R_\nu^{iq\sigma'}(x_L) P_\nu^{iq\sigma}(x) - P_\nu^{iq\sigma'}(x_L) R_\nu^{iq\sigma}(x) \right).$$
(4.42)

This form is essential in order to apply the residue theorem. Whence, the zeta function can be written as a contour integral in the complex u plane

$$
\begin{aligned}
\tilde{\zeta}(z, s) &= 2\mu^{2(s-1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{d_j \, f_{q_n}^2(z)}{\left[q_n^2 + (j + \frac{d-1}{2})^2 \right]^s H^{2s}} \\
&= 2\mu^{2(s-1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{\sigma q_n G(q_n, z)}{\partial_q F(q)|_{q=q_n}} \frac{d_j}{\left[q_n^2 + (j + \frac{d-1}{2})^2 \right]^s H^{2s}} \\
&= 2\mu^{2(s-1)} \oint_C \frac{du}{2\pi i} \frac{\sigma u G(u, z)}{F(u)|_{u=q_n}} \sum_{j=0}^{\infty} \frac{d_j}{[u^2 + (j + \frac{d-1}{2})^2]^s H^{2s}}, (4.43)
\end{aligned}
$$

where the poles at $u = q_n$ are on the positive side of real axis and therefore, the closed contour C has to be taken around the positive real axis in general. Note that we have introduced a mass scale μ to keep the dimension. This term is in fact the renormalization scale and groups with any divergent terms in the expression for the amplitude. Then, given the fact that there are no poles in the complex u-plane, besides those on the real axis, we can naturally deform the contour C into C' (see Fig. 4.3)

$$\tilde{\zeta}(z, s) = 2\mu^{2(s-1)} \oint_{C'} \frac{du}{2\pi i} \frac{\sigma u G(u, z)}{F(u)|_{u=q_n}} \sum_{j=0}^{\infty} \frac{d_j}{[u^2 + (j + \frac{d-1}{2})^2]^s H^{2s}},$$
(4.44)

which is composed of a line parallel to the imaginary axis with a small real part and a large semi circle on the positive real half of the complex plane, which is depicted in Fig. 4.3. As we mentioned previously, initially keeping s larger than $(d+1)/2$, the contribution from the larger semi-circle becomes negligible.

A similar approach has been used e.g., in [119] for infinitely thin Minkowski branes in a bulk AdS space; however, in our case the contour we have to construct is complicated by the presence of the poles which come from the bound state; and as we shall see, it will be convenient to evaluate the bound state contribution separately. Therefore, as it turns out, we shall only focus on the total amplitude from now on.

In particular, we are primarily interested in calculating the mode func-
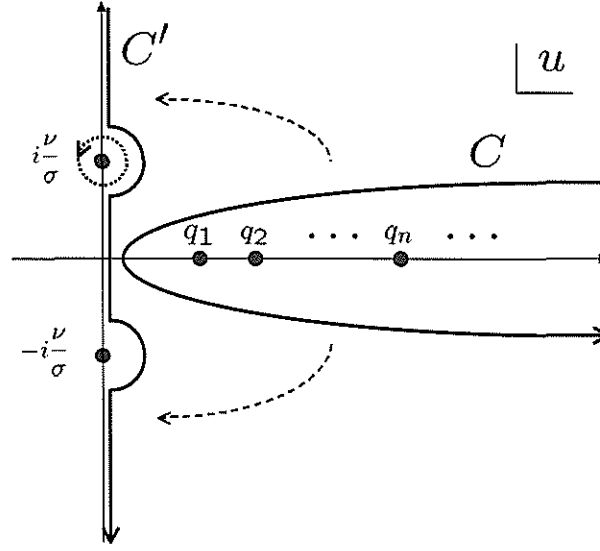
Figure 4.3: The contour C, used to evaluate the KK amplitude. The poles on the real axis $q_i\,(i = 1, 2, \cdots)$ correspond to the KK modes, while those on the imaginary axis correspond to the bound state. We can deform C into C' because there are no poles in the complex plane besides those on the real and imaginary axes. The closed contour depicted by the dotted line is used to evaluate the bound state amplitude.
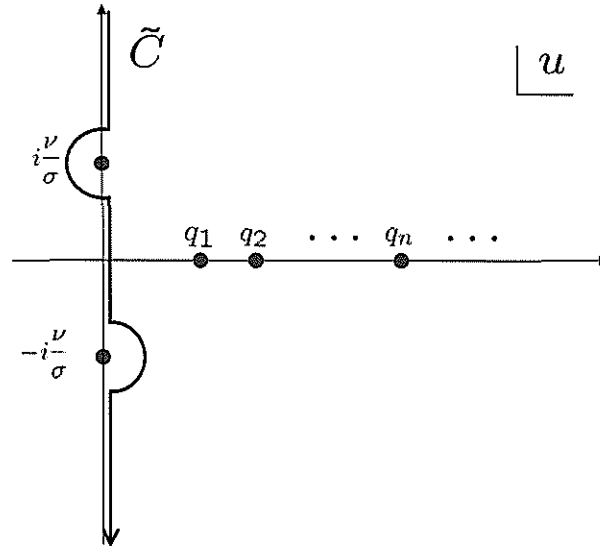
Figure 4.4: The contour $\tilde{C}$, used to evaluate the total amplitude.

tions on the brane at $z = 0$ $(x = 0)$, i.e.,

$$
\left. \frac{G(u,z)}{F(u)} \right|_{z=0}
$$

$$
= \frac{\left(R_\nu^{iu\sigma'}(0) P_\nu^{iu\sigma}(0) - P_\nu^{iu\sigma'}(0) R_\nu^{iu\sigma}(0) \right) \left(R_\nu^{iu\sigma'}(x_L) P_\nu^{iu\sigma}(0) - P_\nu^{iu\sigma'}(x_L) R_\nu^{iu\sigma}(0) \right)}{-\left(R_\nu^{iu\sigma'}(0) P_\nu^{iu\sigma'}(x_L) - P_\nu^{iu\sigma'}(0) R_\nu^{iu\sigma'}(x_L) \right)}
$$

$$
= \frac{P_\nu^{iu\sigma'}(x_L) R_\nu^{iu\sigma}(0) - R_\nu^{iu\sigma}(x_L)' P_\nu^{iu\sigma}(0)}{\left(R_\nu^{iu\sigma'}(0) P_\nu^{iu\sigma'}(x_L) - P_\nu^{iu\sigma'}(0) R_\nu^{iu\sigma'}(x_L) \right)}
$$

$$
= \frac{P_\nu^{iu\sigma'}(x_L) P_\nu^{-iu\sigma}(0) - P_\nu^{-iu\sigma'}(x_L) P_\nu^{iu\sigma}(0)}{\left(P_\nu^{-iu\sigma'}(0) P_\nu^{iu\sigma'}(x_L) - P_\nu^{iu\sigma'}(0) P_\nu^{-iu\sigma'}(x_L) \right)} , \tag{4.45}
$$

where in the first step we used the Wronskian relation Eq. (4.33) and in the final step we specified the second mode function as

$$
R_\nu^{iq\sigma}(x) = -\frac{\pi}{2i \sinh(\pi q \sigma)} P_\nu^{-iq\sigma}(x) . \tag{4.46}
$$

Two types of decomposition are possible:

$$
\frac{P_\nu^{iu\sigma'}(x_L) P_\nu^{-iu\sigma}(0) - P_\nu^{-iu\sigma'}(x_L) P_\nu^{iu\sigma}(0)}{P_\nu^{-iu\sigma'}(0) P_\nu^{iu\sigma'}(x_L) - P_\nu^{iu\sigma'}(0) P_\nu^{-iu\sigma'}(x_L)}
$$

$$
= \frac{P_\nu^{iu\sigma}(0)}{P_\nu^{iu\sigma'}(0)} - \frac{P_\nu^{iu\sigma'}(x_L)}{P_\nu^{iu\sigma'}(0)} \frac{2i \sinh(\pi u \sigma)}{\pi} \frac{1}{P_\nu^{iu\sigma'}(0) P_\nu^{-iu\sigma'}(x_L) - P_\nu^{iu\sigma'}(x_L) P_\nu^{-iu\sigma'}(0)}
$$

$$
= \frac{P_\nu^{-iu\sigma}(0)}{P_\nu^{-iu\sigma'}(0)}
$$

$$
+ \frac{P_\nu^{-iu\sigma'}(x_L)}{P_\nu^{-iu\sigma'}(0)} \frac{2i \sinh(\pi u \sigma)}{\pi} \frac{1}{P_\nu^{-iu\sigma'}(0) P_\nu^{iu\sigma'}(x_L) - P_\nu^{-iu\sigma'}(x_L) P_\nu^{iu\sigma'}(0)} . \tag{4.47}
$$

It is important to note that the second term on the second line is negligible in the $x_L \to 1$ limit on the upper half of the complex u-plane, while the second term on the third line is negligible in the same limit on the lower-half of complex u-plane. Thus, in the single brane limit we use the first term on the second and third lines as the single brane propagator on the upper and lower half of the complex plane, respectively.

In the single brane propagator given above, $P_\nu^{iu\sigma}(0)/P_\nu^{iu\sigma'}(0)$ has poles that are situated on the negative imaginary axis, corresponding to purely decaying modes, plus the bound state contribution at $u = i\nu/\sigma$. However, as we mentioned above, $P_\nu^{iu\sigma}(0)/P_\nu^{iu\sigma'}(0)$ is used for the upper half of the complex plane and we need not worry about the purely decaying modes. Thus, we only need to deal with the bound state mode at $u = i\nu/\sigma$ in the calculation of the KK amplitude. Similarly, the exact opposite occurs for $P_\nu^{-iu\sigma}(0)/P_\nu^{-iu\sigma'}(0)$ and we only need to deal with the pole at $u = -i\nu/\sigma$.

The remaining problem then concerns the avoidance of the bound state poles at $u = \pm i\nu/\sigma$. We avoid the bound state poles by deforming the contour to C', as depicted in Fig. 4.3, when we evaluate the KK amplitude.

However, this contour gives a non-zero contribution (from the bound state poles) when taking the Cauchy principal value on the imaginary axis. This contribution simply corresponds to the subtraction of the bound state from the total amplitude; we can calculate the bound state amplitude separately, see the next section. Thus, it will be rather convenient for us to shift the contour over the upper pole to $\tilde{C}$, as depicted in Fig. 4.4. This is equivalent to adding the bound state contribution with a counter-clockwise contour (the closed dotted line in Fig. 4.3) to C'. Then, by integrating along the contour $\tilde{C}$ and subtracting the bound state contribution, we can obtain the desired KK amplitude. This is the approach we shall take to evaluate the KK amplitude in this article.

4.3   Kaluza-Klein amplitude: $d = 2$ case

To demonstrate the method discussed in the previous subsection as simply as possible, we first evaluate the amplitude of the quantum fluctuations on the brane for $d = 2$. That is, we construct the zeta function for the case of the two-sphere in the transverse dimensions with one non-trivial bulk dimension.

Amplitude of the KK modes

The zeta function for total amplitude at the center of the wall is

$$
\tilde{\zeta}(0, s) = 4\mu^{2(s-1)} \oint_{\tilde{C}} \frac{du}{2\pi i} \frac{\sigma u G(u, 0)}{F(u)} \sum_{j=0}^{\infty} \frac{(j + 1/2)}{[u^2 + (j + 1/2)^2]^s H^{2s}}
$$

$$
= \frac{4\mu^{2(s-1)}}{\pi H^{2s}} \sin[\pi(s - 1)]
$$

$$
\times \ \mathrm{P} \int_0^{2\nu/\sigma} dU U \frac{\sigma G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)} \sum_{j=0}^{\infty} \frac{(j + 1/2)}{\left[U^2 - (j + 1/2)^2\right]^s}, \quad (4.48)
$$

where $U = e^{-\pi i/2}u$ and we use the property

$$
\frac{G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)} = \frac{G(e^{-\pi i/2}U, 0)}{F(e^{-\pi i/2}U)} = \frac{P_\nu^{-U\sigma}(0)}{2P_\nu^{-U\sigma'}(0)}. \quad (4.49)
$$

Here, Roman "P" (not to be confused with the Legendre function of the first kind) means taking the Cauchy principal value in order to deal with the pole at $U = \nu/\sigma$. In Fig. 4.3, the contribution from the anti-clockwise semi-circle around $u = i\nu/\sigma$ cancels with that from the clockwise semi-circle around $u = -i\nu/\sigma$.

In the following, we shall divide the integral into two; i.e., for $U > 2\nu/\sigma$ which we denote as the "UV piece" and that for $0 < U < 2\nu/\sigma$ which we denote as the "IR piece." We emphasize that the reason for this splitting is solely for technical reasons and that the choice of division has no physical significance. We can set the split at any value of $O(1)$.

To begin with, for the UV piece we will use the following asymptotic expansion formula, e.g., see Ref. [120], for large U, i.e.,

$$\sum_{j=0}^{\infty} \frac{(j+1/2)}{[U^2 - (j+1/2)^2]^s} = -\frac{1}{2} U^{-2s+2} \left[\frac{1}{s-1} - \frac{1}{\Gamma(s)} \right.$$
$$+ \left. \sum_{j=1}^{\infty} \frac{\Gamma(j+s-1)}{j!} U^{-2j} \partial_a \zeta_H(-2j, a) \Big|_{a=1/2} \right]. \qquad (4.50)$$

Then, for the IR piece we employ the standard binomial expansions:

$$\sum_{j=0}^{\infty} \frac{j+1/2}{[(j+1/2)^2 - U^2]^s} = \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\Gamma(s)} U^{2J} \zeta_H(2s+2J-1, \tfrac{1}{2}), \qquad (4.51)$$

which is valid for the range $0 < U < 1/2$; while for the range $1/2 < U < 2\nu/\sigma < 1$ we must use [120, 121]

$$\sum_{j=0}^{\infty} \frac{j+1/2}{[(j+1/2)^2 - U^2]^s} = \frac{1}{2} \left(\frac{1}{(\tfrac{1}{4} - U^2)^s} \right.$$
$$+ \left. \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\Gamma(s)} U^{2J} \left(2\zeta_H(2s+2J-1, \tfrac{1}{2}) - \left(\tfrac{1}{2}\right)^{-2s-2J} \right) \right). \qquad (4.52)$$

Then, the total amplitude on the center at the wall is given by the summation of both pieces

$$\tilde{\zeta}(0, s) = \tilde{\zeta}_{\text{UV}}(0, s) + \tilde{\zeta}_{\text{IR}}(0, s). \qquad (4.53)$$

First, let us consider the analytic continuation of the UV piece

$$\tilde{\zeta}_{\text{UV}}(0, s) = -\frac{2\mu^{2(s-1)} \sin[\pi(s-1)]}{\pi H^{2s}}$$
$$\times \int_{2\nu/\sigma}^{\infty} dU \frac{\sigma G(e^{\pi i/2} U, 0)}{F(e^{\pi i/2} U)} U^{-2s+3}$$
$$\times \left[\frac{1}{s-1} - \frac{1}{\Gamma(s)} \sum_{j=1}^{\infty} \frac{\Gamma(j+s-1)}{j!} U^{-2j} \partial_a \zeta_H(-2j, a) \Big|_{a=1/2} \right]. \qquad (4.54)$$

Given the following relation [122]

$$\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma'}(0)} = -\frac{1}{2} \frac{\Gamma(-\tfrac{\nu}{2} + \tfrac{U\sigma}{2})\Gamma(\tfrac{\nu}{2} + \tfrac{U\sigma}{2} + \tfrac{1}{2})}{\Gamma(\tfrac{\nu}{2} + \tfrac{U\sigma}{2} + 1)\Gamma(-\tfrac{\nu}{2} + \tfrac{U\sigma}{2} + \tfrac{1}{2})} \qquad (4.55)$$

and by employing the asymptotic expansion for large U of the Gamma functions [122] we find the following asymptotic series, which in d-dimensions is

$$\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma'}(0)} = \sum_{\ell=0}^{\infty} w_\ell(\sigma, \xi) U^{-1-2\ell}, \qquad (4.56)$$

where ν is given by Eq. (4.18) and

$$w_0(\sigma, \xi) = -\frac{1}{\sigma},$$

$$w_1(\sigma, \xi) = \frac{(2 + \sigma(d-1))(1 + d(-1 + 4\xi))}{8\sigma^2} = \frac{1}{2\sigma}\tilde{V}(0), \tag{4.57}$$

$$w_2(\sigma, \xi) = -\left\{(2 + \sigma(d-1))(1 + d(-1 + 4\xi))\left(8 + 6\sigma(1 + d(-1 + 4\xi))\right.\right.$$

$$\left.\left. + 3\sigma^2(-1 + d)(1 + d(-1 + 4\xi))\right)\right\}/(128\sigma^4) \tag{4.58}$$

The subtraction of the w_0 term just corresponds to that of the trivial background, whereas the w_1 term corresponds to the tadpole graph, see [114, 115]. Here, we require only the term w_0, in order to regularize the $d = 2$ case. For the $d = 4$ case, terms up to w_1 are required.[3]

Thus, after analytic continuation to $s \to 1$, we obtain the UV amplitude

$$H^2 \lim_{s\to 1}\tilde{\zeta}_{\mathrm{UV}}(0, s) = -2\left\{\int_{2\nu/\sigma}^{\infty} dU\, U\, \sigma\left[\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma'}(0)} - \frac{w_0(\sigma, \xi)}{U}\right]\right.$$

$$\left. - w_0(\sigma, \xi)\left(\frac{2\nu}{\sigma}\right)\right\}. \tag{4.59}$$

As for the IR piece it is already finite in the limit $s \to 1$; however, because of the poles on the imaginary axis we make the principle value prescription, i.e.,

$$\tilde{\zeta}_{\mathrm{IR}}(0, s) = \frac{4\mu^{2(s-1)}}{\pi H^{2s}(-1)^s}\sin[\pi(s-1)]$$

$$\times\ \mathrm{P}\int_0^{1/2} dU\frac{\sigma U G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)}\sum_{J=0}^{\infty}\frac{\Gamma(s+J)}{J!\Gamma(s)}U^{2J}\zeta_H(2s + 2J - 1, \frac{1}{2})$$

$$+\ \frac{2\mu^{2(s-1)}}{\pi H^{2s}(-1)^s}\sin[\pi(s-1)]\,\mathrm{P}\int_{1/2}^{2\nu/\sigma} dU\frac{\sigma G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)}$$

$$\times\ \left(\frac{1}{(\frac{1}{4} - U^2)^s} + \sum_{J=0}^{\infty}\frac{\Gamma(s+J)}{J!\Gamma(s)}U^{2J}\right)$$

$$\times\ \left(2\zeta_H(2s + 2J - 1, \frac{1}{2}) - \left(\frac{1}{2}\right)^{-2s-2J}\right), \tag{4.60}$$

where if $2\nu/\sigma < 1/2$ the second term is to be dropped. Then, given the Laurent expansion of the Hurwitz zeta function

$$\zeta_H(2s - 1, \frac{1}{2}) = \frac{1}{2(s-1)} - \psi(1/2) + O(s - 1), \tag{4.61}$$

---

[3]In practice, for better numerical convergence we subtract off more terms than are required to regularize the theory; thus, we include w_1 for $d = 2$ and w_2 for $d = 4$, respectively.

we find that there is only a contribution from $J = 0$ in both terms. Thus, in the limit $s \to 1$, we obtain

$$
\begin{aligned}
H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{IR}}(0, s) &= -\mathrm{P} \int_0^{2\nu/\sigma} dU \frac{2\sigma G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)} U \\
&= 2 \int_0^{2\nu/\sigma} \frac{dU}{U - \nu/\sigma} \Big(U \frac{\Gamma(-\nu/2 + U\sigma/2 + 1)\Gamma(\nu/2 + U\sigma/2 + 1/2)}{\Gamma(\nu/2 + U\sigma/2 + 1)\Gamma(-\nu/2 + U\sigma/2 + 1/2)} \\
&\quad - \frac{1}{\sqrt{\pi}} \frac{\nu}{\sigma} \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu + 1)} \Big) \,,
\end{aligned}
$$

(4.62)

where in the final step, we used the fact that

$$
\begin{aligned}
\mathrm{P} \int_0^{2x_0} dx \frac{f(x)}{x - x_0} &= \mathrm{P} \int_0^{2x_0} dx \frac{f(x)}{x - x_0} - \mathrm{P} \int_0^{2x_0} dx \frac{f(x_0)}{x - x_0} \\
&= \int_0^{2x_0} dx \frac{f(x)}{x - x_0} - \int_0^{2x_0} dx \frac{f(x_0)}{x - x_0} \,,
\end{aligned}
$$

(4.63)

where $f(x)$ is an arbitrary regular function. The second term, which is equal to zero, eliminates the singularity at $x = x_0$ in the first term. This technique will also be used for the $d = 4$ case.

Finally, we obtain the regularized total amplitude

$$
\langle \tilde{\phi}^2(0) \rangle_{\mathrm{tot}} = \lim_{s \to 1} \left(\tilde{\zeta}_{\mathrm{UV}}(0, s) + \tilde{\zeta}_{\mathrm{IR}}(0, s) \right) .
$$

(4.64)

As discussed in the preceding subsection the KK amplitude is obtained by subtracting the bound state amplitude, which is evaluated in Appendix D.2,

$$
\langle \tilde{\phi}^2(0) \rangle_{\mathrm{KK}} = \langle \tilde{\phi}^2(0) \rangle_{\mathrm{tot}} - \langle \tilde{\phi}^2(0) \rangle_{\mathrm{bs}} .
$$

(4.65)

Interestingly, the total amplitude $\langle \tilde{\phi}^2(0) \rangle_{\mathrm{tot}}$ does not depend on the renormalization scale μ, whereas as shown in Appendix D.2 the bound state contribution $\langle \tilde{\phi}^2(0) \rangle_{\mathrm{bs}}$ does depend on it. Thus, the KK amplitude $\langle \tilde{\phi}^2(0) \rangle_{\mathrm{KK}}$ will also depend on μ as can be readily seen from Eq. (4.65). In other words, the dependence on μ in the bound state and KK contribution cancels when they are summed up.

4.3.1 Results of numerical calculations

The total amplitude $\langle \phi^2(0) \rangle_{\mathrm{tot}}$ is shown in Fig. 4.5. For small thicknesses, the UV piece dominates the total amplitude. The leading order divergent behavior can be estimated as follows: by changing variables from U to $x = U\sigma$, the UV piece can be written as

$$
H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{UV}}(0, s) = -2 \left(\frac{1}{\sigma} \int_{2\nu}^{\infty} dx \, x \left(\frac{P_\nu^{-x}(0)}{P_\nu^{-x\prime}(0)} + \frac{1}{x} \right) + \frac{2\nu}{\sigma} \right) .
$$

(4.66)

As shown previously (see Fig. 4.2), ν/σ is almost independent of σ and $\nu = O(\sigma)$ for $\sigma \ll 1$. Then by Taylor expanding the Gamma functions in Eq. (4.55) about ν we find that

$$\frac{P_\nu^{-x}(0)}{P_\nu^{-x\prime}(0)} = -\frac{1}{x} + \frac{\nu}{2x}\left(\psi(\frac{x}{2}) - 2\psi(\frac{x+1}{2}) + \psi(\frac{x}{2} + 1)\right) + O(\sigma^2). \quad (4.67)$$

Therefore,

$$\begin{aligned}
H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{UV}}(0, s) &= -\frac{\nu}{\sigma}\int_{2\nu}^\infty dx \left(\psi(\frac{x}{2}) - 2\psi(\frac{x+1}{2}) + \psi(\frac{x}{2} + 1)\right) \\
&+ O(\sigma^0).
\end{aligned} \quad (4.68)$$

In the case of $d = 2$ the divergence arises only from the leading order. Furthermore, for $x \gg 1$ the integrand behaves as x^{-2} and thus, the contribution from the upper bound vanishes. However, in the opposite limit, $x \ll 1$,

$$\psi(\frac{x}{2}) - 2\psi(\frac{x+1}{2}) + \psi(\frac{x}{2} + 1) = -\frac{2}{x} + (-2\gamma - 2\psi(1/2)) + O(x), \quad (4.69)$$

where $\gamma = 0.57721\cdots$ is Euler's constant, and therefore

$$H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{UV}}(0, s) = -\frac{2\nu}{\sigma}\ln(2\nu) + O(\sigma^0). \quad (4.70)$$

Thus, we find a positive logarithmic divergence in the thin wall limit.

The amplitude of the bound state is derived separately in Appendix D. 2. Here, we recapitulate the final result,

$$\begin{aligned}
H^2 \langle \tilde{\phi}^2(0)\rangle_{\mathrm{bs}} &= \frac{1}{\sigma}\left(2\ln\left(\frac{\mu}{H}\right) - 2\psi(1/2) - \delta_{\xi,0}\left(\frac{1}{2}\right)^{-2}\right. \\
&+ \sum_{J=1}^\infty \left[2\left(\frac{\nu}{\sigma}\right)^{2J}\zeta_H(2J + 1, \frac{1}{2}) - \delta_{\xi,0}\left(\frac{1}{2}\right)^{-2}\right]\right) \\
&\times \left(\int_0^\infty dy \cosh^{-2\nu}(y)\right)^{-1}.
\end{aligned} \quad (4.71)$$

In Fig. 4.6, the amplitude of the bound state is plotted as a function of the brane thickness for each coupling, with $\mu = H$. Interestingly, the resulting amplitude is almost independent of the brane thickness σ and still finite in the thin wall limit.

Thus, as expected, the divergence of the total amplitude in the thin wall limit arises solely from the KK contribution. Regardless, for finite values of $\sigma \sim 0.1$ the total amplitude settles down to finite positive values. The result shows that the surface divergence for the KK modes can be regularized by introducing a finite brane thickness. This is one of the main results of this article.

The bound state amplitude depends on the choice of renormalization scale, μ. In Fig. 4.7, the running of the scale is shown as a function of μ. It is
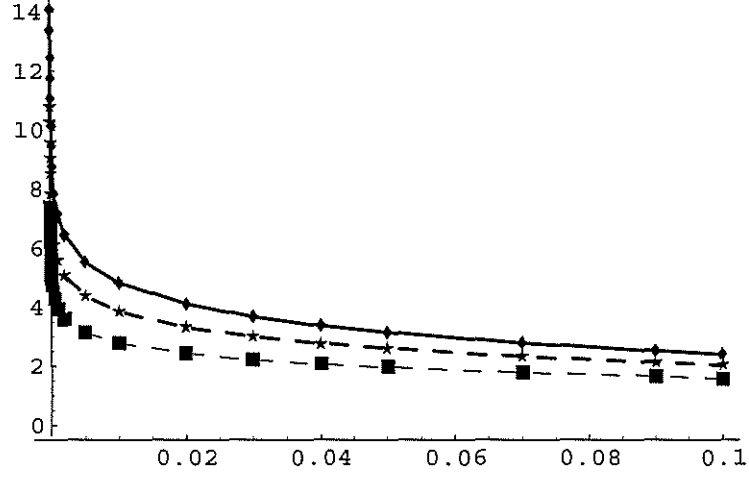
Figure 4.5: The total amplitude is shown as a function of the brane thickness, σ, in the case of $d = 2$. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 1/32, 1/16$, respectively.
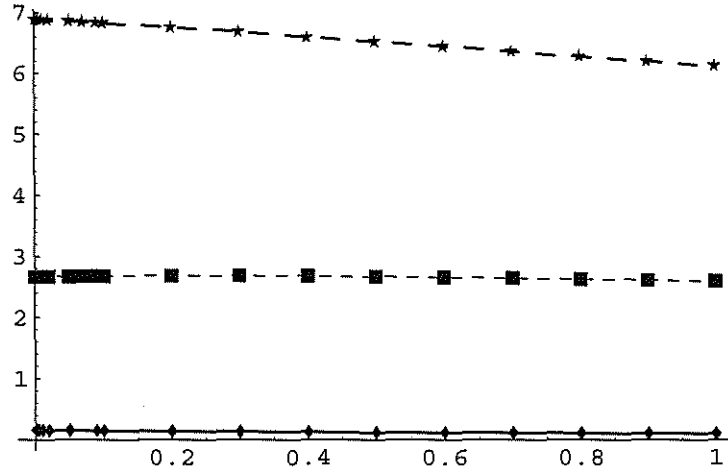
Figure 4.6: The amplitude of the bound state is shown as a function of the brane thickness, σ, in the case of $d = 2$, with $\mu = H$. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 1/32, 1/16$, respectively.
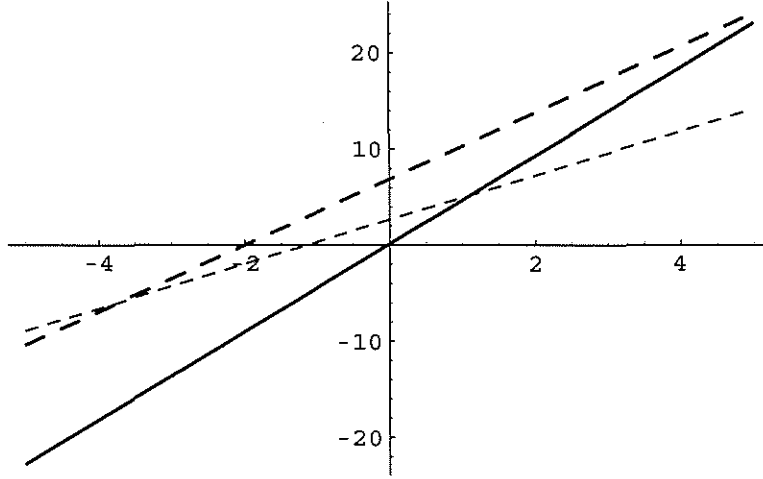
Figure 4.7: The running of the bound state is shown as a function of the renormalization scale ln μ in the case of $d = 2$, with $\sigma = 0.01$. The vertical and horizontal axes show the bound state amplitude and $\log_{10}(\mu/H)$, respectively. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 1/32, 1/16$, respectively.
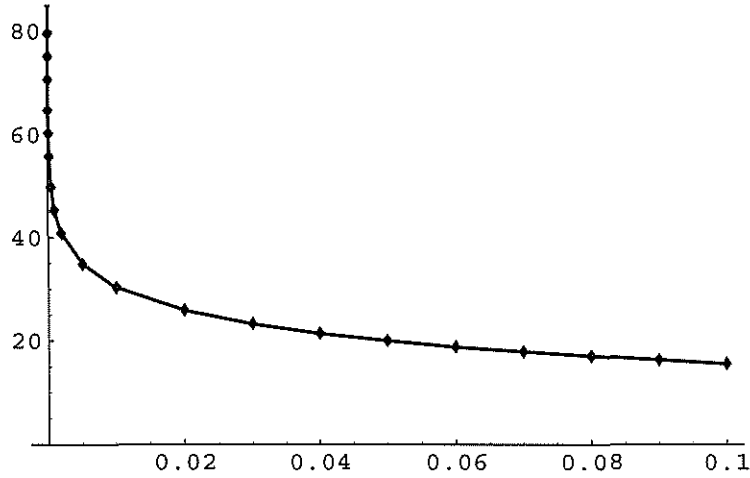
Figure 4.8: The relative amplitude of the KK modes to the bound state mode is shown as a function of the brane thickness, σ, for minimal coupling, $\xi = 0$, for $d = 2$, with $\mu = H$.

essentially proportional to ln μ. The tilt becomes steeper for smaller coupling parameter ξ. There are several possible choices for the renormalization scale, for example, one could choose the expansion rate of the brane $\mu = H$ or another choice is the brane thickness $\mu = H/\sigma$. We still have no signature about braneworlds today and therefore no quantity that we can renormalize into. The renormalzation scale μ should be determined by future observations and/or experiments. In this article we just plot the running of the scale and take the optimal choice $\mu = H$ for cases where one has to make a choice. Note that from Eq. (4.65) the KK amplitude is also proportional to ln(μ) with negative tilts. In this article we just plot the running of the scale and take the optimal choice $\mu = H$ for cases where one has to make a choice. Note that from Eq. (4.65) the KK amplitude is also proportional to ln(μ) with negative tilts.

It is also interesting to compare the relative amplitude of the KK modes to the bound state mode. The relative amplitude is given by

$$r := \frac{\langle \phi^2(0) \rangle_{\text{KK}}}{\langle \phi^2(0) \rangle_{\text{bs}}} = \frac{\langle \tilde{\phi}^2(0) \rangle_{\text{KK}}}{\langle \tilde{\phi}^2(0) \rangle_{\text{bs}}} = \frac{\langle \tilde{\phi}^2(0) \rangle_{\text{tot}}}{\langle \tilde{\phi}^2(0) \rangle_{\text{bs}}} - 1 , \tag{4.72}$$

where in the final step we used Eq. (4.65). The result depends on the choice of the renormalization scale μ and brane thickness, σ. It is meaningful to show the plot for physically reasonable cases. As an example, in Fig. 4.8, we have plotted r as a function of σ for the minimally coupled case, $\xi = 0$, i.e., for tensor perturbations, with $\mu = H$.

4.4 Kaluza-Klein amplitude: $d = 4$ case

In this subsection, we perform the calculation for the more realistic case of $d = 4$. The calculation follows in an identical manner to the $d = 2$ case, if only for more tedium.

Amplitude of the KK modes

In this case the degeneracy factor for the four-sphere ($d = 4$) is

$$d_j = \frac{1}{3} \left(j + \frac{3}{2} \right) (j + 1)(j + 2) \tag{4.73}$$

and hence, the zeta function for the total amplitude can be reduced to

$$\begin{aligned}
\tilde{\zeta}(0, s) &= \frac{2}{3} \mu^{2(s-1)} \oint_{\hat{C}} \frac{du}{2\pi i} \frac{\sigma u G(u, 0)}{F(u)} \sum_{j=0}^{\infty} \frac{(j + 3/2)(j + 1)(j + 2)}{[u^2 + (j + 1/2)^2]^s H^{2s}} \\
&= \frac{2\mu^{2(s-1)}}{3\pi H^{2s}} \sin[\pi(s - 1)] \\
&\quad \times \ P \int_0^{\infty} dU \frac{\sigma U G(e^{\pi i/2} U, 0)}{F(e^{\pi i/2} U)} \sum_{j=0}^{\infty} \frac{(j + 1)(j + 2)(j + \frac{3}{2})}{\left[U^2 - (j + \frac{3}{2})^2 \right]^s} , \tag{4.74}
\end{aligned}$$

where we used the properties of bulk propagator Eq. (4.49). Again, Roman "P" represents taking the Cauchy principal value to deal with the pole at $U = \nu/\sigma$. As for the $d = 2$ case, we divide the total zeta function into a UV piece, i.e., for $U > 2\nu/\sigma$; and an IR piece, i.e., for $0 < U < 2\nu/\sigma$. Similarly, the choice of the division is just for later convenience.

To begin with, for the UV piece we shall use the asymptotic formula [120]

$$
\begin{aligned}
& \frac{2}{3} \sum_{j=0}^{\infty} \frac{(j+3/2)(j+1)(j+2)}{\left[U^2 - (j+3/2)^2\right]^s} \\
=\ & \frac{2}{3} \sum_{j=0}^{\infty} \frac{(j+3/2)^3}{\left[U^2 - (j+3/2)^2\right]^s} - \frac{1}{6} \sum_{j=0}^{\infty} \frac{(j+3/2)}{\left[U^2 - (j+3/2)^2\right]^s} \\
=\ & (-1)^s \left(-\frac{1}{12(s-1)(s-2)(s-3)} \partial_a^3 \theta(-U^2, a, s-2) + \frac{1}{12(s-1)} \partial_a \theta(-U^2, a, s) \right. \\
& \left. -\ \frac{1}{2(s-1)(s-2)} \partial_a \theta(-U^2, a, s-1) \right)_{a=3/2} \\
=\ & -\frac{1}{12\Gamma(s)} \left[U^{-2(s-1)} \sum_{j=0}^{\infty} \frac{\Gamma(j+s-1)}{j!} U^{-2j} \partial_a \zeta_H(-2j, a) \right|_{a=3/2} \\
& -\ U^{-2(s-3)} \sum_{j=0}^{\infty} \frac{\Gamma(j+s-3)}{j!} U^{-2j} \partial_a^3 \zeta_H(-2j, a) \right|_{a=3/2} \\
& +\ 6U^{-2(s-2)} \sum_{j=0}^{\infty} \frac{\Gamma(j+s-2)}{j!} U^{-2j} \partial_a \zeta_H(-2j, a) \right|_{a=3/2} \right],
\end{aligned}
\tag{4.75}
$$

where

$$
\theta(q^2, a, s) := \sum_{j=0}^{\infty} \frac{1}{\left[(j+a)^2 + q^2\right]^{s-1}}.
\tag{4.76}
$$

Then, for the IR piece, we use the following binomial expansions [120, 121]:

$$
\begin{aligned}
& \frac{1}{3} \sum_{j=0}^{\infty} \frac{(j+1)(j+2)(j+3/2)}{\left[(j+3/2)^2 - U^2\right]^s} = \frac{1}{3} \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\Gamma(s)} U^{2J} \\
& \times\ \left(\zeta_H(2s+2J-3, \frac{3}{2}) - \frac{1}{4} \zeta_H(2s+2J-1, \frac{3}{2}) \right),
\end{aligned}
\tag{4.77}
$$

which is valid for the range $0 < U < 3/2$; while for the range $3/2 < U < 5/2$

we must use

$$\frac{1}{3}\sum_{j=0}^{\infty}\frac{(j+1)(j+2)(j+3/2)}{\left[(j+3/2)^2 - U^2\right]^s} = \frac{1}{\left(\frac{9}{4} - U^2\right)^s}$$

$$+ \sum_{J=0}^{\infty}\frac{\Gamma(s+J)}{J!\Gamma(s)}U^{2J}\left[\frac{1}{3}\left(\zeta_H(2s+2J-3,\frac{3}{2}) - \frac{1}{4}\zeta_H(2s+2J-1,\frac{3}{2})\right)\right.$$

$$\left.- \left(\frac{3}{2}\right)^{-2s-2J}\right],$$

(4.78)

and finally, for the range $5/2 < U < \frac{2\nu}{\sigma}$ we have

$$\frac{1}{3}\sum_{j=0}^{\infty}\frac{(j+1)(j+2)(j+3/2)}{\left[(j+3/2)^2 - U^2\right]^s} = \frac{1}{\left(\frac{9}{4} - U^2\right)^s} + \frac{5}{\left(\frac{25}{4} - U^2\right)^s}$$

$$+ \sum_{J=0}^{\infty}\frac{\Gamma(s+J)}{J!\Gamma(s)}U^{2J}\left[\frac{1}{3}\left(\zeta_H(2s+2J-3,\frac{3}{2}) - \frac{1}{4}\zeta_H(2s+2J-1,\frac{3}{2})\right)\right.$$

$$\left.- \left(\frac{3}{2}\right)^{-2s-2J} - 5\left(\frac{5}{2}\right)^{-2s-2J}\right].$$

(4.79)

The total zeta function is obtained from Eq. (4.53).

First, let us focus on the analytic continuation of the UV piece. Some simple manipulations lead to the following expression

$$\tilde{\zeta}_{\rm UV}(0,s) = \frac{1}{12}\frac{\mu^{2(s-1)}}{H^{2s}}\int_{2\nu/\sigma}^{\infty}dU\frac{\sigma G(e^{\pi i/2}U,0)}{F(e^{\pi i/2}U)}$$

$$\times \left\{U^{-2s+3}\sin[\pi(s-2)]\left[\frac{1}{s-1}\partial_a\zeta_H(0,a)\Big|_{a=3/2}\right.\right.$$

$$+ \left.\sum_{j=1}^{\infty}\frac{\Gamma(j+s-1)}{j!\Gamma(s)}U^{-2j}\partial_a\zeta_H(-2j,a)\Big|_{a=3/2}\right]$$

$$- U^{-2s+7}\sin[\pi(s-4)]\left[\frac{1}{(s-1)(s-2)(s-3)}\partial_a^3\zeta_H(0,a)\Big|_{a=3/2}\right.$$

$$+ \frac{U^{-2}}{(s-1)(s-2)}\partial_a^3\zeta_H(-2,a)\Big|_{a=3/2} + \frac{U^{-4}}{2(s-1)}\partial_a^3\zeta_H(-4,a)\Big|_{a=3/2}$$

$$+ \left.\sum_{j=3}^{\infty}\frac{\Gamma(j+s-3)}{j!\Gamma(s)}U^{-2j}\partial_a^3\zeta_H(-2j,a)\Big|_{a=3/2}\right]$$

$$- 6U^{-2s+5}\sin[\pi(s-3)]\left[\frac{1}{(s-1)(s-2)}\partial_a\zeta_H(0,a)\Big|_{a=3/2}\right.$$

$$+ \frac{U^{-2}}{s-1}\partial_a\zeta_H(-2,a)\Big|_{a=3/2}$$

$$+ \left.\left.\sum_{j=2}^{\infty}\frac{\Gamma(j+s-2)}{j!\Gamma(s)}U^{-2j}\partial_a\zeta_H(-2j,a)\Big|_{a=3/2}\right]\right\}.$$

(4.80)

Like for the $d = 2$ case, after analytic continuation to $s \to 1$, this leads to

$$
\begin{aligned}
H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{UV}}(0, s) &= -\frac{1}{3}\sigma \left(\int_{2\nu/\sigma}^{\infty} dU U^3 \left(\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma\prime}(0)} - \sum_{\ell=0}^{1} w_\ell(\sigma, \xi) U^{-1-2\ell} \right) \right. \\
&\quad + \sum_{\ell=0}^{1} \frac{2^{3-2\ell} w_\ell(\sigma, \xi)}{2\ell - 3} \left(\frac{\nu}{\sigma} \right)^{3-2\ell} \right) \\
&\quad + \frac{1}{12}\sigma \left(\int_{2\nu/\sigma}^{\infty} dU U \left(\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma\prime}(0)} - w_0(\sigma, \xi) U^{-1} \right) \right. \\
&\quad \left. - 2 w_0(\sigma, \xi) \left(\frac{\nu}{\sigma} \right) \right),
\end{aligned} \tag{4.81}
$$

where $w_\ell(\xi, \sigma)$ are the coefficients of the asymptotic expansion in Eq. (4.56) given by Eq. (4.57), for $d = 4$.

The IR piece is already finite for $s \to 1$, and some calculation shows that

$$
\begin{aligned}
\tilde{\zeta}_{\mathrm{IR}}(0, s) &= \frac{2\mu^{2(s-1)}}{3\pi H^{2s}(-1)^s} \sin[\pi(s-1)] \, \mathrm{P} \int_0^{3/2} dU \frac{\sigma U G(e^{\pi i/2} U, 0)}{F(e^{\pi i/2} U)} \\
&\quad \times \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J! \Gamma(s)} U^{2J} \left(\zeta_H(2s + 2J - 3, \frac{3}{2}) - \frac{1}{4}\zeta_H(2s + 2J - 1, \frac{3}{2}) \right) \\
&\quad + \frac{2\mu^{2(s-1)}}{\pi H^{2s}(-1)^s} \sin[\pi(s-1)] \, \mathrm{P} \int_{3/2}^{5/2} dU \frac{\sigma U G(e^{\pi i/2} U, 0)}{F(e^{\pi i/2} U)} \left(\frac{1}{\left(\frac{9}{4} - U^2\right)^s} \right. \\
&\quad + \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J! \Gamma(s)} U^{2J} \left[\frac{1}{3} \left(\zeta_H(2s + 2J - 3, \frac{3}{2}) - \frac{1}{4}\zeta_H(2s + 2J - 1, \frac{3}{2}) \right) \right. \\
&\quad \left. \left. - \left(\frac{3}{2} \right)^{-2s-2J} \right] \right) \\
&\quad + \frac{2\mu^{2(s-1)}}{\pi H^{2s}(-1)^s} \sin[\pi(s-1)] \, \mathrm{P} \int_{5/2}^{2\nu/\sigma} dU \frac{\sigma U G(e^{\pi i/2} U, 0)}{F(e^{\pi i/2} U)} \\
&\quad \times \left(\frac{1}{\left(\frac{9}{4} - U^2\right)^s} + \frac{5}{\left(\frac{25}{4} - U^2\right)^s} \right. \\
&\quad + \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J! \Gamma(s)} U^{2J} \left[\frac{1}{3} \left(\zeta_H(2s + 2J - 3, \frac{3}{2}) - \frac{1}{4}\zeta_H(2s + 2J - 1, \frac{3}{2}) \right) \right. \\
&\quad \left. \left. - \left(\frac{3}{2} \right)^{-2s-2J} - \frac{5}{(5/2)^{2s+2J}} \right] \right).
\end{aligned} \tag{4.82}
$$

Note that the number of terms depends on the range of U. For $3/2 < 2\nu/\sigma < 5/2$ the third term should be dropped; while both the second and third terms should be dropped if $2\nu/\sigma < 3/2$.

In the $s \to 1$ limit, as before, just terms with leading order

$$\zeta_H(2s - 1, \frac{3}{2}) = \frac{1}{2(s-1)} - \psi(3/2) + O(s-1) , \qquad (4.83)$$

contribute to the resulting IR amplitude. Thus, taking the limit $s \to 1$, we obtain the IR amplitude as

$$
\begin{aligned}
H^2 \lim_{s \to 1} \tilde{\zeta}_{\mathrm{IR}}(0, s) &= -\frac{1}{3}\sigma\, \mathrm{P} \int_0^{2\nu/\sigma} dU \frac{U^3 G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)} + \frac{1}{12}\sigma\, \mathrm{P} \int_0^{2\nu/\sigma} dU \frac{U G(e^{\pi i/2}U, 0)}{F(e^{\pi i/2}U)} \\
&= \frac{1}{3}\int_0^{2\nu/\sigma} dU \frac{1}{U - \frac{\nu}{\sigma}} \left(U^3 \frac{\Gamma(-\frac{\nu}{2} + \frac{U\sigma}{2} + 1)\Gamma(\frac{\nu}{2} + \frac{U\sigma}{2} + \frac{1}{2})}{\Gamma(\frac{\nu}{2} + \frac{U\sigma}{2} + 1)\Gamma(-\frac{\nu}{2} + \frac{U\sigma}{2} + \frac{1}{2})} \right. \\
&\quad \left. - \frac{1}{\sqrt{\pi}}\left(\frac{\nu}{\sigma}\right)^3 \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} \right) \\
&\quad - \frac{1}{12}\int_0^{2\nu/\sigma} dU \frac{1}{U - \frac{\nu}{\sigma}} \left(U \frac{\Gamma(-\frac{\nu}{2} + \frac{U\sigma}{2} + 1)\Gamma(\frac{\nu}{2} + \frac{U\sigma}{2} + \frac{1}{2})}{\Gamma(\frac{\nu}{2} + \frac{U\sigma}{2} + 1)\Gamma(-\frac{\nu}{2} + \frac{U\sigma}{2} + \frac{1}{2})} \right. \\
&\quad \left. - \frac{1}{\sqrt{\pi}}\left(\frac{\nu}{\sigma}\right) \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} \right) , \qquad (4.84)
\end{aligned}
$$

where in the final step Eq. (4.63) was used.

Finally, we obtain the total regularized amplitude from Eq. (4.64). Furthermore, the KK amplitude is obtained by subtracting the bound state amplitude (evaluated in Appendix D. 2) obtained from Eq. (4.65). Note that the KK amplitude again has a dependence on the renormalization scale μ.

4.4.1 Results of numerical calculations

In Fig. 4.9, a numerical plot of $\langle \phi^2(0) \rangle_{\mathrm{tot}}$ is shown. Again, the divergence for the thin wall limit can be seen. The power of the divergence can be estimated as follows: the dominant contribution in the thin wall limit comes from the first term on the right hand side of Eq. (4.81). By changing variables to $x = U\sigma$ and following the same steps as for $d = 2$, we obtain

$$
\begin{aligned}
&\sigma \left(\int_{2\nu/\sigma}^{\infty} dU U^3 \left(\frac{P_\nu^{-U\sigma}(0)}{P_\nu^{-U\sigma'}(0)} - \sum_{\ell=0}^{1} w_\ell(\sigma, \xi) U^{-1-2\ell} \right) + \sum_{\ell=0}^{1} \frac{2^{3-2\ell} w_\ell(\sigma, \xi)}{2\ell - 3} \left(\frac{\nu}{\sigma}\right)^{3-2\ell} \right) \\
&= \frac{1}{\sigma^2} \int_{2\nu}^{\infty} dx \left(\frac{1}{2}x^2 \left(\frac{\nu}{\sigma}\right) \left(\psi(\frac{x}{2}) - 2\psi(\frac{x+1}{2}) + \psi(\frac{x}{2} + 1) \right) - \frac{(-3 + 16\xi)}{4} \right) \\
&\quad + O(\sigma^{-1}) . \qquad (4.85)
\end{aligned}
$$

In this case, the contribution from the lower bound of the integration does not contribute to any power of σ. Thus, in the thin wall limit the regularized amplitude diverges as σ^{-2}. This is more divergent than the case of $d = 2$
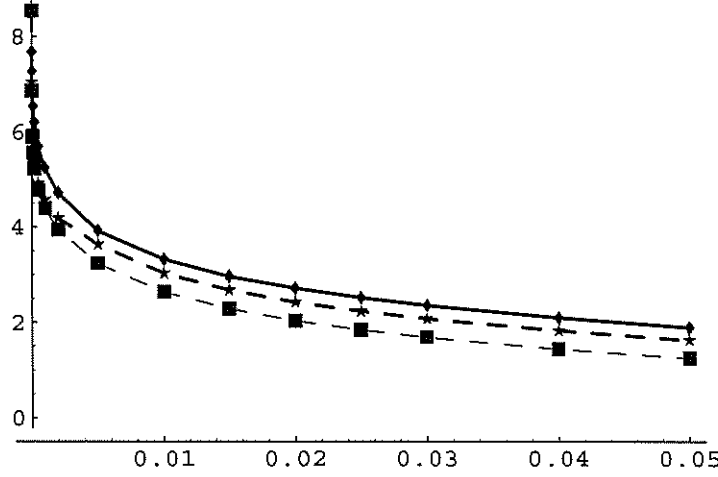
Figure 4.9: The total amplitude is shown as a function of the brane thickness, σ, in the case of $d = 4$. The vertical axis shows $\log_{10}(-H^2\langle\phi^2(0)\rangle_{KK})$. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 3/32, 3/20$, respectively.

and is related to the fact that in higher dimensions we need higher powers of UV subtraction.

The amplitude of the bound state is calculated in Appendix D. 2 and is

$$
\begin{aligned}
H^2\langle\tilde{\phi}^2(0)\rangle_{\rm bs} &= \frac{1}{2\sigma}\left(\int_0^\infty dy\,\cosh^{-2\nu}(y)\right)^{-1} \\
&\times \left\{\left(-\frac{1}{6}+\frac{2}{3}\left(\frac{\nu}{\sigma}\right)^2\right)\ln\left(\frac{\mu}{H}\right)+\frac{2}{3}\zeta_H(-1,\frac{3}{2})+\frac{1}{6}\psi(3/2)\right. \\
&- \left(\frac{\nu}{\sigma}\right)^2\left(-\frac{1}{3}+\frac{2}{3}\psi(3/2)+\frac{1}{6}\zeta_H(3,\frac{3}{2})\right) \\
&+ \frac{2}{3}\sum_{J=2}^\infty\left(\frac{\nu}{\sigma}\right)^{2J}\left(\zeta_H(2J-1,\frac{3}{2})-\frac{1}{4}\zeta_H(2J+1,\frac{3}{2})\right) \\
&- \left.2\sum_{J=0}^\infty\delta_{\xi,0}\left(\frac{3}{2}\right)^{-2}\right\}.
\end{aligned}
\tag{4.86}
$$

This is plotted in Fig. 4.10 and we see that the bound state is almost independent of the brane thickness and still finite in the thin wall limit. Thus, like for $d = 2$, the divergence in the total amplitude arises solely from the KK modes.

Again, the amplitude depends on the choice of the renormalization scale μ. In certain cases the amplitude of the bound state can become negative. In Fig. 4.11, the running of the bound state amplitude is shown as a function
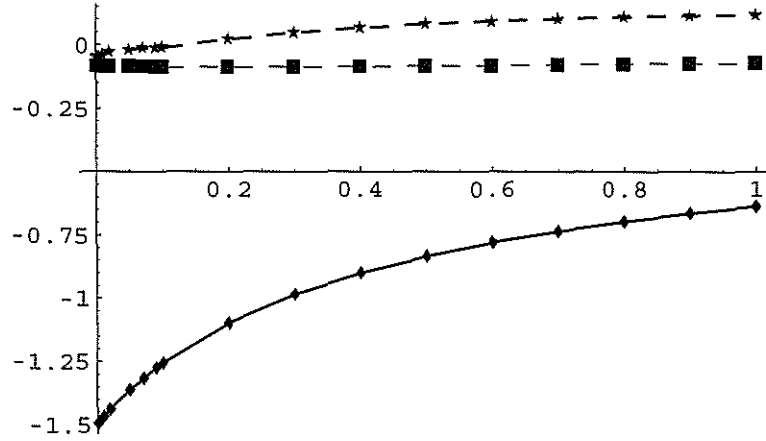
Figure 4.10: The amplitude of the bound state mode is shown as a function of the brane thickness, σ, in the case of $d = 4$, with $\mu = H$. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 3/64, 3/32$, respectively.

of μ. It is basically the same as the case of $d = 2$; however, a new feature is that negative tilts of the running are realized for larger values of coupling ξ which satisfy

$$\xi > \frac{2\sigma + 1}{4(3\sigma + 2)}, \tag{4.87}$$

as can be seen from Eq. (4.86). The critical coupling parameter in Eq. (4.87) is smaller than conformal coupling, $\xi_c = 3/16$, for any choice of brane thickness, σ. This fact means that there always exist coupling parameters which realize negative tilts of the running.

The relative amplitude of the KK to bound state ratio, defined by Eq. (4.72), depends on the choice of renormalization scale μ. As we stated in the previous section, we have no observational singnature about braneworlds and no way to determine the renormalization scale. Again as one of the possible physical choices, in Fig. 4.12, we plot the relative amplitude in the case of $\mu = H$ for minimal coupling, $\xi = 0$.

In this section, we discussed the quantum fluctuations in a thick brane model in order to show that a finite brane thickness can act as a *natural* cut-off of for the Kaluza-Klein (KK) mode spectrum. The thick brane model we examined was supported by a scalar field with an axion-type potential. The thin brane limit of this model is smoothly matched to the system of a de Sitter (dS) brane in a Minkowski bulk.

Here, we investigated the quantum fluctuations in a particular model of thick braneworld. However, the qualitative behavior of the quantum fluc-
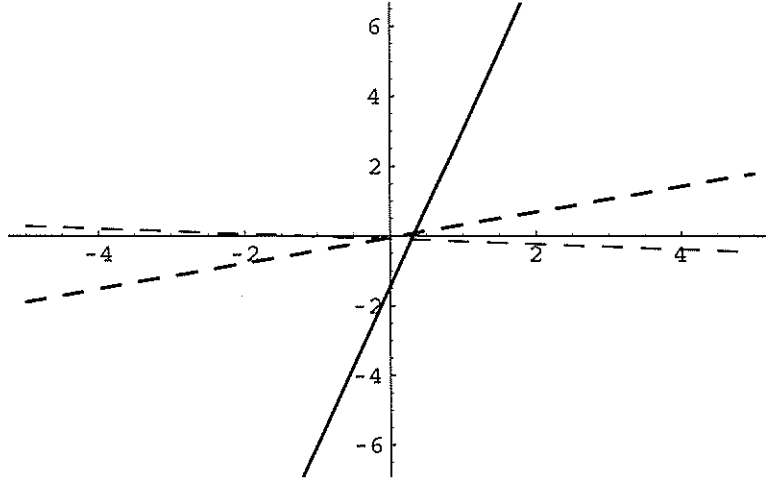
Figure 4.11: The running of the amplitude of the bound state is shown as a function of the brane thickness, σ, in the case of $d = 4$, with $\sigma = 0.01$. The vertical and horizontal axes show the bound state amplitude and $\log_{10}(\mu/H)$, respectively. The thick, thick-dashed and dashed curves correspond to the cases of $\xi = 0, 3/32, 3/20$, respectively.
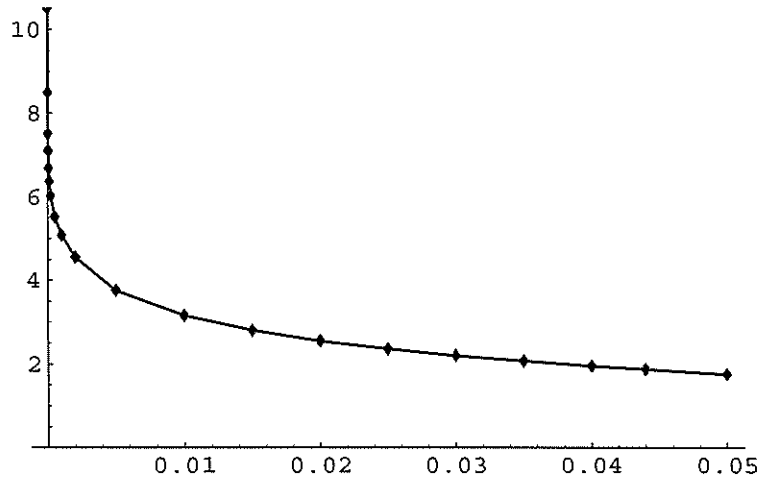
Figure 4.12: The relative amplitude of the KK modes to the bound state mode is shown as a function of the brane thickness, σ, for the minimal coupling, $\xi = 0$, for $d = 4$, with $\mu = H$. The vertical axis shows $\log_{10} |r(\mu)|$ given by Eq. (4.72).

tuations should be independent of the choice of the model. This can be understood as follows. For a thick brane model, for which the spacetime is smooth everywhere, there will be no divergence. Now if we look at the behavior of the background solution, Eq. (4.6), when σ is sufficiently small, we find $\phi \sim \phi_0(Hz/\sigma)$ for $Hz \ll \sigma$. This is a very general behavior that one finds at the center of any domain wall solution, independent of the global features of the bulk potential. Thus the divergence in the thin wall limit is due to the spacetime singularity caused by the divergence of $d\phi/dz = 0$ at $z = 0$, which is common to any thick brane model supported by a bulk scalar field.

4.5 Quantum backreaction and the self-consistency on the brane

In this section, we shall demonstrate that, similarly, a finite thickness also regularizes the quantum backreaction. We give a *theoretical* bound on the thickness in terms of *brane* self-consistency [4] [5] and comment on the realistic case of a four-dimensional brane.

In this subsection, we shall discuss the quantum backreaction of the scalar field ϕ on such a thick-brane background, specifically at $z = 0$. By varying the ϕ-field part of the action, in Eq. (B.1), with respect to the bulk metric we obtain the stress-energy tensor for the ϕ-field;

$$T_{ab} := \phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}g^{cd}\phi_{;c}\phi_{;d}.$$
(4.88)

Furthermore, for simplicity we shall consider the three-dimensional ($d = 2$) case. The method is then based on a dimensional reduction of the higher dimensional canonically quantized fields, see [117]. For a given vacuum, we can calculate the vacuum expectation value of the stress-energy tensor. Hereafter, we work in the Euclideanized space $\gamma_{\mu\nu}^{E}dx^{\mu}dx^{\nu} = H^{-2}(d\theta^2 + \sin^2\theta d\varphi^2)$, where the substitution $\theta \to \pi/2 - iHt$ Wick rotates back to the Lorentzian frame. Choosing the Euclidean vacuum corresponds to a dS invariant vacuum in the original Lorentzian frame. The Hamiltonian density for the field ϕ in this frame is classically defined by $\rho(z, x^i) := -b^2(z)T^{\theta}{}_{\theta}(z, x^i)$.

In general, for one non-trivial extra dimension we can have untwisted, $f^+(-z) = f^+(z)$, and twisted field configurations, $f^-(-z) = -f^-(z)$, [114, 115]. Note that the untwisted and twisted solutions are equivalent to the

---

[4] Self-consistency in the RS (two-brane) model has been investigated in [123] and [124] in terms of how quantum corrections contribute to the gravitational theory in the bulk. Stability of brane solutions including quantum backreaction has also been discussed, see e.g., [125]. What we do here is rather to compare the size of the quantum backreaction with that of the background stress-energy on the brane, i.e., *brane* self-consistency.

[5] Bounds on the brane thickness have also been discussed in terms of phenomenological experiments, see e.g., [126].

mode degeneracy (for one non-trivial dimension). As we shall see, the total Hamiltonian density is given by a combination of untwisted and twisted fields, i.e., $\rho = \rho^{(+)} + \rho^{(-)}$. This quantity diverges when all the modes are naively summed up and we need to employ some kind of regularization scheme. To this end, after a dimensional reduction, we shall employ the point-splitting method in conjunction with zeta function regularization [112, 117], both for untwisted and twisted modes:

$$\zeta^{\pm}(z, x^i, z', x^{i'}; s) := \frac{2\mu^{2(s-1)}H}{b^{1/2}(z)b^{1/2}(z')H^{2(s-1)}} \sum_n f_n^{\pm}(z) f_n^{\pm}(z')$$

$$\times \sum_{j,m} \frac{Y_{jm}(x^i)Y_{jm}^*(x^{i'})}{[q_n^2 + (j + \frac{1}{2})^2]^s}, \tag{4.89}$$

where $f_n^+(z)$ and $f_n^-(z)$ correspond to normalized untwisted and twisted field configurations respectively. The solutions $f_n^{\pm}(z)$ are written in terms of associated Legendre functions and $Y_{jm}(x^i)$ are the usual spherical harmonics defined on the two-sphere, S^2.

Given that we are interested only in the dependence of the backreaction on the bulk coordinate z we integrate out over the trivial dimensions, in this case over S^2:

$$\rho(z) := \int d\Omega_2 \lim_{s \to 1} \lim_{X' \to X} \frac{1}{2} \left(H^2 \partial_\varphi \partial_{\varphi'} + \partial_z \partial_{z'} \right) \zeta(z, x^i, z', x^{i'}; s) \tag{4.90}$$

where $d\Omega_2$ is the volume element of S^2. Note that because of the spherical symmetry transverse to the brane we may focus on the the equatorial plane $\theta = \pi/2$ and remove the dependence on θ [127] Hence, we obtain the angle-integrated Hamiltonian density

$$\rho^{\pm}(z) = 2H \lim_{s \to 1} \sum_{n,j} \frac{j + \frac{1}{2}}{[q_n^2 + (j + \frac{1}{2})^2]^s} \left[\frac{H^2}{2} j(j+1) f_n^{\pm 2}(z) + \left(f_n^{\pm \prime}(z) - \frac{b'(z)}{2b(z)} f_n^{\pm}(z) \right)^2 \right].$$

We are primarily interested in the backreaction on the brane at $z = 0$, given that this is supposed to be where our world is localized. In this case the contribution from the untwisted and twisted parts can be expressed simply as

$$\rho^+(0) = H \lim_{s \to 1} \sum_{n,j} H^2 f_n^{+\,2}(0) \frac{j(j+1)(j+\frac{1}{2})}{[q_n^2 + (j + \frac{1}{2})^2]^s},$$

$$\rho^-(0) = 2H \lim_{s \to 1} \sum_{n,j} f_n^{-\prime 2}(0) \frac{j + \frac{1}{2}}{[q_n^2 + (j + \frac{1}{2})^2]^s} \tag{4.91}$$

respectively. The total backreaction is given by the sum of each:

$$\rho(0) = \rho^+(0) + \rho^-(0). \tag{4.92}$$

We can also derive similar expressions for the pressures normal and parallel to the brane; $T^z{}_z(z, x^i)$ and $T^\varphi{}_\varphi(z, x^i)$, respectively. Like for the Hamiltonian density they are given by a combination of untwisted and twisted fields. In the Euclidean frame these pressures are defined by $p_z(z, x^i) :=$ $b^2(z)T^z{}_z(z, x^i)$ and $p_\varphi(z, x^i) := b^2(z)T^\varphi{}_\varphi(z, x^i)$. Then, as for the case of the Hamiltonian density, the angle-integrated total pressure normal to the brane $p_z(z) = \int d\Omega_2 p_z(z, x^i)$ becomes

$$p_z(0) = -\rho^+(0) + \rho^-(0) \qquad (4.93)$$

on the brane. The angle-integrated pressure parallel to the brane $(p_\varphi(z) = \int d\Omega_2 p_\varphi(z, x^i))$ has the same amplitude with opposite sign, i.e., $p_\varphi(z) = -p_z(z)$. Thus, and hereafter, we may concentrate only on $p_z(0)$. It is also worth mentioning that the angle-integrated trace of the stress-energy tensor in the Euclidean frame has the same amplitude of the Hamiltonian density with the opposite sign; $b^2(z)T^a{}_a(z) = \int d\Omega_2 b^2(z)T^a{}_a(z, x^i) = -\rho(z)$.

The regularization scheme we shall employ is basically the same as the one developed in the prevous sections. Essentially, we convert the mode sums over $\{n\}$ into an integral along the contour as depicted in Fig. 4.4 including the bound state by employing the residue theorem. For the twisted configuration there is no bound state and hence, we need not worry about how the contour approaches the imaginary axis.

Hence, for untwisted configurations we obtain

$$\rho_{\text{UV}}^+(0) = -\frac{\sigma H^3}{2}\left[\int_1^\infty dU U^3\left(\frac{P_{\sigma/2}^{-U\sigma}(0)}{P_{\sigma/2}^{-U\sigma\prime}(0)} - \sum_{\ell=0}^1 \frac{w_\ell(\sigma)}{U^{1+2\ell}}\right)\right.$$
$$\left. - \sum_{\ell=0}^1 \frac{w_\ell(\sigma)}{3-2\ell} - \frac{1}{4}\int_1^\infty dU U\left(\frac{P_{\sigma/2}^{-U\sigma}(0)}{P_{\sigma/2}^{-U\sigma\prime}(0)} - \frac{w_0(\sigma)}{U}\right) - w_0(\sigma)\right],$$

$$\rho_{\text{IR}}^+(0) = \frac{H^3}{2}\int_0^1 dU U\left(U + \frac{1}{2}\right)\frac{\Gamma(-\frac{\sigma}{4} + \frac{U\sigma}{2} + 1)\Gamma(\frac{\sigma}{4} + \frac{U\sigma}{2} + \frac{1}{2})}{\Gamma(\frac{\sigma}{4} + \frac{U\sigma}{2} + 1)\Gamma(-\frac{\sigma}{4} + \frac{U\sigma}{2} + \frac{1}{2})}, \qquad (4.94)$$

where we have split the untwisted contribution into two pieces, i.e., an ultraviolet (UV) and an infrared (IR) piece. Here we used the following asymptotic expansion

$$\frac{P_{\sigma/2}^{-U\sigma}(0)}{P_{\sigma/2}^{-U\sigma\prime}(0)} = \sum_{\ell=0}^\infty w_\ell(\sigma)U^{-1-2\ell}; w_0(\sigma) = -\frac{1}{\sigma}, \quad w_1(\sigma) = -\frac{2+\sigma}{8\sigma^2}, \cdot (4.95)$$

to regularize the UV piece. The regularized untwisted Hamiltonian density is given by $\rho^+ = \rho_{\text{UV}}^+ + \rho_{\text{IR}}^+$.

Similarly, for twisted configurations we find

$$\rho_{\text{UV}}^-(0) = \frac{H^3}{\sigma}\left[\int_1^\infty dU\, U\left(\frac{P_{\sigma/2}^{-U\sigma\prime}(0)}{P_{\sigma/2}^{-U\sigma}(0)} - \sum_{\ell=0}^1 q_\ell(\sigma)U^{1-2\ell}\right) - \sum_{\ell=0}^1 \frac{q_\ell(\sigma)}{3-2\ell}\right],$$

$$\rho_{\text{IR}}^-(0) = -\frac{2H^3}{\sigma}\int_0^1 dU U\frac{\Gamma(\frac{\sigma}{4} + \frac{U\sigma}{2} + 1)\Gamma(-\frac{\sigma}{4} + \frac{U\sigma}{2} + \frac{1}{2})}{\Gamma(-\frac{\sigma}{4} + \frac{U\sigma}{2})\Gamma(\frac{\sigma}{4} + \frac{U\sigma}{2} + \frac{1}{2})}, \qquad (4.96)$$

where this time we used the slightly different asymptotic expansion

$$\frac{P_{\sigma/2}^{-U\sigma\prime}(0)}{P_{\sigma/2}^{-U\sigma}(0)} = \sum_{\ell=0}^{\infty} q_\ell(\sigma)U^{1-2\ell}; \quad q_0(\sigma) = -\sigma, \quad q_1(\sigma) = \frac{2+\sigma}{8}, \quad \cdots \quad (4.97)$$

The twisted Hamiltonian density is also given by $\rho^- = \rho_{UV}^- + \rho_{IR}^-$.

Before considering the total backreaction, we also wish to briefly discuss the backreaction for the bound state mode, which for the untwisted case is given by

$$\rho_0(z,s) = \frac{H^3\mu^{2(s-1)}f_0^{+2}(z)}{H^{2(s-1)}} \sum_j \frac{j(j+1)(j+\frac{1}{2})}{[(j+\frac{1}{2})^2 - (\frac{1}{2})^2]^s} \quad (4.98)$$

where

$$f_0^{+2}(z) = \frac{1}{2\sigma} \frac{\cosh^{-\sigma}(\frac{Hz}{\sigma})}{\int_0^\infty dy \cosh^{-\sigma}(y)}. \quad (4.99)$$

As can easily be verified the twisted solution has no localized bound state. Note that $\rho_0(z,s) = -p_{z,0}(z,s)$. As a result of employing the renormalization discussed in [128], the backreaction for the bound state is given by

$$\rho_0(z) = H^3 f_0^{+2}(z)\Big\{ \zeta_H(-1,\frac{1}{2}) - \frac{1}{16}\zeta_H(3,\frac{1}{2}) + \frac{1}{8}$$
$$+ \sum_{J=2}^{\infty} \Big(\frac{1}{2}\Big)^{2J}\Big[\zeta_H(2J-1,\frac{1}{2}) - \frac{1}{4}\zeta_H(2J+1,\frac{1}{2})\Big]\Big\}.$$

In the case above (of massless, minimal coupling) the backreaction of the bound state mode is found to be independent of the renormalization scale. This happens only for the case of minimal coupling. Also, contrary to the backreaction, the squared amplitude itself does depend on the renormalization scale even for minimal coupling, $\xi = 0$, [112]. If required, the contribution from the KK modes can easily be obtained by employing the relations [112]

$$\rho_{KK}(z) = \rho(z) - \rho_0(z), \quad p_{KK}(z) = p(z) - p_0(z). \quad (4.100)$$

Note that, like for the amplitude [112], ρ_0 is also insensitive to the brane thickness, σ. Particularly in comparison to the KK contribution.

Similar to the calculation for the amplitude [112] in the thin wall limit, $\sigma \to 0$, the leading order behavior for $\rho^\pm(0)$ can easily be obtained

$$\rho^+(0) \to -\frac{H^3}{2\sigma^2}I, \quad \rho^-(0) \to -\frac{H^3}{\sigma^2}I;$$
$$I := \frac{1}{4}\int_0^\infty dx\Big[x^2\Big(\psi(\frac{x}{2}) - 2\psi(\frac{x+1}{2}) + \psi(\frac{x}{2}+1)\Big) + 1\Big]$$
$$\approx 0.213139. \quad (4.101)$$

Thus, from Eq. (4.92) and Eq. (4.93), the thin wall behavior for the Hamiltonian density and pressure are

$$\rho(0) \to -\frac{3H^3}{2\sigma^2} I, \quad p_z(0) \to -\frac{H^3}{2\sigma^2} I, \tag{4.102}$$

respectively. Thus, in the thin wall limit both the total Hamiltonian density and pressure exhibit quadratic divergences.

We now come to discuss the *brane* self-consistency of the quantum corrected Einstein equations: The stress-energy of the backreaction should not become larger than the background stress-energy on the brane. In Figs. 4.13 and 4.14 the Hamiltonian density, $\rho(0)$, and bulk pressure, $p_z(0)$, are compared with respect to their (angle-integrated) classical counterparts:

$$4\pi \left(-\frac{1}{2} (\chi'(z))^2 - b^2(z)V(\chi(z)) \right) \delta^\alpha{}_\beta = -\frac{4\pi H^2}{\sigma \cosh^2\left(\frac{Hz}{\sigma}\right)} \delta^\alpha{}_\beta \xrightarrow{z=0} -\frac{4\pi H^2}{\sigma} \delta^\alpha{}_\beta ,$$

$$4\pi \left(\frac{1}{2} (\chi'(z))^2 - b^2(z)V(\chi(z)) \right) = -\frac{4\pi H^2}{\cosh^2\left(\frac{Hz}{\sigma}\right)} \xrightarrow{z=0} -4\pi H^2$$

for the special case $H = 1 = M_3$. Note, that in Fig. 4.13 the Hamiltonian density is multiplied by a power of σ, in order to easily distinguish between the two, and the three-dimensional Planck mass $M_3 := \kappa_3^{-2}$ is set to unity.

The quantum backreaction scales as H^3/σ^2, whereas the background stress-energy scales as $H^2 M_3/\sigma$. Thus, the ratio of the backreaction to the background energy density scales as $O(H/M_3\sigma)$. From Figs. 4.13 and 4. 14, for the special case $H = M_3$, we can infer that for brane thicknesses with $\sigma \gtrsim 0.3$ the quantum backreaction is at least an order of magnitude smaller than the classical value. Thus, taking these facts into consideration, we obtain a plausible theoretical bound on the brane thickness, $\sigma \gtrsim 0.3 \, (H/M_3)$. Of course, this bound is only valid on the brane not in the whole bulk. In this sense it might not be a sufficient condition, but just a necessary one. However, we are mainly interested in the behavior of the quantum backreaction at $z = 0$, where the backreaction is expected to be largest and our world exists by assumption. Thus, it may be considered as a stringent bound on the brane thickness. We note that the backreaction vanishes in the limit $H \to 0$, because the brane tension (in the sense of the thin wall limit) vanishes in our model. This is in contrast to RS models, where the presence of a negative cosmological constant in the bulk allows for a flat brane with finite tension.

Now let us consider the more realistic case of $d = 4$. Following steps similar to that in (4.101), it is not hard to convince ourselves that in the thin wall limit, $\sigma \to 0$, the quantum backreaction exhibits a quartic divergence proportional to H^5/σ^4 for $d = 4$ (times a numerical factor, like I in Eq. (4.101) for $d = 2$), whereas the background stress-energy scales as $H^2 M_5{}^3/\sigma$. Thus, the ratio of the quantum backreaction to the background stress-energy will be of order $O(H^3/(\sigma M_5)^3)$. Therefore, the brane should satisfy $\sigma \gtrsim H/M_5$ in order to have *brane* self-consistency. Actually, this bound depends on the
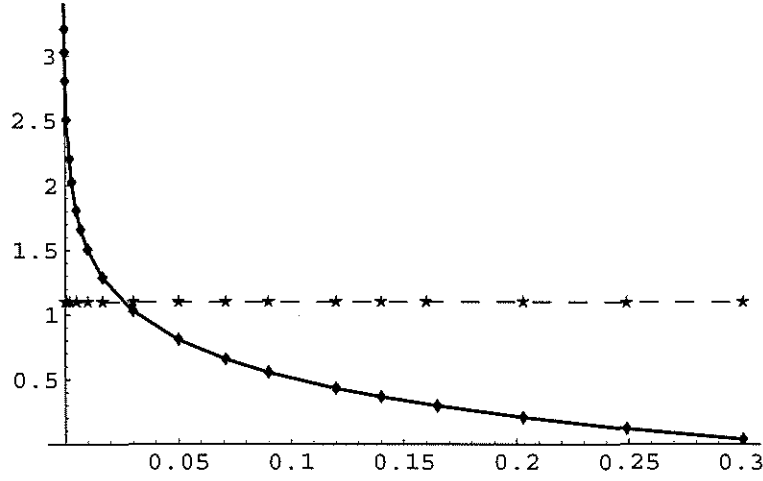
Figure 4.13: The backreaction of the Hamiltonian density (thick curve) and the background energy density (dashed curve), multiplied by a power of the brane thickness, σ, are shown as a function of σ for the case of $H = 1(= M_3)$. Note, the scale of the vertical axis is set to $\log_{10}$.
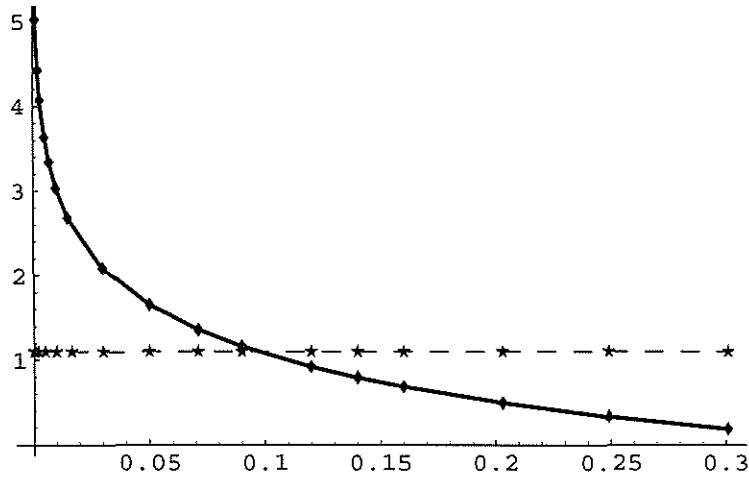
Figure 4.14: The backreaction of the pressure (thick curve) and the background pressure (dashed curve) are shown as a function of the brane thickness, σ, for the case of $H = 1(= M_3)$. Note, the scale of the vertical axis is set to $\log_{10}$.

ratio between the energy scale for brane inflation, H, and M_5. Hence, in order for the bound to be consistent with the assumption $0 < \sigma < 1$ we should require $H \lesssim M_5$. This condition can also be regarded as a bound on the energy scale for brane inflation.

We might ask whether or not this bound on the brane thickness is consistent within the framework of the background model. The thick brane model we have investigated in this letter has an asymptotically flat bulk, which can be regarded as the high-energy limit ($H\ell \to \infty$) of an asymptotically Anti de Sitter (AdS) bulk, where ℓ is curvature radius of AdS spacetime. Quite clearly $\sigma \ll H\ell$ and thus, combining this inequality with the previous theoretical bound, $\sigma \gtrsim H/M_5$, we find that $H/M_5 \ll H\ell$ for *brane* self-consistency. Note that in the RS II set-up the four-dimensional Planck scale on the brane effectively becomes $M_{\mathrm{pl}}{}^2 = M_5{}^3\ell \,(\approx 10^{19}\mathrm{GeV})$, which is determined at low energies ($H\ell \ll 1$) [9, 18]. Then, *brane* self-consistency, $H/M_5 \ll H\ell$, is equivalent to the condition $M_{\mathrm{pl}} \gg M_5$, which seems to be quite a natural one. Indeed, it is not difficult to construct a model with $M_5 \ll M_{\mathrm{pl}}$; just as long as the scale is larger than 10^9 GeV, derived from constraints on the size of any extra-dimensions, $\ell \lesssim 0.1\mathrm{mm}$, which is determined from experimental tests of Newton's law on short distance scales. Thus, we conclude that thick braneworlds, even if they are extremely thin, can be *brane* self-consistent.

4.6 Outlook for the case of KK gravitons

Finally, we shall comment on the quantum backreaction of the KK modes of gravitons on the brane. As we mentioned previously, they are just metric perturbations in the bulk and their existence is rather common in braneworld cosmology, independently of the details of the models. The quantification of the graviton backreaction has been a longstanding issue in brane cosmology. Part of the motivation is that the KK gravitons satisfy the same equation of motion as a massless, minimally coupled scalar field and therefore suffer from a similar pathology on the brane, *surface divergences*. Hence, we expect that the graviton backreaction behaves in a similarly manner to the scalar case, namely the backreaction exhibits quadratic and quartic divergences for $d = 2$ and $d = 4$, respectively. [6] Furthermore, any discussion on the self-consistency of the scalar backreaction should also carry over to the graviton backreaction similarly, though any explicit demonstration of this fact is left for future work.

---

[6] One point that we should note is that the degeneracy for gravitons (i.e., a spin 2 field) on the d-sphere is different from that for a scalar field (i.e., a spin 0 field) 4.31, see, e.g., [129].

5

Linearized gravity in the Einstein Gauss-Bonnet braneworld

In the previous Chapters, we discussed the Kaluza-Klein (KK) modes in braneworld cosmology in the context of the five-dimensional Einstein (or Einstein-scalar) theory. But in reality, in five-dimensions the Einstein theory is not the most general tensor gravitational theory and one may add curvature corrections of quadratic order. In general, these curvature corrections give higher order derivatives in the bulk gravitational equations of motion and may induce instabilities. It is well-known that the Gauss-Bonnet term, which is the special combination of second order curvature corrections, uniquely gives the second order equations of motion in the bulk. In this Chapter, we discuss the linearized effective gravity on a de Sitter brane in the Einstein Gauss-Bonnet (EGB) theory, in order to obtain the implications of the GB term to brane cosmology[130]. The GB term becomes nontrivial in higher-dimensions, whereas in four dimensions it is a toplogical quantity.

First, we derive a dS brane solution in the EGB theory. Then, we solve the tensor metric perturbations in the bulk and derive a closed set of integro-differential equations which describe the effective gravitatioal theory on the brane. We investigate its various limiting cases. Interestingly, the linearized effective gravity on the brane becomes four-dimensional on all distance scales, from short distances to large distances. Also for high energy expanding branes as well as low energy ones, gravity on the brane becomes four-dimensional.

5.1 Braneworlds in the Einstein Gauss-Bonnet theory

We consider a braneworld in the EGB theory with a cosmological constant. As usual, we assume the $\mathbb{Z}_2$ symmetry with respect to the brane. Then we may focus on one of the two identical copies of the spacetime M with the

brane as the boundary ∂M. The action is given by [50, 52];

$$
\begin{aligned}
S = & \int_M d^5 x \sqrt{-g} \frac{1}{2\kappa_5^2} \Big[\overset{(5)}{R} - 2\Lambda_5 \\
& + \alpha \Big(\overset{(5)}{R}{}^2 - 4 \overset{(5)}{R}_{ab} \overset{(5)}{R}{}^{ab} + \overset{(5)}{R}_{abcd} \overset{(5)}{R}{}^{abcd} \Big) \Big] \\
& + \int_{\partial M} d^4 x \sqrt{-q} \Big[-\sigma + \mathcal{L}_m + \frac{1}{\kappa_5^2} \Big(K + 2\alpha \Big(J - 2G^\mu{}_\nu K^\nu{}_\mu \Big) \Big) \Big],
\end{aligned}
\tag{5.1}
$$

where α is the coupling constant for the Gauss-Bonnet term which has dimensions (length)^2, Λ_5 is the negative cosmological constant, g_{ab} and $q_{\mu\nu}$ are the bulk and brane metrics, respectively. $\mathcal{L}_m$ is the Lagrangian density of the matter on the brane, and σ is the brane tension. The second term in the second line in Eq. (5.1) denotes the generalized Gibbons-Hawking term [131] which is added to the boundary action in order to obtain the well-defined boundary value problem. $K_{\mu\nu}$ is extrinsic curvature of the brane and

$$
J^\mu{}_\nu = -\frac{2}{3} K^\mu{}_\rho K^\rho{}_\sigma K^\sigma{}_\nu + \frac{2}{3} K K^\mu{}_\rho K^\rho{}_\nu + \frac{1}{3} K^\mu{}_\nu \Big(K^{\rho\sigma} K_{\rho\sigma} - K^2 \Big).
\tag{5.2}
$$

Extremizing the action S with respect to the bulk metric, the vacuum bulk Einstein Gauss-Bonnet equation is obtained as

$$
\begin{aligned}
& \overset{(5)}{G}_{ab} + \Lambda_5 g_{ab} \\
& + \alpha \Big[2 \Big(\overset{(5)}{R}_a{}^{cde} \overset{(5)}{R}_{bcde} - 2 \overset{(5)}{R}{}^{cd} \overset{(5)}{R}_{acbd} - 2 \overset{(5)}{R}_{ac} \overset{(5)}{R}{}^c{}_b + \overset{(5)}{R}\overset{(5)}{R}_{ab} \Big) \\
& - \frac{1}{2} g_{ab} \Big(\overset{(5)}{R}{}^2 - 4 \overset{(5)}{R}_{cd} \overset{(5)}{R}{}^{cd} + \overset{(5)}{R}_{cdef} \overset{(5)}{R}{}^{cdef} \Big) \Big] = 0.
\end{aligned}
\tag{5.3}
$$

The brane trajectory is determined by the junction condition which is obtained by varying the action S with respect to the brane metric [132, 133]:

$$
B^\mu{}_\nu = K^\mu{}_\nu - K\delta^\mu{}_\nu + 4\alpha \Big(\frac{3}{2} J^\mu{}_\nu - \frac{1}{2} J\delta^\mu{}_\nu - P^\mu{}_{\rho\nu\sigma} K^{\rho\sigma} \Big) = \frac{1}{2} \kappa_5^2 T^\mu{}_\nu,
\tag{5.4}
$$

where

$$
\begin{aligned}
P_{\mu\rho\nu\sigma} := & R_{\mu\rho\nu\sigma} + \Big(R_{\mu\sigma} q_{\rho\nu} - R_{\rho\sigma} q_{\mu\nu} + R_{\rho\nu} q_{\mu\sigma} - R_{\mu\nu} q_{\rho\sigma} \Big) \\
& - \frac{1}{2} R \Big(q_{\mu\sigma} q_{\rho\nu} - q_{\mu\nu} q_{\rho\sigma} \Big),
\end{aligned}
\tag{5.5}
$$

and $T_{\mu\nu}$ is the energy momentum tensor of the matter on the brane, defined as

$$
\delta \Big(\sqrt{-q} \mathcal{L}_m \Big) = -\frac{1}{2} \sqrt{-q} T_{\mu\nu} \delta q^{\mu\nu}.
\tag{5.6}
$$

Note that the extrinsic curvature here is the one for the vector normal to ∂M pointing outward from the side of M.

5.2 de Sitter brane in the Einstein Gauss-Bonnet theory

Let us consider a de Sitter brane in the AdS bulk in the EGB theory, and investigate the linearized gravity on the de Sitter brane.

5.2.1 de Sitter brane in the Einstein Gauss-Bonnet theory

We take the Gaussian normal coordinate with respect to the brane, and assume the bulk metric in the form [18],

$$ds^2 = dy^2 + b^2(y)\gamma_{\mu\nu}dx^\mu dx^\nu \,, \tag{5.7}$$

where $\gamma_{\mu\nu}$ is the metric of the four-dimensional de Sitter spacetime with $R(\gamma) = 12H^2$.

The background Einstein Gauss-Bonnet equation is

$$-3H^2 + 3b''b + 3b'^2 - 12\alpha\frac{b''}{b}\left(b'^2 - H^2\right) = -\Lambda_5 b^2. \tag{5.8}$$

This has a solution,

$$b(y) = H\ell\sinh(y/\ell)\,, \tag{5.9}$$

where ℓ is given by

$$\frac{1}{\ell^2} = \frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda_5}{3}}\right), \tag{5.10}$$

This agrees with the Minkowski brane case [134, 135]. Without loss of generality, we choose the location of the de Sitter brane at

$$b(y_0) = 1\,. \tag{5.11}$$

Thus H is the expansion rate of the de Sitter brane.

5.2.2 Bulk gravitational perturbations

First, we consider gravitational perturbations in the bulk. We take the RS gauge [9, 15, 136],

$$h_{55} = h_{5\mu} = 0\,, \quad h^\mu{}_\mu = D_\nu h^\nu{}_\mu = 0\,, \tag{5.12}$$

where D_α denotes the covariant derivative with respect to $\gamma_{\mu\nu}$, and the perturbed metric is given by

$$ds^2 = dy^2 + b^2(y)\left(\gamma_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu \,. \tag{5.13}$$

The (μ,ν)-components of the linearized Einstein Gauss-Bonnet equation are given by

$$\left(1 - \bar{\alpha}\right)\left[\frac{1}{\sinh^4(y/\ell)}\partial_y\left(\sinh^4(y/\ell)\partial_y\right) + \frac{1}{(H\ell)^2\sinh^2(y/\ell)}\left(\Box_4 - 2H^2\right)\right]h_{\mu\nu}$$
$$= 0,\tag{5.14}$$

where

$$\bar{\alpha} = \frac{4\alpha}{\ell^2},\tag{5.15}$$

and $\Box_4 = D^\mu D_\mu$ is the d'Alembertian with respect to $\gamma_{\mu\nu}$. Throughout this paper, we assume $\bar{\alpha} \neq 1$.

Equation (5.14) is separable. Setting $h_{\mu\nu} = \psi_p(y)Y_{\mu\nu}^{(p,2)}(x^\alpha)$, we obtain

$$\left[\frac{1}{\sinh^4(y/\ell)}\partial_y\left(\sinh^4(y/\ell)\partial_y\right) + \frac{m^2}{\ell^2\sinh^2(y/\ell)}\right]\psi_p(y) = 0,\tag{5.16}$$

$$\left[\Box_4 - (m^2 + 2)H^2\right]Y_{\mu\nu}^{(p,2)} = 0,\tag{5.17}$$

where $p^2 = m^2 - 9/4$ and $Y_{\mu\nu}^{(p,2)}$ are the tensor-type tensor harmonics on the de Sitter spacetime which satisfy the gauge condition [111],

$$Y^{(p,2)\mu}{}_\mu = D_\nu Y^{(p,2)\nu}{}_\mu = 0.\tag{5.18}$$

The properties of these harmonics are discussed in Appendix E.

The equation (5.16) is the same as that for a massless scalar field in the bulk [28]. There exists a mass gap for the eigenvalue $0 < m < 3/2$ [18]. There is a unique bound state at $m = 0$, which gives $\psi_p(y)$ =constant, and it is called the zero mode. For $m > 3/2$, the mass spectrum is continuous and they correspond to the Kaluza-Klein modes. The general solution is

$$\psi_p(y) = \frac{1}{\sinh^{3/2}(y/\ell)}\left[A_p P_{3/2}^{ip}(\cosh(y/\ell)) + B_p Q_{3/2}^{ip}(\cosh(y/\ell))\right],\tag{5.19}$$

where $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are the associated Legendre functions of the 1st and 2nd kinds, respectively.

For $p^2 > 0$ $(m > 3/2)$, we choose those harmonic functions $Y_{\mu\nu}^{(p,2)}$ that behave as e^{-ipt} in the limit $t \to \infty$. Then, assuming that there is no incoming wave from the past infinity $y = 0$, we find we should take $B_p = 0$. In fact, the asymptotic behavior of $P_{3/2}^{ip}$ for $y \to 0$ is [137]

$$\frac{1}{\sinh^{3/2}(y/\ell)}P_{3/2}^{ip}(\cosh(y/\ell)) \xrightarrow[y \to 0]{} \frac{2^{ip}}{\Gamma(1 - ip)}\left(\frac{\sinh(y/\ell)}{\cosh(y/\ell)}\right)^{-ip-3/2}$$

$$\simeq \frac{2^{ip}}{\Gamma(1 - ip)}\left(\frac{y}{\ell}\right)^{-3/2}e^{-ip\ln(y/\ell)},\tag{5.20}$$

which guarantees the no incoming wave (i.e., retarded) boundary condition. Thus the bulk metric perturbations are constructed by

$$h_{\mu\nu} = \oint_D dp\, \psi_p(y) Y_{\mu\nu}^{(p,2)}(x^\mu), \tag{5.21}$$

where the contour of integration D is chosen on the complex p-plane such that it runs from $p = -\infty$ to $p = \infty$ and covers the bound state pole at $p = 3i/2$ below the contour [31].

5.2.3 Linearized effective gravity on the brane

We now investigate the effective gravity on the brane. The position of the brane in the coordinate system is displaced in general as

$$y = y_0 - \ell\varphi(x^\mu), \tag{5.22}$$

where the second term in the right-hand-side describes the brane bending [134, 111]. The induced metric on the brane is given by

$$ds^2\Big|_{(4)} = \left(\gamma_{\mu\nu} + \tilde{h}_{\mu\nu}\right)dx^\mu dx^\nu; \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\coth(y_0/\ell)\,\varphi\gamma_{\mu\nu}. \tag{5.23}$$

The extrinsic curvature on the brane is given by

$$K^\mu_{\ \nu} = \frac{1}{\ell}\coth(y_0/\ell)\delta^\mu_{\ \nu} + \frac{1}{2}h^\mu_{\ \nu,y} + \ell\left(D^\mu D_\nu + H^2\delta^\mu_{\ \nu}\right)\varphi. \tag{5.24}$$

We consider the junction condition (5.4). The background part gives the relation between the brane tension and the location of the brane,

$$\kappa_5^2\sigma = \frac{6}{\ell}\coth(y_0/\ell)\left(1 - \frac{\bar{\alpha}}{3} + \frac{2\bar{\alpha}}{3\sinh^2(y_0/\ell)}\right), \tag{5.25}$$

where

$$\coth(y_0/\ell) = \sqrt{1 + (H\ell)^2}, \quad \sinh(y_0/\ell) = \frac{1}{H\ell}. \tag{5.26}$$

In the limit $H\ell \ll 1$, Eq. (5.25) reduces to the Minkowski tension,

$$\kappa_5^2\sigma \simeq \frac{6}{\ell}\left(1 - \frac{1}{3}\bar{\alpha}\right). \tag{5.27}$$

The perturbative part of the junction condition gives

$$\left(1 + \beta\right)\left(D_\mu D_\nu - \Box_4\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\right)\varphi + \frac{1}{2\ell}\left(1 - \bar{\alpha}\right)h_{\mu\nu,y}$$
$$-\frac{1}{2}\bar{\alpha}\coth(y_0/\ell)\left(\Box_4 - 2H^2\right)h_{\mu\nu} = \frac{\kappa_5^2}{2\ell}S_{\mu\nu}, \tag{5.28}$$

where

$$\beta := \frac{\cosh^2(y_0/\ell) + 1}{\sinh^2(y_0/\ell)}\,\bar{\alpha} = \left(2\coth^2(y_0/\ell) - 1\right)\bar{\alpha} = \left(2(H\ell)^2 + 1\right)\bar{\alpha}\,. \tag{5.29}$$

The trace of Eq. (5.28) gives the equation to determine the brane bending as

$$\left(\Box_4 + 4H^2\right)\varphi = -\frac{\kappa_5^2}{6(1+\beta)\ell}\,S\,, \tag{5.30}$$

where $S = S^\mu{}_\mu$. Note that the field φ seems to be tachyonic, with mass-squared given by $-4H^2$. However, in the case of a de Sitter brane in the Einstein gravity, there was a similar equation for the brane bending, but it was found to be non-dynamical [111]. We shall see below that the situation is quite similar in the present case of the EGB theory.

To find the effective gravitational equation on the brane, we manipulate as follows. Using the expression for the induced metric on the brane, Eq. (5.23), the perturbation of the brane Einstein tensor is given by

$$\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] &= -\frac{1}{2}\Box_4 h_{\mu\nu} - 2H^2 h_{\mu\nu} + 2\coth(y_0/\ell)\left[D_\mu D_\nu - \Box_4\,\gamma_{\mu\nu}\right]\varphi \\
&= -3H^2\left(h_{\mu\nu} - 2\coth(y_0/\ell)\gamma_{\mu\nu}\varphi\right) \\
&\quad - \frac{1}{2}\left(\Box_4 - 2H^2\right)h_{\mu\nu} \\
&\quad + 2\coth(y_0/\ell)\left[D_\mu D_\nu - \Box_4\,\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\right]\varphi\,. \tag{5.31}
\end{aligned}$$

Using the perturbed junction condition (5.28) we can eliminate the term involving φ from the above equation to obtain

$$\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} &= -\frac{1-\bar{\alpha}}{2(1+\beta)}\left(\Box_4 - 2H^2\right)h_{\mu\nu} - \frac{1-\bar{\alpha}}{\ell(1+\beta)}\coth(y_0/\ell)h_{\mu\nu,y} \\
&\quad + \frac{\kappa_5^2}{\ell}\frac{\coth(y_0/\ell)}{(1+\beta)}S_{\mu\nu}\,. \tag{5.32}
\end{aligned}$$

Eliminating the term proportional to $(\Box_4 - 2H^2)h_{\mu\nu}$ from Eqs. (5.31) and (5.32) we obtain

$$\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} &= \frac{\kappa_5^2\tanh(y_0/\ell)}{2\ell\,\bar{\alpha}}S_{\mu\nu} \\
&\quad - \frac{1-\bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell)\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\right)\varphi \\
&\quad - \frac{1-\bar{\alpha}}{2\ell\bar{\alpha}}\tanh(y_0/\ell)h_{\mu\nu,y}\,. \tag{5.33}
\end{aligned}$$

Together with Eq. (5.30), this may be regarded as an effective gravitational equation on the brane. The effect of the bulk gravitational field is contained in the last term proportional to $h_{\mu\nu,y}$. Note that the limit $\bar{\alpha} \to 0$ is singular in the above equation. Thus an Einstein Gauss-Bonnet brane exhibits entirely different effective gravity from an Einstein brane even if $\bar{\alpha} \ll 1$.

5.2.4 Harmonic decomposition

Using the harmonic functions defined in Appendix E. 1, we may obtain a closed (integro-differential) system on the brane. We decompose the perturbations on the brane as

$$
S_{\mu\nu} = S_{\mu\nu}^{(0)} + S_{\mu\nu}^{(2)} ; \quad S_{\mu\nu}^{(0)} = \int_{-\infty}^{\infty} dp \left(S_{(p,0)} Y_{\mu\nu}^{(p,0)} \right) , \quad S_{\mu\nu}^{(2)} = \int_{-\infty}^{\infty} dp \left(S_{(p,2)} Y_{\mu\nu}^{(p,2)} \right) ,
$$

$$
\varphi = \int_{-\infty}^{\infty} dp \, \varphi_{(p)} Y^{(p,0)} ,
$$

$$
h_{\mu\nu} = \int_{-\infty}^{\infty} dp \, h_{(p)} Y_{\mu\nu}^{(p,2)} , \tag{5.34}
$$

where $Y^{(p,0)}$ are the scalar harmonics and $Y_{\mu\nu}^{(p,0)}$ are the scalar-type tensor harmonics given in terms of $Y^{(p,0)}$, as defined in Appendix E. 1. Note that, because of the energy-momentum conservation, $D^\mu S_{\mu\nu} = 0$, there is no contribution from the vector-type tensor harmonics which do not satisfy the divergence free condition. If a bound state exists, we have to deform the contour of integration so that the corresponding pole is covered, as mentioned at the end of the subsection 5.2.2.

With the above decomposition, the metric perturbation on the brane $\tilde{h}_{\mu\nu}$ given by Eq. (5.23) consists of the isotropic scalar-type part and tensor-type part. The scalar-type part is determined by Eq. (5.30), which gives

$$
\begin{aligned}
\varphi_{(p)} &= \frac{\kappa_5^2}{2(1+\beta)\ell} N_p S_{(p,0)} \\
&= \frac{\kappa_5^2}{2(1+\beta)\ell} \frac{1}{\sqrt{3}\sqrt{\left(p^2 + \frac{21}{4}\right)\left(p^2 + \frac{25}{4}\right)}H^2} S_{(p,0)} ,
\end{aligned} \tag{5.35}
$$

where N_p is the normalization factor for the harmonics defined in Appendix E. 1. We see that the propagator part of the above (i.e., the coefficient of $S_{(p,0)}$) do not contain the pole at $p = (5/2)i$ which would corresponds to the tachyonic mode with mass-squared $-4H^2$. Instead, it becomes a branch point and a branch cut appears between the points $p = (\sqrt{21}/2)i$ and $p = (5/2)i$. Thus we find the tachyonic mode is absent and there is no instability associated with the brane bending due to the matter source on the brane.

Before we proceed, it is useful to note the equation,

$$
\left(D_\mu D_\nu - \gamma_{\mu\nu} \Box_4 - 3H^2 \gamma_{\mu\nu} \right) \varphi = \frac{\kappa_5^2}{2(1+\beta)\ell} S_{\mu\nu}^{(0)} , \tag{5.36}
$$

which directly follows from Eq. (5.35) and the definition of the scalar-type tensor harmonics $Y_{\mu\nu}^{(p,0)}$.

There is a free propagating tachyonic mode corresponding to the homogeneous solution of Eq. (5.30), which does not couple to either the scalar or tensor-type matter perturbations on the brane. However, we shall argue

in the next subsection that the mode that corresponds to the exponential growth of the perturbation is unphysical, namely, the only physical mode associated with this tachyonic mode is exponentially decaying with time.

The traceless part of Eq. (5.28) gives

$$
h_{(p)}(y_0) = -\frac{1}{(ip + 3/2)} \frac{\ell^2 \sinh(y_0/\ell)\, P^{ip}_{3/2}(z_0)}{(1 - \bar\alpha)P^{ip}_{1/2}(z_0) + \bar\alpha(-ip + 3/2)(H\ell)^2 \cosh(y_0/\ell)P^{ip}_{3/2}(z_0)}
$$
$$
\times \frac{\kappa_5^2}{\ell} S_{(p,2)} ,
\tag{5.37}
$$

where $z_0 = \cosh(y_0/\ell)$. This shows that the harmonic component of the tensor-type metric perturbations on the brane has a simple pole at $p = (3/2)i$ on the complex p-plane, which corresponds to the zero mode.

For convenience, we also write down the y-derivative of $h_{(p)}$,

$$
\frac{1}{\ell}\partial_y h_{(p)}(y_0) = \frac{P^{ip}_{1/2}(z_0)}{(1 - \bar\alpha)P^{ip}_{1/2}(z_0) + \bar\alpha(-ip + 3/2)(H\ell)^2 \cosh(y_0/\ell)P^{ip}_{3/2}(z_0)}
$$
$$
\times \frac{\kappa_5^2}{\ell} S_{(p,2)} .
\tag{5.38}
$$

Then, Eqs (5.30), (5.33) and (5.38) constitute the effective gravitational equations on the brane that form a closed set of integro-differential equations.

5.2.5   Source-free tachyonic mode

Now, we discuss the source-free tachyonic mode on the brane [136]. This mode corresponds to the homogeneous solution of Eq. (5.30), so does not couple to the matter perturbations on the brane.

On the complex p-plane, the solution corresponds to the pole at $p = (5/2)i$. Thus, the solution is given by

$$
\varphi = \varphi_{(5i/2)} Y^{(5i/2,0)} .
\tag{5.39}
$$

For this mode, the junction condition (5.28) tells us that it is associated with a non-vanishing $h_{\mu\nu}$. The solution in the bulk is given by [136]

$$
h_{\mu\nu} = \phi(y)\mathcal{L}_{\mu\nu}\varphi, \qquad \mathcal{L}_{\mu\nu} = D_\mu D_\nu + H^2 \gamma_{\mu\nu}.
\tag{5.40}
$$

This satisfies the transverse-traceless condition and

$$
\left(\Box_4 - 4H^2\right)h_{\mu\nu} = 0.
\tag{5.41}
$$

Thus, this mode falls within the mass gap between $m = 0$ and $3/2$, with the mass $mH = \sqrt{2}H$.

Let us first analyze the behavior of the function $\phi(y)$. It should satisfy Eq. (5.14), which becomes

$$
\left[\frac{1}{\sinh^4(y/\ell)}\partial_y\left(\sinh^4(y/\ell)\partial_y\right) + \frac{2}{\ell^2 \sinh^2(y/\ell)}\right]\phi(y) = 0.
\tag{5.42}
$$

The general solution is given by

$$\phi(y) = c_1\phi_1(y) + c_2\phi_2(y) ;$$
$$\phi_1(y) = \coth(y/\ell), \quad \phi_2(y) = 1 + \coth^2(y/\ell), \quad (5.43)$$

where the coefficients c_1 and c_2 are related through the junction condition (5.28) as

$$1 - \frac{1}{2}H^2 c_1 - H^2 \coth(y_0/\ell)\frac{1 + \bar{\alpha}\coth^2(y_0/\ell)}{1 + \beta} c_2 = 0. \quad (5.44)$$

As readily seen, this mode diverges badly as $y \to 0$. Therefore, the regularity condition at $y = 0$ will eliminate this mode. Nevertheless, since its effect on the brane seems non-trivial, it is interesting to see the physical meaning of it.

We note that ϕ_1 is a gauge mode. This can be checked by calculating the projected Weyl tensor $E_{\mu\nu} := {}^{(5)}C_{n\mu n\nu}$ [91, 59] which is gauge-invariant. We find that only the coefficient c_2 survives:

$$E^{\mu}_{\nu}(y, x^\alpha) = -\frac{c_2}{\ell^2 \sinh^4(y/\ell)}\mathcal{L}^{\mu}_{\nu}\,\varphi(x^\alpha). \quad (5.45)$$

This means that the junction condition (5.44) does not fix the physical amplitude c_2. It just fixes the gauge amplitude c_1.

To understand the physical meaning of this mode, it is useful to analyze the temporal behavior the projected Weyl tensor. For simplicity, let us consider a spatially homogeneous solution for φ. Choosing the spatially closed chart for the de Sitter brane, for which the scale factor is given by $a(t) = H^{-1}\cosh(Ht)$, we find

$$\varphi = C_1\frac{P^{5/2}_{1/2}(\tanh(Ht))}{\cosh^{3/2}(Ht)} + C_2\frac{P^{-5/2}_{1/2}(\tanh(Ht))}{\cosh^{3/2}(Ht)} \underset{t\to\infty}{\sim} \tilde{C}_1\,e^{Ht} + \tilde{C}_2\,e^{-4Ht} ,(5.46)$$

where $\tilde{C}_1$ and $\tilde{C}_2$ differ from C_1 and C_2, respectively, by unimportant numerical factors. We see that the solution associated with C_1 is the one that shows instability. If we insert this solution to Eq. (5.45), however, this unstable solution disappears. In fact, we obtain

$$E^t_t \simeq \frac{15H^2\tilde{C}_2\,c_2}{\ell^2 \sinh^4(y/\ell)e^{4Ht}} \simeq \frac{15(H\ell)^2\tilde{C}_2\,c_2}{16(H\ell)^4 \sinh^4(y/\ell)a^4(t)}. \quad (5.47)$$

We note that E^t_t on the brane decays as $1/a^4(t)$. This is exactly what one expects for the behavior of the so-called dark radiation. We also note that, although $E_{\mu\nu}$ does not vanish for spatially inhomogeneous modes, they decay as $1/a^3(t)$ [136], giving no instability to the brane.

In the Einstein case, the dark radiation term appears if there exists a black hole in the bulk. This is also true in the EGB case. There also exists a

spherically symmetric black hole solution in the EGB theory [69, 70, 71, 72, 73, 74]. The metric is given by

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_{(3)}^2; \quad f(R) = 1 + \frac{R^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16\alpha\mu}{3R^4} + \frac{4}{3}\alpha\Lambda_5}\right) \tag{5.48}$$

where $\mu = \kappa_5^2 M/(2\pi^2)$ and M is the mass of the black hole. For this solution, the projected Weyl tensor is given by

$$E^t_{\ t} = \frac{\mu}{R^4}\left(1 + \frac{4}{3}\alpha\Lambda_5 + \frac{16\alpha\mu}{3R^4}\right)^{-3/2}\left(1 + \frac{4}{3}\alpha\Lambda_5 + \frac{16\alpha\mu}{9R^4}\right) \simeq \frac{\mu}{R^4}\left(1 + \frac{4}{3}\alpha\Lambda_5\right)^{-1/2} \tag{5.49}$$

for $R \gg (\alpha\mu)^{\frac{1}{4}}$ [59]. Comparing Eq. (5.47) with Eq. (5.49), with the identification $R = \ell\sinh(y/\ell)\cosh(Ht)$, we find

$$c_2\tilde{C}_2 \simeq \frac{16\mu}{15(H\ell)^2}\left(1 + \frac{4}{3}\alpha\Lambda_5\right)^{-1/2}. \tag{5.50}$$

Thus the solution that decays exponentially in time corresponds to adding a small black hole in the bulk [138].

In the two-brane system, the mode discussed here corresponds to the radion, which describes the relative displacement of the branes [111, 136]. As the case of the Einstein gravity, the radion mode is truly tachyonic. However, for the EGB theory, there is a tachyonic bound state mode other than the radionic instability, as in the limit of the Minkowski brane [139], as discussed explicitly in Appendix E. 2. This renders the two-brane system physically unrealistic in the EGB theory without any prescriptions.

5.3   Linearized gravity on a Minkowski brane and its limiting cases

Before investigating limiting cases for a dS brane, we review the results for a Minkowski brane [134, 140]. In the next subsection, we compare these results with those for a dS brane.

5.3.1   Effective equations on the brane

As the same manner in the case of a dS brane, in the RS gauge as before, the perturbed bulk metric in the bulk is written

$$ds^2 = dy^2 + b^2(y)\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu, \quad b(y) = e^{-|y|/\ell}, \tag{5.51}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The brane locates at $y = 0$ in the background. The background part of the Einstein Gauss-Bonnet equation (5.3)

gives the relation of the AdS radius to the bulk cosmological constant, Eq. (5.10). The perturbative part of Eq. (5.3) gives

$$\left(1 - \bar{\alpha}\right)\left(\partial_y^2 - 4\frac{1}{\ell}\partial_y + e^{2y/\ell}\Box_4\right)h_{\mu\nu} = 0\,. \tag{5.52}$$

Again, we consider the case $\bar{\alpha} \neq 1$. The location of the brane is perturbed to be at $y = -\ell\varphi$. Induced metric on the brane is given by

$$ds^2\Big|_{(4)} = \left(\eta_{\mu\nu} + \tilde{h}_{\mu\nu}\right)dx^\mu dx^\nu\,, \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\varphi\eta_{\mu\nu}\,. \tag{5.53}$$

The solution for $h_{\mu\nu}$ on the brane which satisfies the junction condition is given by

$$h_{\mu\nu}\Big|_{y=0} = -\frac{\kappa_5^2}{\ell}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{\ell^2 H_2^{(1)}(q\ell)}{(1 - \bar{\alpha})q\ell H_1^{(1)}(q\ell) + \bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}$$
$$\times\left[S_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)S(p)\right]\,, \tag{5.54}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind and $q^2 = -p^2$. The equation that determines the brane bending is

$$\Box_4\,\varphi = -\frac{\kappa_5^2}{6\ell}\frac{1}{1 + \bar{\alpha}}S. \tag{5.55}$$

As for the dS brane case, the brane bending mode is not dynamical.

The perturbed four-dimensional Einstein tensor is expressed as

$$\delta G_{\mu\nu}[\tilde{h}] = -\frac{1}{2}\Box_4\,h_{\mu\nu} + 2\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi. \tag{5.56}$$

Inserting Eq. (5.54) into Eq. (5.56), we obtain the effective equation on the brane, which reads

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{2\bar{\alpha}\ell}S_{\mu\nu} - \frac{1 - \bar{\alpha}}{\bar{\alpha}}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi + \frac{1 - \bar{\alpha}}{2\ell\bar{\alpha}}h_{\mu\nu,y}\,,$$
$$\frac{1}{\ell}h_{\mu\nu,y}\Big|_{y=0} = -\frac{\kappa_5^2}{\ell}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{H_1^{(1)}(q\ell)}{(1 - \bar{\alpha})H_1^{(1)}(q\ell) + \bar{\alpha}q\ell H_2^{(1)}(q\ell)}$$
$$\times\left[S_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)S(p)\right]\,. \tag{5.57}$$

5.3.2 Short distance limit

In the short distance limit $q\ell \gg 1$, Eq. (5.57) becomes

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{2\bar{\alpha}\ell}S_{\mu\nu} - \left(\frac{1 - \bar{\alpha}}{\bar{\alpha}}\right)\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi. \tag{5.58}$$

Comparing Eqs. (5.55) and (5.58) with the linearized Brans-Dicke gravity,

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{8\pi G_4}{\Phi_0} S_{\mu\nu} + \frac{1}{\Phi_0} \left(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4 \right) \delta\Phi , \quad \Box_4 \, \delta\Phi = \frac{8\pi G_4}{3 + 2\omega} S, \quad (5.59)$$

we find the correspondences,

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\bar{\alpha}\ell}, \quad \frac{\delta\Phi}{\Phi_0} = -\frac{1 - \bar{\alpha}}{\bar{\alpha}}\varphi, \quad \omega = \frac{3\bar{\alpha}}{1 - \bar{\alpha}}. \quad (5.60)$$

The brane bending scalar $\varphi(x^\mu)$ turns to be dynamical. The reason is explained as follows: In neglecting the KK contribution to the effective gravitational equation, the pole at $p = 0$ in Eq. (5.57) seems to dissapear in the theory. This pole contribution just plays the role of the scalar degree of freedom in the effective gravitational theory on the brane. Thus, the scalar degree of freedom is higher-dimensional and comes from KK modes.

The corrections are rewritten as

$$\left(\delta G_{\mu\nu}[\tilde{h}] \right)_{\text{corr}} = -\frac{\kappa_5^2}{2\bar{\alpha}\ell} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{(1 - \bar{\alpha})q\ell H_1^{(1)}(q\ell)}{(1 - \bar{\alpha})q\ell H_1^{(1)}(q\ell) + \bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}$$

$$\times \left[S_{\mu\nu} - \frac{1}{3} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) S \right] . \quad (5.61)$$

5.3.3   Large distance limit

In the large distance limit $q\ell \ll 1$, Eq. (5.57) becomes

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{\ell} \frac{1}{1 + \bar{\alpha}} S_{\mu\nu}. \quad (5.62)$$

Thus we obtain the Einstein gravity with

$$8\pi G_4 = \frac{\kappa_5^2}{\ell} \frac{1}{1 + \bar{\alpha}}. \quad (5.63)$$

For $\bar{\alpha} \to 0$, this reduces to the result in the Einstein theory.

5.4   Linearized gravity on a de Sitter brane in limiting cases

In this subsection, we discuss the effective gravity on the brane in various limiting cases. We find the effective gravity reduces to four-dimensional theories in all the limiting cases.

5.4.1 High energy brane: $H\ell \gg 1$

For a high energy brane, i.e., $H\ell \gg 1$ limit, we have $\tanh(y_0/\ell) \simeq 1/(H\ell)$ and $\beta \simeq 2(H\ell)^2$. We assume that matter perturbations on the brane are dominated by the modes $p \sim O(1)$. Namely, we consider the case $H\ell \gg p$. Then, from Eq. (5.36) and Eq. (5.38), we find that the second and the third terms in the right-hand-side of Eq. (5.33) are suppressed by the small factor $1/(H\ell)^2$ relative to the first term,

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} = \frac{\kappa_5^2 \tanh(y_0/\ell)}{2\ell\bar{\alpha}} \left(S_{\mu\nu} + O\big((H\ell)^{-2}\big) \right). \quad (5.64)$$

Thus, we obtain Einstein gravity with the cosmological constant $3H^2$, with the gravitational constant G_4 given by

$$8\pi G_4 = \frac{\kappa_5^2}{2\ell\,\bar{\alpha}} \tanh(y_0/\ell) \approx \frac{\kappa_5^2}{2(H\ell)\bar{\alpha}\ell}. \quad (5.65)$$

The terms we have neglected give the low energy non-local corrections:

$$\begin{aligned}
\left(\delta G_{\mu\nu}[\tilde{h}] \right)_{\mathrm{corr},H\ell} &= -\frac{\kappa_5^2(1-\bar{\alpha})}{2\ell\bar{\alpha}} \tanh(y_0/\ell) \\
&\times \int_{-\infty}^{\infty} dp \left\{ Y_{\mu\nu}^{(p,2)} S_{(p,2)} \right. \\
&\times \left[\frac{P_{1/2}^{ip}(z_0)}{(1-\bar{\alpha})P_{1/2}^{ip}(z_0) + \bar{\alpha}(-ip+3/2)(H\ell)^2 \cosh(y_0/\ell)P_{3/2}^{ip}(z_0)} \right] \\
&\left. + Y_{\mu\nu}^{(p,0)} S_{(p,0)} \frac{1}{(1+\beta)} \right\}. \quad (5.66)
\end{aligned}$$

5.4.2 Short and large distance limits

In order to discuss short and large distance limits, it is convenient to start from the expression (5.31) for the perturbed Einstein tensor, and Eq. (5.36) which relates the brane bending scalar φ to the scalar part of the energy momentum tensor $S_{\mu\nu}^{(0)}$. Let us recapitulate these expressions:

$$\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} &= 2\coth(y_0/\ell)\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu} \right)\varphi \\
&\quad - \frac{1}{2}\left(\Box_4 - 2H^2 \right)h_{\mu\nu}, \quad (5.67)
\end{aligned}$$

$$\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu} \right)\varphi = \frac{\kappa_5^2}{2(1+\beta)\ell} S_{\mu\nu}^{(0)}. \quad (5.68)$$

1. Short distance limit: $r \ll \min\{\ell, H^{-1}\}$

For the short distance limit $p \to \infty$, using Eq. (5.37), we find

$$
-\frac{1}{2}\left(\Box_4 - 2H^2\right)h_{\mu\nu}
$$

$$
= \frac{\kappa_5^2}{2\ell}\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)}\, S_{(p,2)}\frac{(H\ell)\,(-ip+3/2)\,P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}{\left(1-\bar\alpha\right)+\bar\alpha(H\ell)^2\cosh(y_0/\ell)\left(-ip+3/2\right)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}
$$

$$
\xrightarrow[p\to\infty]{} \frac{\kappa_5^2}{2\ell\bar\alpha}\tanh(y_0/\ell)\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)} S_{(p,2)}\,. \tag{5.69}
$$

Also, using Eq. (5.68), we manipulate as

$$
2\coth(y_0/\ell)\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\right)\varphi
$$

$$
= \frac{\kappa_5^2}{2\ell\bar\alpha}\tanh(y_0/\ell)\int_{-\infty}^{\infty} dp\, S_{(p,0)}Y_{\mu\nu}^{(p,0)}
$$

$$
-\frac{1-\bar\alpha}{\bar\alpha}\tanh(y_0/\ell)\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\right)\varphi\,, \tag{5.70}
$$

where we have used an identity,

$$
2\coth(y_0/\ell) = 2\coth(y_0/\ell) - \frac{1+\beta}{\bar\alpha}\tanh(y_0/\ell) + \frac{1+\beta}{\bar\alpha}\tanh(y_0/\ell)
$$

$$
= -\frac{1-\bar\alpha}{\bar\alpha}\tanh(y_0/\ell) + \frac{1+\beta}{\bar\alpha}\tanh(y_0/\ell)\,, \tag{5.71}
$$

which follows from the definition of the parameter β, Eq. (5.29).

Substituting Eqs. (5.69) and (5.70) in Eq. (5.67), the linearized gravity on the brane at short distances becomes

$$
\delta G_{\mu\nu}[\tilde h] + 3H^2\tilde h_{\mu\nu} = \frac{\kappa_5^2}{2\ell\bar\alpha}\tanh(y_0/\ell)\, S_{\mu\nu}
$$

$$
-\frac{(1-\bar\alpha)}{\bar\alpha}\tanh(y_0/\ell)\left(D_\mu D_\nu - \Box_4\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\right)\varphi\,, \tag{5.72}
$$

with

$$
\left(\Box_4 + 4H^2\right)\varphi = -\frac{\kappa_5^2}{6(1+\beta)\ell}\, S\,. \tag{5.73}
$$

This is a scalar-tensor type theory.

As in the case of a Minkowski brane, the scalar field φ which describes the brane bending degree of freedom turns to be dynamical. As we have seen in the previous subsection, there is no intrinsically dynamical mode associated with the brane bending. Therefore, this emergence of a dynamical degree of freedom is due to an accumulative effect of the whole Kaluza-Klein modes, like a collective mode. Furthermore, because of the tachyonic mass, the system appears to be unstable. However, this is not the case. Since we have

taken the limit $p \to \infty$, all the perturbations have energy much larger than H, and the tachyonic mass-squared $-4H^2$ is completely negligible. In other words, the spacetime appears to be flat at sufficiently short distance scales.

We can rewrite Eq. (5.72) in the form,

$$\delta G_{\mu\nu}[\tilde{h}] + \Lambda_4 \tilde{h}_{\mu\nu} = \frac{1}{\Phi_0}\Big(D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu}\Big)\delta\Phi + \frac{8\pi G_4}{\Phi_0}S_{\mu\nu},$$

$$\Big(\Box_4 + 4H^2\Big)\delta\Phi = \frac{8\pi G_4}{3 + 2\omega}S, \tag{5.74}$$

with the identifications,

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell), \qquad \frac{\delta\Phi}{\Phi_0} = -\frac{1-\bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell)\varphi,$$

$$\omega = \frac{3\bar{\alpha}}{1-\bar{\alpha}}\coth^2(y_0/\ell), \qquad \Lambda_4 = 3H^2. \tag{5.75}$$

Neglecting the tachyonic mass of $\delta\Phi$, as justified above, this is the linearized Brans-Dicke gravity with a cosmological constant. For $H\ell \ll 1$, we have $\tanh(y_0/\ell) \simeq \coth(y_0/\ell) \simeq 1$. Then

$$\frac{8\pi G_4}{\Phi_0} \simeq \frac{\kappa_5^2}{2\bar{\alpha}\ell}, \qquad \frac{\delta\Phi}{\Phi_0} \simeq -\frac{1-\bar{\alpha}}{\bar{\alpha}}\varphi, \qquad \omega \simeq \frac{3\bar{\alpha}}{1-\bar{\alpha}}. \tag{5.76}$$

This is in agreement with the Minkowski brane case investigated recently [140].

The corrections are written as

$$\Big(\delta G_{\mu\nu}[\tilde{h}]\Big)_{\text{corr},p\gg1}$$

$$= -\frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\int_{-\infty}^{\infty}dp\, Y_{\mu\nu}^{(p,2)}S_{(p,2)}$$

$$\times\left[\frac{(1-\bar{\alpha})P_{1/2}^{ip}(z_0)}{(1-\bar{\alpha})P_{1/2}^{ip}(z_0) + \bar{\alpha}(H\ell)^2\big(-ip+3/2\big)\cosh(y_0/\ell)P_{3/2}^{ip}(z_0)}\right]. \tag{5.77}$$

2. Large distance limit: $r \gg \max\{\ell, H^{-1}\}$

For the limit $p \to 0$, using Eq. (5.37), we have

$$-\frac{1}{2}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}dp\, Y_{\mu\nu}^{(p,2)}S_{(p,2)}\frac{\kappa_5^2\ell\sinh(y_0/\ell)H^2(-ip+3/2)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}{1-\bar{\alpha}+\bar{\alpha}(H\ell)^2\cosh(y_0/\ell)\big(-ip+3/2\big)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}$$

$$\simeq \frac{3\kappa_5^2}{4\ell}\frac{(H\ell)P_{3/2}(z_0)/P_{1/2}(z_0)}{(1-\bar{\alpha}) + (3/2)(H\ell)\coth(y_0/\ell)\bar{\alpha}P_{3/2}(z_0)/P_{1/2}(z_0)}$$

$$\times \int_{-\infty}^{\infty}dp\, S_{(p,2)}Y_{\mu\nu}^{(p,2)}. \tag{5.78}$$

As for the term involving φ, we pull out the part that takes the same form as the above equation. Using Eq. (5.35), we find

$$2 \coth(y_0/\ell)\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu} \right)\varphi$$

$$= \frac{3\kappa_5^2}{4\ell} \frac{(H\ell)P_{3/2}(z_0)/P_{1/2}(z_0)}{(1-\bar{\alpha}) + (3/2)(H\ell)\coth(y_0/\ell)\bar{\alpha}P_{3/2}(z_0)/P_{1/2}(z_0)}$$

$$\times \int_{-\infty}^{\infty} dp\, S_{(p,0)}Y_{\mu\nu}^{(p,0)}$$

$$- \frac{(H\ell)(1-\bar{\alpha})P_{-1/2}(z_0)}{2(1+\beta)P_{1/2}(z_0) - (H\ell)\coth(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0)}$$

$$\times\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu} \right)\varphi\,, \tag{5.79}$$

where we have used the recursion relation,

$$\frac{3}{2}P_{3/2}(z_0) = 2z_0 P_{1/2}(z_0) - \frac{1}{2}P_{-1/2}(z_0)\,. \tag{5.80}$$

Thus, the effective gravitational equation is expressed as

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = \frac{\kappa_5^2}{\ell}F_T\, S_{\mu\nu} - F_S\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu} \right)\tilde{\varphi}\,,$$

$$\left(\Box_4 + 4H^2 \right)\tilde{\varphi} = -\frac{\kappa_5^2}{\ell}S\,, \tag{5.81}$$

where we have rescaled φ to $\tilde{\varphi} = 6(1+\beta)\varphi$, and F_T and F_S are constants that represent the tensor and scalar coupling strengths, respectively, given by

$$F_T = \frac{(H\ell)\left(4\cosh(y_0/\ell)P_{1/2}(z_0) - P_{-1/2}(z_0) \right)}{2\left(2(1+\beta)P_{1/2}(z_0) - (H\ell)^2\cosh(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0) \right)}\,,$$

$$F_S = \frac{(H\ell)(1-\bar{\alpha})P_{-1/2}(z_0)}{6(1+\beta)\left(2(1+\beta)P_{1/2}(z_0) - (H\ell)^2\cosh(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0) \right)} \tag{5.82}$$

In the intermediate range of $H\ell$, i.e., when $H\ell = O(1)$, then F_T and F_S are comparable and we obtain a Brans-Dicke type theory given by Eq. (5.74) with the identifications,

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{\ell}F_T\,, \quad \frac{\delta\Phi}{\Phi_0} = -F_S\tilde{\varphi}\,, \quad \Lambda_4 = 3H^2\,, \tag{5.83}$$

$$\omega = \frac{F_T - 3F_S}{2F_S} = \frac{6(1+\beta)\cosh(y_0/\ell)P_{1/2}(z_0) - 3\left(1 + (H\ell)^2\bar{\alpha}\right)P_{-1/2}(z_0)}{(1-\bar{\alpha})P_{-1/2}(z_0)}\,.$$

A potential problem in this case is that the tachyonic mass of the scalar field seems to make the system unstable. However, as discussed in subsection

5.2.4, the tachyonic pole is not excited by the matter source. Further, as discussed in subsection 5.2.5, the source-free tachyonic mode do not cause an instability either.

For $H\ell \ll 1$, $\omega \gg 1$ and the scalar field decouples to yield

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} = \frac{\kappa_5^2}{\ell} \frac{\coth(y_0/\ell)}{1+\beta} S_{\mu\nu}. \tag{5.84}$$

Thus we obtain the Einstein gravity with

$$8\pi G_4 = \frac{\kappa_5^2}{\ell} \frac{\coth(y_0/\ell)}{1+\beta}. \tag{5.85}$$

In the limit $H\ell \to 0$,

$$8\pi G_4 \simeq \frac{\kappa_5^2}{\ell} \frac{1}{1+\bar{\alpha}}. \tag{5.86}$$

This is the result for the Minkowski brane.

· In the limit $H\ell \gg 1$, $\omega \gg 1$ and we recover the four-dimensional Einstein gravity on the brane with

$$8\pi G_4 = \frac{\kappa_5^2}{2(H\ell)\bar{\alpha}\ell}. \tag{5.87}$$

Note that this is just a special case of the high energy brane case.

Thus we conclude that despite the presence of the tachyonic mass, the system is stable and well-behaved for all ranges of $H\ell$.

5.5   Summary of this Chapter

We have investigated the linear perturbations of a de Sitter brane in an Ant-de Sitter bulk in the five-dimensional Einstein Gauss-Bonnet (EGB) theory. We have derived the effective theory on the brane which is described by a set of integro-differential equations.

Then, we have investigated the behavior of the theory in various limiting cases. In contrast to the case of braneworld in the five-dimensional Einstein theory, in which both the short distance and high energy brane limits exhibit five-dimensional behavior, we have found that gravity on the brane is effectively four-dimensional for all the limiting cases.

For a high energy brane, i.e., in the limit $H\ell \gg 1$, the Einstein gravity is recovered, provided that the length scale of fluctuations is of order H^{-1}. It is found that the low energy corrections are suppressed by the factor $O((H\ell)^{-2})$.

In the short distance limit $r \ll \min\{\ell, H^{-1}\}$, the scalar field that describes brane bending becomes dynamical, and we obtain the Brans-Dicke gravity. This is consistent with the case of the Minkowski brane. A slight complication

is that this brane-bending scalar field is tachyonic, with mass-squared $-4H^2$. Therefore, if it becomes dynamical, one would naively expect the theory to become unstable. However, since the energy scale of fluctuations are much larger than H, the fluctuations actually do not see this tachyonic mass, hence there is no instability.

In the large distance limit $r \gg \max\{\ell, H^{-1}\}$, the Einstein gravity is obtained in both limits $H\ell \ll 1$ and $H\ell \gg 1$, while a Brans-Dicke type theory is obtained for $H\ell = O(1)$. Although the scalar field of this Brans-Dicke gravity is tachyonic with mass-squared given by $-4H^2$, we have shown that this mode is not excited by the matter source, hence does not lead to an instability of the system.

In the limit $H\ell \to 0$, the previous results for the Minkowski brane have been recovered, that is, the Brans-Dicke gravity at short distances and the Einstein gravity at large distances.

In all the cases, the effective four-dimensional gravitational constant depends non-trivially on the values of $H\ell$ and $\bar{\alpha}$, where $\bar{\alpha}$ is the non-dimensional coupling constant for the Gauss-Bonnet term. This indicates the time variation of the gravitational constant in the course of the cosmological evolution of a brane in the EGB theory. It will be interesting to investigate in more details the cosmological implications of the braneworld in the EGB theory.

6

Summary and discussion

6.1 Summary of this thesis

We have investigated the dynamics and effects of Kaluza-Klein (KK) modes in the context of the second Randall-Sundrum (RS) braneworld. The RS model realizes four-dimensional general relativity on the brane, where we are living, by warping of the extra-dimension, which is called the bulk, not by the conventional compactifications as in KK theory. KK modes correspond to waves in the bulk and are observed as an infinite number of massive modes on the brane. These modes usually give corrections to four-dimensional gravity. In this sense, their contribution should not be so large in order for the braneworld cosmology to be viable. On the other hand, as the self-accelerating solution in the DGP model, which is briefly introduced in Chapter 1, KK modes may give new possibilities for difficulties in four-dimensional cosmology.

For mathematical set-up, we first gave effectve gravitational equations on the brane by the covariant geometrical projection method. We also generalize several local conservation laws, which are well known in the case of four-dimensional spherically symmetric spacetimes, to the case of higher dimensions, in order to discuss the dynamics of the brane with a cosmological symmetry, i.e., homogeneity and isotropy, in the bulk.

Then, we discussed the backreaction of KK graviton modes which are *naturally* produced at the early stage of the brane universe. They are just the bulk metric perturbations and their existence is rather generic in braneworld, independently of the detailed assumptions. KK graviton modes are considered to be produced mainly by two different mechanisms: The first possibility is that they are produced by high energy particle interactions on the brane. The second one is that they are produced quantum mechanically in the whole bulk during brane inflation. First, we consider KK gravitons produced by particle interactions on the brane. We treat the emission of KK gravitons as ingoing null dust flux. Then the metric of the bulk spacetime is given by an ingoing Vaidya-type solution. We discussed the geometry in the bulk and obtained a closed set of equations which represent the trajectory of the

radiating brane in the bulk. We also found that in the case that the convn-
tional three-space is closed (i.e., a three-sphere) and the flux of KK gravitons
increases eventually, then a null, strong and visible naked singularity can be
formed in the bulk.

Next, we discussed the backreaction of gravitons of a KK mode produced
quantum mechanically in the whole bulk. In order to discuss the backreac-
tion of these KK gravitons on the brane correctly, we derived their effective
stress-energy tensor for gravitons of a KK modes by computing the curva-
ture tensors up to the second order of perturbations and averaging them,
by taking the existence of the infinitely thin brane into consideration. The
averaging scheme is discussed in Appendix A. 2 in details. Essentially, the
averaging is done only for the derivatives in the direction of the usual four
dimensions (namely parallel to the brane) and for the derivatives in the direc-
tion of the extra-dimension, we used the boundary conditions and equation
of motion in the bulk, in order to eliminate them. Then, by the geometrical
projection, we derived the effective stress tensor of gravitons of a KK mode
on the brane. As a result, we found that a KK graviton mode behaves as
cosmic dust, but the energy density becomes negative. The negativity of the
effective energy density results from the pressure of the KK modes onto the
brane and the energy density in the bulk is still finite.

In reality, however, what we observe is the sum of all KK modes. We
also need to determine the amplitude and the amount of backreaction of all
KK modes quantum mechanically. As is well-known, however, the sum of
all the KK modes suffers from divergences as one approaches the brane from
the bulk, even after a conventional UV regularization. Then, we proposed a
new regularization scheme for this type of divergence by taking a finite brane
thickness into account. As a demonstration, we considered a thick de Sitter
brane model which is supported by a bulk scalar field. As a probe, we con-
sider another, quantized, massless scalar field, which is coupled to the bulk
scalar curvature. Especially, for the case of the minimal coupling, the evolu-
tion of the text scalar field in the bulk is the same as that of KK gravitons
and thus we may obtain physical insight about the case of KK gravitons. We
have computed the amplitude of the quantum fluctuations and the amount
backreaction by employing zeta-function regularization. We showed that a
finite brane thickness can regularize these on the brane. Though we have
investigated only one explicit model, the behavior of the quantum fluctua-
tions at the center of the wall should be independent of the choice of the
model because the behavior of a supporting scalar field is generic, indepen-
dent of the global feature of the model. Then, we compared the amount of
the quantum backreaction for the minimally coupled case with that of the
background stress-energy tensor and found that the former can be naturally
reduced to below the latter. From these discussions, we obtained a theoreti-
cal bound on the brane thickness parameter and showed that this bound is
realized without contradicting the framework of the model. This bound can
be also intepretted as that on the energy scale of the brane expansion.

In the previous Chapters, we have discussed brane models in the five-dimensional Einstein (and Einstein-scalar) theory. However, in five-dimensions the gravitational theory with curvature corrections of quadratic order is more natural. Especially, among them, the Einstein Gauss-Bonnet (EGB) theory uniquely gives second order equations of motion as in the Einstein theory. In four dimensions, the GB term is just a topological quantity, but in higher dimensions it becomes dynamical. In order to find cosmological implications of braneworld in the EGB theory, we investigated the linearized effective gravity on a dS brane in an AdS bulk in the five-dimensional EGB theory. We solved the tensor metric perturbations in the bulk and then derived the effective theory on the brane which is described by a closed set of integro-differential equations.

In contrast to the case of the five-dimensional Einstein theory, we have found that gravity on the brane is effectively four-dimensional for all of distance scales, from short distances to large distances. In the short distance limit, the scalar field that describes brane bending becomes non-trivial, and we obtain the scalar-tensor (Brans-Dicke) type gravity. In the large distance limit, the Einstein gravity is obtained in both low energy and high energy limits, while a Brans-Dicke type theory is obtained for intermediate energy scales. On high energy expanding branes as well as on low energy ones, four-dimensional Einstein gravity is obtained.

6.2 Related issues and future works

There are several remaining issues which are related to our work. One of these issues is the quantification of the backreaction of the KK gravitons. As we discussed, KK gravitons are considered to be produced quantum mechanically and they may affect brane cosmology non-trivially as discussed in this thesis. As is mentioned before, because they are corresponding to the bulk metric perturbations, the existence of them is generic in RS-type braneworld, independent of the detailed assumptions. To determine the amount of backreaction of KK gravitons is important work from the observational point of view. The main problem is the divergence of the sum of all KK modes as one approach the brane from the bulk. In Chapter 4, we have proposed a new regularization scheme for such a divergence by taking a finite brane thickness into account. We expect that this regularization scheme can be applied also to the case of the KK gravitons because the evolution of the KK gravitons are quite similar as that of a massless, minimally coupled scalar KK modes, apart from the only difference of the degeneracy on a sphere in Euclideanized space. Thus, for a more realistic brane model, we can easily quantify the backreaction of the KK modes. The explicit determination of the backreaction is left for future work.

Another issue is about KK modes in a higher-codimensional braneworld. In this thesis, we have assumed that there is one non-trivial extra-dimension.

However, for instance, string theory, which brane cosmology is based upon, predicts that there are ten or eleven spacetime dimensions. The other dimensions are usually considered to be compactified. But the idea of the RS braneworld tells us that there may be other infinitely extended extra-dimensions, as long as they are warped.

In embedding a singular, self-gravitating brane into a higher-codimensional spacetime, what we should note is that the gravitational field around a brane becomes more singular in the bulk. This fact is understood intuitively from the following considerations: Assuming that there are n-codimensions, the gravitational potential in extra-dimension around a brane is scaled as r^{n-2}, where r is radial distance from the brane. In the case $n = 2$, the potential exhibits a logarithmic dependence on r and there is a conical singularity at the brane position. In this case the junction condition is marginally tractable and there have been studies about codimension two brane solutions, see e.g., [141, 142, 143, 144, 145, 146, 147, 148]. In the cases of codimensions more than three, i.e., $n \geq 3$, the gravitational potential is proportional to inverse powers of the distance r and singular at the brane position. This fact implies that the brane becomes a black hole as long as it is assumed to be infinitesimally thin. Thus, taking a finite brane thickness into consideration may be essential to realize braneworlds where we can live. Constructing explicit solutions is an intriguing future issue.

Investigations of quantum effects in higher-codimensional braneworld also are not considered even for the case of codimension two. Divergences of the sum of the KK modes as one approaches the brane from the bulk are also expected. It should be checked whether our regularization scheme by a finite brane thickness works or not. Stability of higher co-dimensional braneworlds should also be analyzed from classical and quantum mechanical point of views. Anyway, a finite brane thickness should become an essential key to extend (self-gravitating) brane models into higher-codimensions and investigate the behavior of the KK modes in these models.

Other than the brane thickness, in higher dimensinals one may add higher curvature terms into the bulk gravitational theory, like the Gauss-Bonnet term discussed in Chapter 5. These higher curvature terms may be also essential in these higher-codimensional cases.

We hope to report the results of investigations about these issues in our future publications.

Appendix A

Curvature tensors

In this Appendix, we list components of curvature tensors and quantities related to them.

A.1 Curvature tensors and locally conserved quantities

We give useful formulas in an $(n + 2)$-dimensional spacetime with constant curvature n-space, and generalize the expression for the local mass and the charge associated with Weyl tensor.

We consider the metric in the double-null form,

$$ds^2 = \frac{4r_{,u}r_{,v}}{\Omega}dudv + r(u,v)^2 d\Sigma^2_{(K,n)}, \tag{A.1}$$

where $K = +1, 0$, or -1, corresponding to the sphere, flat space and hyperboloid, respectively. We denote the metric tensor of the constant curvature space as γ_{ij}. The explicit expressions for the geometrical quantities in this spacetime are as follows.

- **Christoffel symbol**

$$\overset{(n+2)}{\Gamma}{}^{u}_{uu} = \left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,u}, \quad \overset{(n+2)}{\Gamma}{}^{v}_{vv} = \left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,v},$$

$$\overset{(n+2)}{\Gamma}{}^{u}_{ij} = -\frac{r\,\Omega}{2r_{,u}}\gamma_{ij}, \quad \overset{(n+2)}{\Gamma}{}^{v}_{ij} = -\frac{r\,\Omega}{2r_{,v}}\gamma_{ij},$$

$$\overset{(n+2)}{\Gamma}{}^{i}_{uj} = \frac{r_{,u}}{r}\delta_{ij}, \quad \overset{(n+2)}{\Gamma}{}^{i}_{vj} = \frac{r_{,v}}{r}\delta_{ij}, \quad \overset{(n+2)}{\Gamma}{}^{i}_{jk} = \overset{(\Sigma)}{\Gamma}{}^{i}_{jk}. \tag{A.2}$$

- **Riemann tensor**

$$\overset{(n+2)}{R}{}^{u}{}_{uuv} = R^{v}_{vvu} = -\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv},$$

$$\overset{(n+2)}{R}{}^{u}{}_{iuj} = \left[-\frac{1}{2}r\left(\frac{\Omega}{r_{,u}}\right)_{,u} - \frac{r\,\Omega}{2r_{,u}}\left(\log\left|\frac{r_{,u}\,r_{,v}}{\Omega}\right|\right)_{,u}\right]\gamma_{ij},$$

$$\overset{(n+2)}{R}{}^{v}{}_{ivj} = \left[-\frac{1}{2}r\left(\frac{\Omega}{r_{,v}}\right)_{,v} - \frac{r\,\Omega}{2r_{,v}}\left(\log\left|\frac{r_{,u}\,r_{,v}}{\Omega}\right|\right)_{,v}\right]\gamma_{ij},$$

$$\overset{(n+2)}{R}{}^{i}{}_{uju} = \left[-\frac{r_{,uu}}{r} + \frac{r_{,u}}{r}\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,u}\right]\delta^{i}_{j},$$

$$\overset{(n+2)}{R}{}^{i}{}_{vjv} = \left[-\frac{r_{,vv}}{r} + \frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,v}\right]\delta^{i}_{j},$$

$$\overset{(n+2)}{R}{}^{i}{}_{vju} = \overset{(n+2)}{R}{}^{i}{}_{ujv} = -\frac{r_{,uv}}{r}\delta^{i}_{j},$$

$$\overset{(n+2)}{R}{}^{i}{}_{jkl} = \left(K - \Omega\right)\left(\delta^{i}_{k}\gamma_{jl} - \delta^{i}_{l}\gamma_{jk}\right). \tag{A.3}$$

- **Ricci tensor**

$$\overset{(n+2)}{R}{}_{uu} = n\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,v}}{\Omega}\right|\right)_{,u}, \qquad \overset{(n+2)}{R}{}_{vv} = n\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}}{\Omega}\right|\right)_{,v},$$

$$\overset{(n+2)}{R}{}_{uv} = -\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv} - n\frac{r_{,uv}}{r},$$

$$\overset{(n+2)}{R}{}_{ij} = \left[-\frac{rr_{,uv}}{r_{,u}r_{,v}}\Omega + 2\left(n-1\right)\left(K - \Omega\right)\right]\gamma_{ij}. \tag{A.4}$$

- **scalar curvature**

$$\overset{(n+2)}{R} = -\frac{\Omega}{r_{,u}r_{,v}}\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv}$$

$$-2n\frac{\Omega r_{,uv}}{rr_{,u}r_{,v}} + \frac{n(n-1)}{r^{2}}\left(K - \Omega\right). \tag{A.5}$$

- **Einstein tensor**

$$\overset{(n+2)}{G}{}_{uu} = \overset{(n+2)}{R}{}_{uu}, \qquad \overset{(n+2)}{G}{}_{vv} = \overset{(n+2)}{R}{}_{vv},$$

$$\overset{(n+2)}{G}{}_{uv} = n(n-1)\frac{r_{,u}r_{,v}}{r^{2}}\left(1 - \frac{K}{\Omega}\right) + n\frac{r_{,uv}}{r},$$

$$\overset{(n+2)}{G}{}_{ij} = \left\{\frac{r^{2}\Omega}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv} + 2(n-1)\frac{r_{,uv}}{r}\right]\right.$$

$$\left. -\frac{(n-2)(n-1)}{2}\left(K - \Omega\right)\right\}\gamma_{ij}. \tag{A.6}$$

• **Weyl tensor**

$$\overset{(n+2)}{C}{}_{uvu}{}^{u} = \frac{n-1}{n+1}\left(\log\left|\frac{r_{,u}r_{,v}}{\Omega}\right|\right)_{,uv} - \frac{r_{,uv}}{r} - \frac{r_{,u}r_{,v}}{r^2\Omega}\left(K - \Omega\right),$$

$$\overset{(n+2)}{C}{}_{iujv} = \frac{1}{n}r^2\gamma_{ij}C_{uvu}{}^{u}$$

$$\overset{(n+2)}{C}{}_{ijkl} = -\frac{2}{n(n-1)}r^4\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right)\overset{(n+2)}{C}{}_{uv}{}^{vu}. \tag{A.7}$$

From these formulas, we can show the existence of a conserved current in the same way as given in the text. Namely, with the timelike vector field ξ^a defined by Eq. (2.13), the currents $\tilde{S}^a = \xi^b\tilde{T}_b{}^a$ and $S^a = \xi^b T_b{}^a$ are separately conserved, and the corresponding local masses are given, respectively, by

$$\tilde{M} = r^{n-1}\left(K - \Omega\right), \tag{A.8}$$

and

$$M = \tilde{M} - \frac{2}{(n-1)(n-2)}\Lambda_{n+2}r^{n-1}. \tag{A.9}$$

The v and u derivatives of M are given by the energy-momentum tensor as

$$M_{,v} = \kappa_{n+2}^2\frac{2r^n}{n}\left(T^u{}_v r_{,u} - T^v{}_v r_{,v}\right),$$

$$M_{,u} = \kappa_{n+2}^2\frac{2r^n}{n}\left(T^v{}_u r_{,v} - T^v{}_v r_{,u}\right). \tag{A.10}$$

Let us now turn to the conserved current associated with the Weyl tensor. We start from the equation that results from the Bianchi identities [97],

$$\overset{(n+2)}{C}{}_{abcd}{}^{;d} = J_{abc}, \tag{A.11}$$

where

$$J_{abc} = \frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]} + \frac{1}{(n+1)}g_{c[b}T_{;a]}\right). \tag{A.12}$$

From this equation, we can show the existence of a locally conserved current Q^a given by

$$Q^a = r\,\ell_b n_c J^{bca}, \quad Q^a{}_{;a} = 0, \tag{A.13}$$

where ℓ^a and n^a are the null vectors defined in Eqs. (2.28). The non-zero components are explicitly written as

$$Q^u = -rJ^{vu}{}_v, \quad Q^v = -rJ^{vu}{}_u. \tag{A.14}$$

We then find the following relations,

$$\left(r^{n+1}\overset{(n+2)}{C}{}_{vu}{}^{vu}\right)_{,v} = r^{n+1}J^{v}{}_{vv}\,,\tag{A.15}$$

$$\left(r^{n+1}\overset{(n+2)}{C}{}_{vu}{}^{vu}\right)_{,u} = r^{n+1}J^{u}{}_{uu}\,.$$

These relations are generalization of Eqs. (2.30), and imply that the Weyl component $r^{n+1}\overset{(n+2)}{C}{}_{vu}{}^{vu}$ is the local charge associated with this conserved current.

Using the explicit form of $\overset{(n+2)}{C}{}_{vu}{}^{vu}$ in Eqs. (A.7) and the Einstein equations, we can relate the this charge to the local mass. We find

$$\begin{aligned}
r^{n+1}\overset{(n+2)}{C}{}_{vu}{}^{vu} &= \frac{n(n-1)\tilde{M}}{2} - \frac{n-1}{n(n+1)}r^{n+1}\left(\overset{(n+2)}{G}{}^{i}{}_{i} - 2n\overset{(n+2)}{G}{}^{v}{}_{v}\right) \\
&= \frac{n(n-1)M}{2} \\
&\quad - \frac{n-1}{n(n+1)}\kappa_{n+2}^{2}\,r^{n+1}\left(T^{i}{}_{i} - 2nT^{v}{}_{v}\right).
\end{aligned}\tag{A.16}$$

Finally, we note that this equation implies that the linear combination of the energy-momentum tensor,

$$r^{n+1}\left(T^{i}{}_{i} - 2nT^{v}{}_{v}\right),\tag{A.17}$$

plays the role of a local charge as well. Therefore, the behavior of this quantity is constrained non-locally by the integral of the flux given by the corresponding linear combination of the currents S^{a} and Q^{a}.

A.2 Second order curvature tensors of tensor perturbations

A.2.1 Second order curvature tensors

Here we spell out the components of the curvature tensors up to quadratic order in the bulk metric perturbation in order to derive effective stress-energy tensor of KK gravitons. We consider the $(d+1)$-dimensional perturbed metric in the form,

$$ds^{2} = dy^{2} + b^{2}(y)\left(\gamma_{\mu\nu} + h_{\mu\nu}\right)dx^{\mu}dx^{\nu}\,,\tag{A.18}$$

where $\gamma_{\mu\nu}$ is the metric of the background d-dimensional spacetime subsection. In the text, we identify $\gamma_{\mu\nu}$ with the metric of a de Sitter spacetime. We impose the following gauge conditions on the perturbation:

$$h^{\alpha}{}_{\alpha} = h_{\alpha}{}^{\beta}{}_{|\beta} = 0\,,\tag{A.19}$$

where the vertical bar ($|$) denotes the covariant derivative associated with the d-dimensional metric $\gamma_{\mu\nu}$, and the tensor indices of $h_{\mu\nu}$ are raised or lowered by the metric $\gamma_{\mu\nu}$ (not by the five-dimensional metric).

The non-trivial components of the connection are given by

$$\overset{(d+1)}{\Gamma}{}^{\mu}{}_{y\nu} = \frac{b'}{b}\delta^{\mu}{}_{\nu} + \frac{1}{2}h^{\mu}{}_{\nu}{}' - \frac{1}{2}h^{\mu\alpha}h'_{\alpha\nu}\,,$$

$$\overset{(d+1)}{\Gamma}{}^{y}{}_{\mu\nu} = -b\,b'\gamma_{\mu\nu} - \frac{1}{2}b^2 h'_{\mu\nu} - b\,b' h_{\mu\nu}$$

$$\overset{(d+1)}{\Gamma}{}^{\mu}{}_{\alpha\beta} = \overset{(d)}{\Gamma}{}^{\mu}{}_{\alpha\beta} + \frac{1}{2}\gamma^{\mu\rho}\Big(h_{\alpha\rho|\beta} + h_{\beta\rho|\alpha} - h_{\alpha\beta|\rho}\Big)$$
$$- \frac{1}{2}h^{\mu\rho}\Big(h_{\alpha\rho|\beta} + h_{\beta\rho|\alpha} - h_{\alpha\beta|\rho}\Big), \tag{A.20}$$

where the prime ($'$) denotes the y-derivative.

The non-trivial components of the Riemann tensor are given by

$$\overset{(d+1)}{R}{}^{\mu}{}_{y\nu y} = -\frac{b''}{b}\delta^{\mu}{}_{\nu} - \frac{1}{2}h^{\mu}{}_{\nu}{}'' - \frac{b'}{b}h^{\mu}{}_{\nu}{}' + \frac{1}{4}h^{\mu\alpha'}h'_{\alpha\nu} + \frac{b'}{b}h^{\mu\rho}h'_{\rho\nu} + \frac{1}{2}h^{\mu\alpha}h''_{\alpha\nu}\,,$$

$$\overset{(d+1)}{R}{}^{y}{}_{\mu y\nu} = -b\,b''\gamma_{\mu\nu} - b\,b'' h_{\mu\nu} - \frac{1}{2}b^2 h''_{\mu\nu} - b\,b' h'_{\mu\nu} + \frac{1}{4}b^2 h'_{\alpha\nu}h^{\alpha}{}_{\mu}{}'\,,$$

$$\overset{(d+1)}{R}{}^{\alpha}{}_{y\mu\nu} = \Big(\frac{1}{2}h^{\alpha}{}_{\nu}{}' - \frac{1}{2}h^{\alpha\beta}h'_{\beta\nu}\Big)_{|\mu} - \Big(\frac{1}{2}h^{\alpha}{}_{\mu}{}' - \frac{1}{2}h^{\alpha\beta}h'_{\beta\mu}\Big)_{|\nu}$$
$$+ \frac{1}{4}h^{\rho}{}_{\nu}{}'\Big(h^{\alpha}{}_{\rho|\mu} + h^{\alpha}{}_{\mu|\rho} - h_{\mu\rho}{}^{|\alpha}\Big) - \frac{1}{4}h^{\rho}{}_{\mu}{}'\Big(h^{\alpha}{}_{\rho|\nu} + h^{\alpha}{}_{\nu|\rho} - h_{\nu\rho}{}^{|\alpha}\Big)\,,$$

$$\overset{(d+1)}{R}{}^{y}{}_{\alpha\mu\nu} = -\frac{1}{2}b^2\Big(h_{\alpha\nu|\mu} - h_{\alpha\mu|\nu}\Big)' - \frac{b^2}{4}h'_{\rho\mu}\Big(h^{\rho}{}_{\alpha|\nu} + h^{\rho}{}_{\nu|\alpha} - h_{\alpha\nu}{}^{|\rho}\Big)$$
$$+ \frac{b^2}{4}h'_{\rho\nu}\Big(h^{\rho}{}_{\alpha|\mu} + h^{\rho}{}_{\mu|\alpha} - h_{\alpha\mu}{}^{|\rho}\Big)\,,$$

$$\overset{(d+1)}{R}{}^{\mu}{}_{\alpha\nu\beta} = \overset{(d)}{R}{}^{\mu}{}_{\alpha\nu\beta} - b'^2\Big(\delta^{\mu}{}_{\nu}\gamma_{\alpha\beta} - \delta^{\mu}{}_{\beta}\gamma_{\alpha\nu}\Big)$$
$$+ \frac{1}{2}\Big(h_{\alpha}{}^{\mu}{}_{|\beta\nu} + h_{\beta}{}^{\mu}{}_{|\alpha\nu} - h_{\alpha\beta}{}^{|\mu}{}_{|\nu} - h_{\alpha}{}^{\mu}{}_{|\nu\beta} - h_{\nu}{}^{\mu}{}_{|\alpha\beta} + h_{\alpha\nu}{}^{|\mu}{}_{|\beta}\Big)$$
$$- \frac{1}{2}b\,b'\Big(\delta^{\mu}{}_{\nu}h'_{\alpha\beta} + \gamma_{\alpha\beta}h^{\mu}{}_{\nu}{}' - \delta^{\mu}{}_{\beta}h'_{\alpha\nu} - \gamma_{\alpha\nu}h^{\mu}{}_{\beta}{}'\Big) - b'^2\Big(\delta^{\mu}{}_{\nu}h_{\alpha\beta} - \delta^{\mu}{}_{\beta}h_{\alpha\nu}\Big)$$
$$- \frac{1}{2}\Big[h^{\mu\rho}\Big(h_{\alpha\rho|\beta} + h_{\beta\rho|\alpha} - h_{\alpha\beta|\rho}\Big)\Big]_{|\nu} + \frac{1}{2}\Big[h^{\mu\rho}\Big(h_{\alpha\rho|\nu} + h_{\nu\rho|\alpha} - h_{\alpha\nu|\rho}\Big)\Big]_{|\beta}$$
$$- \frac{b^2}{4}\Big(h^{\mu'}{}_{\nu}h'_{\alpha\beta} - h^{\mu'}{}_{\beta}h'_{\alpha\nu}\Big)$$
$$- \frac{1}{2}b\,b'\Big(h^{\mu}{}_{\nu}{}'h_{\alpha\beta} - h^{\mu}{}_{\beta}{}'h_{\alpha\nu}\Big) + \frac{1}{2}b\,b'\Big(\gamma_{\alpha\beta}h^{\mu\rho}h'_{\rho\nu} - \gamma_{\alpha\nu}h^{\mu\rho}h'_{\rho\beta}\Big)$$
$$+ \frac{1}{4}\Big(h^{\mu}{}_{\rho|\nu}h^{\rho}{}_{\alpha|\beta} + h^{\mu}{}_{\rho|\nu}h^{\rho}{}_{\beta|\alpha} - h^{\mu}{}_{\rho|\nu}h_{\alpha\beta}{}^{|\rho} + h^{\mu}{}_{\nu|\rho}h^{\rho}{}_{\alpha|\beta} + h^{\mu}{}_{\nu|\rho}h^{\rho}{}_{\beta|\alpha} - h^{\mu}{}_{\nu|\rho}h_{\alpha\beta}{}^{|\rho}$$
$$- h_{\rho\nu}{}^{|\mu}h^{\rho}{}_{\alpha|\beta} - h_{\rho\nu}{}^{|\mu}h^{\rho}{}_{\beta|\alpha} + h_{\rho\nu}{}^{|\mu}h_{\alpha\beta}{}^{|\rho} - h^{\mu}{}_{\rho|\beta}h^{\rho}{}_{\alpha|\nu} - h^{\mu}{}_{\rho|\beta}h^{\rho}{}_{\nu|\alpha} + h^{\mu}{}_{\rho|\beta}h_{\alpha\nu}{}^{|\rho}$$
$$- h^{\mu}{}_{\beta|\rho}h^{\rho}{}_{\alpha|\nu} - h^{\mu}{}_{\beta|\rho}h^{\rho}{}_{\nu|\alpha} + h^{\mu}{}_{\beta|\rho}h_{\alpha\nu}{}^{|\rho}$$
$$+ h_{\rho\beta}{}^{|\mu}h^{\rho}{}_{\alpha|\nu} + h_{\rho\beta}{}^{|\mu}h^{\rho}{}_{\nu|\alpha} - h_{\rho\beta}{}^{|\mu}h_{\alpha\nu}{}^{|\rho}\Big). \tag{A.21}$$

The mixed components of the Ricci tensor are given by

$$\overset{(d+1)}{R}{}^{y}{}_{y} = -d\frac{b''}{b} + \frac{1}{4}h^{\rho\sigma\prime}h'_{\rho\sigma} + \frac{b'}{b}h^{\rho\sigma}h'_{\rho\sigma} + \frac{1}{2}h^{\rho\sigma}h''_{\rho\sigma},$$

$$\overset{(d+1)}{R}{}^{y}{}_{\nu} = -\frac{1}{2}h^{\rho\sigma}h'_{\rho\nu|\sigma} + \frac{1}{4}h^{\rho\sigma}{}_{|\nu}h'_{\rho\sigma} + \frac{1}{2}h^{\rho\sigma}h'_{\rho\sigma|\nu}$$

$$\overset{(d+1)}{R}{}^{\nu}{}_{y} = \frac{1}{b^2}\left(-\frac{1}{2}h^{\rho\sigma}h^{\nu}{}_{\rho|\sigma}{}' + \frac{1}{4}h^{\rho\sigma|\nu}h'_{\rho\sigma} + \frac{1}{2}h^{\rho\sigma}h_{\rho\sigma}{}^{|\nu\prime}\right)$$

$$\overset{(d+1)}{R}{}^{\alpha}{}_{\beta} = -\frac{b''}{b}\delta^{\alpha}{}_{\beta} + \frac{1}{b^2}\overset{(d)}{R}{}^{\alpha}{}_{\beta} - (d-1)\left(\frac{b'}{b}\right)^2\delta^{\alpha}{}_{\beta}$$

$$-\frac{1}{2}h^{\alpha}{}_{\beta}{}'' - \frac{1}{2}d\frac{b'}{b}h^{\alpha}{}_{\beta}{}' + \frac{1}{2b^2}\left(h^{\alpha\rho}{}_{|\beta\rho} + h_{\beta}{}^{\rho|\alpha}{}_{|\rho} - \Box_d h^{\alpha}{}_{\beta}\right) - \frac{1}{b^2}\overset{(d)}{R}{}_{\rho\beta}h^{\alpha\rho}$$

$$+\frac{1}{2}h^{\alpha\rho\prime}h'_{\rho\beta} + \frac{1}{2}\frac{b'}{b}\delta^{\alpha}{}_{\beta}h^{\rho\sigma}h'_{\rho\sigma} + \frac{1}{2}h^{\alpha\rho}h''_{\rho\beta} + \frac{1}{2}d\frac{b'}{b}h^{\alpha\rho}h'_{\rho\beta}$$

$$-\frac{1}{2b^2}h^{\rho\sigma}\left(h_{\beta\rho}{}^{|\alpha}{}_{|\sigma} + h^{\alpha}{}_{\rho|\beta\sigma} - h^{\alpha}{}_{\beta|\rho\sigma} - h_{\rho\sigma}{}^{|\alpha}{}_{|\beta}\right)$$

$$+\frac{1}{4b^2}\left(h^{\rho\sigma|\alpha}h_{\rho\sigma|\beta} - 2h^{\alpha\rho|\sigma}h_{\beta\sigma|\rho} + 2h^{\alpha\rho|\sigma}h_{\beta\rho|\sigma}\right)$$

$$-\frac{1}{2b^2}h^{\alpha\rho}\left(h_{\rho}{}^{\mu}{}_{|\beta\mu} + h_{\beta}{}^{\mu}{}_{|\rho\mu} - \Box_d h_{\rho\beta}\right) + \frac{1}{b^2}\overset{(d)}{R}{}_{\gamma\beta}h^{\alpha\rho}h_{\rho}{}^{\gamma}. \tag{A.22}$$

The Ricci scalar is given by

$$\overset{(d+1)}{R} = -2d\frac{b''}{b} + \frac{1}{b^2}\overset{(d)}{R} - d(d-1)\left(\frac{b'}{b}\right)^2 - \frac{1}{b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho\sigma}$$

$$+\frac{3}{4}h^{\rho\sigma\prime}h'_{\rho\sigma} + (d+1)\frac{b'}{b}h^{\rho\sigma}h'_{\rho\sigma} + h^{\rho\sigma}h''_{\rho\sigma} + \frac{1}{b^2}h^{\rho\sigma}\Box_d h_{\rho\sigma}$$

$$+\frac{1}{b^2}\left(\frac{3}{4}h^{\rho\sigma|\mu}h_{\rho\sigma|\mu} - \frac{1}{2}h^{\rho\sigma|\mu}h_{\rho\mu|\sigma}\right)$$

$$-\frac{1}{b^2}h^{\mu}{}_{\rho|\sigma\mu}h^{\rho\sigma} + \frac{1}{b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho}{}_{\mu}h^{\mu\sigma}. \tag{A.23}$$

Using these results, the components of the Einstein tensor are given by

$$
\overset{(d+1)}{G}{}^{y}{}_{y} = -\frac{1}{2b^2}\overset{(d)}{R} + \frac{1}{2}d(d-1)\left(\frac{b'}{b}\right)^2 + \frac{1}{2b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho\sigma}
$$

$$
-\frac{1}{8}h^{\rho\sigma\,\prime}h'_{\rho\sigma} - \frac{1}{2}(d-1)\frac{b'}{b}h^{\rho\sigma}h'_{\rho\sigma} - \frac{1}{2b^2}h^{\rho\sigma}\Box_d\,h_{\rho\sigma}
$$

$$
-\frac{1}{2b^2}\left(\frac{3}{4}h^{\rho\sigma|\mu}h_{\rho\sigma|\mu} - \frac{1}{2}h^{\rho\sigma|\mu}h_{\rho\mu|\sigma}\right) + \frac{1}{2b^2}h^{\mu}{}_{\rho|\sigma\mu}h^{\rho\sigma} - \frac{1}{2b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho}{}_{\mu}h^{\mu\sigma}
$$

$$
\overset{(d+1)}{G}{}^{y}{}_{\nu} = \overset{(d+1)}{R}{}^{y}{}_{\nu}
$$

$$
\overset{(d+1)}{G}{}^{\nu}{}_{y} = \overset{(d+1)}{R}{}^{\nu}{}_{y}
$$

$$
\overset{(d+1)}{G}{}^{\alpha}{}_{\beta} = (d-1)\frac{b''}{b}\delta^{\alpha}{}_{\beta} + \frac{1}{b^2}\overset{(d)}{G}{}^{\alpha}{}_{\beta} + \frac{1}{2}(d-1)(d-2)\left(\frac{b'}{b}\right)^2\delta^{\alpha}{}_{\beta}
$$

$$
-\frac{1}{2}h^{\alpha}{}_{\beta}{}'' - \frac{1}{2}d\frac{b'}{b}h^{\alpha}{}_{\beta}{}' + \frac{1}{2b^2}\left(h^{\alpha\rho}{}_{|\beta\rho} + h_{\beta\rho}{}^{|\alpha\rho} - \Box_d\,h^{\alpha}{}_{\beta}\right)
$$

$$
-\frac{1}{b^2}\overset{(d)}{R}{}_{\rho\beta}h^{\alpha\rho} + \frac{1}{2b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho\sigma}\delta^{\alpha}{}_{\beta}
$$

$$
+\frac{1}{2}h^{\alpha\rho\,\prime}h'_{\beta\rho} + \frac{1}{2}h^{\alpha\rho}h''_{\rho\beta} + \frac{1}{2}d\frac{b'}{b}h^{\alpha\rho}h'_{\rho\beta}
$$

$$
-\frac{1}{2b^2}h^{\rho\sigma}\left(h_{\beta\rho}{}^{|\alpha}{}_{|\sigma} + h^{\alpha}{}_{\rho|\beta\sigma} - h^{\alpha}{}_{\beta|\rho\sigma} - h_{\rho\sigma}{}^{|\alpha}{}_{|\beta}\right)
$$

$$
+\frac{1}{4b^2}\left(h^{\rho\sigma|\alpha}h_{\rho\sigma|\beta} - 2h^{\alpha\rho|\sigma}h_{\beta\sigma|\rho} + 2h^{\alpha\rho|\sigma}h_{\beta\rho|\sigma}\right) + \frac{1}{b^2}\overset{(d)}{R}{}_{\rho\beta}h^{\alpha\sigma}h_{\sigma}{}^{\rho}
$$

$$
-\frac{1}{2b^2}h^{\rho\alpha}\left(h_{\rho}{}^{\sigma}{}_{|\beta\sigma} + h_{\beta}{}^{\sigma}{}_{|\rho\sigma} - \Box_d\,h_{\rho\beta}\right)
$$

$$
-\frac{1}{2}\delta^{\alpha}{}_{\beta}\left[\frac{3}{4}h^{\rho\sigma\,\prime}h'_{\rho\sigma} + d\frac{b'}{b}h^{\rho\sigma}h'_{\rho\sigma} + h^{\rho\sigma}h''_{\rho\sigma} + \frac{1}{b^2}h^{\rho\sigma}\Box_d\,h_{\rho\sigma}\right.
$$

$$
+\frac{1}{b^2}\left(\frac{3}{4}h^{\rho\sigma|\mu}h_{\rho\sigma|\mu} - \frac{1}{2}h^{\rho\sigma|\mu}h_{\rho\mu|\sigma}\right)
$$

$$
\left.-\frac{1}{b^2}h^{\mu}{}_{\rho|\sigma\mu}h^{\rho\sigma} + \frac{1}{b^2}\overset{(d)}{R}{}_{\rho\sigma}h^{\rho}{}_{\mu}h^{\mu\sigma}\right] . \tag{A.24}
$$

A.2.2   Computational rules for averaging

Here, we describe the computational rules for averaging the components of the second order part of the curvature tensors listed previously. As we have noted in the main text, the notation $\langle A \rangle$ includes both the averaging along the ordinary spatial dimensions which are assumed to be homogeneous and isotropic, and the small-scale time averaging as defined in (B.17). In both cases, the computational rules are similar. However, we do not apply the same rules for terms with derivatives in the bulk direction, because we are dealing with a braneworld and the averaging along the bulk direction is ill-defined.

First, we note that we are interested in massive KK modes. So, we can

neglect terms coupled to the background curvature tensor as

$$\left\langle \overset{(d)}{R}_{\rho\sigma} h^{\rho}{}_{\mu} h^{\sigma}{}_{\nu} \right\rangle,$$ (A.25)

which are of order $\mathcal{O}(h^2/L^2)$, where L is the d-dimensional characteristic background curvature radius, in comparison with terms as

$$\left\langle h^{\rho\sigma}{}_{|\mu} h_{\rho\sigma|\nu} \right\rangle, \quad \left\langle h^{\rho\sigma} h_{\rho\sigma|\mu\nu} \right\rangle,$$
$$\left\langle h^{\rho}{}_{\mu}{}' h'_{\rho\nu} \right\rangle, \quad \left\langle h^{\rho}{}_{\mu} h''_{\rho\nu} \right\rangle, \cdots,$$ (A.26)

which are of order $\mathcal{O}(m^2 h^2)$. For instance for a cosmological brane with expansion rate H, we have $L = \mathcal{O}(1/H)$. Thus $m \gg H$ implies $m \gg L^{-1}$, and we can safely neglect corrections of the form (A.25).

As a consequence, when taking the average, we are allowed to freely interchange the order of the covariant derivatives. For example,

$$\left\langle h^{\rho\sigma}{}_{|\mu\nu} h_{\rho\sigma} \right\rangle \simeq \left\langle h^{\rho\sigma}{}_{|\nu\mu} h_{\rho\sigma} \right\rangle,$$ (A.27)

where corrections of order $\mathcal{O}(h^2/L^2)$ are neglected. From now on, as in the main text we will use "=" instead of "$\simeq$" by neglecting the corrections.

Another computation rule is that total derivative terms can be neglected. For example,

$$\left\langle h^{\rho\sigma}{}_{|\mu} h_{\rho\sigma|\nu} \right\rangle = \left\langle \left(h^{\rho\sigma} h_{\rho\sigma|\nu} \right)_{|\mu} \right\rangle - \left\langle h^{\rho\sigma} h_{\rho\sigma|\nu\mu} \right\rangle = -\left\langle h^{\rho\sigma} h_{\rho\sigma|\nu\mu} \right\rangle.$$ (A.28)

This is because the total derivative term can be cast into the surface integral which is smaller in magnitude than the volume term by a factor $mR (\gg 1)$, where R is the length scale of the averaging volume which is taken to satisfy $R \gg m^{-1}$.

As mentioned above, we do not apply the same rules for terms with derivatives with respect to the bulk coordinate y. However, when one considers projections onto the brane, some simplifications occur. On the brane, we have the boundary condition $h'_{\alpha\beta}|_{\text{brane}} = 0$, which enables us to neglect all the first derivative terms, e.g.,

$$\left\langle h^{\alpha\rho}{}' h_{\rho\beta}{}' \right\rangle\Big|_{\text{brane}} = \frac{b'}{b} \left\langle h^{\alpha\rho} h_{\rho\beta}{}' \right\rangle\Big|_{\text{brane}} = \left\langle h^{\alpha\rho|\mu} h_{\rho\beta}{}' \right\rangle\Big|_{\text{brane}} = 0.$$ (A.29)

In addition, using the bulk equation of motion (3.49), we have

$$\left\langle h^{\rho\sigma} h''_{\rho\sigma} \right\rangle\Big|_{\text{brane}} = -\left\langle h^{\rho\sigma} \Box_d h_{\rho\sigma} \right\rangle\Big|_{\text{brane}}.$$ (A.30)

Appendix B

Backreaction of Kaluza-Klein modes of a bulk scalar field

In this Appendix, we discuss the backreaction of KK modes of a homogeneous scalar field, in order to discuss the backreaction of KK gravitons in Chapter 3. We assume its amplitude ϕ to be small so that its effect can be treated perturbatively: in particular, the backreaction of the scalar field on the metric will be of order $\mathcal{O}(\phi^2)$. The equation of motion of KK gravitons is the same as the case of the scalar field in the separable case and the latter case is more tracteble than the former case because of no brane contribution.

It is known that the field equations for general cases are not separable and the notion of a KK mode cannot be well defined. The separability property is satisfied only for two limiting cases. One is the case of a dS brane with an expansion rate H and one finds a mass gap $\Delta m = 3H/2$ between the zero mode and KK modes. Thus the continuum of KK modes starts above the mass $3H/2$. The other case is a low energy cosmological brane, in which case the dependence on the extra dimension can be approximated by the profile obtained for a static brane, i.e. the RS brane.

We start from the five-dimensional action which consists of the Einstein-Hilbert term, a cosmological constant Λ_5 and a bulk scalar field, complemented by the four-dimensional action for the brane:

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\left(\overset{(5)}{R} - 2\Lambda_5\right) + \int d^5x \sqrt{-g}\left(-\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right)$$
$$+ \int d^4x \sqrt{-q}\left(-\sigma + \mathcal{L}_m\right), \tag{B.1}$$

where q is the determinant of the induced metric on the brane, which we denote by $q_{\alpha\beta}$, and $\mathcal{L}_m$ is the Lagrangian density of the matter confined on the brane. The Latin indices $\{a, b, \cdots\}$ and the Greek indices $\{\alpha, \beta, \cdots\}$ are used for tensors defined in the bulk and on the brane, respectively. We will assume that the brane tension on the brane is tuned to its RS value so that $\kappa_5^4\sigma^2 = -6\Lambda_5$. We also take a constant bulk potential

$$V(\phi) = V_0 > 0, \tag{B.2}$$

so that the scalar field is effectively massless.

We consider backgrounds given by a fixed value of the scalar field which we choose $\phi = 0$. For a non-zero V_0, one has a de Sitter brane background, which will be discussed in subsection B.1. For $V_0 = 0$, one has a low energy cosmological brane, discussed in subsection B.2.

The field equation for the bulk scalar field is linear and given by

$$\Box_5\,\phi = 0. \tag{B.3}$$

Since we consider a background configuration with $\phi = 0$, the solution of the above equation can be seen as a perturbation. This perturbation will induce a bulk energy-momentum tensor, of order $\mathcal{O}(\phi^2)$, which embodies the backreaction of the scalar field on the metric. This is the effect we wish to calculate explicitly.

The variation of the action (B.1) yields the five-dimensional Einstein equations

$$\overset{(5)}{G}_{ab} + \Lambda_5 g_{ab} = -\kappa_5^2 V_0 g_{ab} + \kappa_5^2 T_{ab} + \left(-\sigma q_{ab} + \tau_{ab}\right)\delta(y - y_0) \tag{B.4}$$

where we have implicitly assumed a coordinate system in which the brane stays at a fixed location $y = y_0$ and where

$$\tau_{\alpha\beta} = \frac{2}{\sqrt{-q}}\frac{\delta}{\delta q^{\alpha\beta}}\left(\sqrt{-q}\mathcal{L}_m\right) \tag{B.5}$$

represents the energy-momentum tensor of matter confined on the brane. The stress energy tensor of the bulk scalar field, *not* including the constant potential V_0, is given by

$$T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}g^{cd}\phi_{,c}\phi_{,d}\,. \tag{B.6}$$

It is useful to consider the projection of the gravitational equations on the brane[91]. Taking into account the bulk energy-momentum tensor, one finds

$$G^{\alpha}{}_{\beta} = -\frac{1}{2}\kappa_5^2 V_0 \delta^{\alpha}{}_{\beta} + \frac{1}{6}\kappa_5^4 \sigma \tau^{\alpha}{}_{\beta} + \kappa_5^2 T^{(b)\alpha}{}_{\beta} - E^{\alpha}{}_{\beta}\,, \tag{B.7}$$

where

$$\overset{(b)}{T}{}^{\alpha}{}_{\beta} = \frac{2}{3}\left[\phi^{,\alpha}\phi_{,\beta} + \delta^{\alpha}{}_{\beta}\left(\frac{3}{8}\phi^2_{,y} - \frac{5}{8}q^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}\right)\right]\,, \tag{B.8}$$

and $E_{\alpha\beta}$ is the projection on the brane of the bulk Weyl tensor and is traceless by construction. If, in addition, one assumes the brane geometry to be homogeneous and isotropic then the components of $E_{\alpha\beta}$ (in an appropriate coordinate system) reduce to $E^t{}_t$ and

$$E^i{}_j = -\frac{1}{3}\delta^i{}_j E^t{}_t. \tag{B.9}$$

By using the four-dimensional Bianchi identities, and assuming that the brane matter content is conserved, one is able to express the component $E^t{}_t$ in terms of the values on the brane of the bulk scalar field and its derivatives [29]:

$$E^t{}_t = \frac{\kappa_5^2}{a^4} \int_{t_0}^{t} dt' a^4 \left(\partial_t \stackrel{(b)}{T}{}^t{}_t + 3\frac{\dot{a}}{a} \stackrel{(b)}{T}{}^t{}_t - \frac{\dot{a}}{a} \stackrel{(b)}{T}{}^i{}_i \right), \qquad (B.10)$$

B.1 KK mode on a de Sitter brane

First, we consider the case of a de Sitter brane. The bulk metric around a de Sitter brane can be expressed as

$$ds^2 = dy^2 + b^2(y)\gamma_{\mu\nu}dx^\mu dx^\nu , \qquad (B.11)$$

where the warp factor $b(y)$ is given by

$$b(y) = H\ell \sinh(y/\ell) , \qquad (B.12)$$

and $\gamma_{\mu\nu}$ is the four-dimensional de Sitter metric, which may be expressed by using a flat slicing for simplicity:

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j ,$$
$$a(t) = e^{Ht}, \qquad H^2 = \frac{1}{6}\kappa_5^2 V_0 . \qquad (B.13)$$

The brane is located at $y = y_0$ such that $b(y_0) = 1$, that is,

$$\sinh(y_0/\ell) = \frac{1}{H\ell} .$$

In this geometry, the equation of motion for the scalar field is

$$\frac{1}{b^4}\partial_y \left(b^4 \partial_y \phi \right) - \frac{1}{b^2}\left(\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\stackrel{(3)}{\Delta}\phi \right) = 0 . \qquad (B.14)$$

This equations is separable and one can solve it by looking for a solution of the form $\phi = f(y)\varphi(t, x^i)$, with

$$\frac{1}{b^2}\partial_y \left(b^4 \partial_y f \right) + m^2 f = 0 ,$$
$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\stackrel{(3)}{\Delta}\varphi + m^2\varphi = 0 . \qquad (B.15)$$

The separation constant m^2 corresponds to the square of the KK mass, as measured by an observer on the brane.

Since there is no coupling between the brane and the bulk scalar field, the boundary condition for the scalar field at the brane location is simply $\partial_y \phi = 0$, and therefore $\partial_y f = 0$. The equation along the y-direction implies that the mass spectrum is characterized by a mass gap $3H/2$ [17]. The

corresponding eigenfunctions f can be written in terms of the associated Legendre functions.

Let us now focus on a single KK mode, which is spatially homogeneous and sufficiently massive: $m \gg H$. One finds from (B.15)

$$\varphi(t) = \frac{1}{a^{3/2}} \cos(mt) \, . \tag{B.16}$$

If we take a time average over a time scale much longer than the period of oscillation m^{-1}, we can ignore the oscillatory behavior and use

$$\langle \sin^2(mt) \rangle = \langle \cos^2(mt) \rangle = \frac{1}{2} \, , \quad \text{etc.} \tag{B.17}$$

From Eq. (B.8), we thus find

$$\overset{(b)}{T}{}^t{}_t = -\frac{1}{8}|f_m|^2 m^2 \frac{1}{a^3} \, ,$$
$$\overset{(b)}{T}{}^i{}_j = \frac{5}{24}|f_m|^2 m^2 \frac{1}{a^3}\delta^i_j \, , \tag{B.18}$$

where f_m is the value of $f(y)$ on the brane for the eigenvalue m^2. From Eqs. (B.10) and (B.16), and from the fact that $\partial_y^2 \phi = -m^2 \phi$ on the brane, we can evaluate $E_{\mu\nu}$ as

$$-E^t{}_t = \frac{5}{8}\kappa_5^2|f_m|^2 m^2 \frac{1}{a^3} \, ,$$
$$-E^i{}_j = -\frac{5}{24}\kappa_5^2|f_m|^2 m^2 \frac{1}{a^3}\delta^i_j \, , \tag{B.19}$$

where we have neglected the terms that depend on the initial data, which behave as a^{-4} and thus become negligible at late times.

The above results show that the Weyl term $E_{\mu\nu}$ contributes *negatively* to the effective energy density and pressure on the brane for a massive mode. Moreover, if one computes the total effective contribution of the bulk, i.e., the sum of $\overset{(b)}{T}{}_{\alpha\beta}$ and of the Weyl term $E_{\alpha\beta}$, one finds for the effective energy density and pressure on the brane

$$\kappa_4^2 \rho_{(\text{eff})} = -\left(\kappa_5^2 \overset{(b)}{T}{}^t{}_t - E^t{}_t \right) = -\frac{1}{2}\kappa_5^2|f_m|^2 m^2 \frac{1}{a^3} \, ,$$
$$\kappa_4^2 p_{(\text{eff})} = \frac{1}{3}\left(\kappa_5^2 \overset{(b)}{T}{}^i{}_i - E^i{}_i \right) = 0 \, . \tag{B.20}$$

This represents the backreaction effects of the bulk scalar field, which are of order $\mathcal{O}(\phi^2)$. Whereas the effective pressure due to the KK mode vanishes, because the bulk component and the Weyl component exactly cancel each other, the effective energy, remarkably, is negative.

B.2 KK mode for a low energy cosmological brane

We next calculate the effective energy density and pressure of a KK mode for a low energy cosmological brane whose metric in the Gaussian normal coordinate is approximately given by Eq. (1.16). If one considers the evolution of a massless, minimally coupled scalar field in the the low energy cosmological background metric, one finds that the field equation is separable and thus admits a solution of the form $\phi(t, y) = f(y)\varphi(t)$ with

$$\partial_y^2 f - \frac{4}{\ell}\partial_y f + m^2 e^{2y/\ell} f = 0\,,$$

$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0\,, \tag{B.21}$$

where the function $f(y)$ is assumed to be $\mathbb{Z}_2$-symmetric.

The solution for $f(y)$ with the appropriate Neumann boundary condition on the brane, $f'(0) = 0$ is given in terms of the Hankel functions. There is a zero mode corresponding to $m = 0$ as well as a continuum of KK modes with $m > 0$. For a massive KK mode $m \gg H$, the four-dimensional part evolves as

$$\varphi = \frac{1}{a^{3/2}}\cos(mt)\,, \tag{B.22}$$

Similarly to the de Sitter brane case, one can compute the projection of the bulk energy-momentum tensor on the brane and one finds for its components:

$$\overset{(b)}{T}{}^t{}_t = -\frac{1}{4a^3(t)}|f_m|^2 m^2\langle\sin^2(mt)\rangle = -\frac{1}{8a^3(t)}|f_m|^2 m^2\,,$$

$$\overset{(b)}{T}{}^i{}_i = \frac{5}{4a^3(t)}|f_m|^2 m^2\langle\sin^2(mt)\rangle = \frac{5}{8a^3(t)}|f_m|^2 m^2\,. \tag{B.23}$$

This gives

$$\kappa_5^{-2}E^t{}_t = \frac{1}{a^4}\int_{t_0}^t dt' a^4\left(\partial_t \overset{(b)}{T}{}^t{}_t + 3\frac{\dot{a}}{a}\overset{(b)}{T}{}^t{}_t - \frac{\dot{a}}{a}\overset{(b)}{T}{}^i{}_i\right)$$

$$= -\frac{5}{8a^3(t)}|f_m|^2 m^2\left(1 - \frac{a(t_0)}{a(t)}\right)\,, \tag{B.24}$$

and $E^t{}_t = -E^i{}_i$. Thus we obtain

$$\overset{(b)}{T}{}^t{}_t - \kappa_5^{-2}E^t{}_t = \frac{1}{2a^3(t)}|f_m|^2 m^2\,,$$

$$\overset{(b)}{T}{}^i{}_i - \kappa_5^{-2}E^i{}_i = 0\,, \tag{B.25}$$

at late times. Therefore, the effective energy density and pressure for a KK mode becomes

$$\kappa_4^2\rho_{(\mathrm{eff})} = -\left(\kappa_5^2 \overset{(b)}{T}{}^t{}_t - E^t{}_t\right) = -\frac{\kappa_5^2}{2a^3(t)}|f_m|^2 m^2\,,$$

$$\kappa_4^2 p_{(\mathrm{eff})} = \frac{1}{3}\left(\kappa_5^2 \overset{(b)}{T}{}^i{}_i - E^i{}_i\right) = 0\,. \tag{B.26}$$

This means that, also for a low energy cosmological brane, a massive KK mode behaves as cosmic dust with negative energy density.

The analyses given above imply that the result is independent of the existence of a mass gap and the essential factor is the background expansion of the brane. A KK mode can be approximately defined only for a cosmological brane which slightly deviates from the dS geometry and for a low energy brane, thus we expect that our result can be applied at least for these cases. However, for intermediate energy scales a KK mode is not well-defined in general and it is not clear how our result might be applied.

Finally, we note that the bulk energy density of a KK mode on the brane remains positive as

$$\kappa_5^2 \rho_{(\text{bulk})} := -\kappa_5^2 T^t{}_t = \frac{1}{4}\kappa_5^2 |f_m|^2 m^2 \frac{1}{a^3} > 0, \tag{B.27}$$

for both de Sitter and low energy branes (with the understanding that the time average over scales greater than m^{-1} is taken). It shows that there is no singular effect in the bulk in contrast to the peculiar behavior on the brane.

Appendix C

Classical stability against tensor and scalar perturbations

In this Appendix, we analyze the classical stability against the tensor and scalar perturbations on the thick brane background which is discussed in Chapter 4.

C.1 Tensor perturbations

We first discuss the tensor perturbations about the domain wall background. Here we shall assume a Randall-Sundrum (RS) gauge [9] in which the components of the extra-dimension are zero, i.e.,

$$ds^2 = b^2(z) \left[dz^2 + \left(\gamma_{\mu\nu} + h_{\mu\nu} \right) dx^\mu dx^\nu \right] , \tag{C.1}$$

where $h_{\mu\nu}$ satisfies the usual transverse-traceless gauge about the background dS metric; $D^\mu h_{\mu\nu} = h^\mu{}_\mu = 0$, where D^μ is the covariant derivative associated with $\gamma_{\mu\nu}$.

In this case, the perturbation is separable and we obtain the equation of motion in the bulk direction, which can be written in the standard quantum mechanical form as

$$\left[-\frac{d^2}{dz^2} + V_T(z) \right] \psi(z) = m^2 \psi(z) , \tag{C.2}$$

where $\psi(z) \propto b(z)^{-(d-1)/2} h_{\mu\nu}$ and

$$V_T(z) = \frac{(d-1)^2}{4} H^2 - \frac{d-1}{4} H^2 \frac{d-1+\frac{2}{\sigma}}{\cosh^2(Hz/\sigma)} . \tag{C.3}$$

The thin wall limit can be obtained from the limit $\sigma \to 0$, which leads to a system composed of a thin dS brane embedded in a flat Minkowski bulk. The potential for the tensor perturbations in the thin wall limit is then

$$V_T(z) = \frac{(d-1)^2}{4} H^2 - (d-1) H \delta(z) , \tag{C.4}$$

116

where we used for $\sigma \to 0$

$$\frac{1}{2\sigma \cosh^2(x/\sigma)} \to \delta(x) \,. \tag{C.5}$$

In this limit the solution for the tensor perturbations reduces to the standard exponential form.

The general solution can be decomposed into a zero mode with mass $m = 0$, which may realize four-dimensional gravity on the brane, and a continuous spectrum of Kaluza-Klein (KK) modes with $m > 3/2$ (in the five-dimensional case). Thus, the model is classically stable against the tensor perturbations.

C.2   Scalar perturbations

Next, we discuss the stability of the model against scalar perturbations. We consider a scalar metric perturbation of the form

$$
\begin{aligned}
ds^2 &= b(z)^2 \Big[\Big(1 + 2A\Big) dz^2 + 2D_\mu B dx^\mu dz \\
&\quad + \Big(\gamma_{\mu\nu}\Big(1 + 2\mathcal{R}\Big) + 2D_\mu D_\nu E\Big) dx^\mu dx^\nu \Big] \,,
\end{aligned} \tag{C.6}
$$

and also a perturbation of the field $\chi(z) \to \chi(z) + \delta\chi(x)$, which supports the domain wall.

In the bulk longitudinal gauge, $B = E = 0$, the perturbed Einstein equations can be written as follows:

$$
\begin{aligned}
&d(d-1)\frac{b'}{b}\mathcal{R}' + (d-1)\Box\mathcal{R} + d(d-1)H^2\mathcal{R} - d(d-1)\Big(\frac{b'}{b}\Big)^2 A \\
&= \chi'\delta\chi' - A\chi'^2 - b^2\frac{\partial V}{\partial\chi}\delta\chi \,, \\
&-(d-1)D_\mu\Big(\mathcal{R}' - \frac{b'}{b}A\Big) = \phi' D_\mu\delta\chi \,, \\
&(d-1)\mathcal{R}'' + (d-1)^2\frac{b'}{b}\mathcal{R}' + (d-2)\Box\mathcal{R} + (d-2)(d-1)H^2\mathcal{R} \\
&- (d-1)\frac{b'}{b}A' - 2(d-1)\frac{b''}{b}A + \Box A \\
&= A\chi'^2 - \phi'\delta\chi' - b^2\frac{\partial V}{\partial\chi}\delta\chi \,, \\
&D^\alpha D_\beta\Big((d-2)\mathcal{R} + A\Big) = 0 \,.
\end{aligned} \tag{C.7}
$$

The perturbed equation of motion of the scalar field is found to be

$$
\begin{aligned}
&\delta\chi'' + (d-1)\frac{b'}{b}\delta\chi' + \Box\delta\chi - 2A\Big(\chi'' + (d-1)\frac{b'}{b}\chi'\Big) \\
&+ \Big(-A' + d\mathcal{R}'\Big)\phi' - b^2\frac{\partial^2 V}{\partial\chi^2}\delta\chi = 0 \,.
\end{aligned} \tag{C.8}
$$

Next, we derive the evolution equation for the curvature perturbation $\mathcal{R}$. By defining

$$\Psi = \mathcal{R}\Big(\frac{\chi'^2}{b^{d-1}}\Big)^{-1/2} ,\tag{C.9}$$

the equation for the curvature perturbations can be reduced to the form

$$-\Psi'' + V_S\Psi = \Box_d\Psi ,\tag{C.10}$$

with potential

$$V_S = \frac{d^2 + 4d - 13}{4}\Big(\frac{b'}{b}\Big)^2 - \frac{3d - 7}{2}\frac{b''}{b} + (d-3)\frac{b'}{b}\frac{\chi''}{\chi'}$$
$$- \frac{\phi'''}{\chi'} + 2\Big(\frac{\chi''}{\chi'}\Big)^2 - 2(d-1)H^2 .\tag{C.11}$$

For the dS thick brane case, which is considered in this article, we obtain

$$V_S = \frac{\beta^2}{4\cosh^2(\beta z)}\Big\{2[2 + (3d - 7)\sigma - (4d - 4)\sigma^2]$$
$$+ \big[4 + 4(d-3)\sigma + (d^2 - 10d + 9)\sigma^2\big]\sinh^2(\beta z)\Big\} .\tag{C.12}$$

Thus, it is simple to see that at least both for the cases of interest, $d = 2$ and $d = 4$, $V_S > 0$ and therefore, the model is always stable against scalar perturbations. The $d = 4$ case was originally derived in [48].

Appendix D

Appendices for analyzing quantum effects on the thick brane model

In this Appedix, we make mathematical preparations for evaluating quantum effects of a test scalar field on the thick brane model, which is discussed in Chapter 4. We first derive the normalized mode functions of Kaluza-Klein (KK) modes. We also calculate the amplitude of the bound state quantum fluctuations.

D.1 Normalized mode functions for Kaluza-Klein modes

The normalization constants and normalized mode functions for KK modes on the thick brane model which is discussed in Chapter 4 are derived. The perturbations are not needed to be $\mathbb{Z}_2$ symmetric with respect to the center of the brane in general, even if the background geometry is $\mathbb{Z}_2$ symmetric. So, we can have two types of configurations of mode functions: One is the untwisted configuration with respect to the center of the thick brane $f(-z) = f(z)$, which corresponds to the Neumann boundary condition and the other is twisted configuration $f(-z) = -f(z)$, which corresponds to the Dirichlet boundary condition. If we impose the Neumann boundary conditions at the center of the brane, the twisted mode functions are not relevant for the amplitude of quantum fluctuations on the brane and only contribute to the quantum backreactions, whereas the untwisted mode functions are relevant both for the amplitude of fluctuations and quantum backreactions on the brane.

D.1.1   The untwisted case

We consider the case of the untwisted configuration, i.e., $f(-z) = f(z)$. We first derive the normalization constant for each KK mode and then the normalized mode functions. We start from the following quantity:

$$
(q^2 - q_n^2) \int_0^L (H\,dz) \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)
$$

$$
= \int_0^L (H\,dz) \left[\left(\alpha_q q^2 P_\nu^{iq\sigma}(x) - \beta_q q^2 R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right) \right.
$$

$$
\left. - \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} q_n^2 P_\nu^{iq_n\sigma}(x) - \beta_{q_n} q_n^2 R_\nu^{iq_n\sigma}(x) \right) \right]. \tag{D.1}
$$

Using the equations of motion for mode functions Eq. (4.15), Eq.(D.1) becomes

$$
= - \int_0^L (H\,dz) \left[\left(\alpha_q \frac{d^2}{d(Hz)^2} P_\nu^{iq\sigma}(x) - \beta_q \frac{d^2}{d(Hz)^2} R_\nu^{iq\sigma}(x) \right) \right.
$$

$$
\times \ \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)
$$

$$
\left. - \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} \frac{d^2}{d(Hz)^2} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} \frac{d^2}{d(Hz)^2} R_\nu^{iq_n\sigma}(x) \right) \right]
$$

$$
= - \left(\alpha_q \frac{d}{d(Hz)} P_\nu^{iq\sigma}(x) - \beta_q \frac{d}{d(Hz)} R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right) \Big|_{z=L}
$$

$$
+ \ \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} \frac{d}{d(Hz)} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} \frac{d}{d(Hz)} R_\nu^{iq_n\sigma}(x) \right) \Big|_{z=L}
$$

$$
+ \ \left(\alpha_q \frac{d}{d(Hz)} P_\nu^{iq\sigma}(x) - \beta_q \frac{d}{d(Hz)} R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right) \Big|_{z=0}
$$

$$
- \ \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} \frac{d}{d(Hz)} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} \frac{d}{d(Hz)} R_\nu^{iq_n\sigma}(x) \right) \Big|_{z=0}.
$$

$$\tag{D.2}$$

Note, that in the second line, only the boundary terms survive. Furthermore, using Eq. (4.38),we can deform Eq. (D.2)

$$
= - \frac{1}{\sigma} \left(1 - x_L^2 \right) \left(\alpha_q P_\nu^{iq\sigma\prime}(x_L) - \beta_q R_\nu^{iq\sigma\prime}(x_L) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x_L) - \beta_{q_n} R_\nu^{iq_n\sigma}(x_L) \right)
$$

$$
+ \ \frac{1}{\sigma} \left(\alpha_q P_\nu^{iq\sigma\prime}(0) - \beta_q R_\nu^{iq\sigma\prime}(0) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(0) - \beta_{q_n} R_\nu^{iq_n\sigma}(0) \right). \tag{D.3}
$$

Without loss of generality, we can set $\alpha_q = R_\nu^{iq\sigma\prime}(0)$ and $\beta_q = P_\nu^{iq\sigma\prime}(0)$ and obtain the following equation from Eq. (D.3)

$$
= - \frac{1}{\sigma} \left(1 - x_L^2 \right) \left(\alpha_q P_\nu^{iq\sigma\prime}(x_L) - \beta_q R_\nu^{iq\sigma\prime}(x_L) \right)
$$

$$
\times \ \left(R_\nu^{iq_n\sigma\prime}(0) P_\nu^{iq_n\sigma}(x_L) - P_\nu^{iq_n\sigma\prime}(0) R_\nu^{iq_n\sigma}(x_L) \right)
$$

$$
= - \frac{1}{\sigma} \left(\alpha_q P_\nu^{iq\sigma\prime}(x_L) - \beta_q R_\nu^{iq\sigma\prime}(x_L) \right) \frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}. \tag{D.4}
$$

Note, that we have used the Wronskian normalization of Eq.(4.33), which corresponds to the definition of the second independent solution of equation of motion Eq. (4.15) $R_\nu^{iq\sigma}(x)$:

$$R_\nu^{iq_n\sigma\prime}(0)P_\nu^{iq_n\sigma}(x_L) - P_\nu^{iq_n\sigma\prime}(0)R_\nu^{iq_n\sigma}(x_L)$$

$$= R_\nu^{iq_n\sigma\prime}(0)\left(P_\nu^{iq_n\sigma}(x_L) - \frac{P_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(0)}R_\nu^{iq_n\sigma}(x_L)\right)$$

$$= R_\nu^{iq_n\sigma\prime}(0)\left(P_\nu^{iq_n\sigma}(x_L) - \frac{P_\nu^{iq_n\sigma\prime}(x_L)}{R_\nu^{iq_n\sigma\prime}(x_L)}R_\nu^{iq_n\sigma}(x_L)\right)$$

$$= \frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\left(R_\nu^{iq_n\sigma\prime}(x_L)P_\nu^{iq_n\sigma}(x_L) - P_\nu^{iq_n\sigma\prime}(x_L)R_\nu^{iq_n\sigma}(x_L)\right)$$

$$= \frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\frac{1}{1-x_L^2}\,. \tag{D.5}$$

As a result, we obtain the following relation for our choice of the coefficients

$$\left(q^2-q_n^2\right)\int_0^L (H\,dz)\left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x)\right)\left(\alpha_{q_n}P_\nu^{iq_n\sigma}(x) - \beta_{q_n}R_\nu^{iq_n\sigma}(x)\right)$$

$$= -\frac{1}{\sigma}\left(R_\nu^{iq\sigma\prime}(0)P_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(0)R_\nu^{iq\sigma\prime}(x_L)\right)\frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\,. \tag{D.6}$$

Taking the derivative with respect to q and then setting to $q = q_n$ (namely, a KK mode), we obtain the following equation

$$\partial_q\left[\left(q^2-q_n^2\right)\int_0^L (H\,dz)\left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x)\right)\right.$$

$$\times \left.\left(\alpha_{q_n}P_\nu^{iq_n\sigma}(x) - \beta_{q_n}R_\nu^{iq_n\sigma}(x)\right)\right]_{q=q_n}$$

$$= 2q_n\int_0^L (H\,dz)\left(\alpha_{q_n}P_\nu^{iq_n\sigma}(x) - \beta_{q_n}R_\nu^{iq_n\sigma}(x)\right)^2$$

$$= \partial_q\left[-\frac{1}{\sigma}\left(R_\nu^{iq\sigma\prime}(0)P_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(0)R_\nu^{iq\sigma\prime}(x_L)\right)\frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\right]_{q=q_n}\,. \tag{D.7}$$

Thus, the normalization constant in our choice of coefficients becomes

$$N_{q_n}^{-2}$$

$$= 2\int_0^L (H\,dz)\left(R_\nu^{iq\sigma\prime}(0)P_\nu^{iq_n\sigma}(x) - P_\nu^{iq\sigma\prime}(0)R_\nu^{iq_n\sigma}(x)\right)^2$$

$$= \frac{1}{\sigma q_n}\partial_q\left[-\left(R_\nu^{iq\sigma\prime}(0)P_\nu^{iq\sigma\prime}(x_L) - P_\nu^{iq\sigma\prime}(0)R_\nu^{iq\sigma\prime}(x_L)\right)\frac{R_\nu^{iq_n\sigma\prime}(0)}{R_\nu^{iq_n\sigma\prime}(x_L)}\right]_{q=q_n}\,. \tag{D.8}$$

Finally, we obtain the desired normalized mode functions

$$
\begin{aligned}
f_{q_n}^2(z) &= N_{q_n}^2 \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)^2 \\
&= \frac{\sigma q_n}{\partial_q \left(P_\nu^{iq\sigma\prime}(0) R_\nu^{iq\sigma\prime}(x_L) - R_\nu^{iq\sigma\prime}(0) P_\nu^{iq\sigma\prime}(x_L) \right)_{q=q_n}} \\
&\quad \times \left(R_\nu^{iq_n\sigma\prime}(0) P_\nu^{iq_n\sigma}(x) - P_\nu^{iq_n\sigma\prime}(0) R_\nu^{iq_n\sigma}(x) \right) \\
&\quad \times \left(R_\nu^{iq_n\sigma\prime}(x_L) P_\nu^{iq_n\sigma}(x) - P_\nu^{iq_n\sigma\prime}(x_L) R_\nu^{iq_n\sigma}(x) \right) .
\end{aligned}
\tag{D.9}
$$

D.1.2   The case of the twisted configuration

Similarly, we discuss the case of the twisted configuration, i.e., $f(-z) = -f(z)$. The derivation is essentially the same as the previous case of the untwisted configuration and thus we omit the detailed derivations. Here, we introduce the essential results.

Because of the Dirichlet boundary condition, instead of Eq. (4.38), we have the relation between coefficients as

$$
\frac{\alpha_{q_n}}{\beta_{q_n}} = \frac{R_\nu^{iq_n\sigma}(0)}{P_\nu^{iq_n\sigma}(0)} = \frac{R_\nu^{iq_n\sigma}(x_L)}{P_\nu^{iq_n\sigma}(x_L)} .
\tag{D.10}
$$

Without of generality, we may set

$$
\alpha_q = R_\nu^{iq\sigma}(0), \quad \beta_q = P_\nu^{iq\sigma}(0).
\tag{D.11}
$$

Along the lines in the case of the untwisted configuration, we obtain the following relation:

$$
\begin{aligned}
&\left(q^2 - q_n^2 \right) \int_0^L (H\,dz) \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right) \\
&= -\frac{1}{\sigma} \left(R_\nu^{iq\sigma}(0) P_\nu^{iq\sigma}(x_L) - P_\nu^{iq\sigma}(0) R_\nu^{iq\sigma}(x_L) \right) \frac{R_\nu^{iq_n\sigma}(0)}{R_\nu^{iq_n\sigma}(x_L)} .
\end{aligned}
\tag{D.12}
$$

Taking the derivative with respect to q and setting to $q = q_n$ (i.e., a KK mode), we obtain

$$
\begin{aligned}
&\partial_q \left[\left(q^2 - q_n^2 \right) \int_0^L (H\,dz) \left(\alpha_q P_\nu^{iq\sigma}(x) - \beta_q R_\nu^{iq\sigma}(x) \right) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right) \right]_{q=q_n} \\
&= 2q_n \int_0^L (H\,dz) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)^2 \\
&= \partial_q \left[-\frac{1}{\sigma} \left(R_\nu^{iq\sigma}(0) P_\nu^{iq\sigma}(x_L) - P_\nu^{iq\sigma}(0) R_\nu^{iq\sigma}(x_L) \right) \frac{R_\nu^{iq_n\sigma}(0)}{R_\nu^{iq_n\sigma}(x_L)} \right]_{q=q_n} .
\end{aligned}
\tag{D.13}
$$

The normalization constant in our choice of the coefficients becomes

$$
\begin{aligned}
N_{q_n}^{-2} &= 2 \int_0^L (H\,dz) \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)^2 \\
&= \frac{1}{\sigma q_n} \partial_q \left[-\left(R_\nu^{iq\sigma}(0) P_\nu^{iq\sigma}(x_L) - P_\nu^{iq\sigma}(0) R_\nu^{iq\sigma}(x_L) \right) \frac{R_\nu^{iq_n\sigma}(0)}{R_\nu^{iq_n\sigma}(x_L)} \right]_{q=q_n} .
\end{aligned}
\tag{D.14}
$$

Finally, we obtain the desired normalized mode functions

$$
\begin{aligned}
&f_{q_n}^2(z) \\
&= N_{q_n}^2 \left(\alpha_{q_n} P_\nu^{iq_n\sigma}(x) - \beta_{q_n} R_\nu^{iq_n\sigma}(x) \right)^2 \\
&= \frac{\sigma q_n}{\partial_q \left(P_\nu^{iq\sigma}(0) R_\nu^{iq\sigma}(x_L) - R_\nu^{iq\sigma}(0) P_\nu^{iq\sigma}(x_L) \right)_{q=q_n}} \\
&\quad \times \left(R_\nu^{iq_n\sigma}(0) P_\nu^{iq_n\sigma}(x) - P_\nu^{iq_n\sigma}(0) R_\nu^{iq_n\sigma}(x) \right) \\
&\quad \times \left(R_\nu^{iq_n\sigma}(x_L) P_\nu^{iq_n\sigma}(x) - P_\nu^{iq_n\sigma}(x_L) R_\nu^{iq_n\sigma}(x) \right) .
\end{aligned}
$$

$$(D.15)$$

D.2   Bound state amplitude

Next, we evaluate the amplitude of the bound state zero mode. The integration here is doing along the closed contour with the dotted line as depicted in Fig. 4.3. In order to obtain the KK amplitude, we need to subtract the bound state amplitudes which are evaluated here from the total amplitude derived in Chapter 4.

D.2.1   On the two-sphere (two-dimensional de Sitter brane)

First, we note that the bound state for the minimally coupled case is given by

$$
q_0 = \frac{i\nu}{\sigma} ,
$$

$$(D.16)$$

where the bound state zeta function is defined as

$$
\begin{aligned}
\tilde{\zeta}_{\rm bs}(s) &= 4 f_0^2(0) \frac{\mu^{2(s-1)}}{H^{2s}} \sum_{j=0}^{\infty} (j + 1/2) \left((j + 1/2)^2 - \left(\frac{\nu}{\sigma} \right)^2 \right)^{-s} \\
&= 4 f_0^2(0) \frac{\mu^{2(s-1)}}{H^{2s}} \bar{\zeta}_{\rm bs}(s) .
\end{aligned}
$$

$$(D.17)$$

Here, $f_0(z)$ is the normalized mode function of the bound state. Quite clearly we have a zero mode (by zero mode we mean that the lowest eigenvalue λ_0 is a null eigenvalue, i.e., $\lambda_0 = 0$) and in such a case we have to project out this mode to evaluate the bound state contribution. However, in general the bound state varies from the top of the mass gap at $(\nu/\sigma)^2 = 1/4$ down to $\nu/\sigma = 0$, which is for the massless conformally coupled case. In the following we shall focus on a general bound state mass ν/σ taking care when dealing with the bound state zero mode.

It is straightforward to evaluate the above ζ-function by employing the binomial expansion method which follows identically to that of Allen [149],

see [121] for the case when null eigenvalues are present. Thus, subtracting out the null eigenvalue we obtain the following:

$$\bar{\zeta}_{\text{bs}}(s) = \frac{1}{2} \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J! \, \Gamma(s)} \left[\left(\frac{\nu}{\sigma}\right)^{2J} \zeta_c(s+J) - \delta_{\xi,0} \left(\frac{1}{2}\right)^{-2s} \right], \qquad (\text{D.18})$$

with $\zeta_c(s)$ for S^2 defined by

$$\zeta_c(s) = 2\zeta_H\left(2s-1,\frac{1}{2}\right), \qquad (\text{D.19})$$

which is the zeta function for the bound state mode in the conformally coupled case, evaluated explicitly in [117] in the case of $d+1 = 3$. Essentially the minimally coupled case requires summing from $J = 1$ instead of $J = 0$ in (D.18), i.e. we have to subtract out the null eigenvalue.

Similar to the case discussed in [121] (subsection 11.3, Eq. (11.73), pp. 80) there is a pole in the above Hurwitz zeta function at $s = 1$, which can be simply inferred from the relation Eq. (4.61). As discussed in [117] a suitable way to deal with the the pole at $s = 1$ is to apply the improved zeta function method, described in [150, 151], which leads to a expression for the amplitude

$$\langle \phi^2(x) \rangle_{\text{bs}} = \lim_{s \to 1} \frac{d}{ds} \left[(s-1)\, \zeta_{\text{bs}}(s)(x) \right] . \qquad (\text{D.20})$$

Note, the above expression agrees with the usual definition when there is no pole at $s = 1$.

Applying the above equation to our case we obtain

$$H^2 \langle \tilde{\phi}^2(0) \rangle_{\text{bs}} = 2 f_0^2(0) \left(2\ln\left(\frac{\mu}{H}\right) - 2\psi(1/2) - \delta_{\xi,0} \left(\frac{1}{2}\right)^{-2} \right.$$

$$\left. + \sum_{J=1}^{\infty} \left[2\left(\frac{\nu}{\sigma}\right)^{2J} \zeta_H(2J+1,\frac{1}{2}) - \delta_{\xi,0} \left(\frac{1}{2}\right)^{-2} \right] \right).$$

$$(\text{D.21})$$

Next, we determine $f_0(0)$. The normalized bound state solution is

$$f_0^2(z) = \frac{1}{2\sigma} \cosh^{-2\nu}(x) \left(\int_0^{\infty} dy \, \cosh^{-2\nu}(y) \right)^{-1} . \qquad (\text{D.22})$$

Thus,

$$f_0^2(0) = \frac{1}{2\sigma} \left(\int_0^{\infty} dy \, \cosh^{-2\nu}(y) \right)^{-1} . \qquad (\text{D.23})$$

Note that for the conformally coupled case $\xi = \xi_c$, $\nu = 0$ and therefore, the amplitude of the bound state vanishes. This agrees with the result found in [117], for the thin brane case. In fact numerical plots of the amplitude for

the bound state mode versus the brane thickness show that the bound state mode is independent of the brane thickness.

Finally, for the bound state mode, we obtain the normalized amplitude as

$$
\begin{aligned}
H^2 \langle \tilde{\phi}^2(0) \rangle_{\text{bs}} &= \frac{1}{\sigma} \Big(2\ln\Big(\frac{\mu}{H}\Big) - 2\psi(1/2) - \delta_{\xi,0}\Big(\frac{1}{2}\Big)^{-2} \\
&\quad + \sum_{J=1}^{\infty} \Big[2\Big(\frac{\nu}{\sigma}\Big)^{2J} \zeta_H(2J+1,\tfrac{1}{2}) - \delta_{\xi,0}\Big(\frac{1}{2}\Big)^{-2} \Big] \Big) \\
&\quad \times \Big(\int_0^{\infty} dy \cosh^{-2\nu}(y) \Big)^{-1} .
\end{aligned}
\tag{D.24}
$$

This can now be compared with the result for that of the KK modes.

D.2.2 On the four-sphere (four-dimensional de Sitter brane)

The zeta function for the bound state can be written as

$$
\tilde{\zeta}_{\text{bs}}(s) = 2\frac{1}{H^2}\Big(\frac{\mu}{H}\Big)^{2(s-1)} f_0^2(0)\bar{\zeta}_{\text{bs}}(s),
\tag{D.25}
$$

where

$$
\bar{\zeta}_{\text{bs}}(s) := \frac{1}{3}\sum_{j=0}^{\infty} \frac{(j+3/2)(j+1)(j+2)}{[(j+3/2)^2 - (\nu/\sigma)^2]^s}
\tag{D.26}
$$

is the zeta function for a massive scalar field on S^4. For the S^4 geometry, the zeta function for a massless, conformally coupled scalar field is given by

$$
\zeta_c(z) = \frac{1}{3}\Big[\zeta_H(2s-3,\tfrac{3}{2}) - \frac{1}{4}\zeta_H(2s-1,\tfrac{3}{2})\Big].
\tag{D.27}
$$

Thus, the dS zeta function for a general mass can be written as a summation over the massless conformal zeta functions (by employing the binomial

expansion)

$$
\begin{aligned}
\bar{\zeta}_{\text{bs}}(s) &= \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\,\Gamma(s)} \left[\left(\frac{\nu}{\sigma}\right)^{2J} \zeta_c(s+J) - \delta_{\xi,0} \left(\frac{3}{2}\right)^{-2s} \right] \\
&= \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\,\Gamma(s)} \left[\frac{1}{3} \left(\frac{\nu}{\sigma}\right)^{2J} \left\{ \zeta_H(2s-3+2J,\frac{3}{2}) - \frac{1}{4}\zeta_H(2s-1+2J,\frac{3}{2}) \right\} \right. \\
&\quad \left. - \delta_{\xi,0} \left(\frac{3}{2}\right)^{-2s} \right] \\
&= \frac{1}{3} \left[\zeta_H(2s-3,\frac{3}{2}) + s\left(\frac{\nu}{\sigma}\right)^2 \zeta_H(2s-1,\frac{3}{2}) \right. \\
&\quad - \frac{1}{4}\zeta_H(2s-1,\frac{3}{2}) - \frac{1}{4}s\left(\frac{\nu}{\sigma}\right)^2 \zeta_H(2s+1,\frac{3}{2}) \\
&\quad + \sum_{J=2}^{\infty} \frac{\Gamma(s+J)}{J!\,\Gamma(s)} \left(\frac{\nu}{\sigma}\right)^{2J} \zeta_H(2s-3+2J,\frac{3}{2}) \\
&\quad \left. - \frac{1}{4}\sum_{J=2}^{\infty} \frac{\Gamma(s+J)}{J!\,\Gamma(s)} \left(\frac{\nu}{\sigma}\right)^{2J} \zeta_H(2s-1+2J,\frac{3}{2}) \right] \\
&\quad - \sum_{J=0}^{\infty} \frac{\Gamma(s+J)}{J!\Gamma(s)} \delta_{\xi,0} \left(\frac{3}{2}\right)^{-2s} .
\end{aligned}
\tag{D.28}
$$

Now, we can evaluate the squared amplitude of the bound state from Eq. (D.20). The normalization of the bulk mode is the same as the $d = 2$ case and at $z = 0$ we obtain Eq. (D.23) with ν for $d = 4$,

$$
\nu = \frac{1}{2}\left(\sqrt{1 + (3 - 16\xi)(3\sigma + 2\sigma^2)} - 1\right) .
\tag{D.29}
$$

The resultant bound state amplitude is

$$
\begin{aligned}
H^2 \langle \tilde{\phi}^2(0) \rangle_{\text{bs}} &= \frac{1}{2\sigma} \left(\int_0^{\infty} dy \, \cosh^{-2\nu}(y) \right)^{-1} \\
&\times \left\{ \left(-\frac{1}{6} + \frac{2}{3}\left(\frac{\nu}{\sigma}\right)^2 \right) \ln\left(\frac{\mu}{H}\right) + \frac{2}{3}\zeta_H(-1,\frac{3}{2}) + \frac{1}{6}\psi(3/2) \right. \\
&\quad - \left(\frac{\nu}{\sigma}\right)^2 \left(-\frac{1}{3} + \frac{2}{3}\psi(3/2) + \frac{1}{6}\zeta_H(3,\frac{3}{2}) \right) \\
&\quad + \frac{2}{3}\sum_{J=2}^{\infty} \left(\frac{\nu}{\sigma}\right)^{2J} \left(\zeta_H(2J-1,\frac{3}{2}) - \frac{1}{4}\zeta_H(2J+1,\frac{3}{2}) \right) \\
&\quad \left. - 2\sum_{J=0}^{\infty} \delta_{\xi,0} \left(\frac{3}{2}\right)^{-2} \right\} .
\end{aligned}
\tag{D.30}
$$

Appendix E

Appendices for analyzing linearized gravity in the Einstein Gauss-Bonnet braneworld

In this Appendix, we give harmonic functions to analyze the linearized gravity on a dS brane in the EGB theory. We also show that there is a tachyonic (unstable) bound state in a dS two-brane system in the EGB theory.

E.1 Harmonic Functions on a de Sitter geometry

Here, we consider the harmonics on the de Sitter spacetime with curvature radius H^{-1}. They are obtained by the Lorentzian generalization of the tensor harmonics on an n-dimensional constant curvature Riemannian space [152]. We focus on the tensor-type and scalar-type harmonics.

E.1.1 Tensor-type harmonics

The tensor-type tensor harmonics satisfy

$$\left(\Box_4 - \left(p^2 + 17/4\right)H^2\right)Y_{\mu\nu}^{(p,2)}(x^\mu) = 0, \tag{E.1}$$

which corresponds to the four-dimensional massive gravitons with mass-squared $m^2 H^2 = (p^2 + 9/4)H^2$. They satisfy the transverse-traceless condition,

$$Y^{(p,2)\mu}{}_\mu = Y^{(p,2)\nu}{}_{\mu}{}_{|\nu} = 0. \tag{E.2}$$

In reality, the tensor harmonics have three more indices for the spatial eigenvalues. If we adopt the flat slicing,

$$ds^2 = -dt^2 + H^{-2}e^{2Ht}\delta_{ij}dx^i dx^j, \tag{E.3}$$

127

we can use the standard Fourier modes $e^{i k \cdot x}$, and the spatial indices will be continuous. In addition, we also have discrete indices σ that describe the polarization degrees of freedom (five in four-dimensions). However, for notational simplicity, we omit these indices.

We ortho-normalize the tensor harmonics as

$$\int d^4 x \sqrt{-\gamma}\, Y^{(p,2)}_{\mu\nu} Y^{*(p',2)\mu\nu} = \delta(p - p')\delta^3(\mathbf{k} - \mathbf{k}')\delta_{\sigma,\sigma'}. \tag{E.4}$$

Although we have no explicit proof for the completeness, due to our poor knowledge, we assume that $Y^{(p,2)}_{\mu\nu}$ for $-\infty < p < \infty$ constitute a complete set for the space of transverse-traceless tensors.

E.1.2 Scalar-type harmonics

The scalar-type harmonics $Y^{(p,0)}(x^\mu)$ satisfy the equation for a scalar field with mass-squared $m^2 H^2 = (p^2 + 9/4)H^2$,

$$\left(\Box_4 - \left(p^2 + \frac{9}{4}\right)H^2\right)Y^{(p,0)}(x^\mu) = 0. \tag{E.5}$$

We assume they satisfy the ortho-normality condition,

$$\int d^4 x \sqrt{-\gamma}\, Y^{(p,0)} Y^{*(p',0)} = \delta(p - p')\delta^3(\mathbf{k} - \mathbf{k}'). \tag{E.6}$$

From $Y^{(p,0)}$, the ortho-normalized scalar-type vector harmonics are constructed as

$$Y^{(p,0)}_\mu = \frac{i}{H\sqrt{p^2 + 9/4}} D_\mu Y^{(p,0)}, \tag{E.7}$$

which satisfy

$$\int d^4 x \sqrt{-\gamma}\, Y^{(p,0)}_\mu Y^{*(p',0)\mu} = \delta(p - p')\delta^3(\mathbf{k} - \mathbf{k}'). \tag{E.8}$$

The trace-free and divergence-free scalar-type tensor harmonics are constructed, respectively, as

$$\bar{Y}^{(p,0)}_{\mu\nu} = N_p\left[D_\mu D_\nu Y^{(p,0)} - \frac{1}{4}\left(p^2 + \frac{9}{4}\right)\gamma_{\mu\nu}H^2 Y^{(p,0)}\right]$$

$$Y^{(p,0)}_{\mu\nu} = N_p\left[D_\mu D_\nu Y^{(p,0)} - \left(p^2 + \frac{21}{4}\right)\gamma_{\mu\nu}H^2 Y^{(p,0)}\right]$$

$$= \bar{Y}^{(p,0)}_{\mu\nu} - \frac{3}{4}N_p\left(p^2 + \frac{25}{4}\right)H^2 \gamma_{\mu\nu} Y^{(p,0)}, \tag{E.9}$$

where

$$|N_p|^2 = \frac{1}{3\left(p^2 + 21/4\right)\left(p^2 + 25/4\right)H^4}, \tag{E.10}$$

Without loss of generality, we assume that N_p is real and positive. The scalar-type divergence-free tensor harmonics $Y_{\mu\nu}^{(p,0)}$ satisfy the ortho-normality condition,

$$\int d^4 x \sqrt{-\gamma}\, Y_{\mu\nu}^{(p,0)} Y^{*(p',0)\mu\nu} = \delta(p - p')\delta^3(k - k'). \qquad (\text{E.11})$$

E.2   Tachyonic bound state in de Sitter two-brane system

In [139], Charmousis and Dufaux showed that for the Minkowski two-brane system there exists a tachyonic bound state on the negative tension brane. This fact implies that the Minkowski two-brane system is unstable under the linear perturbation. Following [139], we show that there exits a tachyonic bound state also for the de Sitter two-brane system.

E.2.1   Possibility of a negative norm state

We consider a de Sitter two-brane system. One of the branes located at a smaller radius of the AdS space has a negative tension. We discuss only the bulk gravitational perturbations. The matter perturbations on each brane are not taken into account.

The bulk component of the perturbed Einstein Gauss-Bonnet equation including the boundary branes are written in the Sturm-Liouville form as

$$\left\{ \left(b^4 - \bar{\alpha}\,\ell^2 \left(b^2 b'^2 - b^2 H^2 \right) \right) \psi_{p,y} \right\}_{,y}$$

$$= -b^2 \left(1 - \bar{\alpha}\,\ell^2 \frac{b''}{b} \right) \left(p^2 + \frac{9}{4} \right) H^2 \psi_p. \qquad (\text{E.12})$$

Using Eq. (E.12), the boundary condition on each brane is derived. For $H = 0$ and $b(y) = e^{-|y|/\ell}$, Eq. (E.12) naturally reduces to the Minkowski version, Eq. (8) in [139].

1) **On positive tension brane**

Imposing the $\mathbb{Z}_2$ symmetry, the warp factor around the positive tension brane is expressed as

$$b(y) = H\ell \, \sinh\left(\frac{y_+ - |y - y_+|}{\ell} \right). \qquad (\text{E.13})$$

Integrating Eq. (E.12) around $y = y_+$ and using the $\mathbb{Z}_2$-symmetry,

$$\partial_y \psi_p(y_+ - 0) = \frac{\eta}{\ell} \frac{(p^2 + 9/4)\cosh(y_+/\ell)}{\sinh^3(y_+/\ell)} \psi_p(y_+), \qquad (\text{E.14})$$

where

$$\eta := \frac{\bar{\alpha}}{1 - \bar{\alpha}}. \qquad (\text{E.15})$$

2) On negative tension brane

Similarly, the $\mathbb{Z}_2$ symmetry gives the warp factor around the negative tension brane as

$$b(y) = H\ell \, \sinh\left(\frac{|y - y_-| + y_-}{\ell}\right).$$ (E.16)

Integrating Eq. (E.12) around $y = y_-$ and using the $\mathbb{Z}_2$-symmetry,

$$\partial_y \psi_p(y_- + 0) = \frac{\eta}{\ell} \frac{(p^2 + 9/4) \cosh(y_-/\ell)}{\sinh^3(y_-/\ell)} \psi_p(y_-).$$ (E.17)

For both branes, the boundary conditions are of a mixed (Robin) type. This renders us impossible to prove the positivity of the norm. Namely, we have

$$\int_{y_-}^{y_+} dy \left(b^4 - \bar{\alpha}\,\ell^2\left(b^2 b'^2 - b^2 H^2\right)\right)(\partial_y \psi_p)^2$$

$$= (H\ell)^4 \left(p^2 + \frac{9}{4}\right)\left[\frac{\bar{\alpha}}{2\ell}\left(\sinh(2y_+/\ell)\psi_p^2(y_+) - \sinh(2y_-/\ell)\psi_p^2(y_-)\right)\right.$$

$$\left.+\frac{(1 - \bar{\alpha})}{\ell^2} \int_{y_-}^{y_+} dy \, \sinh^2(y/\ell)\psi_p^2(y)\right].$$ (E.18)

Thus the norm is no longer positive definite for $p^2 + 9/4 > 0$.

E.2.2 Condition for the existence of tachyonic bound state

In order to determine whether a tachyonic bound state exists, we need to analyze the mass spectrum. The tachyonic eigenmode, if it exists, is written by

$$\psi_q(y) = \frac{1}{\sinh^{3/2}(y/\ell)}\left[A_q P_{3/2}^{-q}(\cosh(y/\ell)) + B_q P_{3/2}^q(\cosh(y/\ell))\right],$$ (E.19)

where $m^2 = -\mu^2$, $q := \sqrt{\mu^2 + 9/4}$, and $q^2 = -p^2$. The y-derivative of it is

$$\partial_y \psi_q = -\frac{1}{\ell \sinh^{5/2}(y/\ell)}\left[\left(\frac{3}{2} - q\right)A_q P_{1/2}^{-q}(\cosh(y/\ell)) + \left(\frac{3}{2} + q\right)B_q P_{1/2}^q(\cosh(y/\ell))\right].$$

(E.20)

Using the boundary condition on each brane, Eqs. (E.14) and (E.17), we

obtain

$$A_q\left(\frac{3}{2} - q\right)\left((z_+^2 - 1)P_{1/2}^{-q}(z_+) + \eta\left(\frac{3}{2} + q\right)z_+P_{3/2}^{-q}(z_+)\right)$$

$$+B_q\left(\frac{3}{2} + q\right)\left((z_+^2 - 1)P_{1/2}^{q}(z_+) + \eta\left(\frac{3}{2} - q\right)z_+P_{3/2}^{q}(z_+)\right) = 0,$$

$$A_q\left(\frac{3}{2} - q\right)\left((z_-^2 - 1)P_{1/2}^{-q}(z_-) + \eta\left(\frac{3}{2} + q\right)z_-P_{3/2}^{-q}(z_-)\right)$$

$$+B_q\left(\frac{3}{2} + q\right)\left((z_-^2 - 1)P_{1/2}^{q}(z_-) + \eta\left(\frac{3}{2} - q\right)z_-P_{3/2}^{q}(z_-)\right) = 0 \quad\text{(E.21)}$$

where $z_\pm = \cosh(y_\pm/\ell)$.

For a non-trivial solution for A_q and B_q to exist, the determinant must vanish. Thus

$$\left((z_+^2 - 1)P_{1/2}^{-q}(z_+) + \eta\left(\frac{3}{2} + q\right)z_+P_{3/2}^{-q}(z_+)\right)$$

$$\left((z_-^2 - 1)P_{1/2}^{q}(z_-) + \eta\left(\frac{3}{2} - q\right)z_-P_{3/2}^{q}(z_-)\right)$$

$$-\left((z_-^2 - 1)P_{1/2}^{-q}(z_-) + \eta\left(\frac{3}{2} + q\right)z_-P_{3/2}^{-q}(z_-)\right)$$

$$\left((z_+^2 - 1)P_{1/2}^{q}(z_+) + \eta\left(\frac{3}{2} - q\right)z_+P_{3/2}^{q}(z_+)\right) = 0. \quad\text{(E.22)}$$

The pole at $q = 3/2$, which corresponds to the zero mode, is divided out in deriving Eq. (E.22). If there exists a solution of Eq. (E.22) at $q > 3/2$, it implies the existence of a tachyonic bound state.

E.2.3 Existence of a tachyonic bound state

From Eq. (E.22),

$$\frac{(z_-^2 - 1)P_{1/2}^{q}(z_-) + \eta\left(\frac{3}{2} - q\right)z_-P_{3/2}^{q}(z_-)}{(z_-^2 - 1)P_{1/2}^{-q}(z_-) + \eta\left(\frac{3}{2} + q\right)z_-P_{3/2}^{-q}(z_-)}$$

$$= \frac{(z_+^2 - 1)P_{1/2}^{q}(z_+) + \eta\left(\frac{3}{2} - q\right)z_-P_{3/2}^{q}(z_-)}{(z_+^2 - 1)P_{1/2}^{-q}(z_+) + \eta\left(\frac{3}{2} + q\right)z_+P_{3/2}^{-q}(z_+)}. \quad\text{(E.23)}$$

Using the definition of the Legendre functions [137],

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1 - \mu)}\left(\frac{z + 1}{z - 1}\right)^{\mu/2} {}_2F_1\left[-\nu, \nu + 1; 1 - \mu; \frac{1 - z}{2}\right], \quad\text{(E.24)}$$

we see that the left-hand-side of Eq. (E.23) is generally much larger than the right-hand-side for $q \gg 1$ for fixed z_+ and z_-. Therefore, in order for this equation to be satisfied, we must have

$$q - \frac{3}{2} \simeq \frac{(z_-^2 - 1)P_{1/2}^{q}(z_-)}{\eta z_-P_{3/2}^{q}(z_-)} \rightarrow \frac{z_-^2 - 1}{\eta z_-} \quad\text{for } q \rightarrow \infty. \quad\text{(E.25)}$$

This is a consistent solution for $\eta \ll 1$. Thus a tachyonic bound state exists in the de Sitter brane case as well.

The tachyon mass is given by

$$\mu H = \sqrt{q^2 - 9/4} \, H \simeq \frac{(z_-^2 - 1)H\ell}{\eta z_- \ell} \, . \tag{E.26}$$

In the low energy limit, we have $z_+ > z_- \gg 1$ and $H\ell \simeq 1/z_+ \ll 1$. Hence, the above reduces to

$$\mu H \simeq \frac{\Omega}{\eta \ell} , \tag{E.27}$$

where

$$\Omega := \frac{b(z_-)}{b(z_+)} \simeq \frac{z_-}{z_+} \sim e^{-(y_+ - y_-)/\ell} . \tag{E.28}$$

This agrees with the result for the Minkowski brane [139].

On the other hand, in the high energy limit, $H\ell \gg 1$, we have

$$\mu H \simeq \frac{\Omega^2 H}{\eta (H\ell)^2} \ll \frac{H}{\eta} . \tag{E.29}$$

Thus the high background expansion rate of the brane suppresses the tachyonic mass, giving a tendency to stabilize the two-brane system.

Bibliography

[1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921).

[2] O. Klein, Z. Phys. **37**, 895 (1926).

[3] P. Horava and E. Witten, Nucl. Phys. B **460**, 506 (1996) [arXiv:hep-th/9510209].

[4] P. Horava and E. Witten, Nucl. Phys. B **475**, 94 (1996) [arXiv:hep-th/9603142].

[5] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].

[6] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. D **59**, 086004 (1999) [arXiv:hep-ph/9807344].

[7] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].

[8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[9] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[10] D. Langlois, Prog. Theor. Phys. Suppl. **148**, 181 (2003) [arXiv:hep-th/0209261].

[11] P. Brax and C. van de Bruck, Class. Quant. Grav. **20**, R201 (2003) [arXiv:hep-th/0303095];

[12] R. Maartens, Living Rev. Rel. **7**, 7 (2004) [arXiv:gr-qc/0312059].

[13] P. Brax, C. van de Bruck and A. C. Davis, Rept. Prog. Phys. **67**, 2183 (2004) [arXiv:hep-th/0404011].

[14] W. Israel, Nuovo Cim. B **44S10**, 1 (1966) [Erratum-ibid. B **48**, 463 (1967 NUCIA,B44,1.1966)].

[15] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000) [arXiv:hep-th/9911055].

[16] S. Kanno and J. Soda, Phys. Rev. D **66**, 043526 (2002) [arXiv:hep-th/0205188].

[17] D. Langlois, R. Maartens and D. Wands, Phys. Lett. B **489**, 259 (2000) [arXiv:hep-th/0006007].

[18] J. Garriga and M. Sasaki, Phys. Rev. D **62**, 043523 (2000) [arXiv:hep-th/9912118].

[19] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565**, 269 (2000) [arXiv:hep-th/9905012].
P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477** (2000) 285 [arXiv:hep-th/9910219].

[20] P. Kraus, JHEP **9912**, 011 (1999) [arXiv:hep-th/9910149].

[21] D. Ida, JHEP **0009**, 014 (2000) [arXiv:gr-qc/9912002].

[22] V. Barger, J. P. Kneller, H. S. Lee, D. Marfatia and G. Steigman, Phys. Lett. B **566**, 8 (2003) [arXiv:hep-ph/0305075].

[23] K. Ichiki, M. Yahiro, M. Orito and G. J. Mathews, Phys. Rev. D **66**, 043521 (2002) [arXiv:astro-ph/0203272]
M. Orito, T. Kajino, K. Ichiki, M. Yahiro, G. J. Mathews and Y. Wang, Nucl. Phys. A **719**, 1 (2003).

[24] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999) [arXiv:hep-ph/9907447].

[25] T. Tanaka and X. Montes, Nucl. Phys. B **582**, 259 (2000) [arXiv:hep-th/0001092].

[26] K. i. Maeda and D. Wands, Phys. Rev. D **62**, 124009 (2000) [arXiv:hep-th/0008188].

[27] E. E. Flanagan, S. H. H. Tye and I. Wasserman, Phys. Lett. B **522**, 155 (2001) [arXiv:hep-th/0110070].

[28] S. Kobayashi, K. Koyama and J. Soda, Phys. Lett. B **501**, 157 (2001) [arXiv:hep-th/0009160].

[29] Y. Himemoto and M. Sasaki, Phys. Rev. D **63**, 044015 (2001) [arXiv:gr-qc/0010035].

[30] N. Sago, Y. Himemoto and M. Sasaki, Phys. Rev. D **65**, 024014 (2002) [arXiv:gr-qc/0104033].

[31] Y. Himemoto, T. Tanaka and M. Sasaki, Phys. Rev. D **65**, 104020 (2002) [arXiv:gr-qc/0112027].

[32] D. Langlois and M. Sasaki, Phys. Rev. D **68**, 064012 (2003) [arXiv:hep-th/0302069].

[33] S. Kanno and J. Soda, Gen. Rel. Grav. **36**, 689 (2004) [arXiv:hep-th/0303203].

[34] M. Minamitsuji, Y. Himemoto and M. Sasaki, Phys. Rev. D **68**, 024016 (2003) [arXiv:gr-qc/0303108].

[35] K. Koyama and K. Takahashi, Phys. Rev. D **67**, 103503 (2003) [arXiv:hep-th/0301165].

[36] K. Koyama and K. Takahashi, Phys. Rev. D **68**, 103512 (2003) [arXiv:hep-th/0307073].

[37] T. Kobayashi and T. Tanaka, Phys. Rev. D **69**, 064037 (2004) [arXiv:hep-th/0311197].

[38] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D **62**, 046008 (2000) [arXiv:hep-th/9909134].

[39] M. Gremm, Phys. Lett. B **478**, 434 (2000) [arXiv:hep-th/9912060].

[40] C. Csaki, J. Erlich, T. J. Hollowood and Y. Shirman, Nucl. Phys. B **581**, 309 (2000) [arXiv:hep-th/0001033].

[41] I. Y. Arefeva, I. V. Volovich, W. Muck, K. S. Viswanathan and M. G. Ivanov, "Quantization, gauge theory, and strings," Vol. 1, pp. 271 (Moscow, 2000).

[42] M. Giovannini, Phys. Rev. D **65**, 064008 (2002) [arXiv:hep-th/0106131].

[43] S. Kobayashi, K. Koyama and J. Soda, Phys. Rev. D **65**, 064014 (2002) [arXiv:hep-th/0107025].

[44] N. Sasakura, JHEP **0202**, 026 (2002) [arXiv:hep-th/0201130].

[45] N. Sasakura, Phys. Rev. D **66**, 065006 (2002) [arXiv:hep-th/0203032].

[46] K. Ghoroku and M. Yahiro, arXiv:hep-th/0305150.

[47] K. A. Bronnikov and B. E. Meierovich, Grav. Cosmol. **9**, 313 (2003) [arXiv:gr-qc/0402030].

[48] A. Z. Wang, Phys. Rev. D **66**, 024024 (2002) [arXiv:hep-th/0201051].

[49] R. Guerrero, R. Ortiz, R. O. Rodriguez and R. Torrealba, arXiv:gr-qc/0504080.

[50] R. C. Myers, Phys. Rev. D **36**, 392 (1987).

[51] S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D **65**, 023521 (2002) [arXiv:hep-th/0108172].

[52] N. Deruelle and J. Madore, arXiv:gr-qc/0305004.

[53] J. E. Kim, B. Kyae and H. M. Lee, Nucl. Phys. B **582** (2000) 296 [Erratum-ibid. B **591** (2000) 587] [arXiv:hep-th/0004005].

[54] B. Abdesselam and N. Mohammedi, Phys. Rev. D **65**, 084018 (2002) [arXiv:hep-th/0110143].

[55] C. Charmousis and J. F. Dufaux, Class. Quant. Grav. **19**, 4671 (2002) [arXiv:hep-th/0202107].

[56] E. Gravanis and S. Willison, Phys. Lett. B **562**, 118 (2003) [arXiv:hep-th/0209076].

[57] J. E. Lidsey and N. J. Nunes, Phys. Rev. D **67**, 103510 (2003) [arXiv:astro-ph/0303168].

[58] M. H. Dehghani, arXiv:hep-th/0404118.

[59] K. i. Maeda and T. Torii, Phys. Rev. D **69**, 024002 (2004) [arXiv:hep-th/0309152].

[60] S. Tsujikawa, M. Sami and R. Maartens, Phys. Rev. D **70**, 063525 (2004) [arXiv:astro-ph/0406078].

[61] K. s. Aoyanagi and K. i. Maeda, Phys. Rev. D **70**, 123506 (2004) [arXiv:hep-th/0408008].

[62] M. Sami, N. Savchenko and A. Toporensky, Phys. Rev. D **70**, 123528 (2004) [arXiv:hep-th/0408140].

[63] Ph. Brax, N. Chatillon and D. A. Steer, Phys. Lett. B **608**, 130 (2005) [arXiv:hep-th/0411058].

[64] M. C. Bento, R. Gonzalez Felipe and N. M. C. Santos, Phys. Rev. D **71**, 123517 (2005) [arXiv:hep-ph/0504113].

[65] R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005) [arXiv:hep-th/0505186].

[66] L. Amendola, C. Charmousis and S. C. Davis, arXiv:hep-th/0506137.

[67] G. L. Alberghi and A. Tronconi, Phys. Rev. D **73**, 027702 (2006) [arXiv:hep-ph/0510267].

[68] G. Panotopoulos, arXiv:hep-ph/0511040.

[69] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 084013 (2000) [arXiv:hep-th/0003271].

[70] R. G. Cai, Phys. Rev. D **65**, 084014 (2002) [arXiv:hep-th/0109133].

[71] M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002) [arXiv:hep-th/0112045].

[72] M. Cvitan, S. Pallua and P. Prester, Phys. Lett. B **546**, 119 (2002) [arXiv:hep-th/0207265].

[73] M. Cvitan, S. Pallua and P. Prester, Phys. Lett. B **555**, 248 (2003) [arXiv:hep-th/0212029].

[74] T. Clunan, S. F. Ross and D. J. Smith, Class. Quant. Grav. **21**, 3447 (2004) [arXiv:gr-qc/0402044].

[75] R. Konoplya, Phys. Rev. D **71**, 024038 (2005) [arXiv:hep-th/0410057].

[76] T. Torii and H. Maeda, Phys. Rev. D **71**, 124002 (2005) [arXiv:hep-th/0504127].

[77] T. Torii and H. Maeda, Phys. Rev. D **72**, 064007 (2005) [arXiv:hep-th/0504141].

[78] M. H. Dehghani and R. B. Mann, Phys. Rev. D **72**, 124006 (2005) [arXiv:hep-th/0510083].

[79] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000) [arXiv:hep-th/0005016].

[80] K. i. Maeda, S. Mizuno and T. Torii, Phys. Rev. D **68**, 024033 (2003) [arXiv:gr-qc/0303039].

[81] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002) [arXiv:astro-ph/0105068].

[82] M. A. Luty, M. Porrati and R. Rattazzi, JHEP **0309**, 029 (2003) [arXiv:hep-th/0303116].

[83] K. Koyama, Phys. Rev. D **72**, 123511 (2005) [arXiv:hep-th/0503191].

[84] R. A. Brown, R. Maartens, E. Papantonopoulos and V. Zamarias, JCAP **0511**, 008 (2005) [arXiv:gr-qc/0508116].

[85] A. Lue, Phys. Rept. **423**, 1 (2006) [arXiv:astro-ph/0510068].

[86] K. Koyama and R. Maartens, arXiv:astro-ph/0511634.

[87] J. S. Alcaniz and N. Pires, Phys. Rev. D **70**, 047303 (2004) [arXiv:astro-ph/0404146].

[88] A. Lue, R. Scoccimarro and G. D. Starkman, Phys. Rev. D **69**, 124015 (2004) [arXiv:astro-ph/0401515].

[89] V. Sahni, arXiv:astro-ph/0502032.

[90] I. Sawicki and S. M. Carroll, arXiv:astro-ph/0510364.

[91] T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D **62**, 024012 (2000) [arXiv:gr-qc/9910076].

[92] M. Minamitsuji and M. Sasaki, Phys. Rev. D **70**, 044021 (2004) [arXiv:gr-qc/0312109].

[93] M. Sasaki, T. Shiromizu and K. i. Maeda, Phys. Rev. D **62**, 024008 (2000) [arXiv:hep-th/9912233].

[94] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).

[95] S. A. Hayward, Phys. Rev. D **53**, 1938 (1996) [arXiv:gr-qc/9408002].

[96] I. Racz, Class. Quant. Grav. **23**, 115 (2006) [arXiv:gr-qc/0511052].

[97] See e.g.,
S. W. Hawking, G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973).
R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

[98] M. Minamitsuji, M. Sasaki and D. Langlois, Phys. Rev. D **71**, 084019 (2005) [arXiv:gr-qc/0501086].

[99] P. C. Vaidya Nature **171**, 260 (1953)
R. W. Lindqiist, R. A. Schwarz, C. W. Misner, Phys. Rev. **137**, B1364 (1965).

[100] D. Langlois and L. Sorbo, Phys. Rev. D **68**, 084006 (2003) [arXiv:hep-th/0306281].

[101] D. Langlois, L. Sorbo and M. Rodriguez-Martinez, Phys. Rev. Lett. **89**, 171301 (2002) [arXiv:hep-th/0206146].

[102] E. Leeper, R. Maartens and C. F. Sopuerta, Class. Quant. Grav. **21**, 1125 (2004) [arXiv:gr-qc/0309080].

[103] L. A. Gergely, E. Leeper and R. Maartens, Phys. Rev. D **70**, 104025 (2004) [arXiv:gr-qc/0408084].

[104] I. R. Vernon and D. Jennings, JCAP **0507**, 011 (2005) [arXiv:hep-th/0412083].

[105] I. H. Dwivedi and P. S. Joshi, Class. Quant. Grav. **6**, 1599 (1989).
I. H. Dwivedi and P. S. Joshi, Class. Quant. Grav. **8**, 1339 (1989).

[106] K. Lake, Phys. Rev. D **43**, 1416 (1991)

[107] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D **45**, 2147 (1992)

[108] S. M. Wagh and S. D. Maharaj, Gen. Rel. Grav. **31**, 975 (1999)

[109] S. G. Ghosh and A. Beesham, Phys. Rev. D **61**, 067502 (2000)

[110] S. G. Ghosh and N. Dadhich, Phys. Rev. D **64**, 047501 (2001) [arXiv:gr-qc/0105085].

[111] U. Gen and M. Sasaki, Prog. Theor. Phys. **105**, 591 (2001) [arXiv:gr-qc/0011078].

[112] M. Minamitsuji, W. Naylor and M. Sasaki, Nucl. Phys. B, **737**, 121(2006) [arXiv:hep-th/0508093].

[113] M. Minamitsuji, W. Naylor and M. Sasaki, Phys. Lett. B **633**, 607 (2006) [arXiv:hep-th/0510117].

[114] K. D. Olum and N. Graham, Phys. Lett. B **554**, 175 (2003) [arXiv:gr-qc/0205134].

[115] N. Graham and K. D. Olum, Phys. Rev. D **67**, 085014 (2003) [Erratum-ibid. D **69**, 109901 (2004)] [arXiv:hep-th/0211244].

[116] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, Nucl. Phys. B **645** (2002) 49 [arXiv:hep-th/0207120].

[117] W. Naylor and M. Sasaki, Prog. Theor. Phys. **113**, 535 (2005) [arXiv:hep-th/0411155].

[118] O. Pujolas and M. Sasaki, JCAP **0509**, 002 (2005) [arXiv:hep-th/0507239].

[119] A. A. Saharian, Phys. Rev. D **70**, 064026 (2004) [arXiv:hep-th/0406211].

[120] E. Elizalde, Lect. Notes Phys. **M35** (1995) 1.

[121] R. Camporesi, Phys. Rept. **196** (1990) 1.

[122] M. Abramowtz and I. A. Stegun, "Handbook of Mathematical Functions," (Dover, New York, 1972).

[123] A. Flachi and D. J. Toms, Nucl. Phys. B **610**, 144 (2001) [arXiv:hep-th/0103077],

[124] A. Knapman and D. J. Toms, Phys. Rev. D **69**, 044023 (2004) [arXiv:hep-th/0309176].

[125] R. Hofmann, P. Kanti and M. Pospelov, Phys. Rev. D **63**, 124020 (2001) [arXiv:hep-ph/0012213].

[126] R. Casadio, A. Gruppuso and G. Venturi, Phys. Lett. B **495**, 378 (2000) [arXiv:hep-th/0010065].

[127] D. Schwartz-Perlov and K. D. Olum, Phys. Rev. D **72**, 065013 (2005) [arXiv:hep-th/0507013].

[128] D. Iellici and V. Moretti, Phys. Lett. B **425**, 33 (1998) [arXiv:gr-qc/9705077].

[129] M. A. Rubin and C. R. Ordonez, J. Math. Phys. **26**, 65 (1985).

[130] M. Minamitsuji and M. Sasaki, Prog. Theor. Phys. **112**, 451 (2004) [arXiv:hep-th/0404166].

[131] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).

[132] W. Israel, Nuovo Cim. B **44S10**, 1 (1966) [Erratum-ibid. B **48**, 463 (1967 NU-CIA,B44,1.1966)].

[133] S. C. Davis, Phys. Rev. D **67**, 024030 (2003) [arXiv:hep-th/0208205].

[134] N. Deruelle and M. Sasaki, Prog. Theor. Phys. **110**, 441 (2003) [arXiv:gr-qc/0306032].

[135] Y. M. Cho, I. P. Neupane and P. S. Wesson, Nucl. Phys. B **621**, 388 (2002) [arXiv:hep-th/0104227].

[136] U. Gen and M. Sasaki, Prog. Theor. Phys. **108**, 471 (2002) [arXiv:gr-qc/0201031].

[137] See e.g., H. Bateman, *Higher Transcendental Functions* Vol. I (McGraw-Hill, New York, 1953).

[138] H. Yoshiguchi and K. Koyama, Phys. Rev. D **70**, 043513 (2004) [arXiv:hep-th/0403097].

[139] C. Charmousis and J. F. Dufaux, Phys. Rev. D **70**, 106002 (2004) [arXiv:hep-th/0311267].

[140] S. C. Davis, arXiv:hep-th/0402151.

[141] I. Navarro, JCAP **0309**, 004 (2003) [arXiv:hep-th/0302129].

[142] P. Bostock, R. Gregory, I. Navarro and J. Santiago, Phys. Rev. Lett. **92**, 221601 (2004) [arXiv:hep-th/0311074].

[143] S. Kanno and J. Soda, JCAP **0407**, 002 (2004) [arXiv:hep-th/0404207].

[144] J. Vinet and J. M. Cline, Phys. Rev. D **70**, 083514 (2004) [arXiv:hep-th/0406141].

[145] J. Garriga and M. Porrati, JHEP **0408**, 028 (2004) [arXiv:hep-th/0406158].

[146] P. Wang and X. H. Meng, Phys. Rev. D **71**, 024023 (2005) [arXiv:hep-th/0406170].

[147] H. M. Lee and A. Papazoglou, Nucl. Phys. B **705**, 152 (2005) [arXiv:hep-th/0407208].

[148] S. Mukohyama, Y. Sendouda, H. Yoshiguchi and S. Kinoshita, JCAP **0507**, 013 (2005) [arXiv:hep-th/0506050].

[149] B. Allen, Nucl. Phys. B **226** (1983) 228.

[150] D. Iellici and V. Moretti, Phys. Lett. B **425** (1998), 33 [arXiv:gr-qc/9705077].

[151] G. Cognola, E. Elizalde and S. Zerbini, Phys. Rev. D **65**, 085031 (2002), [arXiv:hep-th/0201152].

[152] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

Local conservation law and dark radiation in cosmological braneworld

Masato Minamitsuji[1,2] and Misao Sasaki[2]

[1]*Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan*
[2]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
(Received 26 January 2004; published 18 August 2004)

In the context of the Randall-Sundrum (RS) single-brane scenario, we discuss the bulk geometry and dynamics of a cosmological brane in terms of the local energy conservation law which exists for the bulk that allows slicing with a maximally symmetric three-space. This conservation law enables us to define a local mass in the bulk. We show that there is a unique generalization of the dark radiation on the brane, which is given by the local mass. We find there also exists a conserved current associated with the Weyl tensor, and the corresponding local charge, which we call the Weyl charge, is given by the sum of the local mass and a certain linear combination of the components of the bulk energy-momentum tensor. This expression of the Weyl charge relates the local mass to the projected Weyl tensor, $E_{\mu\nu}$, which plays a central role in the geometrical formalism of the RS braneworld. On the brane, in particular, this gives a decomposition of the projected Weyl tensor into the local mass and the bulk energy-momentum tensor. Then, as an application of these results, we consider a null dust model for the bulk energy-momentum tensor and discuss the black hole formation in the bulk. We investigate the causal structure by identifying the locus of the apparent horizon and clarify possible brane trajectories in the bulk. We find that the brane stays always outside the black hole as long as it is expanding. We also find an upper bound on the value of the Hubble parameter in terms of the matter energy density on the brane, irrespective of the energy flux emitted from the brane.

I. INTRODUCTION

The braneworld scenario has attracted much attention in recent years [1]. In this scenario, our Universe is assumed to be on a (mem)brane embedded in a higher-dimensional spacetime. There are many models of the braneworld scenario and corresponding cosmologies. One of them that has been extensively studied is the braneworld cosmology based on a model proposed by Randall and Sundrum (RS) [2], in which a single positive tension brane exists in a five-dimensional spacetime (called the bulk) with negative cosmological constant, the so-called RS2 model. In this paper, we focus our discussion on this single-brane model.

In many cases, the five-dimensional bulk geometry is assumed to be anti–de Sitter (AdS) or AdS-Schwarzschild [3–5]:

$$ds^2 = -\left(K + \frac{r^2}{\ell^2} - \frac{M_0}{r^2}\right)dt^2 + \left(K + \frac{r^2}{\ell^2} - \frac{M_0}{r^2}\right)^{-1}dr^2$$
$$+ r^2 d\Omega^2_{(K,3)}, \qquad (1.1)$$

where $\ell := \sqrt{-6/\Lambda_5}$ is the AdS curvature radius, M_0 is the black hole mass, and $d\Omega^2_{(K,3)}$ is the maximally symmetric (constant curvature) three-space with $K = -1$, 0, or $+1$. The brane trajectory in the bulk, $(t,r) = (t(\tau), r(\tau))$, is determined by the junction condition [6]. As usual, we impose the reflection symmetry with respect to the brane. Then, we obtain the effective Friedmann equation on the brane as [4,5]

$$\left(\frac{\dot{r}}{r}\right)^2 + \frac{K}{r^2} = \left(\frac{\kappa_5^4}{36}\sigma^2 - \frac{1}{\ell^2}\right) + \frac{\kappa_5^4}{18}(2\sigma\rho + \rho^2) + \frac{M_0}{r^4},$$
$$(1.2)$$

where σ and ρ are the brane tension and energy density of the matter on the brane, respectively, and $\dot{r} = dr/d\tau$ with τ being the proper time on the brane. The final term is proportional to the mass of the bulk black hole and is often called the "dark radiation" since it behaves as the ordinary radiation. Geometrically, it comes from the projected Weyl tensor in the bulk, denoted commonly by $E_{\mu\nu}$ [7]. If we apply Eq. (1.2) to the real Universe, the values of σ, ℓ, and M_0 are constrained by observations of the cosmological parameters [8].

When the bulk ceases to be pure AdS-Schwarzschild, or when there exists a dynamical degree of freedom other than the metric, the parameter M_0 is no longer constant in general, but becomes dynamical. For instance, this is the case of the so-called bulk inflaton model [9–13], or when the brane radiates gravitons into the bulk [15]. In particular, in Ref. [10], the dynamics of a bulk scalar field is investigated in the context of the bulk inflaton model under the assumption that the backreaction of the scalar field on the geometry is small, and it is found that there exists an interesting integral expression for the projected Weyl tensor in terms of the energy-momentum tensor of the scalar field. This suggests the existence of a local conservation law in the bulk that directly relates the dark radiation on the brane to the dynamics in the bulk.

In this paper, we investigate the case when there is non-trivial dynamics in the bulk, and clarify the relation between the bulk geometry and the dynamics of the brane. We focus on the case of isotropic and homogeneous branes and hence assume the existence of slicing by the maximally symmetric three-space as in Eq. (1.1). In this case, we can derive a local energy conservation law in the bulk, in analogy with spherical symmetric spacetimes in four dimensions [16]. Then, this

conservation law can be used to relate the brane dynamics to the geometrical properties of the bulk, especially with the projected Weyl tensor in the bulk.

The paper is organized as follows. In Sec. II, we derive the local energy conservation law in the bulk and discuss the general property of the bulk geometry and cosmology on the brane. We show that there exists a unique generalization of the dark radiation that is directly related to the local mass in the bulk. We also find that there exists another conserved current associated with the Weyl tensor, as a nonlinear version of what was found in Ref. [10]. In a vacuum (Ricci flat) spacetime, the local charge for this current is found to be equivalent to the local mass. Let us call this the Weyl charge. The difference between the local mass and Weyl charge is given by the linear combination of certain components of the bulk energy-momentum tensor, and the projected Weyl tensor that appears in the effective Friedmann equation on the brane is indeed given by this Weyl charge. Thus we have a unique decomposition of the projected Weyl tensor term into the part due to the bulk mass that generalizes the dark radiation term and the part due to the bulk energy-momentum tensor. In Sec. III, as an application of the conservation law derived in Sec. II, we consider a simple null dust model and discuss the black hole formation in the bulk. We identify the location of an apparent horizon and analyze possible trajectories of the brane in the bulk. We show that the brane stays always outside of the apparent horizon of the black hole as long as the brane is expanding. In Sec. IV, we summarize our work and mention future issues.

II. LOCAL CONSERVATION LAW IN A SPACETIME WITH MAXIMALLY SYMMETRIC THREE-SPACE

In this section, we discuss the general property of a dynamical bulk spacetime with a maximally symmetric three-space, and consider cosmology on the brane. First, we derive a local conservation law in the bulk, as a generalization of the local energy conservation law in a spherically symmetric spacetime in four dimensions [16]. Namely, we show that a locally conserved energy flux vector exists in spite of the absence of a timelike Killing vector field. This enables us to define a local mass in the bulk spacetime. We also show that there exists a conserved current associated with the Weyl tensor. This gives rise to a locally defined Weyl charge. It is shown that the Weyl charge and the local mass are closely related to each other.

Next, we introduce the brane as a boundary of the dynamical spacetime. The effective Friedmann equation, is determined via the junction condition, and it is shown that the local mass corresponds to the generalized dark radiation. Finally, we show that the projected Weyl tensor on the brane is uniquely related to the local mass.

A. Local conservation law

We assume that the bulk allows slicing by a maximally symmetric three-space. Then, the bulk metric can written in the double-null form

$$ds^2 = \frac{4 r_{,u} r_{,v}}{\Phi} du\, dv + r(u,v)^2 d\Omega^2_{(K,3)}, \qquad (2.1)$$

where we refer to v and u as the advanced and retarded time coordinates, respectively. In Appendix A, the explicit components of the connection and curvature in an $(n+2)$-dimensional spacetime with maximally symmetric n-space are listed.

The five-dimensional Einstein equations are given by

$$G_{ab} + \Lambda_5 g_{ab} = \kappa_5^2 T_{ab} + S_{ab}\delta(y - y_0), \qquad (2.2)$$

where the indices $\{a,b\}$ run from 0 to 3, and 5, and Λ_5 and κ_5^2 are the five-dimensional cosmological constant and gravitational constant, respectively. The brane is introduced as a singular hypersurface located at $y = y_0$, where y denotes a Gaussian normal coordinate in the direction of the extra dimension in the vicinity of the brane, and S_{ab} denotes the energy-momentum tensor on the brane. The spacetime is assumed to be reflection symmetric with respect to the brane.

First, we consider the Einstein equations in the bulk. They are given by

$$3\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,v}}{\Phi}\right|\right)_{,u} = \kappa_5^2 T_{uu}, \quad 3\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}}{\Phi}\right|\right)_{,v} = \kappa_5^2 T_{vv},$$

$$6\frac{r_{,u}r_{,v}}{r^2}\left(1 - \frac{K}{\Phi}\right) + 3\frac{r_{,uv}}{r} = \kappa_5^2 T_{uv} - \frac{2r_{,u}r_{,v}}{\Phi}\Lambda_5, \quad (2.3)$$

$$\left\{\frac{r^2\Phi}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv} + 4\frac{r_{,uv}}{r}\right] - (K - \Phi)\right\}\gamma_{ij}$$

$$= \kappa_5^2 T_{ij} - r^2\gamma_{ij}\Lambda_5,$$

where γ_{ij} is the intrinsic metric of the maximally symmetric three-space.

Now, we derive the local conservation law. We introduce a vector field in five-dimensional spacetime as

$$\xi^a = \frac{1}{2}\Phi\left(-\frac{1}{r_{,v}}\frac{\partial}{\partial v} + \frac{1}{r_{,u}}\frac{\partial}{\partial u}\right)^a. \qquad (2.4)$$

From the form of the metric (2.1), we can readily see that ξ^a is conserved:

$$\sqrt{-g}\,\xi^a{}_{;a} = (\sqrt{-g}\,\xi^a)_{,a} = 2\sqrt{\gamma}[(r^3 r_{,u})_{,v} - (r^3 r_{,v})_{,u}] = 0, \qquad (2.5)$$

where $\gamma = \det\gamma_{ij}$. Note that, for an asymptotically constant curvature spacetime, the vector field ξ^a becomes asymptotically the timelike Killing vector field $-(\partial/\partial t)^a$.

With this vector field ξ^a, we define a new vector field,

$$\tilde{S}^a = \xi^b \tilde{T}_b{}^a, \qquad (2.6)$$

where

$$\tilde{T}_{ab}=T_{ab}-\frac{1}{\kappa_5^2}\Lambda_5 g_{ab}. \tag{2.7}$$

Using the Einstein equations, the components of the vector field $\tilde{S}^a$ are given by

$$\kappa_5^2\sqrt{-g}\,\tilde{S}^v=\frac{3}{2}[r^2(K-\Phi)]_{,u}\sqrt{\gamma},$$

$$\kappa_5^2\sqrt{-g}\,\tilde{S}^u=-\frac{3}{2}[r^2(K-\Phi)]_{,v}\sqrt{\gamma}. \tag{2.8}$$

Then, we have the local conservation law as

$$\tilde{S}^a{}_{;a}=0. \tag{2.9}$$

Since ξ^a is conserved separately, the conservation of $\tilde{S}^a$ implies that we have another conserved current S^a defined by

$$S^a:=\xi^b T_b{}^a \quad\left(=\tilde{S}^a+\frac{1}{\kappa_5^2}\Lambda_5\xi^a\right). \tag{2.10}$$

Thus we have the local conservation law for the energy-momentum tensor in the bulk.

From Eqs. (2.8), we readily see the local mass corresponding to $\tilde{S}^a$ is given by [16]

$$\tilde{M}:=(K-\Phi)r^2, \tag{2.11}$$

where the factor 3/2 in the original expression for $\tilde{S}^a$ is eliminated for later convenience. Alternatively, corresponding to S^a, we have another local mass that excludes the contribution of the bulk cosmological constant,

$$M:=\tilde{M}-\frac{1}{6}\Lambda_5 r^4=(K-\Phi)r^2-\frac{1}{6}\Lambda_5 r^4. \tag{2.12}$$

In what follows, we focus on the matter part M, rather than on the whole mass $\tilde{M}$. It may be noted, however, that this decomposition of $\tilde{M}$ to the cosmological constant part and the matter part is rather arbitrary, as in the case of a bulk scalar field. Here we adopt this decomposition just for convenience. For example, this decomposition is more useful when we consider small perturbations on the static AdS-Schwarzschild bulk. We note that, in the case of a spherically symmetric asymptotic flat spacetime in four dimensions (hence $K=+1$ and with no cosmological constant), this function M agrees with the Arnowitt-Deser-Misner (ADM) energy or the Bondi energy in the appropriate limits.

B. Local mass and Weyl charge

From the five-dimensional Einstein equations (2.3), we can write down the local conservation equation for M in terms of the bulk energy-momentum tensor explicitly as

$$M_{,v}=\frac{2}{3}\kappa_5^2 r^3(T^u{}_v r_{,u}-T^v{}_v r_{,v}),$$

$$M_{,u}=\frac{2}{3}\kappa_5^2 r^3(T^v{}_u r_{,v}-T^v{}_v r_{,u}), \tag{2.13}$$

or in a bit more concise form,

$$dM=\frac{2}{3}\kappa_5^2 r^3(T^u{}_v r_{,u}dv+T^v{}_u r_{,v}du-T^v{}_v dr). \tag{2.14}$$

Using the above, we can immediately write down two integral expressions for M given in terms of flux crossing the $u=$ const hypersurfaces from v_1 to v_2, and flux crossing the $v=$ const hypersurfaces from u_1 to u_2, respectively, as

$$M(v_2,u)-M(v_1,u)$$

$$=\frac{2}{3}\kappa_5^2\int_{v_1}^{v_2}dv\,r^3(T^u{}_v r_{,u}-T^v{}_v r_{,v})\bigg|_{u=\mathrm{const.}},$$

$$M(v,u_2)-M(v,u_1)$$

$$=\frac{2}{3}\kappa_5^2\int_{u_1}^{u_2}du\,r^3(T^v{}_u r_{,v}-T^v{}_v r_{,u})\bigg|_{v=\mathrm{const.}}. \tag{2.15}$$

Finally, let us consider the Weyl tensor in the bulk. In the present case of a five-dimensional spacetime with maximally symmetric three-space, there exists only one nontrivial component of the Weyl tensor, say $C_{vu}{}^{vu}$. The explicit expressions for the components of the Weyl tensor are given in Appendix A, Eqs. (A7). Using the Bianchi identities and the Einstein equations, we have [26]

$$C_{abcd}{}^{;d}=J_{abc}, \tag{2.16}$$

where

$$J_{abc}=\frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]}+\frac{1}{(n+1)}g_{c[b}T_{;a]}\right). \tag{2.17}$$

From this, we can show that there exists a conserved current,

$$Q^a=r\ell_b n_c J^{bca}, \quad Q^a{}_{;a}=0, \tag{2.18}$$

where ℓ_a and n_a are a set of two hypersurface orthogonal null vectors,

$$\ell_a=\sqrt{\frac{2}{\Phi}}(-r_{,v}dv)_a, \quad \ell^a=-\sqrt{\frac{1}{2}\Phi}\,\frac{1}{r_{,u}}\left(\frac{\partial}{\partial u}\right)^a,$$

$$n_a=\sqrt{\frac{2}{\Phi}}(r_{,u}du)_a, \quad n^a=\sqrt{\frac{1}{2}\Phi}\,\frac{1}{r_{,v}}\left(\frac{\partial}{\partial v}\right)^a. \tag{2.19}$$

The nonzero components are written explicitly as

$$Q^u=-rJ^{vu}{}_v, \quad Q^v=-rJ^{vu}{}_u, \tag{2.20}$$

and we have

$$(r^4 C_{vu}{}^{vu})_{,v} = r^4 J^v{}_{vv},$$

$$(r^4 C_{vu}{}^{vu})_{,u} = r^4 J^u{}_{uu}. \qquad (2.21)$$

These are very similar to Eqs. (2.8). It is clear that $r^4 C_{vu}{}^{vu}$ defines a local charge associated with this conserved current, that is, the Weyl charge.

Using the Einstein equations, we then find that the Weyl charge can be expressed in terms of M and the energy-momentum tensor as

$$r^4 C_{vu}{}^{vu} = 3\tilde{M} + \frac{r^4}{6}(6 G^v{}_v - G^i{}_i) = 3M + \frac{\kappa_5^2}{6} r^4 (6 T^v{}_v - T^i{}_i). \qquad (2.22)$$

This is one of the most important results in this paper. As we shall see below, the Weyl component $C_{vu}{}^{vu}$ is directly related to the projected Weyl tensor $E_{\mu\nu}$, and hence this relation gives explicitly how the local mass M and the local value of the energy-momentum tensor affects the brane dynamics.

C. Apparent horizons

As in the conventional four-dimensional gravity, the gravitational dynamics may lead to the formation of a black hole in the bulk. Rigorously speaking, the black hole formation can be discussed only by analyzing the global causal structure of a spacetime. Nevertheless, we discuss the black hole formation by studying the formation of an apparent horizon.

In four dimensions, an apparent horizon is defined as a closed two-sphere on which the expansion of an outgoing (or ingoing) null geodesic congruence vanishes. Here, we extend the definition to our case and define an apparent horizon as a three-surface on which the expansion of a radial null geodesic congruence vanishes. Note that "radial" here means simply those congruences that have only the (v,u) components; hence an apparent horizon will not be a closed surface if $K = 0$.

The expansions of the congruence of null geodesics forming the $u = \text{const}$ and $v = \text{const}$ hypersurfaces, respectively, are given by [16]

$$\rho_u = -\frac{1}{2} u^{;a}{}_{;a} = -\frac{1}{2r} \frac{\Phi}{r_{,u}}, \qquad \rho_v = -\frac{1}{2} v^{;a}{}_{;a} = -\frac{1}{2r} \frac{\Phi}{r_{,v}}. \qquad (2.23)$$

Naively, if $\Phi = 0$, one might think that both ρ_u and ρ_v vanish. However, from the regularity condition of the metric (2.1), we have

$$-4\frac{r_{,u} r_{,v}}{\Phi} > 0. \qquad (2.24)$$

Hence, it must be that $r_{,u} = 0$ or $r_{,v} = 0$ if $\Phi = 0$. If $\Phi = r_{,v} = 0$, we have $\rho_u = 0$ and an apparent horizon for the outgo-

ing null geodesics is formed, whereas if $\Phi = r_{,u} = 0$, we have $\rho_v = 0$ and an apparent horizon for the ingoing null geodesics is formed.

D. Brane cosmology

We now consider the dynamics of a brane in a dynamical bulk with maximally symmetric three-space [3]. The brane trajectory is parametrized as $(v,u) = (v(\tau), u(\tau))$. Taking τ to be the proper time on the brane, we have

$$4\frac{r_{,u} r_{,v}}{\Phi} \dot{u}\dot{v} = -1 \qquad (2.25)$$

on the brane, where $\dot{u} = du/d\tau$ and so on. The unit vector tangent to the brane (i.e., the five-velocity of the brane) is given by

$$v^a = \left(\dot{v} \frac{\partial}{\partial v} + \dot{u} \frac{\partial}{\partial u} \right)^a, \qquad v_a = \frac{2 r_{,u} r_{,v}}{\Phi} (\dot{u} dv + \dot{v} du)_a, \qquad (2.26)$$

and the unit normal to the brane is given by

$$n^a = \left(-\dot{v} \frac{\partial}{\partial v} + \dot{u} \frac{\partial}{\partial u} \right)^a, \qquad n_a = \frac{2 r_{,u} r_{,v}}{\Phi} (\dot{u} dv - \dot{v} du)_a. \qquad (2.27)$$

The components of the induced metric on the brane are calculated as

$$q_{\mu\nu} = \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} g_{ab}, \qquad (2.28)$$

where μ, ν run from 0 to 3 and y^μ are the intrinsic coordinates on the brane with $y^0 = \tau$ and $y^i = x^i$ ($i = 1,2,3$). Then the induced metric on the brane is given by

$$ds_{(4)}^2 = -d\tau^2 + r(\tau)^2 d\Omega_{(K,3)}^2. \qquad (2.29)$$

The trajectory of the brane is determined by the junction condition under the Z_2 symmetry with respect to the brane. The extrinsic curvature on the brane is determined as

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2} \left(S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu} \right), \qquad (2.30)$$

where $S_{\mu\nu}$ is assumed to take the form

$$S^\mu_\nu = \text{diag}(-\rho, p, p, p) - \sigma \delta^\mu_\nu, \qquad (2.31)$$

with σ and ρ being the tension and energy density of the matter on the brane, respectively, as introduced previously, and p being the isotropic pressure of the matter on the brane. Substituting the induced metric (2.29) in Eq. (2.30), we obtain

$$r_{,u} \dot{u} = -\frac{r}{2} \left[\frac{\kappa_5^2}{6} (\rho + \sigma) - H \right], \qquad (2.32)$$

$$r_{,v}\dot{v} = \frac{r}{2}\left[\frac{\kappa_5^2}{6}(\rho + \sigma) + H\right], \tag{2.33}$$

where $H = \dot{r}/r$. Multiplying the above two equations and using the normalization condition (2.25), we then obtain the effective Friedmann equation on the brane:

$$H^2 + \frac{K}{r^2} = \left(\frac{\kappa_5^4}{36}\sigma^2 - \frac{1}{l^2}\right) + \frac{\kappa_5^4}{18}(2\sigma\rho + \rho^2) + \frac{M}{r^4}. \tag{2.34}$$

We see that M is a natural generalization of the dark radiation in the AdS-Schwarzschild case to a dynamical bulk.

For a dynamical bulk, M varies in time. The evolution of M is determined by Eq. (2.14), and on the brane it gives

$$\dot{M} = M_{,v}\dot{v} + M_{,u}\dot{u}$$

$$= \frac{2}{3}\kappa_5^2 r^4 \left[T_{vv}\left(\frac{1}{6}\kappa_5^2(\rho + \sigma) - H\right)\dot{v}^2 \right.$$

$$\left. - T_{uu}\left(\frac{1}{6}\kappa_5^2(\rho + \sigma) + H\right)\dot{u}^2 \right] - \frac{2}{3}\kappa_5^2 r^4 H T^v{}_v. \tag{2.35}$$

This result is consistent with Refs. [12,15]. From the Codacci equation on the brane [7],

$$D_\nu K^\nu{}_\mu - D_\mu K^\nu{}_\nu = \kappa_5^2 T_{ab}n^b q^a{}_\mu, \tag{2.36}$$

where D_μ is the covariant derivative with respect to $q_{\mu\nu}$ and $K_{\mu\nu}$ is the extrinsic curvature of the brane, we obtain the equation for the energy transfer of the matter on the brane to the bulk,

$$\dot{\rho} + 3H(\rho + p) = 2(-T_{vv}\dot{v}^2 + T_{uu}\dot{u}^2). \tag{2.37}$$

Equations (2.34), (2.35), and (2.37) determine the cosmological evolution on the brane, once the bulk geometry is solved. These equations will be applied to a null dust model in the next section. The case of the Einstein-scalar theory in the bulk is briefly discussed in Appendix B.

Now we relate the above result to the geometrical approach developed in Ref. [7], in particular with the $E_{\mu\nu}$ term on the brane. The projected Weyl tensor

$$E_{\mu\nu} = C_{a\mu b\nu}n^a n^b \tag{2.38}$$

has only one nontrivial component as

$$E_{\tau\tau} = C_{abcd}n^a n^c v^b v^d = 4C_{uvuv}\dot{u}^2 \dot{v}^2 = -C_{vu}{}^{vu}. \tag{2.39}$$

Using Eq. (2.22), this can be uniquely decomposed into the part proportional to M and the part due to the projection of the bulk energy-momentum tensor on the brane. We find

$$E_{\tau\tau} = -\frac{3\tilde{M}}{r^4} + \frac{1}{6}(G^i{}_i - 6G^v{}_v) = -\frac{3M}{r^4} + \frac{\kappa_5^2}{6}(T^i{}_i - 6T^v{}_v). \tag{2.40}$$

If we eliminate the M/r^4 term from Eq. (2.34) by using this equation, we recover the effective Friedmann equation on the brane in the geometrical approach [7],

$$H^2 + \frac{K}{r^2} = \left(\frac{\kappa_5^4}{36}\sigma^2 - \frac{1}{l^2}\right) + \frac{\kappa_5^4}{18}(2\sigma\rho + \rho^2) + \kappa_5^2 T^{(b)}_{\tau\tau} - \frac{E_{\tau\tau}}{3}, \tag{2.41}$$

where $T^{(b)}_{\tau\tau}$ comes from the projection of the bulk energy-momentum tensor on the brane and is given in the present case by

$$T^{(b)}_{\tau\tau} = \frac{1}{6}T^i{}_i - T^v{}_v. \tag{2.42}$$

Finally, from the brane point of view, it may be worthwhile to give the expressions for the effective total energy density and pressure on the brane. They are given by

$$\rho^{(\text{tot})} = \rho^{(\text{brane})} + \rho^{(\text{bulk})}, \qquad p^{(\text{tot})} = p^{(\text{brane})} + p^{(\text{bulk})}, \tag{2.43}$$

where

$$\kappa_4^2 \rho^{(\text{brane})} = 3\left[\frac{1}{6}\kappa_5^2(\rho + \sigma)\right]^2,$$

$$\kappa_4^2 p^{(\text{brane})} = \frac{1}{12}\kappa_5^4(\rho + \sigma)(\rho - \sigma + 2p),$$

$$\kappa_4^2 \rho^{(\text{bulk})} = \frac{3\tilde{M}}{r^4},$$

$$\kappa_4^2 p^{(\text{bulk})} = \frac{\tilde{M}}{r^4} + \frac{1}{3}\kappa_5^2\left(-\frac{\dot{u}}{v}\tilde{T}^v{}_u - \frac{\dot{v}}{u}\tilde{T}^u{}_v + 2\tilde{T}^v{}_v\right), \tag{2.44}$$

where $\tilde{M}$ is given by Eq. (2.11) and $\tilde{T}^a{}_b$ is defined by Eq. (2.7), and both contain the contribution from the bulk cosmological constant. It may be noted that, unlike the effective energy density, the effective pressure contains a part coming from the bulk that cannot be described by the local mass alone. The contracted Bianchi identity implies the conservation law for the total effective energy-momentum on the brane:

$$\dot{\rho}^{(\text{bulk})} + 3H(\rho^{(\text{bulk})} + p^{(\text{bulk})})$$

$$= -\dot{\rho}^{(\text{brane})} - 3H(\rho^{(\text{brane})} + p^{(\text{brane})}). \tag{2.45}$$

This is mathematically equivalent to Eq. (2.35). However, these two equations have different interpretations. From the bulk point of view, Eq. (2.35) is more relevant, which describes the energy exchange between the brane and the bulk, whereas a natural interpretation of Eq. (2.45) is that it describes the energy exchange between two different matters on the brane: the intrinsic matter on the brane and the bulk matter induced on the brane. The important point is, as mentioned above, that the pressure of the bulk matter has contri-

butions not only from the local mass but also from a projection of the bulk energy-momentum tensor, which makes the equation of state different from $p^{(\text{bulk})}=\rho^{(\text{bulk})}/3$, i.e., that of a simple dark radiation.

III. APPLICATION TO THE NULL DUST MODEL

In this section, by using the local mass derived in the preceding section, we discuss the bulk geometry and brane cosmology in the context of a null dust model. Especially, we pay attention to the gravitational collapse due to the emission of energy from the brane. Namely, we consider an ingoing null dust fluid emitted from the brane [15,17,18].

A. Setup

The energy-momentum tensor of a null dust fluid takes the form [24],

$$T_{ab}=\mu_1\ell_a\ell_b+\mu_2 n_a n_b, \qquad (3.1)$$

where ℓ_a and n_a are the ingoing and outgoing null vectors, respectively, introduced in Eqs. (2.19). If we require that the energy-momentum conservation law is satisfied for the ingoing and outgoing null dust independently, we have

$$\mu_1=\frac{\Phi}{(r_{,v})^2 r^3}\frac{f(v)}{2}, \qquad \mu_2=\frac{\Phi}{(r_{,u})^2 r^3}\frac{g(u)}{2}, \qquad (3.2)$$

where $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively, and have the dimension $(G_5\times\text{mass})^{-1}$. We assume the positive energy density, i.e., $f(v)\geq 0$ and $g(u)\geq 0$. Thus, the nontrivial components of the energy-momentum tensor are

$$T_{vv}=\frac{f(v)}{r^3}, \qquad T_{uu}=\frac{g(u)}{r^3}. \qquad (3.3)$$

To satisfy the local conservation law in an infinitesimal interval $(u,u+du)$ and $(v,v+dv)$, we find that the intensity functions $f(v)$ and $g(u)$ have to satisfy the relation

$$f(v)\left(\frac{\Phi}{r_{,v}}\right)_{,u}=g(u)\left(\frac{\Phi}{r_{,u}}\right)_{,v}. \qquad (3.4)$$

In general, if both $f(v)$ and $g(u)$ are nonzero, it seems almost impossible to find an analytic solution that satisfies Eq. (3.4). Hence we choose to set either $f(v)=0$ or $g(u)=0$. In the following discussion, we focus on the case that $g(u)=0$, that is, the ingoing null dust.

B. Bulk geometry of the null dust collapse

For $g(u)=0$, Eqs. (2.14) give

$$M_{,v}=\frac{1}{3}\kappa_5^2\frac{\Phi}{r_{,v}}f(v), \qquad M_{,u}=0. \qquad (3.5)$$

The second equation implies $M=M(v)$. Substituting Eq. (3.3) into the Einstein equations (2.3), we find

$$\frac{\Phi}{r_{,v}}=e^{F(v)}, \qquad (3.6)$$

where the function $F(v)$ describes the freedom in the rescaling off the null coordinate v. This equation is consistent with Eq. (3.4). Thus, we obtain the solution as

$$\Phi=r_{,v}e^{F(v)}=K+\frac{r^2}{\ell^2}-\frac{M(v)}{r^2},$$

$$M(v)=\frac{1}{3}\kappa_5^2\int_{v_0}^{v}dv\,e^{F(v)}f(v)+M_0, \qquad (3.7)$$

where we have assumed that $f(v)=0$ for $v<v_0$, that is, v_0 is the epoch at which the ingoing flux is turned on. For definiteness, we assume that the bulk is pure AdS at $v<v_0$ and set $M_0=0$ in what follows.

Transforming the double-null coordinates (v,u) to the half-null coordinates (v,r) as

$$r_{,u}du=dr-r_{,v}dv, \qquad (3.8)$$

the solution is expressed as

$$ds^2=-4\Phi(r,v)e^{-2F(v)}dv^2+4e^{-F(v)}dv\,dr+r^2d\Omega_{(K,3)}^2, \qquad (3.9)$$

where Φ is given by the first of Eqs. (3.7). This is an ingoing Vaidya solution with a negative cosmological constant [15,17]. For an arbitrary intensity function $f(v)$, this is an exact solution for the bulk geometry. Note that if we rescale v as $dv\to d\bar{v}=e^{-F}dv$, $f(v)$ scales as $f(v)\to\bar{f}(v)=e^{-2F}f(v)$, which manifestly shows the invariance of the solution under this rescaling.

An apparent horizon for the outgoing radial null congruence is located on the three-space, satisfying

$$\Phi=r_{,v}=0, \quad \text{while} \quad r_{,u}=\text{finite}. \qquad (3.10)$$

This gives

$$r^2=\frac{\ell^2}{2}\left(\sqrt{K^2+4\frac{M(v)}{\ell^2}}-K\right). \qquad (3.11)$$

The direction of the trajectory of the apparent horizon is given by

$$\frac{dr}{dv}=\frac{M_{,v}\ell^2 r}{2(r^4+M\ell^2)}=\frac{\kappa_5^2 f(v)e^{F(v)}\ell^2 r}{6(r^4+M\ell^2)}. \qquad (3.12)$$

Thus, for $f(v)>0$, dr/dv is positive, which implies that the trajectory of the apparent horizon is spacelike.

For the case of $K=+1$ or $K=0$, the apparent horizon originates from $r=0$, while it originates from $r=\ell$ for $K=-1$. A schematic view of the null dust collapse is shown in Fig. 1. We assume that the brane emits the ingoing flux during a finite interval (bounded by the dashed lines in the figures) and no naked singularity is formed. For all the cases,
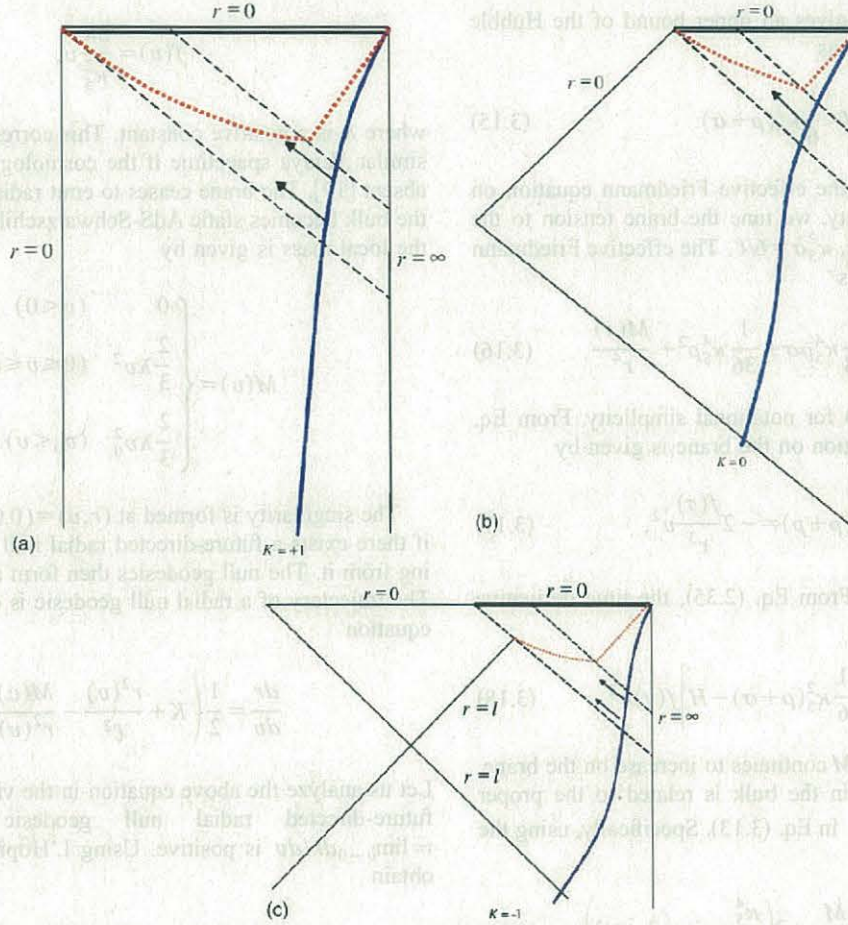
FIG. 1. (Color online) Causal structure of a spacetime with ingoing null dust for the cases of $K=+1$, 0 and -1. In each figure, The (almost vertical) wavy curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. The thick horizontal line at $r=0$ represents the spacelike curvature singularity formed there. The ingoing flux is assumed to be emitted during a finite interval bounded by the dashed lines.

the causal structures after the onset of emission are very similar. The spacelike singularity is formed at $r=0$, but it is hidden inside the apparent horizon.

C. Brane trajectory in the bulk

In the null dust model, using Eq. (2.25), the proper time on the brane is related to the advanced time in the bulk as [18]

$$\dot{v}_\pm = e^{F(v)} \frac{\dot{r} \pm \sqrt{\dot{r}^2 + \Phi}}{2\Phi}. \tag{3.13}$$

To determine the appropriate sign in the above, we require that the brane trajectory is timelike, hence $\dot{v}>0$, and examine the signs of $\dot{v}_\pm$ for all possible cases:

(1) $\dot{r}>0$, $\Phi>0 \rightarrow \dot{v}_+>0$, $\dot{v}_-<0$.

(2) $\dot{r}>0$, $\Phi<0 \rightarrow \dot{v}_+<0$, $\dot{v}_-<0$.

(3) $\dot{r}<0$, $\Phi>0 \rightarrow \dot{v}_+>0$, $\dot{v}_-<0$.

(4) $\dot{r}<0$, $\Phi<0 \rightarrow \dot{v}_+>0$, $\dot{v}_->0$.

From these, we can conclude the following. For an expanding brane, $\dot{r}>0$, the brane exists always outside the horizon, $\Phi>0$, and $\dot{v}$ is given by $\dot{v}_+$. On the other hand, a contracting brane, $\dot{r}<0$, can exist either outside or inside of the horizon. Thus, if the brane is expanding initially, the trajectory is given by $\dot{v}=\dot{v}_+$, and it stays outside the horizon until it starts to recollapse, if ever. If the brane universe starts to recollapse, which is possible only in the case $K=+1$, by continuity, the trajectory is still given by $\dot{v}=\dot{v}_+$, and the brane universe is eventually swallowed into the black hole.

From the above result, we find

$$r_{,u}\dot{u}=\dot{r}-r_{,v}\dot{v}=\frac{\dot{r}-\sqrt{\dot{r}^2+\Phi}}{2}<0. \tag{3.14}$$

Using Eq. (2.32), this gives an upper bound of the Hubble parameter on the brane as

$$H < \frac{1}{6} \kappa_5^2 (\rho + \sigma). \tag{3.15}$$

Let us now turn to the effective Friedmann equation on the brane. For simplicity, we tune the brane tension to the Randall-Sundrum value, $\kappa_5^2 \sigma = 6/\ell$. The effective Friedmann equation on the brane is

$$H^2 + \frac{K}{r^2} = \frac{1}{18} \kappa_5^4 \rho \sigma + \frac{1}{36} \kappa_5^4 \rho^2 + \frac{M(\tau)}{r^4}, \tag{3.16}$$

where $M(\tau) = M(v(\tau))$ for notational simplicity. From Eq. (2.37), the energy equation on the brane is given by

$$\dot{\rho} + 3\frac{\dot{r}}{r}(\rho + p) = -2\frac{f(\tau)}{r^3}\dot{v}^2, \tag{3.17}$$

where $f(\tau) = f(v(\tau))$. From Eq. (2.35), the time derivative of M is given by

$$\dot{M} = \frac{2}{3} r \kappa_5^2 \left[\frac{1}{6} \kappa_5^2 (\rho + \sigma) - H \right] f(\tau)\dot{v}^2. \tag{3.18}$$

Thus, from Eq. (3.15), M continues to increase on the brane.

The advanced time in the bulk is related to the proper time on the brane by $\dot{v}_+$ in Eq. (3.13). Specifically, using the equality,

$$\Phi = K + \frac{r^2}{\ell^2} - \frac{M}{r^2} = r^2 \left(\frac{\kappa_5^4}{36} (\rho + \sigma)^2 - H^2 \right) \tag{3.19}$$

on the brane, we have

$$\dot{v} = \frac{e^{F(v)}}{2r} \left(\frac{\kappa_5^2}{6} (\rho + \sigma) - H \right)^{-1}. \tag{3.20}$$

Note that the product $f\dot{v}^2$ is invariant under the rescaling of v. Once $f(\tau)$ is given, we can solve the system of equations (3.16)–(3.18) self-consistently for a given initial condition, and determine the bulk geometry and the brane dynamics at the same time [15]. A quantitative analysis of the brane cosmology is left for future work.

D. Formation of a naked singularity

In the previous subsections, we assumed that there is no naked singularity in the bulk. However, it has been shown that a naked singularity can be formed in the null dust collapse [19–25]. For instance, a naked singularity exists in a Vaidya spacetime when the flux of radiation rises from zero sufficiently slowly. We expect the same is true in the present case.

Without loss of generality, we set $e^{F(v)} = 2$. We consider the following situation. For $v < 0$, the bulk geometry is purely AdS. The radiative emission from the brane begins at $v = 0$. We choose the intensity function as

$$f(v) = \frac{2\lambda}{\kappa_5^2} v, \tag{3.21}$$

where λ is a positive constant. This corresponds to the self-similar Vaidya spacetime if the cosmological constant were absent [19]. The brane ceases to emit radiation at $v = v_0$ and the bulk becomes static AdS-Schwarzschild for $v > v_0$. Thus the local mass is given by

$$M(v) = \begin{cases} 0 & (v < 0) \\ \dfrac{2}{3} \lambda v^2 & (0 \le v \le v_0) \\ \dfrac{2}{3} \lambda v_0^2 & (v_0 < v). \end{cases} \tag{3.22}$$

The singularity is formed at $(r, v) = (0, 0)$, and it is naked if there exists a future-directed radial null geodesic emanating from it. The null geodesics then form a Cauchy horizon. The trajectory of a radial null geodesic is determined by the equation

$$\frac{dr}{dv} = \frac{1}{2} \left(K + \frac{r^2(v)}{\ell^2} - \frac{M(v)}{r^2(v)} \right). \tag{3.23}$$

Let us analyze the above equation in the vicinity of $v = 0$. A future-directed radial null geodesic exists if $x := \lim_{v \to 0} dr/dv$ is positive. Using L'Hôpital's theorem, we obtain

$$x = \lim_{v \to 0} \frac{r(v)}{v} = \lim_{v \to 0} \frac{dr}{dv} = \frac{1}{2} \left(K - \frac{2\lambda}{3x^2} \right). \tag{3.24}$$

It is clear that the above equation has no solution when $K = 0$ or $K = -1$. Hence no naked singularity is formed for $K = 0$ or $K = -1$. Therefore, we consider the case $K = 1$. We introduce a function,

$$Q(x) = 3x^3 - \frac{3}{2}x^2 + \lambda. \tag{3.25}$$

Then, the condition for the naked singularity formation is that $Q(x) = 0$ has a solution for a positive x. The function $Q(x)$ has a minimal point at $x = 1/3$. Therefore, the singularity is naked if

$$Q(1/3) = -\frac{1}{18} + \lambda \le 0, \tag{3.26}$$

that is,

$$0 < \lambda \le \frac{1}{18}. \tag{3.27}$$

Thus, the bulk has a naked singularity for small values of λ, i.e., for the flux of radiation which rises slowly enough.
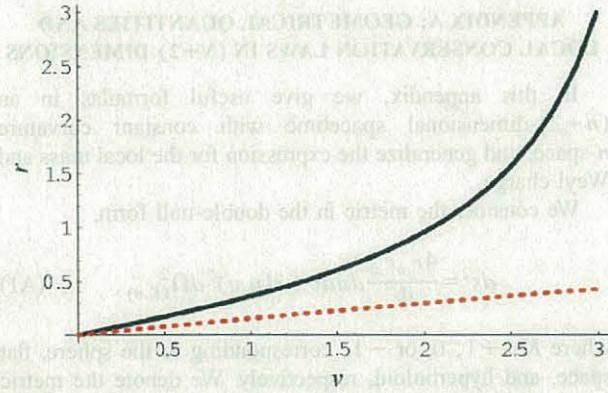
FIG. 2. (Color online) The loci of the null geodesic (the solid curve) and the apparent horizon (the dotted curve) on the (v,r) plane, scaled in units of the AdS radius ℓ, in the critical case $\lambda = 1/18$. Their behaviors are qualitatively the same for all the other values of λ in the range $0 < \lambda < 1/18$.

Our next interest is whether the naked singularity is local or global. If it is globally naked, it may be visible on the brane. To examine this, we integrate Eq. (3.23). In the vicinity of $v=0$, we find

$$r_{\text{null}}(v) = x_0 v \left(1 + b \frac{v^2}{\ell^2} + \cdots \right), \quad (3.28)$$

where x_0 is the largest positive root of $Q(x)=0$;

$$x_0 = \frac{1}{6}\{1 + [1 - 36\lambda + i6\sqrt{2\lambda(1-18\lambda)}]^{1/3}$$

$$+ [1 - 36\lambda - i6\sqrt{2\lambda(1-18\lambda)}]^{1/3}\} \quad (3.29)$$

and

$$b = \frac{x_0^2}{2(5x_0 - 1)}. \quad (3.30)$$

From the form of $Q(x)$, we readily see that x_0 monotonically decreases from 1/2 to 1/3 as λ increases from 0 to 1/18, and hence b is positive definite. We compare this trajectory with the trajectory of the apparent horizon. It is given by Eq. (3.11) with $K=+1$. In the vicinity of $v=0$, it gives

$$r_{\text{app}}(v) = \sqrt{\frac{2\lambda}{3}} v \left(1 - \frac{\lambda}{8} \frac{v^2}{\ell^2} + \cdots \right). \quad (3.31)$$

Since $x_0 > \sqrt{2\lambda/3}$ for all the values of λ in the range $0 < \lambda \leq 1/18$, and dr_{app}/dv is a decreasing function of v while dr_{null}/dv is an increasing function of v, it follows that the null geodesic lies in the exterior of the apparent horizon and the difference in the radius at the same v increases as v increases, at least when v is small. This suggests that the singularity is globally naked.

In Fig. 2, we plot the loci of the null geodesic and the apparent horizon. The result is clear. The null geodesic always stays outside of the apparent horizon, thus outside of
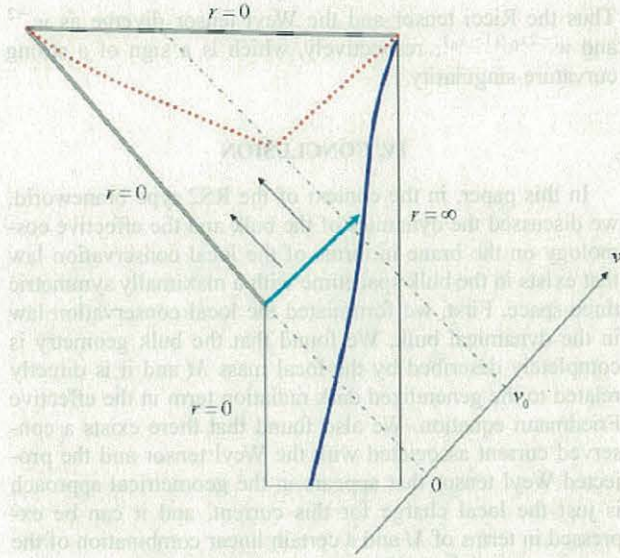

FIG. 3. (Color online) Causal structure of a spacetime with ingoing null dust when a naked singularity is formed. The wavy and almost vertical curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. A naked singularity is formed at $r=0$ along the $v=0$ null line. A radial, future-directed null geodesic originating from the naked singularity (the right-pointed thick line) stays outside of the apparent horizon and reaches the brane.

the final event horizon at $v=v_0$. Mathematically, this is due to the cosmological constant term in Eq. (3.23), which strongly drives the null geodesic trajectory to larger values of r. Thus, we conclude that the naked singularity is global and visible on the brane. The causal structure in this case is illustrated in Fig. 3. Investigations on the effect of the visible singularity on the brane are necessary, but they are left for future work.

Finally, let us mention the strength of the naked singularity as we approach it along a radial null geodesic. Let w be an affine parameter of the geodesic, $w=0$ be the singularity, and the tangent vector be denoted by $k^a = dx^a/dw$. We examine $R_{ab}k^a k^b$ and $C_{vu}{}^{vu}$. From Eq. (3.3) and the Einstein equations, we have

$$R_{ab}k^a k^b = \frac{\kappa_5^2 f(v)}{r^3}\left(\frac{dv}{dw}\right)^2 = \frac{2\lambda v}{r^3}\left(\frac{dv}{dw}\right)^2$$

$$\xrightarrow[w\to 0]{} \frac{2\lambda}{x_0(1-x_0)^2} w^{-2}, \quad (3.32)$$

Also, from Eq. (2.22), we have

$$C_{vu}{}^{vu} = \frac{3M}{r^4} = \frac{2\lambda v^2}{r^4} \xrightarrow[w\to 0]{} \frac{2\lambda}{x_0^4} v^{-2} \propto w^{-2x_0/(1-x_0)}.$$

$$(3.33)$$

Thus the Ricci tensor and the Weyl tensor diverge as w^{-2} and $w^{-2x_0/(1-x_0)}$, respectively, which is a sign of a strong curvature singularity.

IV. CONCLUSION

In this paper, in the context of the RS2 type braneworld, we discussed the dynamics of the bulk and the effective cosmology on the brane in terms of the local conservation law that exists in the bulk spacetime with a maximally symmetric three-space. First, we formulated the local conservation law in the dynamical bulk. We found that the bulk geometry is completely described by the local mass M and it is directly related to the generalized dark radiation term in the effective Friedmann equation. We also found that there exists a conserved current associated with the Weyl tensor and the projected Weyl tensor that appears in the geometrical approach is just the local charge for this current, and it can be expressed in terms of M and a certain linear combination of the components of the bulk energy-momentum tensor.

Next, as an application of our formalism, we adopted a simple null dust model, in which the energy emitted by the brane is approximated by an ingoing null dust fluid, and investigated the general properties of the bulk geometry and the brane trajectory in the bulk. Usually, the ingoing null dust forms a black hole in the bulk. However, in the case of $K = +1$, a naked singularity can be formed in the bulk when the flux rises from zero slower than a critical rate. We show that the naked singularity is global and thus it can be visible to an observer on the brane. Studies on the implications of a visible naked singularity on the brane is left for future work.

Also, we found that the brane can never enter the black hole horizon as long as it is expanding. In addition, we found an upper bound on the Hubble expansion rate, given by the energy density of the matter on the brane, for arbitrary but non-negative energy flux emitted by the brane. We also presented a set of equations that completely determine the brane dynamics as well as the bulk geometry.

Finally, let us briefly comment on some future issues. In this paper, we only discussed the case of null dust. However, this is too simplified to be realistic. As a realistic situation, it will be interesting to consider a bulk scalar field such as a dilaton or a moduli field. In this case, it will be necessary to solve the bulk and brane dynamics numerically in general. Another interesting issue will be the evaporation of a bulk black hole by the Hawking radiation and its effect on the brane dynamics. We plan to come back to these issues in future publications.

ACKNOWLEDGMENT

APPENDIX A: GEOMETRICAL QUANTITIES AND LOCAL CONSERVATION LAWS IN $(N+2)$ DIMENSIONS

In this appendix, we give useful formulas in an $(n+2)$-dimensional spacetime with constant curvature n-space, and generalize the expression for the local mass and Weyl charge.

We consider the metric in the double-null form,

$$ds^2 = \frac{4r_{,u}r_{,v}}{\Phi} du\,dv + r(u,v)^2 d\Omega_{(K,n)}^2, \qquad (A1)$$

where $K = +1$, 0, or -1, corresponding to the sphere, flat space, and hyperboloid, respectively. We denote the metric tensor of the constant curvature space as γ_{ij}. The explicit expressions for the geometrical quantities in this spacetime are as follows.

The Christoffel symbol is

$$\Gamma^u_{uu} = \left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,u}, \qquad \Gamma^v_{vv} = \left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,v},$$

$$\Gamma^u_{ij} = -\frac{r\Phi}{2r_{,u}}\gamma_{ij}, \qquad \Gamma^v_{ij} = -\frac{r\Phi}{2r_{,v}}\gamma_{ij}, \qquad (A2)$$

$$\Gamma^i_{uj} = \frac{r_{,u}}{r}\delta_{ij}, \qquad \Gamma^i_{vj} = \frac{r_{,v}}{r}\delta_{ij}, \qquad \Gamma^i_{jk} = {}^n\Gamma^i_{jk}.$$

The Riemann tensor is

$$R^u_{uuv} = R^v_{vvu} = -\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,uv},$$

$$R^u_{iuj} = \left[-\frac{1}{2}r\left(\frac{\Phi}{r_{,u}} \right)_{,u} - \frac{r\Phi}{2r_{,u}}\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,u} \right]\gamma_{ij},$$

$$R^v_{ivj} = \left[-\frac{1}{2}r\left(\frac{\Phi}{r_{,v}} \right)_{,v} - \frac{r\Phi}{2r_{,v}}\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,v} \right]\gamma_{ij},$$

$$R^i_{uju} = \left[-\frac{r_{,uu}}{r} + \frac{r_{,u}}{r}\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,u} \right]\delta^i_j, \qquad (A3)$$

$$R^i_{vjv} = \left[-\frac{r_{,vv}}{r} + \frac{r_{,v}}{r}\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,v} \right]\delta^i_j,$$

$$R^i_{vju} = R^i_{ujv} = -\frac{r_{,uv}}{r}\delta^i_j,$$

$$R^i_{jkl} = (K - \Phi)(\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}).$$

The Ricci tensor is

$$R_{uu} = n\frac{r_{,u}}{r}\left(\log\left| \frac{r_{,v}}{\Phi} \right| \right)_{,u}, \qquad R_{vv} = n\frac{r_{,v}}{r}\left(\log\left| \frac{r_{,u}}{\Phi} \right| \right)_{,v},$$

$$R_{uv} = -\left(\log\left| \frac{r_{,u}r_{,v}}{\Phi} \right| \right)_{,uv} - n\frac{r_{,uv}}{r}, \qquad (A4)$$

$$R_{ij} = \left[-\frac{rr_{,uv}}{r_{,u}r_{,v}}\Phi + 2(n-1)(K-\Phi) \right]\gamma_{ij}.$$

The scalar curvature is

$$R = -\frac{\Phi}{r_{,u}r_{,v}}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right| \right)_{,uv} - 2n\frac{\Phi r_{,uv}}{rr_{,u}r_{,v}}$$

$$+ \frac{n(n-1)}{r^2}(K-\Phi). \tag{A5}$$

The Einstein tensor is

$$G_{uu} = R_{uu}, \quad G_{vv} = R_{vv},$$

$$G_{uv} = n(n-1)\frac{r_{,u}r_{,v}}{r^2}\left(1 - \frac{K}{\Phi} \right) + n\frac{r_{,uv}}{r}, \tag{A6}$$

$$G_{ij} = \left\{ \frac{r^2\Phi}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv} + 2(n-1)\frac{r_{,uv}}{r} \right] \right.$$

$$\left. - \frac{(n-2)(n-1)}{2}(K-\Phi) \right\}\gamma_{ij}.$$

The Weyl tensor is

$$C_{uvu}{}^u = \frac{n-1}{n+1}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right| \right)_{,uv} - \frac{r_{,uv}}{r} - \frac{r_{,u}r_{,v}}{r^2\Phi}(K-\Phi),$$

$$C_{iuju} = \frac{1}{n}r^2\gamma_{ij}C_{uvu}{}^u, \tag{A7}$$

$$C_{ijkl} = -\frac{2}{n(n-1)}r^4(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})C_{uv}{}^{vu}.$$

From these formulas, we can show the existence of a conserved current in the same way as given in the text. Namely, with the timelike vector field ξ^a defined by Eq. (2.4), the currents $\tilde{S}^a = \xi^b\tilde{T}_b{}^a$ and $S^a = \xi^bT_b{}^a$ are separately conserved, and the corresponding local masses are given, respectively, by

$$\tilde{M} = r^{n-1}(K-\Phi) \tag{A8}$$

and

$$M = \tilde{M} - \frac{2}{(n-1)(n-2)}\Lambda_{n+2}r^{n-1}. \tag{A9}$$

The v and u derivatives of M are given by the energy-momentum tensor as

$$M_{,v} = \kappa_{n+2}^2\frac{2r^n}{n}(T^u_v r_{,u} - T^v_v r_{,v}), \tag{A10}$$

$$M_{,u} = \kappa_{n+2}^2\frac{2r^n}{n}(T^v_u r_{,v} - T^v_v r_{,u}).$$

Let us now turn to the conserved current associated with the Weyl tensor. We start from the equation that results from the Bianchi identities [26],

$$C_{abcd}{}^{;d} = J_{abc}, \tag{A11}$$

where

$$J_{abc} = \frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]} + \frac{1}{(n+1)}g_{c[b}T_{;a]} \right). \tag{A12}$$

From this equation, we can show the existence of a locally conserved current Q^a given by

$$Q^a = r\ell_b n_c J^{bca}, \quad Q^a{}_{;a} = 0, \tag{A13}$$

where ℓ^a and n^a are the null vectors defined in Eqs. (2.19). The nonzero components are explicitly written as

$$Q^u = -rJ^{vu}{}_v, \quad Q^v = -rJ^{vu}{}_u. \tag{A14}$$

We then find the following relations,

$$(r^{n+1}C_{vu}{}^{vu})_{,v} = r^{n+1}J^v{}_{vv},$$

$$(r^{n+1}C_{vu}{}^{vu})_{,u} = r^{n+1}J^u{}_{uu}. \tag{A15}$$

These relations are generalization of Eqs. (2.21), and imply that the Weyl component $r^{n+1}C_{vu}{}^{vu}$ is the local charge associated with this conserved current.

Using the explicit form of $C_{vu}{}^{vu}$ in Eqs. (A7) and the Einstein equations, we can relate the Weyl charge to the local mass. We find

$$r^{n+1}C_{vu}{}^{vu}$$

$$= \frac{n(n-1)\tilde{M}}{2} - \frac{n-1}{n(n+1)}r^{n+1}(G^i{}_i - 2nG^v{}_v)$$

$$= \frac{n(n-1)M}{2} - \frac{n-1}{n(n+1)}\kappa_{n+2}^2 r^{n+1}(T^i{}_i - 2nT^v{}_v). \tag{A16}$$

Finally, we note that this equation implies that the linear combination of the energy-momentum tensor,

$$r^{n+1}(T^i{}_i - 2nT^v{}_v), \tag{A17}$$

plays the role of a local charge as well. Therefore, the behavior of this quantity is constrained nonlocally by the integral of the flux given by the corresponding linear combination of the currents S^a and Q^a. Although we do not explore it here, this fact may be useful in an analysis of the behavior of the bulk matter.

APPENDIX B: EINSTEIN-SCALAR THEORY IN THE BULK

In this appendix, we apply the local conservation law to the five-dimensional Einstein-scalar theory. We assume that there is no matter on the brane, but we take account of a coupling of the bulk scalar field to the brane tension. In this

case, the energy exchange between the brane and the bulk, hence the time evolution of M, occurs through the coupling.

We first consider a general bulk scalar field. Then, as a special case, we analyze the local mass on the brane for the exact dilatonic solution discussed by Koyama and Takahashi [13]. Finally, we clarify the relation between the local mass and the term that is identified as the dark radiation term in the effective four-dimensional approach in which the contribution of the scalar field energy-momentum to the brane is required to take the standard four-dimensional form [11].

1. Setup

We consider a theory described by the action

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{2\kappa_5^2} R - \frac{1}{2} \partial_a \phi \partial^a \phi - V(\phi) \right]$$

$$- \int d^4x \sqrt{-q} \, \sigma(\phi). \tag{B1}$$

For the bulk with the metric given by Eq. (2.1), the energy-momentum tensor in the bulk is given by

$$T_{vv} = \phi_{,v}^2, \quad T_{uu} = \phi_{,u}^2,$$

$$T_{uv} = - \frac{2r_{,u} r_{,v}}{\Phi} V(\phi),$$

$$T_{ij} = -r^2 \left[\frac{\Phi}{2r_{,u} r_{,v}} \phi_{,u} \phi_{,v} + V(\phi) \right] \gamma_{ij}. \tag{B2}$$

On the brane, the first derivatives of the scalar field tangent and normal to the brane are expressed, respectively, as

$$\phi' := \phi_{,a} n^a = -\phi_{,v} \dot{v} + \phi_{,u} \dot{u},$$

$$\dot{\phi} := \phi_{,a} v^a = \phi_{,v} \dot{v} + \phi_{,u} \dot{u}. \tag{B3}$$

The Codacci equation (2.36) gives, via the coupling to the brane tension, the boundary condition at the brane,

$$\phi' = \frac{1}{2} \frac{d}{d\phi} \sigma(\phi). \tag{B4}$$

In the present case, the effective Friedmann equation induced on the brane, Eq. (2.34), becomes

$$3 \left[H^2 + \frac{K}{r^2} \right] = \frac{1}{12} \kappa_5^2 \sigma^2 + \frac{3M}{r^4}. \tag{B5}$$

The time evolution of the local mass M on the brane is given by

$$\dot{M} = -\frac{1}{3} \kappa_5^2 r^4 H \left[\dot{\phi}^2 - 2V + \frac{1}{4} \left(\frac{d}{d\phi} \sigma \right)^2 \right] - \frac{1}{36} \kappa_5^4 r^4 \dot{\phi} \frac{d}{d\phi} \sigma^2. \tag{B6}$$

From the brane point of view, as given by Eqs. (2.43) in the text, the effective energy density and pressure are composed of the brane tension and the bulk matter induced on the brane as

$$\rho^{(\text{tot})} = \rho^{(T)} + \rho^{(B)}, \quad p^{(\text{tot})} = p^{(T)} + p^{(B)}, \tag{B7}$$

where

$$\kappa_4^2 \rho^{(T)} = \frac{1}{12} \kappa_5^4 \sigma^2, \quad \kappa_4^2 p^{(T)} = -\frac{1}{12} \kappa_5^4 \sigma^2,$$

$$\kappa_4^2 \rho^{(B)} = \frac{3M}{r^4},$$

$$\kappa_4^2 p^{(B)} = \frac{M}{r^4} + \frac{1}{3} \kappa_5^2 \left[\dot{\phi}^2 - 2V + \frac{1}{4} \left(\frac{d}{d\phi} \sigma \right)^2 \right], \tag{B8}$$

where Eq. (B4) is used. From the Bianchi identity on the brane, the conservation law for the total effective energy-momentum on the brane is obtained as

$$\dot{\rho}^{(B)} + 3H(\rho^{(B)} + p^{(B)}) = -\dot{\rho}^{(T)}. \tag{B9}$$

The above relation is mathematically equivalent to Eq. (B6).

As discussed after Eq. (2.45) in the text, Eq. (B9) gives the point of view from the brane, and it is naturally interpreted as the equation describing the energy exchange between the brane tension and the bulk matter induced on the brane. On the other hand, the time variation of the local mass along the brane, Eq. (B6), gives the point of view from the bulk, and it contains not only the energy transfer from the brane tension to the bulk (the last term) but also the energy flow of the bulk scalar field at the location of the brane, which is nonvanishing in general even if the scalar field has no coupling to the brane tension.

2. Dilatonic exact solution

In the case $K = 0$, and for special forms of $V(\phi)$ and $\sigma(\phi)$, an exact cosmological solution is known, as a realization of the bulk inflaton model [13]. The forms of the potential and brane tension are

$$\kappa_5^2 V(\phi) = \left(\frac{\Delta}{8} + \delta \right) \lambda_0^2 e^{-2\sqrt{2}b\kappa_5\phi}, \tag{B10}$$

$$\kappa_5^2 \sigma(\phi) = \sqrt{2} \lambda_0 e^{-\sqrt{2}b\kappa_5\phi}, \tag{B11}$$

where δ, b, and λ_0 are constant and are all assumed to be non-negative, and

$$\Delta = 4b^2 - \frac{8}{3}. \tag{B12}$$

If $\delta = 0$, there exists a static, Minkowski brane solution [14]. In order to avoid the presence of a naked singularity, the dilatonic coupling b^2 is assumed to be smaller than $1/6$ [13]. This implies that Δ is negative and is in the range

$$2 \leqslant (-\Delta) \leqslant \frac{8}{3}. \tag{B13}$$

The exact solution takes the form

$$ds^2 = e^{2W(z)}(-d\tau^2 + e^{2\alpha(\tau)}\delta_{ij}dx^i dx^j + e^{2\sqrt{2}b\kappa_5\phi(\tau)}dz^2),$$

$$\phi = \phi(\tau) + \Xi(z), \tag{B14}$$

with the brane located at $z = z_0$ and it is assumed that $\Xi(z_0) = 0$ without loss of generality. The scale factor of the brane and the scalar field on the brane are given by

$$r(\tau) = e^{\alpha(\tau)} = (H_0\tau)^{1/6b^2}, \quad e^{\sqrt{2}b\kappa_5\phi(\tau)} = H_0\tau, \tag{B15}$$

where

$$H_0 = \left(\Delta + \frac{8}{3}\right)\lambda_0 \sqrt{\frac{\delta}{-\Delta}} = 4b^2\lambda_0\sqrt{\frac{\delta}{-\Delta}}. \tag{B16}$$

As seen from the first of Eqs. (B15), the power-law inflation is realized on the brane for $b^2 < 1/6$.

Let us consider the time evolution of the energy content in this model. From the brane point of view, the time derivative of the brane tension $\rho^{(T)}$ is always negative:

$$\dot{\rho}^{(T)} = \frac{\Delta}{48b^4\delta\tau^3} < 0. \tag{B17}$$

Thus, from Eq. (B9), for an observer on the brane, there is one-way energy transfer from the brane tension to the bulk matter induced on the brane. From the bulk point of view, however, the situation is slightly more complicated. The time derivative of the local mass (or the generalized dark radiation) on the brane, Eq. (B6), is evaluated as

$$\frac{\dot{M}}{r^4} = \frac{1}{18b^6\delta\tau^3}\left(\frac{1}{3} - b^2\right)\left(\frac{\Delta}{8} + \delta\right). \tag{B18}$$

The sign of $\dot{M}$ is determined by the sign of $\Delta/8 + \delta$. Note that the sign of $\Delta/8 + \delta$ determines the sign of the bulk potential as well, as seen from Eq. (B10). If $\Delta/8 + \delta > 0$, i.e., $\delta > (-\Delta)/8 = (b^2/2) - 1/3$, we have $\dot{M} > 0$. Since M is the total bulk mass integrated up to the location of the brane, the increase in M implies an energy flow from the brane to the bulk. Therefore, in this case, the energy in the brane tension is transferred to the bulk scalar field and it flows out into the bulk. In contrast, if $\delta < (-\Delta)/8$, we have $\dot{M} < 0$. In this case, although there is still energy transfer from the brane tension to the bulk scalar field, the bulk energy flows onto the brane. In other words, there is a localization process of the bulk energy onto the brane that overwhelms the energy released from the brane tension.

3. Local mass and the effective four-dimensional description

In the bulk inflaton model with a quadratic potential [9–12], it has been shown that the bulk scalar field projected on the brane behaves exactly like a four-dimensional field in

the low energy limit, $H^2\ell^2 \ll 1$, where H is the Hubble parameter of the brane, and the leading order correction gives the gradual energy loss from the scalar field to the bulk, giving rise to the dark radiation term [9,11]. Here, we discuss the relation between the dark radiation term appearing in this effective four-dimensional description and the generalized dark radiation term given by the local mass in the bulk.

From the geometrical description [7], the induced Einstein equation on the brane is written as

$$^{(4)}G_{\mu\nu} = -\frac{1}{12}\kappa_5^4\sigma^2 q_{\mu\nu} + \kappa_5^2\tilde{T}^{(b)}_{\mu\nu} - E_{\mu\nu}, \tag{B19}$$

where

$$\tilde{T}^{(b)}_{\mu\nu} = \frac{2}{3}\left[\tilde{T}_{ab}q^a_\mu q^b_\nu - \left(\tilde{T}_{ab}n^a n^b - \frac{1}{4}\tilde{T}_{ab}g^{ab}\right)q_{\mu\nu}\right], \tag{B20}$$

is the projected tensor of the bulk energy-momentum onto the brane that includes the contribution of the cosmological constant; see Eq. (2.7). For a homogeneous and isotropic brane, the nonvanishing components are

$$\tilde{T}^{(b)}_{\tau\tau} = -\tilde{T}^v_v + \frac{1}{6}\tilde{T}^i_i,$$

$$\tilde{T}^{(b)i}_{\ i} = \frac{1}{6}\tilde{T}^i_i - \frac{\dot{u}}{v}\tilde{T}^v_u - \frac{\dot{v}}{u}\tilde{T}^u_v + \tilde{T}^v_v. \tag{B21}$$

Let us decompose $E_{\mu\nu}$ as

$$E_{\mu\nu} = E^{(b)}_{\mu\nu} + E^{(d)}_{\mu\nu}, \tag{B22}$$

where $E^{(b)}_{\mu\nu}$ is to be expressed in terms of the bulk energy-momentum tensor in such a way that the effective four-dimensional description is recovered, and $E^{(d)}_{\mu\nu}$ is the part that should be identified as the dark radiation term in this effective four-dimensional approach. To be in accordance with Ref. [11], we choose the components of $E^{(b)}_{\mu\nu}$ as

$$-E^{(b)}_{\tau\tau} = -\frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{v}\tilde{T}^v_u + \frac{\dot{v}}{u}\tilde{T}^u_v\right) + \frac{1}{12}\kappa_5^2(\tilde{T}^i_i - 3\tilde{T}^v_v)$$

$$= -E^{(b)i}_{\ i}, \tag{B23}$$

and identify the remaining part with the dark radiation term, X,

$$-E^{(d)}_{\tau\tau} = X = -E^{(d)i}_{\ i}. \tag{B24}$$

In the effective four-dimensional description, the Einstein equation on the brane takes the form,

$$^{(4)}G_{\mu\nu} = \kappa_4^2 T^{(\text{eff})}_{\mu\nu} - E^{(d)}_{\mu\nu}, \tag{B25}$$

where $T^{(\text{eff})}_{\mu\nu}$ is the effective energy-momentum tensor on the brane,

$$\kappa_4^2 T_{\mu\nu}^{(\mathrm{eff})} = -\frac{1}{12}\kappa_5^4 \sigma^2 q_{\mu\nu} + \kappa_5^2 \tilde{T}_{\mu\nu}^{(b)} - E_{\mu\nu}^{(b)}, \qquad (\mathrm{B}26)$$

and κ_4^2 is the four-dimensional gravitational constant that should be appropriately defined to agree with the conventional four-dimensional Einstein gravity in the low energy limit. In the present case of homogeneous and isotropic cosmology, the only nontrivial components are the effective energy density and pressure, which are given explicitly by

$$\kappa_4^2 \rho^{(\mathrm{eff})} = -\kappa_4^2 \tilde{T}^{(\mathrm{eff})\tau}{}_\tau = \frac{1}{12}\kappa_5^4 \sigma^2 - \frac{5}{4}\kappa_5^2 \tilde{T}^v{}_v + \frac{1}{4}\kappa_5^2 \tilde{T}^i{}_i$$

$$-\frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{v}\tilde{T}^v{}_u + \frac{\dot{v}}{u}\tilde{T}^u{}_v\right),$$

$$\kappa_4^2 p^{(\mathrm{eff})} = \frac{1}{3}\kappa_4^2 \tilde{T}^{(\mathrm{eff})i}{}_i = -\frac{1}{12}\kappa_5^4 \sigma^2 + \frac{1}{4}\kappa_5^2 \tilde{T}^v{}_v + \frac{1}{12}\kappa_5^2 \tilde{T}^i{}_i$$

$$-\frac{3}{8}\kappa_5^2\left(\frac{\dot{u}}{u}\tilde{T}^v{}_u + \frac{\dot{v}}{u}\tilde{T}^u{}_v\right). \qquad (\mathrm{B}27)$$

The effective Friedmann equation on the brane is written as

$$3\left[H^2 + \frac{K}{r^2}\right] = \kappa_4^2 \rho^{(\mathrm{eff})} - E_{\tau\tau}^{(d)}$$

$$= \frac{1}{12}\kappa_5^4 \sigma^2 - \frac{5}{4}\kappa_5^2 \tilde{T}^v{}_v + \frac{1}{4}\kappa_5^2 \tilde{T}^i{}_i$$

$$-\frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{v}\tilde{T}^v{}_u + \frac{\dot{v}}{u}\tilde{T}^u{}_v\right) + X. \qquad (\mathrm{B}28)$$

Applying this to a bulk scalar field with the action (B1), we find $p^{(\mathrm{eff})}$ and $p^{(\mathrm{eff})}$ are given by those of a four-dimensional scalar field φ with the potential,

$$\kappa_4^2 V^{(\mathrm{eff})}(\varphi) = \kappa_5^2\left(\frac{1}{2}V(\phi) + \frac{\kappa_5^2}{12}\sigma^2(\phi) - \frac{1}{16}\sigma'^2(\phi)\right),$$
$$\qquad (\mathrm{B}29)$$

where $\varphi = \sqrt{\kappa_5^2/\kappa_4^2}\,\phi$. From the contracted Bianchi identity, we obtain

$$D^\mu T_{\mu\tau}^{(\mathrm{eff})} = \dot{\rho}^{(\mathrm{eff})} + 3H(\rho^{(\mathrm{eff})} + p^{(\mathrm{eff})}) = -\frac{1}{\kappa_5^2}\frac{1}{r^4}(r^4 X).$$
$$\qquad (\mathrm{B}30)$$

Unfortunately, as we can see from Eqs. (B27), there is no simplification in the energy equation in terms of the five-dimensional energy-momentum tensor.

From the effective four-dimensional point of view, what happens is the conversion of the scalar field energy on the brane to the dark radiation via the coupling to the brane tension. From the bulk point of view, a natural interpretation is to regard the local mass M on the brane as the generalized dark radiation. These two different identifications of the dark radiation term on the brane coincide only when the bulk is in vacuum and M is constant. Comparison of the above decomposition of $E_{\tau\tau}$ with Eq. (2.40), we find the difference between the dark radiation in the four-dimensional description and the generalized dark radiation in terms of the local mass M as

$$\frac{3\tilde{M}}{r^4} = \frac{1}{4}\kappa_5^2(\tilde{T}^i{}_i - 5\tilde{T}^v{}_v) - \frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{v}\tilde{T}^v{}_u + \frac{\dot{v}}{u}\tilde{T}^u{}_v\right) + X.$$
$$\qquad (\mathrm{B}31)$$

[1] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996); **B475**, 94 (1996); N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, *ibid.* **436**, 257 (1998); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[3] D. Ida, J. High Energy Phys. **09**, 014 (2000); P. Kraus, *ibid.* **12**, 011 (1999).

[4] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000); J. Garriga and M. Sasaki, Phys. Rev. D **62**, 043523 (2000); H.A. Chamblin and H.S. Reall, Nucl. Phys. **B562**, 133 (1999).

[5] As the nice review of the brane cosmology, see, e.g., R. Maartens, gr-qc/0312059; D. Langlois, Suppl. Prog. Theor. Phys. **148**, 181 (2003).

[6] W. Israel, Nuovo Cimento Soc. Ital. Fis., B **44S10**, 1 (1966); **48**, 463(E) (1967).

[7] T. Shiromizu, K.i. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).

[8] K. Ichiki, M. Yahiro, M. Orito, and G.J. Mathews, Phys. Rev. D **66**, 043521 (2002); M. Orito, T. Kajino, K. Ichiki, M. Yahiro, G.J. Mathews, and Y. Wang, Nucl. Phys. **A719**, 1 (2003).

[9] Y. Himemoto and M. Sasaki, Phys. Rev. D **63**, 044015 (2001); Suppl. Prog. Theor. Phys. **148**, 235 (2003).

[10] M. Minamitsuji, Y. Himemoto, and M. Sasaki, Phys. Rev. D **68**, 024016 (2003).

[11] D. Langlois and M. Sasaki, Phys. Rev. D **68**, 064012 (2003); S. Kanno and J. Soda, Gen. Rel. Grav. **36**, 689 (2004).

[12] T. Tanaka and Y. Himemoto, Phys. Rev. D **67**, 104007 (2003); Y. Himemoto and T. Tanaka, *ibid.* **67**, 084014 (2003); S.L. Dubovsky, V.A. Rubakov, and P.G. Tinyakov, *ibid.* **62**, 105011 (2000); K. Ichiki, P.M. Garnavich, T. Kajino, G.J. Mathews, and M. Yahiro, *ibid.* **68**, 083518 (2003).

[13] K. Koyama and K. Takahashi, Phys. Rev. D **67**, 103503 (2003); **68**, 103512 (2003); T. Kobayashi and T. Tanaka, Phys. Rev. D **69**, 064037 (2004).

[14] M. Cvetic, H. Lu, and C.N. Pope, Phys. Rev. D **63**, 086004 (2001).

[15] D. Langlois and L. Sorbo, Phys. Rev. D **68**, 084006 (2003); D. Langlois, L. Sorbo, and M. Rodriguez-Martinez, Phys. Rev. Lett. **89**, 171301 (2002); A. Hebecker and J. March-Russell, Nucl. Phys. **B608**, 375 (2001); E. Leeper, R. Maartens, and C.F. Sopuerta, Class. Quantum Grav. **21**, 1125 (2004); T. Hiramatsu, K. Koyama, and A. Taruya, Phys. Lett. B **578**, 269

(2004); D. Langlois, R. Maartens, and D. Wands, *ibid.* **489**, 259 (2000); T. Kobayashi, H. Kudoh, and T. Tanaka, Phys. Rev. D **68**, 044025 (2003).

[16] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980); S.A. Hayward, Phys. Rev. D **53**, 1938 (1996).

[17] P.C. Vaidya, Nature (London) **171**, 260 (1953); R.W. Lindqiist, R.A. Schwarz, and C.W. Misner, Phys. Rev. **137**, B1364 (1965); A.z. Wang, gr-qc/9803038; B. Waugh and K. Lake, Phys. Rev. D **34**, 2978 (1986).

[18] A. Nayeri, Int. J. Mod. Phys. A **16S1C**, 1040 (2001); A. Chamblin, A. Karch, and A. Nayeri, Phys. Lett. B **509**, 163 (2001).

[19] I.H. Dwivedi and P.S. Joshi, Class. Quantum Grav. **6**, 1599 (1989); **8**, 1339 (1989).

[20] K. Lake, Phys. Rev. D **43**, 1416 (1991); P.S. Joshi and I.H. Dwivedi, *ibid.* **45**, 2147 (1992).

[21] S.M. Wagh and S.D. Maharaj, Gen. Relativ. Gravit. **31**, 975 (1999); S.G. Ghosh and N. Beesham, Phys. Rev. D **61**, 067502

(2000); S.G. Ghosh and N. Dadhich, *ibid.* **64**, 047501 (2001).

[22] T. Tanaka, Prog. Theor. Phys. Suppl. **148**, 307 (2003).

[23] S. Mukohyama, T. Shiromizu, and K.i. Maeda, Phys. Rev. D **62**, 024028 (2000); **63**, 029901(E) (2001).

[24] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990); Phys. Rev. Lett. **63**, 1663 (1989); R.G. Cai, Phys. Rev. D **53**, 5698 (1996); P.R. Brady, D. Nunez, and S. Sinha, *ibid.* **47**, 4239 (1993); S. Hod and T. Piran, Phys. Rev. Lett. **81**, 1554 (1998); C.M. Chambers and I.G. Moss, Class. Quantum Grav. **11**, 1035 (1994).

[25] C. Vaz, L. Witten, and T.P. Singh, Phys. Rev. D **65**, 104016 (2002).

[26] For mathematical aspects of general relativity, see, e.g., S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973); R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

Linearized Gravity on the de Sitter Brane in the Einstein Gauss-Bonnet Theory

Masato MINAMITSUJI[1,2,*)] and Misao SASAKI[2,**)]

[1] *Department of Earth and Space Science, Graduate School of Science,*
Osaka University, Toyonaka 560-0043, Japan
[2] *Yukawa Institute for Theoretical Physics,*
Kyoto University, Kyoto 606-8502, Japan

We investigate linearized gravity on a single de Sitter brane in the anti-de Sitter (AdS) bulk in the Einstein Gauss-Bonnet (EGB) theory. We find that Einstein gravity is recovered for a high energy brane, i.e., in the limit of a large expansion rate, i.e., for $H\ell \gg 1$, where H is the de Sitter expansion rate and ℓ is the curvature radius of the AdS bulk. We also show that, in the short distance limit, $r \ll \min\{\ell, H^{-1}\}$, Brans-Dicke gravity is obtained, whereas in the large distance limit, $r \gg \max\{\ell, H^{-1}\}$, a Brans-Dicke-type theory is obtained for $H\ell = O(1)$, and Einstein gravity is recovered both for $H\ell \gg 1$ and $H\ell \ll 1$. In the limit $H\ell \to 0$, these results smoothly match the results known for the Minkowski brane.

§1. Introduction

Recent progress in string theory suggests that our universe is not a 4-dimensional spacetime in reality, but is a 4-dimensional submanifold "brane" embedded in a higher-dimensional spacetime called "bulk". As a simple realization of this braneworld, the model proposed by Randall and Sundrum (RS)[1)] has attracted much attention because of its interesting feature that gravity is localized on the brane not through compactification but through warping of the extra dimension. This model is a solution of the 5-dimensional Einstein equations with a negative cosmological constant, where a Minkowski brane is embedded in the 5-dimensional anti-de Sitter (AdS) bulk. The linear perturbation theory in the RS model reveals that Einstein gravity is realized on the brane in the large distance limit. However, in the short distance limit, gravity on the brane becomes essentially 5-dimensional, which can be attributed to the large contribution of the Kaluza-Klein corrections.[2)] The cosmological extension of this model, the inclusion of black holes, and so on, have been discussed by various authors.[3)]

From the stringy point of view, it is plausible that there exist many fields and higher-order curvature corrections in addition to the bulk cosmological constant. In this paper, we consider the gravitational action with the Gauss-Bonnet term added to the usual Einstein-Hilbert term. This type of correction appears as low energy corrections in the perturbative approach to string theory, and it is a natural extension of the Einstein-Hilbert action from 4 dimensions to higher dimensions.[4) 6)]

---

*) E-mail: masato@vega.ess.sci.osaka-u.ac.jp
**) E-mail: misao@yukawa.kyoto-u.ac.jp

Cosmological braneworld models in the Einstein Gauss-Bonnet (EGB) theory are treated in Refs. 7)–13), and black holes in the EGB theory are studied in Refs. 14)–19).

Recently, Deruelle and Sasaki showed that in the EGB theory, the linearized gravitational force on the Minkowski brane behaves like a 4-dimensional one even in the short distance limit.[20] Then, Davis showed that Brans-Dicke gravity[21] is realized on the Minkowski brane in the short distance limit.[22] Although the effective gravitational theory in the nonlinear regime is completely unknown, these results imply that the experimental constraint on the maximum size (curvature radius) of the extra dimension is drastically weaker than in the RS model, in which the size of the extra dimension must be less than $\sim 0.1\,\mathrm{mm}$. Thus, the EGB theory deserves more detailed investigation from various points of view.

In this paper, as a step toward the understanding cosmological implications of the EGB theory, we investigate linear perturbations of a single de Sitter brane embedded in the AdS bulk. This paper is organized as follows. In §2, we describe our formulation in the EGB theory. We consider an AdS bulk with a single de Sitter brane as the background spacetime. In §3, we analyze the linear perturbation theory in the bulk and on the de Sitter brane. In §4, we consider the effective gravity theory on the brane in various limits. In the case $H\ell \gg 1$, where H is the expansion rate of the de Sitter brane and ℓ is the AdS curvature radius, we find that Einstein gravity with a cosmological constant is recovered on the de Sitter brane. We also show that Brans-Dicke gravity is obtained in the short distance limit, whereas in the large distance limit a Brans-Dicke type theory is obtained for $H\ell = O(1)$ and Einstein gravity both for $H\ell \gg 1$ and $H\ell \ll 1$. Furthermore, it is shown that the results for the Minkowski brane are recovered in the limit $H\ell \to 0$, namely, Brans-Dicke gravity in the short distance limit and Einstein gravity in the large distance limit.[22] In §5, we briefly summarize our results. In Appendix A, we review the results for the Minkowski brane.[22] In Appendix B, we define harmonic functions on the de Sitter spacetime that correspond to the Fourier modes in the Minkowski spacetime. In Appendix C, we consider the case of two de Sitter branes and show that there exists a tachyonic bound state mode that makes the system unstable, just as in the Minkowski case discussed in Ref. 23).

§2. Einstein Gauss-Bonnet braneworld

We consider a braneworld in the EGB theory with a cosmological constant. As usual, we assume mirror symmetry with respect to the brane. Then, we can focus on one of the two identical copies of the spacetime M with the brane as the boundary ∂M. The action is given by [4),6)]

$$
S = \int_M d^5x\sqrt{-g}\frac{1}{2\kappa_5^2}\left[{}^{(5)}R - 2\Lambda_5 + \alpha\left({}^{(5)}R^2 - 4\,{}^{(5)}R_{ab}\,{}^{(5)}R^{ab} + {}^{(5)}R_{abcd}\,{}^{(5)}R^{abcd}\right)\right]
$$
$$
+ \int_{\partial M} d^4x\sqrt{-q}\left[-\sigma + \mathcal{L}_m + \frac{1}{\kappa_5^2}\left(K + 2\alpha\left(J - 2G^{\mu}{}_{\nu}K^{\nu}{}_{\mu}\right)\right)\right], \tag{2·1}
$$

where α is the coupling constant for the Gauss-Bonnet term, which has dimensions of (length)^2, Λ_5 is the negative cosmological constant, g_{ab} and $q_{\mu\nu}$ are the bulk and brane metrics, respectively. Here, $\mathcal{L}_m$ is the Lagrangian density of the matter on the brane, and σ is the brane tension. The second term on the second line of Eq. (2·1) is the generalized Gibbons-Hawking term,[24] which is added to the boundary action in order to obtain a well-defined boundary value problem. Further, $K_{\mu\nu}$ is extrinsic curvature of the brane, and

$$J^\mu{}_\nu = -\frac{2}{3}K^\mu{}_\rho K^\rho{}_\sigma K^\sigma{}_\nu + \frac{2}{3}K K^\mu{}_\rho K^\rho{}_\nu + \frac{1}{3}K^\mu{}_\nu\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right). \qquad (2\cdot2)$$

The Latin indices $\{a, b, \cdots\}$ and the Greek indices $\{\mu, \nu, \cdots\}$ are used for tensors defined in the bulk and on the brane, respectively.

Extremizing the action S with respect to the bulk metric, the vacuum bulk Einstein Gauss-Bonnet equation is obtained as

$$\begin{aligned}
&{}^{(5)}G_{ab} + \Lambda_5\, g_{ab} \\
&\quad + \alpha\Big[2\Big({}^{(5)}R_a{}^{cde\,(5)}R_{bcde} - 2^{(5)}R^{cd\,(5)}R_{acbd} - 2^{(5)}R_{ac}{}^{(5)}R^c{}_b + {}^{(5)}R^{(5)}R_{ab}\Big) \\
&\quad - \frac{1}{2}g_{ab}\Big({}^{(5)}R^2 - 4^{(5)}R_{cd}{}^{(5)}R^{cd} + {}^{(5)}R_{cdef}{}^{(5)}R^{cdef}\Big)\Big] = 0.
\end{aligned} \qquad (2\cdot3)$$

The brane trajectory is determined by the junction condition, which is obtained by varying the action S with respect to the brane metric,[25],[26]

$$B^\mu{}_\nu = K^\mu{}_\nu - K\delta^\mu{}_\nu + 4\alpha\Big(\frac{3}{2}J^\mu{}_\nu - \frac{1}{2}J\delta^\mu{}_\nu - P^\mu{}_{\rho\nu\sigma}K^{\rho\sigma}\Big) = \frac{1}{2}\kappa_5^2 T^\mu{}_\nu, \qquad (2\cdot4)$$

where

$$\begin{aligned}
P_{\mu\rho\nu\sigma} &:= R_{\mu\rho\nu\sigma} + \Big(R_{\mu\sigma}q_{\rho\nu} - R_{\rho\sigma}q_{\mu\nu} + R_{\rho\nu}q_{\mu\sigma} - R_{\mu\nu}q_{\rho\sigma}\Big) \\
&\quad - \frac{1}{2}R\Big(q_{\mu\sigma}q_{\rho\nu} - q_{\mu\nu}q_{\rho\sigma}\Big),
\end{aligned} \qquad (2\cdot5)$$

and $T_{\mu\nu}$ is the energy momentum tensor of the matter on the brane, defined as

$$\delta\Big(\sqrt{-q}\mathcal{L}_m\Big) = -\frac{1}{2}\sqrt{-q}T_{\mu\nu}\delta q^{\mu\nu}. \qquad (2\cdot6)$$

Note that the extrinsic curvature here is that for the vector normal to ∂M pointing outward from the side of M.

§3. de Sitter brane in the Einstein Gauss-Bonnet theory

We next consider a de Sitter brane in the AdS bulk in the EGB theory and investigate the linearized gravity on the de Sitter brane.

3.1. *de Sitter brane in the Einstein Gauss-Bonnet theory*

We employ the Gaussian normal coordinates with respect to the brane and assume that the bulk metric takes the form[27]

$$ds^2 = dy^2 + b^2(y)\gamma_{\mu\nu}dx^\mu dx^\nu, \qquad (3\cdot1)$$

where $\gamma_{\mu\nu}$ is the metric of the 4-dimensional de Sitter spacetime with $R(\gamma) = 12H^2$. The background Einstein Gauss-Bonnet equation is

$$-3H^2 + 3b''b + 3b'^2 - 12\alpha\frac{b''}{b}(b'^2 - H^2) = -\Lambda_5 b^2. \qquad (3\cdot2)$$

This has the solution

$$b(y) = H\ell \sinh(y/\ell), \qquad (3\cdot3)$$

where ℓ is given by

$$\frac{1}{\ell^2} = \frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda_5}{3}}\right). \qquad (3\cdot4)$$

This is identical to the Minkowski brane case.[20],[28] Without loss of generality, we choose the location of the de Sitter brane to be

$$b(y_0) = 1. \qquad (3\cdot5)$$

Thus H is the expansion rate of the de Sitter brane.

3.2. *Bulk gravitational perturbations*

First, we consider gravitational perturbations in the bulk. We take the RS gauge,[1),2),29)]

$$h_{55} = h_{5\mu} = 0, \quad h^\mu_{\ \mu} = D_\nu h^\nu_{\ \mu} = 0, \qquad (3\cdot6)$$

where D_α denotes the covariant derivative with respect to $\gamma_{\mu\nu}$, and the perturbed metric is given by

$$ds^2 = dy^2 + b^2(y)\left(\gamma_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu. \qquad (3\cdot7)$$

The (μ,ν)-components of the linearized Einstein Gauss-Bonnet equation are given by

$$\left(1 - \bar{\alpha}\right)\left[\frac{1}{\sinh^4(y/\ell)}\partial_y\left(\sinh^4(y/\ell)\partial_y\right)\right.$$
$$\left. + \frac{1}{(H\ell)^2\sinh^2(y/\ell)}\left(\Box_4 - 2H^2\right)\right]h_{\mu\nu} = 0, \qquad (3\cdot8)$$

where

$$\bar{\alpha} = \frac{4\alpha}{\ell^2}, \qquad (3\cdot9)$$

and $\Box_4 = D^\mu D_\mu$ is the d'Alembertian with respect to $\gamma_{\mu\nu}$. Throughout this paper, we assume $\bar{\alpha} \neq 1$.

Equation (3·8) is separable. Setting $h_{\mu\nu} = \psi_p(y) Y_{\mu\nu}^{(p,2)}(x^\alpha)$, we obtain

$$\left[\frac{1}{\sinh^4(y/\ell)} \partial_y \Big(\sinh^4(y/\ell) \partial_y \Big) + \frac{m^2}{\ell^2 \sinh^2(y/\ell)} \right] \psi_p(y) = 0, \qquad (3·10)$$

$$\left[\Box_4 - (m^2 + 2) H^2 \right] Y_{\mu\nu}^{(p,2)} = 0, \qquad (3·11)$$

where $p^2 = m^2 - 9/4$ and $Y_{\mu\nu}^{(p,2)}$ are the tensor-type tensor harmonics on the de Sitter spacetime, which satisfy the gauge condition[30]

$$Y^{(p,2)\mu}{}_\mu = D_\nu Y^{(p,2)\nu}{}_\mu = 0. \qquad (3·12)$$

The properties of these harmonics are discussed in Appendix B.

Equation (3·10) is the same as that for a massless scalar field in the bulk.[31] There exists a mass gap for eigenvalues in the range $0 < m < 3/2$.[27] There is a unique bound state at $m = 0$, which gives $\psi_p(y) =$ constant and is called the zero mode. For $m > 3/2$, the mass spectrum is continuous and they are called the Kaluza-Klein modes. The general solution is

$$\psi_p(y) = \frac{1}{\sinh^{3/2}(y/\ell)} \left[A_p P_{3/2}^{ip}(\cosh(y/\ell)) + B_p Q_{3/2}^{ip}(\cosh(y/\ell)) \right], \qquad (3·13)$$

where $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are the associated Legendre functions of the first and second kinds, respectively.

For $p^2 > 0$ $(m > 3/2)$, we choose those harmonic functions $Y_{\mu\nu}^{(p,2)}$ that behave as e^{-ipt} in the limit $t \to \infty$. Then, assuming that there is no incoming wave from the past infinity $y = 0$, we find that we should set $B_p = 0$. In fact, the asymptotic behavior of $P_{3/2}^{ip}$ for $y \to 0$ is[32]

$$\frac{1}{\sinh^{3/2}(y/\ell)} P_{3/2}^{ip}(\cosh(y/\ell)) \xrightarrow[y \to 0]{} \frac{2^{ip}}{\Gamma(1 - ip)} \left(\frac{\sinh(y/\ell)}{\cosh(y/\ell)} \right)^{-ip - 3/2}$$

$$\simeq \frac{2^{ip}}{\Gamma(1 - ip)} \left(\frac{y}{\ell} \right)^{-3/2} e^{-ip \ln(y/\ell)}, \qquad (3·14)$$

which guarantees the boundary conditions with no incoming wave (i.e., retarded). Thus the bulk metric perturbations are constructed by

$$h_{\mu\nu} = \oint_C dp \, \psi_p(y) Y_{\mu\nu}^{(p,2)}(x^\mu), \qquad (3·15)$$

where the contour of integration C is chosen on the complex p-plane such that it runs from $p = -\infty$ to $p = \infty$ and covers the bound state pole at $p = 3i/2$ below the contour.[33]

3.3. *Linearized effective gravity on the brane*

We now investigate effective gravity on the brane. The position of the brane in the coordinate system is displaced in general as

$$y = y_0 - \ell \varphi(x^\mu), \qquad (3·16)$$

where the second term on the right-hand side describes the brane bending.[20),30)] The induced metric on the brane is given by

$$ds^2\Big|_{(4)} = \Big(\gamma_{\mu\nu} + \tilde{h}_{\mu\nu}\Big)dx^\mu dx^\nu; \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\coth(y_0/\ell)\,\varphi\gamma_{\mu\nu}\,. \tag{3.17}$$

The extrinsic curvature on the brane is given by

$$K^\mu{}_\nu = \frac{1}{\ell}\coth(y_0/\ell)\delta^\mu{}_\nu + \frac{1}{2}h^\mu{}_{\nu,y} + \ell\Big(D^\mu D_\nu + H^2\delta^\mu{}_\nu\Big)\varphi\,. \tag{3.18}$$

We consider the junction condition (2.4). The background part gives the relation between the brane tension and the location of the brane,

$$\kappa_5^2\sigma = \frac{6}{\ell}\coth(y_0/\ell)\left(1 - \frac{\bar{\alpha}}{3} + \frac{2\bar{\alpha}}{3\sinh^2(y_0/\ell)}\right), \tag{3.19}$$

where

$$\coth(y_0/\ell) = \sqrt{1 + (H\ell)^2}\,, \quad \sinh(y_0/\ell) = \frac{1}{H\ell}\,. \tag{3.20}$$

In the limit $H\ell \to 0$, Eq. (3.19) reduces to the Minkowski tension,

$$\kappa_5^2\sigma \simeq \frac{6}{\ell}\Big(1 - \frac{1}{3}\bar{\alpha}\Big). \tag{3.21}$$

The perturbative part of the junction condition gives

$$\Big(1 + \beta\Big)\Big(D_\mu D_\nu - \Box_4\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\Big)\varphi + \frac{1}{2\ell}\Big(1 - \bar{\alpha}\Big)h_{\mu\nu,y}$$

$$-\frac{1}{2}\bar{\alpha}\coth(y_0/\ell)\Big(\Box_4 - 2H^2\Big)h_{\mu\nu} = \frac{\kappa_5^2}{2\ell}S_{\mu\nu}\,, \tag{3.22}$$

where $S_{\mu\nu}$ is the perturbation of $T_{\mu\nu}$, namely

$$T^\mu{}_\nu = -\sigma\delta^\mu{}_\nu + S^\mu{}_\nu\,, \tag{3.23}$$

and

$$\beta := \frac{\cosh^2(y_0/\ell) + 1}{\sinh^2(y_0/\ell)}\,\bar{\alpha} = \big(2\coth^2(y_0/\ell) - 1\big)\bar{\alpha} = \big(2(H\ell)^2 + 1\big)\bar{\alpha}\,. \tag{3.24}$$

The trace of Eq. (3.22) gives the equation determining the brane bending as

$$\Big(\Box_4 + 4H^2\Big)\varphi = -\frac{\kappa_5^2}{6(1+\beta)\ell}\,S\,, \tag{3.25}$$

where $S = S^\mu{}_\mu$. Note that the field φ seems to be tachyonic, with mass-squared given by $-4H^2$. However, in the case of a de Sitter brane in Einstein gravity, there is a similar equation for the brane bending, but it was found to be non-dynamical.[30)] We see below that the situation is quite similar in the present case of the EGB theory.

To find the effective gravitational equation on the brane, we proceed as follows. Using the expression for the induced metric on the brane, Eq. (3·17), the perturbation of the brane Einstein tensor is given by

$$
\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] &= -\frac{1}{2}\Box_4 h_{\mu\nu} - 2H^2 h_{\mu\nu} + 2\coth(y_0/\ell)\Big[D_\mu D_\nu - \Box_4\,\gamma_{\mu\nu}\Big]\varphi \\
&= -3H^2\Big(h_{\mu\nu} - 2\coth(y_0/\ell)\gamma_{\mu\nu}\varphi\Big) - \frac{1}{2}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu} \\
&\quad +2\coth(y_0/\ell)\Big[D_\mu D_\nu - \Box_4\,\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\Big]\varphi\,.
\end{aligned}
\tag{3·26}
$$

Using the perturbed junction condition (3·22), we can eliminate the term involving φ from the above equation to obtain

$$
\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} &= -\frac{1-\bar{\alpha}}{2(1+\beta)}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu} \\
&\quad -\frac{1-\bar{\alpha}}{\ell(1+\beta)}\coth(y_0/\ell)h_{\mu\nu,y} + \frac{\kappa_5^2}{\ell}\frac{\coth(y_0/\ell)}{(1+\beta)}\,S_{\mu\nu}\,.
\end{aligned}
\tag{3·27}
$$

Eliminating the term proportional to $(\Box_4 - 2H^2)h_{\mu\nu}$ from Eqs. (3·26) and (3·27), we obtain

$$
\begin{aligned}
\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} &= \frac{\kappa_5^2\tanh(y_0/\ell)}{2\ell\,\bar{\alpha}}\,S_{\mu\nu} \\
&\quad -\frac{1-\bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell)\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi \\
&\quad -\frac{1-\bar{\alpha}}{2\ell\bar{\alpha}}\tanh(y_0/\ell)h_{\mu\nu,y}\,.
\end{aligned}
\tag{3·28}
$$

Together with Eq. (3·25), this can be regarded as an effective gravitational equation on the brane. The effect of the bulk gravitational field is contained in the last term, proportional to $h_{\mu\nu,y}$. Note that the limit $\bar{\alpha} \to 0$ is singular in the above equation. Thus an Einstein Gauss-Bonnet brane exhibits entirely different effective gravity from an Einstein brane even if $\bar{\alpha} \ll 1$.

3.4. *Harmonic decomposition*

Using the harmonic functions defined in Appendix B, we can obtain a closed (integro-differential) system on the brane. We decompose the perturbations on the brane as

$$
S_{\mu\nu} = S_{\mu\nu}^{(0)} + S_{\mu\nu}^{(2)}\,; \quad S_{\mu\nu}^{(0)} = \int_{-\infty}^{\infty} dp\,\Big(S_{(p,0)}Y_{\mu\nu}^{(p,0)}\Big)\,, \quad S_{\mu\nu}^{(2)} = \int_{-\infty}^{\infty} dp\,\Big(S_{(p,2)}Y_{\mu\nu}^{(p,2)}\Big)\,,
$$

$$
\varphi = \int_{-\infty}^{\infty} dp\,\varphi_{(p)}Y^{(p,0)}\,,
$$

$$
h_{\mu\nu} = \int_{-\infty}^{\infty} dp\,h_{(p)}Y_{\mu\nu}^{(p,2)}\,,
\tag{3·29}
$$

where $Y^{(p,0)}$ are the scalar harmonics and $Y_{\mu\nu}^{(p,0)}$ are the scalar-type tensor harmonics given in terms of $Y^{(p,0)}$, as defined in Appendix B. Note that, because of energy-momentum conservation, $D^\mu S_{\mu\nu} = 0$, there is no contribution from the vector-type

tensor harmonics that do not satisfy the divergence-free condition. If a bound state exists, we have to deform the contour of integration so that the corresponding pole is covered, as mentioned at the end of §3.2.

With the above decomposition, the metric perturbation on the brane $\bar{h}_{\mu\nu}$ given by Eq. (3·17) consists of the isotropic scalar-type part and tensor-type part. The scalar-type part is determined by Eq. (3·25), which gives

$$
\begin{aligned}
\varphi_{(p)} &= \frac{\kappa_5^2}{2(1+\beta)\ell} N_p \, S_{(p,0)} \\
&= \frac{\kappa_5^2}{2(1+\beta)\ell} \frac{1}{\sqrt{3}\sqrt{\left(p^2 + \frac{21}{4}\right)\left(p^2 + \frac{25}{4}\right)}H^2} S_{(p,0)} ,
\end{aligned}
\tag{3.30}
$$

where N_p is the normalization factor for the harmonics defined in Appendix B. We see that the propagator part of the above (i.e., the coefficient of $S_{(p,0)}$) does not contain a pole at $p = (5/2)i$, which would correspond to the tachyonic mode with mass-squared $-4H^2$. Instead, it becomes a branch point, and a branch cut appears between the points $p = (\sqrt{21}/2)i$ and $p = (5/2)i$. Thus we find that the tachyonic mode is absent and there is no instability associated with the brane bending due to the matter source on the brane.

Before we proceed, it is useful to note the equation

$$
\left(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\right)\varphi = \frac{\kappa_5^2}{2(1+\beta)\ell} S^{(0)}_{\mu\nu} ,
\tag{3.31}
$$

which directly follows from Eq. (3·30) and the definition of the scalar-type tensor harmonics $Y^{(p,0)}_{\mu\nu}$.

There is a free propagating tachyonic mode corresponding to the homogeneous solution of Eq. (3·25), which couples to neither the scalar nor tensor-type matter perturbations on the brane. However, we argue in the next subsection that the mode that corresponds to the exponential growth of the perturbation is unphysical; that is, the only physical mode associated with this tachyonic mode is exponentially decaying in time.

The traceless part of Eq. (3·22) gives

$$
\begin{aligned}
h_{(p)}(y_0) = &-\frac{1}{(ip + 3/2)} \\
&\times \frac{\ell^2 \sinh(y_0/\ell)\, P^{ip}_{3/2}(z_0)}{\left(1-\bar{\alpha}\right)P^{ip}_{1/2}(z_0) + \bar{\alpha}\left(-ip + 3/2\right)(H\ell)^2 \cosh(y_0/\ell)P^{ip}_{3/2}(z_0)} \\
&\times \frac{\kappa_5^2}{\ell} S_{(p,2)} ,
\end{aligned}
\tag{3.32}
$$

where $z_0 = \cosh(y_0/\ell)$. This shows that the harmonic component of the tensor-type metric perturbations on the brane has a simple pole at $p = (3/2)i$ in the complex p-plane, which corresponds to the zero mode.

For convenience, we also present the y-derivative of $h_{(p)}$:

$$\frac{1}{\ell}\partial_y h_{(p)}(y_0) = \frac{P^{ip}_{1/2}(z_0)}{(1-\bar{\alpha})P^{ip}_{1/2}(z_0) + \bar{\alpha}(-ip+3/2)(H\ell)^2\cosh(y_0/\ell)P^{ip}_{3/2}(z_0)}$$
$$\times \frac{\kappa_5^2}{\ell} S_{(p,2)}. \tag{3.33}$$

Then, Eqs. (3·25), (3·28) and (3·33) constitute the effective gravitational equations on the brane, which form a closed set of integro-differential equations.

3.5. *Source-free tachyonic mode*

Now, we consider the source-free tachyonic mode on the brane.[29] This mode corresponds to the homogeneous solution of Eq. (3·25), and therefore it does not couple to the matter perturbations on the brane.

On the complex p-plane, the solution corresponds to a pole at $p = (5/2)i$. Thus, the solution is given by

$$\varphi = \varphi_{(5i/2)}Y^{(5i/2,0)}. \tag{3.34}$$

For this mode, the junction condition (3·22) reveals that it is associated with a non-vanishing $h_{\mu\nu}$. The solution in the bulk is given by[29]

$$h_{\mu\nu} = \phi(y)\mathcal{L}_{\mu\nu}\varphi, \quad \mathcal{L}_{\mu\nu} = D_\mu D_\nu + H^2\gamma_{\mu\nu}. \tag{3.35}$$

This satisfies the transverse-traceless condition and the relation

$$\left(\Box_4 - 4H^2\right)h_{\mu\nu} = 0. \tag{3.36}$$

Thus, this mode falls within the mass gap between $m = 0$ and $3/2$, with mass $mH = \sqrt{2}H$.

Let us first analyze the behavior of the function $\phi(y)$. It should satisfy Eq. (3·8), which becomes

$$\left[\frac{1}{\sinh^4(y/\ell)}\partial_y\left(\sinh^4(y/\ell)\partial_y\right) + \frac{2}{\ell^2\sinh^2(y/\ell)}\right]\phi(y) = 0. \tag{3.37}$$

The general solution of this equation is given by

$$\phi(y) = c_1\phi_1(y) + c_2\phi_2(y),$$
$$\phi_1(y) = \coth(y/\ell), \quad \phi_2(y) = 1 + \coth^2(y/\ell), \tag{3.38}$$

where the coefficients c_1 and c_2 are related through the junction condition (3·22) as

$$1 - \frac{1}{2}H^2 c_1 - H^2\coth(y_0/\ell)\frac{1+\bar{\alpha}\coth^2(y_0/\ell)}{1+\beta}c_2 = 0. \tag{3.39}$$

As is readily seen, this mode diverges badly as $y \to 0$. Therefore, the regularity condition at $y = 0$ eliminates this mode. Nevertheless, because its effect on the brane seems to be non-trivial, it is interesting to investigate its physical meaning.

We note that ϕ_1 is a gauge mode. This can be checked by calculating the projected Weyl tensor $E_{\mu\nu} := {}^{(5)}C_{n\mu n\nu}$[13),34)] which is gauge invariant. We find that only the coefficient c_2 survives:

$$E^\mu{}_\nu(y, x^\alpha) = -\frac{c_2}{\ell^2 \sinh^4(y/\ell)} \mathcal{L}^\mu{}_\nu \varphi(x^\alpha).$$

(3·40)

This means that the junction condition (3·39) does not fix the physical amplitude c_2. It just fixes the gauge amplitude c_1.

To understand the physical meaning of this mode, it is useful to analyze the temporal behavior of the projected Weyl tensor. For simplicity, let us consider a spatially homogeneous solution for φ. Choosing a spatially closed chart for the de Sitter brane, for which the scale factor is given by $a(t) = H^{-1}\cosh(Ht)$, we find

$$\varphi = C_1 \frac{P_{1/2}^{5/2}(\tanh(Ht))}{\cosh^{3/2}(Ht)} + C_2 \frac{P_{1/2}^{-5/2}(\tanh(Ht))}{\cosh^{3/2}(Ht)} \underset{t\to\infty}{\sim} \tilde{C}_1 e^{Ht} + \tilde{C}_2 e^{-4Ht},$$

(3·41)

where $\tilde{C}_1$ and $\tilde{C}_2$ differ from C_1 and C_2, respectively, only by unimportant numerical factors. We see that the solution associated with C_1 is that which exhibits instability. If we insert this solution into Eq. (3·40), however, this unstable solution disappears. In fact, we obtain

$$E^t{}_t \simeq \frac{15H^2\tilde{C}_2 \, c_2}{\ell^2 \sinh^4(y/\ell)e^{4Ht}} \simeq \frac{15(H\ell)^2\tilde{C}_2 \, c_2}{16(H\ell)^4 \sinh^4(y/\ell)a^4(t)}.$$

(3·42)

We note that $E^t{}_t$ on the brane decays as $1/a^4(t)$. This is exactly what one expects for the behavior of so-called dark radiation. We also note that, although $E_{\mu\nu}$ does not vanish for spatially inhomogeneous modes, they decay as $1/a^3(t)$,[29)] giving no instability to the brane.

In the Einstein case, the dark radiation term appears if there exists a black hole in the bulk. This is also true in the EGB case. There also exists a spherically symmetric black hole solution in the EGB theory.[14)-19)] The metric is given by

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_{(3)}^2;$$

$$f(R) = 1 + \frac{R^2}{4\alpha}\left(1 - \sqrt{1 + \frac{16\alpha\mu}{3R^4} + \frac{4}{3}\alpha\Lambda_5}\right),$$

(3·43)

where $\mu = \kappa_5^2 M/(2\pi^2)$ and M is the mass of the black hole. For this solution, the projected Weyl tensor is given by

$$E^t{}_t = \frac{\mu}{R^4}\left(1 + \frac{4}{3}\alpha\Lambda_5 + \frac{16\alpha\mu}{3R^4}\right)^{-3/2}\left(1 + \frac{4}{3}\alpha\Lambda_5 + \frac{16\alpha\mu}{9R^4}\right)$$

$$\simeq \frac{\mu}{R^4}\left(1 + \frac{4}{3}\alpha\Lambda_5\right)^{-1/2},$$

(3·44)

for $R \gg (\alpha\mu)^{\frac{1}{4}}$.[13] Comparing Eq. (3·42) with Eq. (3·44), with the identification $R = \ell \sinh(y/\ell)\cosh(Ht)$, we find

$$c_2 \tilde{C}_2 \simeq \frac{16\mu}{15(H\ell)^2}\left(1 + \frac{4}{3}\alpha\Lambda_5\right)^{-1/2}. \qquad (3·45)$$

Thus the solution that decays exponentially in time corresponds to adding a small black hole in the bulk.[35]

In the two-brane system, the mode discussed here corresponds to the radion, which describes the relative displacement of the branes.[29),30] As in the case of Einstein gravity, the radion mode is truly tachyonic. However, for the EGB theory, there is a tachyonic bound state mode other than the radionic instability, as in the limit of the Minkowski brane,[23] as discussed explicitly in Appendix C. This renders the two-brane system physically unrealistic in the EGB theory.

§4. Linearized gravity in limiting cases

In this section, we discuss the effective gravity on the brane in various limiting cases. We find that the effective gravity reduces to 4-dimensional theories in all the limiting cases.

4.1. *High energy brane: $H\ell \gg 1$*

For a high energy brane, i.e., in the $H\ell \gg 1$ case, we have $\tanh(y_0/\ell) \simeq 1/(H\ell)$ and $\beta \simeq 2(H\ell)^2$. We assume that matter perturbations on the brane are dominated by modes for which $p \sim O(1)$. Specifically, we consider the case $H\ell \gg p$. Then, from Eqs. (3·31) and (3·33), we find that the second and third terms on the right-hand side of Eq. (3·28) are suppressed by a factor of $1/(H\ell)^2$ relative to the first term,

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} = \frac{\kappa_5^2 \tanh(y_0/\ell)}{2\ell\bar{\alpha}}\left(S_{\mu\nu} + O\big((H\ell)^{-2}\big)\right). \qquad (4·1)$$

Thus, we obtain Einstein gravity with the cosmological constant $3H^2$, with the gravitational constant G_4 given by

$$8\pi G_4 = \frac{\kappa_5^2}{2\ell\,\bar{\alpha}}\tanh(y_0/\ell) \approx \frac{\kappa_5^2}{2(H\ell)\bar{\alpha}\ell}. \qquad (4·2)$$

The terms we have ignored give the low energy non-local corrections:

$$\left(\delta G_{\mu\nu}[\tilde{h}]\right)_{\mathrm{corr},H\ell} = -\frac{\kappa_5^2(1 - \bar{\alpha})}{2\ell\bar{\alpha}}\tanh(y_0/\ell)$$

$$\times \int_{-\infty}^{\infty} dp \left\{ Y_{\mu\nu}^{(p,2)} S_{(p,2)} \right.$$

$$\times \left[\frac{P_{1/2}^{ip}(z_0)}{(1 - \bar{\alpha})P_{1/2}^{ip}(z_0) + \bar{\alpha}\big(-ip + 3/2\big)(H\ell)^2\cosh(y_0/\ell)P_{3/2}^{ip}(z_0)}\right]$$

$$\left. + Y_{\mu\nu}^{(p,0)} S_{(p,0)}\frac{1}{(1 + \beta)}\right\}. \qquad (4·3)$$

4.2. *Short and large distance limits*

In order to study short and large distance limits, it is convenient to start from the expression (3·26) for the perturbed Einstein tensor and Eq. (3·31), which relates the brane bending scalar φ to the scalar part of the energy momentum tensor $S_{\mu\nu}^{(0)}$. Let us recapitulate these expressions:

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = 2\coth(y_0/\ell)\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi$$

$$-\frac{1}{2}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu}, \tag{4·4}$$

$$\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi = \frac{\kappa_5^2}{2(1+\beta)\ell}\,S_{\mu\nu}^{(0)}. \tag{4·5}$$

1. Short distance limit: $r \ll \min\{\ell, H^{-1}\}$

For the short distance limit, $p \to \infty$, using Eq. (3·32), we find

$$-\frac{1}{2}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu}$$

$$= \frac{\kappa_5^2}{2\ell}\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)} S_{(p,2)} \frac{(H\ell)\,(-ip + 3/2)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}{\big(1 - \bar{\alpha}\big) + \bar{\alpha}(H\ell)^2\cosh(y_0/\ell)\big(-ip + 3/2\big)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}$$

$$\xrightarrow[p\to\infty]{} \frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)} S_{(p,2)}. \tag{4·6}$$

Also, using Eq. (4·5), we manipulate as

$$2\coth(y_0/\ell)\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi$$

$$= \frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\int_{-\infty}^{\infty} dp\, S_{(p,0)}Y_{\mu\nu}^{(p,0)}$$

$$-\frac{1-\bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell)\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi, \tag{4·7}$$

where we have used the identity

$$2\coth(y_0/\ell) = 2\coth(y_0/\ell) - \frac{1+\beta}{\bar{\alpha}}\tanh(y_0/\ell) + \frac{1+\beta}{\bar{\alpha}}\tanh(y_0/\ell)$$

$$= -\frac{1-\bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell) + \frac{1+\beta}{\bar{\alpha}}\tanh(y_0/\ell), \tag{4·8}$$

which follows from the definition of the parameter β, given in Eq. (3·24).

Substituting Eqs. (4·6) and (4·7) into Eq. (4·4), the linearized gravity on the brane at short distances becomes

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = \frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\,S_{\mu\nu}$$

$$-\frac{(1-\bar{\alpha})}{\bar{\alpha}}\tanh(y_0/\ell)\Big(D_\mu D_\nu - \Box_4\gamma_{\mu\nu} - 3H^2\gamma_{\mu\nu}\Big)\varphi, \tag{4·9}$$

with

$$\left(\Box_4 + 4H^2\right)\varphi = -\frac{\kappa_5^2}{6(1+\beta)\ell}\, S\,.$$ (4·10)

This is a scalar-tensor type theory.

Interestingly, the scalar field φ that describes the brane bending degree of freedom turns to be dynamical. As we have seen in the previous subsection, there is no intrinsically dynamical mode associated with the brane bending. Therefore, this emergence of a dynamical degree of freedom is due to the accumulative effect of all of the Kaluza-Klein modes, like a collective mode. Furthermore, because of the tachyonic mass, the system appears to be unstable. However, this is not the case. Because we have taken the limit $p \to \infty$, all the perturbations have energies much larger than H, and the tachyonic mass-squared $-4H^2$ is completely negligible. In other words, the spacetime appears to be flat at sufficiently short distance scales.

We can rewrite Eq. (4·9) in the form

$$\delta G_{\mu\nu}[\tilde{h}] + \Lambda_4 \tilde{h}_{\mu\nu} = \frac{1}{\Phi_0}\left(D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu}\right)\delta\Phi + \frac{8\pi G_4}{\Phi_0} S_{\mu\nu},$$

$$\left(\Box_4 + 4H^2\right)\delta\Phi = \frac{8\pi G_4}{3 + 2\omega}S\,,$$ (4·11)

with the identifications

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\,, \qquad \frac{\delta\Phi}{\Phi_0} = -\frac{1 - \bar{\alpha}}{\bar{\alpha}}\tanh(y_0/\ell)\varphi,$$

$$\omega = \frac{3\bar{\alpha}}{1 - \bar{\alpha}}\coth^2(y_0/\ell)\,, \qquad \Lambda_4 = 3H^2.$$ (4·12)

Ignoring the tachyonic mass of $\delta\Phi$, which is justified for the reason stated above, this is the linearized Brans-Dicke gravity with a cosmological constant.[21] For $H\ell \ll 1$, we have $\tanh(y_0/\ell) \simeq \coth(y_0/\ell) \simeq 1$. Thus we obtain

$$\frac{8\pi G_4}{\Phi_0} \simeq \frac{\kappa_5^2}{2\bar{\alpha}\ell}, \qquad \frac{\delta\Phi}{\Phi_0} \simeq -\frac{1 - \bar{\alpha}}{\bar{\alpha}}\varphi, \qquad \omega \simeq \frac{3\bar{\alpha}}{1 - \bar{\alpha}}.$$ (4·13)

This is in agreement with the Minkowski brane case investigated recently.[22]

The corrections are written

$$\left(\delta G_{\mu\nu}[\tilde{h}]\right)_{\mathrm{corr},p\gg1}$$

$$= -\frac{\kappa_5^2}{2\ell\bar{\alpha}}\tanh(y_0/\ell)\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)} S_{(p,2)}$$

$$\times\left[\frac{\left(1 - \bar{\alpha}\right)P_{1/2}^{ip}(z_0)}{\left(1 - \bar{\alpha}\right)P_{1/2}^{ip}(z_0) + \bar{\alpha}(H\ell)^2\left(-ip + 3/2\right)\cosh(y_0/\ell)P_{3/2}^{ip}(z_0)}\right].$$ (4·14)

2. Large distance limit: $r \gg \max\{\ell, H^{-1}\}$

For the limit $p \to 0$, using Eq. (3·32), we have

$$
-\frac{1}{2}\Big(\Box_4 - 2H^2\Big)h_{\mu\nu}
$$
$$
= \frac{1}{2}\int_{-\infty}^{\infty} dp\, Y_{\mu\nu}^{(p,2)} S_{(p,2)} \frac{\kappa_5^2 \ell \sinh(y_0/\ell) H^2(-ip + 3/2)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}{1 - \bar{\alpha} + \bar{\alpha}(H\ell)^2 \cosh(y_0/\ell)\big(-ip + 3/2\big)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0)}
$$
$$
\simeq \frac{3\kappa_5^2}{4\ell} \frac{(H\ell)P_{3/2}(z_0)/P_{1/2}(z_0)}{(1 - \bar{\alpha}) + (3/2)(H\ell)\coth(y_0/\ell)\bar{\alpha}P_{3/2}(z_0)/P_{1/2}(z_0)}
$$
$$
\times \int_{-\infty}^{\infty} dp\, S_{(p,2)} Y_{\mu\nu}^{(p,2)}\,. \tag{4·15}
$$

For the term involving φ, we pull out the part that takes the same form as the above equation. Using Eq. (3·30), we find

$$
2\coth(y_0/\ell)\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi
$$
$$
= \frac{3\kappa_5^2}{4\ell} \frac{(H\ell)P_{3/2}(z_0)/P_{1/2}(z_0)}{(1 - \bar{\alpha}) + (3/2)(H\ell)\coth(y_0/\ell)\bar{\alpha}P_{3/2}(z_0)/P_{1/2}(z_0)} \int_{-\infty}^{\infty} dp\, S_{(p,0)} Y_{\mu\nu}^{(p,0)}
$$
$$
- \frac{(H\ell)(1 - \bar{\alpha})P_{-1/2}(z_0)}{2(1 + \beta)P_{1/2}(z_0) - (H\ell)\coth(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0)}
$$
$$
\times \Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\varphi\,, \tag{4·16}
$$

where we have used the recursion relation

$$
\frac{3}{2}P_{3/2}(z_0) = 2z_0 P_{1/2}(z_0) - \frac{1}{2}P_{-1/2}(z_0)\,. \tag{4·17}
$$

Thus, the effective gravitational equation is expressed as

$$
\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = \frac{\kappa_5^2}{\ell}F_T\, S_{\mu\nu} - F_S\Big(D_\mu D_\nu - \gamma_{\mu\nu}\Box_4 - 3H^2\gamma_{\mu\nu}\Big)\tilde{\varphi}\,,
$$
$$
\Big(\Box_4 + 4H^2\Big)\tilde{\varphi} = -\frac{\kappa_5^2}{\ell}S\,, \tag{4·18}
$$

where we have rescaled φ to $\tilde{\varphi} = 6(1 + \beta)\varphi$, and F_T and F_S are constants that represent the tensor and scalar coupling strengths, respectively, given by

$$
F_T = \frac{(H\ell)\Big(4\cosh(y_0/\ell)P_{1/2}(z_0) - P_{-1/2}(z_0)\Big)}{2\Big(2(1 + \beta)P_{1/2}(z_0) - (H\ell)^2\cosh(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0)\Big)}\,,
$$
$$
F_S = \frac{(H\ell)(1 - \bar{\alpha})P_{-1/2}(z_0)}{6(1 + \beta)\Big(2(1 + \beta)P_{1/2}(z_0) - (H\ell)^2\cosh(y_0/\ell)\bar{\alpha}P_{-1/2}(z_0)\Big)}\,. \tag{4·19}
$$

In the intermediate range of $H\ell$, i.e., when $H\ell = O(1)$, F_T and F_S are comparable and we obtain a Brans-Dicke-type theory given by Eq. (4·11) with the identifications

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{\ell}F_T, \quad \frac{\delta\Phi}{\Phi_0} = -F_S\tilde{\varphi}, \quad \Lambda_4 = 3H^2,$$

$$\omega = \frac{F_T - 3F_S}{2F_S}$$

$$= \frac{6(1+\beta)\cosh(y_0/\ell)P_{1/2}(z_0) - 3\left(1 + (H\ell)^2\bar{\alpha}\right)P_{-1/2}(z_0)}{(1-\bar{\alpha})P_{-1/2}(z_0)}. \quad (4·20)$$

A potential problem in this case is that the tachyonic mass of the scalar field seems to make the system unstable. However, as discussed in §3.4, the tachyonic pole is not excited by the matter source. Further, as discussed in §3.5, the source-free tachyonic mode does not cause an instability either.

For $H\ell \ll 1$, we have $\omega \gg 1$, and the scalar field decouples to yield

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = \frac{\kappa_5^2}{\ell}\frac{\coth(y_0/\ell)}{1+\beta}S_{\mu\nu}. \quad (4·21)$$

Thus we obtain Einstein gravity with

$$8\pi G_4 = \frac{\kappa_5^2}{\ell}\frac{\coth(y_0/\ell)}{1+\beta}. \quad (4·22)$$

In the limit $H\ell \to 0$, we have

$$8\pi G_4 \simeq \frac{\kappa_5^2}{\ell}\frac{1}{1+\bar{\alpha}}. \quad (4·23)$$

This is the result for the Minkowski brane.

In the case $H\ell \gg 1$, we have $\omega \gg 1$, and we recover 4-dimensional Einstein gravity on the brane with

$$8\pi G_4 = \frac{\kappa_5^2}{2(H\ell)\bar{\alpha}\ell}. \quad (4·24)$$

Note that this is just a special case of the high energy brane case discussed in §4.1.

Thus we conclude that despite the presence of the tachyonic mass, the system is stable and well-behaved for all ranges of $H\ell$.

§5. Summary and discussion

We have investigated the linear perturbation of a de Sitter brane in an anti-de Sitter bulk in the 5-dimensional Einstein Gauss-Bonnet (EGB) theory. We have derived the effective theory on the brane which is described by a set of integro-differential equations.

To understand the nature of this theory in more detail, we have investigated its behavior in various limiting cases. In contrast to the case of a braneworld in

5-dimensional Einstein gravity, in which both the short distance and high energy brane limits exhibit 5-dimensional behavior, we have found that the gravity on the brane is effectively 4-dimensional for all the limiting cases.

For a high energy brane, i.e., in the case $H\ell \gg 1$, Einstein gravity is recovered, provided that the length scale of the fluctuations is of order H^{-1}. It is found that the low energy corrections are suppressed by an $O\left((H\ell)^{-2}\right)$ factor.

In the short distance case, $r \ll \min\{\ell, H^{-1}\}$, the scalar field that describes brane bending becomes dynamical, and we obtain Brans-Dicke gravity. This is consistent with the case of the Minkowski brane. A slight complication is that this brane-bending scalar field is tachyonic, with mass-squared $-4H^2$. Therefore, if it becomes dynamical, one would naively expect the theory to become unstable. However, because the energy scale of fluctuations is much larger than H, the fluctuations actually do not see this tachyonic mass, and hence there is no instability.

In the large distance case, $r \gg \max\{\ell, H^{-1}\}$, Einstein gravity is obtained in both cases $H\ell \ll 1$ and $H\ell \gg 1$, while a Brans-Dicke type theory is obtained for $H\ell = O(1)$. Although the scalar field of this Brans-Dicke gravity is tachyonic with mass-squared given by $-4H^2$, we have shown that this mode is not excited by the matter source, and hence it does not lead to an instability of the system.

In the limit $H\ell \to 0$, the previous results for the Minkowski brane have been recovered, that is, Brans-Dicke gravity at short distances and Einstein gravity at large distances.

In all the cases, the effective 4-dimensional gravitational constant depends non-trivially on the values of $H\ell$ and $\bar{\alpha}$, where $\bar{\alpha}$ is the non-dimensional coupling constant for the Gauss-Bonnet term. This indicates a variation in time of the gravitational constant in the course of the cosmological evolution of a brane in the EGB theory. It will be interesting to investigate in more detail the cosmological implications of the braneworld in the EGB theory.

Acknowledgements

Appendix A
——— *The Results for the Minkowski Brane* ———

Here, we summarize the results for the Minkowski brane.[22]

A.1. *Effective equations on the brane*

In the RS gauge, the perturbed metric in the bulk is written

$$ds^2 = dy^2 + b^2(y)\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu, \quad b(y) = e^{-|y|/\ell}, \tag{A·1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The brane is located at $y = 0$ in the background. The background part of the Einstein Gauss-Bonnet equation (2·3) gives the relation of the AdS radius to the bulk cosmological constant, Eq. (3·4). The perturbative part of Eq. (2·3) gives

$$\left(1 - \bar{\alpha}\right)\left(\partial_y^2 - 4\frac{1}{\ell}\partial_y + e^{2y/\ell}\Box_4\right)h_{\mu\nu} = 0\,. \tag{A·2}$$

Again, we consider the case $\bar{\alpha} \neq 1$. The location of the brane is perturbed to be at $y = -\ell\varphi$. The induced metric on the brane is given by

$$ds^2\Big|_{(4)} = \left(\eta_{\mu\nu} + \tilde{h}_{\mu\nu}\right)dx^\mu dx^\nu, \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\varphi\eta_{\mu\nu}\,. \tag{A·3}$$

The solution for $h_{\mu\nu}$ on the brane which satisfies the junction condition is given by

$$h_{\mu\nu}\Big|_{y=0} = -\frac{\kappa_5^2}{\ell}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{\ell^2 H_2^{(1)}(q\ell)}{(1-\bar{\alpha})q\ell H_1^{(1)}(q\ell) + \bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}$$
$$\times\left[S_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)S(p)\right], \tag{A·4}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind and $q^2 = -p^2$. The equation that determines the brane bending is

$$\Box_4\,\varphi = -\frac{\kappa_5^2}{6\ell}\frac{1}{1+\bar{\alpha}}S. \tag{A·5}$$

The perturbed 4-dimensional Einstein tensor is expressed as

$$\delta G_{\mu\nu}[\tilde{h}] = -\frac{1}{2}\Box_4\,h_{\mu\nu} + 2\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi. \tag{A·6}$$

Inserting Eq. (A·4) into Eq. (A·6), we obtain the effective equation on the brane, which reads

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{2\bar{\alpha}\ell}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{\bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}{(1-\bar{\alpha})q\ell H_1^{(1)}(q\ell) + \bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}$$
$$\times\left[S_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)S(p)\right]$$
$$+2\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi. \tag{A·7}$$

A.2. *Short distance limit*

In the short distance case, $q\ell \gg 1$, Eq. (A·7) becomes

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{2\bar{\alpha}\ell}S_{\mu\nu} - \left(\frac{1-\bar{\alpha}}{\bar{\alpha}}\right)\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\varphi. \tag{A·8}$$

Comparing Eqs. (A·5) and (A·8) with linearized Brans-Dicke gravity

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{8\pi G_4}{\Phi_0} S_{\mu\nu} + \frac{1}{\Phi_0}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box_4\right)\delta\Phi, \quad \Box_4\,\delta\Phi = \frac{8\pi G_4}{3 + 2\omega}S, \quad \text{(A·9)}$$

we find the correspondences

$$\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\bar{\alpha}\ell}, \quad \frac{\delta\Phi}{\Phi_0} = -\frac{1 - \bar{\alpha}}{\bar{\alpha}}\varphi, \quad \omega = \frac{3\bar{\alpha}}{1 - \bar{\alpha}}. \quad \text{(A·10)}$$

The corrections can be rewritten as

$$\left(\delta G_{\mu\nu}[\tilde{h}]\right)_{\text{corr}} = -\frac{\kappa_5^2}{2\bar{\alpha}\ell}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{(1 - \bar{\alpha})q\ell H_1^{(1)}(q\ell)}{(1 - \bar{\alpha})q\ell H_1^{(1)}(q\ell) + \bar{\alpha}q^2\ell^2 H_2^{(1)}(q\ell)}$$

$$\times\left[S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)S\right]. \quad \text{(A·11)}$$

A.3. *Large distance limit*

In the large distance case, $q\ell \ll 1$, Eq. (A·7) becomes

$$\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa_5^2}{\ell}\frac{1}{1 + \bar{\alpha}}S_{\mu\nu}. \quad \text{(A·12)}$$

Thus we obtain Einstein gravity with

$$8\pi G_4 = \frac{\kappa_5^2}{\ell}\frac{1}{1 + \bar{\alpha}}. \quad \text{(A·13)}$$

Appendix B
—— *Harmonic Functions on de Sitter Geometry* ——

In this appendix, we consider the harmonics on the de Sitter spacetime with curvature radius H^{-1}. They are obtained by the Lorentzian generalization of the tensor harmonics on an n-dimensional constant curvature Riemannian space.[36] We focus on the tensor-type and scalar-type harmonics.

B.1. *Tensor-type harmonics*

The tensor-type tensor harmonics satisfy the relation

$$\left(\Box_4 - \left(p^2 + 17/4\right)H^2\right)Y_{\mu\nu}^{(p,2)}(x^\mu) = 0, \quad \text{(B·1)}$$

which corresponds to 4-dimensional massive gravitons with mass-squared $m^2 H^2 = (p^2 + 9/4)H^2$. They satisfy the transverse-traceless condition,

$$Y^{(p,2)\mu}{}_\mu = Y^{(p,2)\nu}{}_{\mu|\nu} = 0. \quad \text{(B·2)}$$

In reality, the tensor harmonics have 3 more indices for the spatial eigenvalues. If we adopt flat slicing,

$$ds^2 = -dt^2 + H^{-2}e^{2Ht}\delta_{ij}dx^i dx^j, \quad \text{(B·3)}$$

we can use the standard Fourier modes $e^{ik \cdot x}$, and the spatial indices will be continuous. In addition, we also have discrete indices σ that describe the polarization degrees of freedom (5 in 4-dimensions). However, for notational simplicity, we omit these indices.

We ortho-normalize the tensor harmonics as

$$\int d^4x \sqrt{-\gamma}\, Y_{\mu\nu}^{(p,2)}\, Y^{*(p',2)\mu\nu} = \delta(p - p')\delta^3(k - k')\delta_{\sigma,\sigma'}\,. \tag{B·4}$$

Although we have no explicit proof of completeness, due to our poor knowledge, we assume that $Y_{\mu\nu}^{(p,2)}$ for $-\infty < p < \infty$ constitute a complete set for the space of transverse-traceless tensors.

B.2. *Scalar-type harmonics*

The scalar-type harmonics $Y^{(p,0)}(x^\mu)$ satisfy the equation for a scalar field with mass-squared $m^2 H^2 = (p^2 + 9/4)H^2$,

$$\left(\Box_4 - \left(p^2 + \frac{9}{4}\right)H^2\right) Y^{(p,0)}(x^\mu) = 0. \tag{B·5}$$

We assume that they satisfy the ortho-normality condition,

$$\int d^4x \sqrt{-\gamma}\, Y^{(p,0)} Y^{*(p',0)} = \delta(p - p')\delta^3(k - k'). \tag{B·6}$$

From $Y^{(p,0)}$, the ortho-normalized scalar-type vector harmonics are constructed as

$$Y_\mu^{(p,0)} = \frac{i}{H\sqrt{p^2 + 9/4}} D_\mu Y^{(p,0)}, \tag{B·7}$$

which satisfy the relation

$$\int d^4x \sqrt{-\gamma}\, Y_\mu^{(p,0)} Y^{*(p',0)\mu} = \delta(p - p')\delta^3(k - k'). \tag{B·8}$$

The trace-free and divergence-free scalar-type tensor harmonics are constructed, respectively, as

$$\overline{Y}_{\mu\nu}^{(p,0)} = N_p\left[D_\mu D_\nu Y^{(p,0)} - \frac{1}{4}\left(p^2 + \frac{9}{4}\right)\gamma_{\mu\nu} H^2 Y^{(p,0)}\right],$$

$$Y_{\mu\nu}^{(p,0)} = N_p\left[D_\mu D_\nu Y^{(p,0)} - \left(p^2 + \frac{21}{4}\right)\gamma_{\mu\nu} H^2 Y^{(p,0)}\right]$$

$$= \overline{Y}_{\mu\nu}^{(p,0)} - \frac{3}{4}N_p\left(p^2 + \frac{25}{4}\right)H^2 \gamma_{\mu\nu} Y^{(p,0)}, \tag{B·9}$$

where

$$|N_p|^2 = \frac{1}{3\left(p^2 + 21/4\right)\left(p^2 + 25/4\right)H^4}. \tag{B·10}$$

Without loss of generality, we assume that N_p is real and positive. The scalar-type divergence-free tensor harmonics $Y_{\mu\nu}^{(p,0)}$ satisfy the ortho-normality condition

$$\int d^4x \sqrt{-\gamma}\, Y_{\mu\nu}^{(p,0)} Y^{*(p',0)\mu\nu} = \delta(p - p')\delta^3(k - k'). \tag{B·11}$$

Appendix C

—— *Tachyonic Bound State in de Sitter Two-Brane System* ——

In Ref. 23), Charmousis and Dufaux showed that for a Minkowski two-brane system, there exists a tachyonic bound state on the negative tension brane. This fact implies that the Minkowski two-brane system is unstable with respect to linear perturbations. Following Ref. 23), we show that there exists a tachyonic bound state also for the de Sitter two-brane system.

C.1. *Possibility of a negative norm state*

We consider a de Sitter two-brane system. The brane located at smaller radius in the AdS space has negative tension. We discuss only the bulk gravitational perturbations. The matter perturbations on each brane are not taken into account.

The bulk component of the perturbed Einstein Gauss-Bonnet equation including the boundary branes is written in Sturm-Liouville form as

$$\left\{\left(b^4 - \bar{\alpha}\ell^2\left(b^2 b'^2 - b^2 H^2\right)\right)\psi_{p,y}\right\}_{,y} = -b^2\left(1 - \bar{\alpha}\ell^2\frac{b''}{b}\right)\left(p^2 + \frac{9}{4}\right)H^2\psi_p. \quad \text{(C·1)}$$

Using Eq. (C·1), the boundary condition on each brane is derived. For $H = 0$ and $b(y) = e^{-|y|/\ell}$, Eq. (C·1) naturally reduces to the Minkowski version, Eq. (8) in Ref. 23).

1. **On the positive tension brane**

   Imposing Z_2 symmetry, the warp factor around the positive tension brane can be expressed as

$$b(y) = H\ell \sinh\left(\frac{y_+ - |y - y_+|}{\ell}\right). \quad \text{(C·2)}$$

Integrating Eq. (C·1) around $y = y_+$ and using the Z_2-symmetry,

$$\partial_y\psi_p(y_+ - 0) = \frac{\zeta}{\ell}\frac{(p^2 + 9/4)\cosh(y_+/\ell)}{\sinh^3(y_+/\ell)}\psi_p(y_+), \quad \text{(C·3)}$$

where

$$\zeta := \frac{\bar{\alpha}}{1 - \bar{\alpha}}. \quad \text{(C·4)}$$

2. **On the negative tension brane**

   Similarly, the Z_2 symmetry gives the warp factor around the negative tension brane as

$$b(y) = H\ell \sinh\left(\frac{|y - y_-| + y_-}{\ell}\right). \quad \text{(C·5)}$$

Integrating Eq. (C·1) around $y = y_-$ and using the Z_2-symmetry, we have

$$\partial_y\psi_p(y_- + 0) = \frac{\zeta}{\ell}\frac{(p^2 + 9/4)\cosh(y_-/\ell)}{\sinh^3(y_-/\ell)}\psi_p(y_-). \quad \text{(C·6)}$$

For both branes, the boundary conditions are of a mixed (Robin) type. This makes it impossible for us to prove the positivity of the norm. Explicitly, we have

$$\int_{y_-}^{y_+} dy \left(b^4 - \bar{\alpha}\ell^2 (b^2 b'^2 - b^2 H^2) \right) (\partial_y \psi_p)^2$$

$$= (H\ell)^4 \left(p^2 + \frac{9}{4} \right) \left[\frac{\bar{\alpha}}{2\ell} \left(\sinh(2y_+/\ell)\psi_p^2(y_+) - \sinh(2y_-/\ell)\psi_p^2(y_-) \right) \right.$$

$$\left. + \frac{(1 - \bar{\alpha})}{\ell^2} \int_{y_-}^{y_+} dy \, \sinh^2(y/\ell)\psi_p^2(y) \right]. \tag{C·7}$$

Thus the norm is no longer positive definite for $p^2 + 9/4 > 0$.

C.2. *Condition for the existence of a tachyonic bound state*

In order to determine whether a tachyonic bound state exists, we need to analyze the mass spectrum. The tachyonic eigenmode, if it exists, is given by

$$\psi_q(y) = \frac{1}{\sinh^{3/2}(y/\ell)} \left[A_q P_{3/2}^{-q}(\cosh(y/\ell)) + B_q P_{3/2}^q(\cosh(y/\ell)) \right], \tag{C·8}$$

where $m^2 = -\mu^2$, $q := \sqrt{\mu^2 + 9/4}$, and $q^2 = -p^2$. The y-derivative of the quantity is

$$\partial_y \psi_q = -\frac{1}{\ell \sinh^{5/2}(y/\ell)} \left[\left(\frac{3}{2} - q \right) A_q P_{1/2}^{-q}(\cosh(y/\ell)) \right.$$

$$\left. + \left(\frac{3}{2} + q \right) B_q P_{1/2}^q(\cosh(y/\ell)) \right]. \tag{C·9}$$

Using the boundary conditions on the two branes, Eqs. (C·3) and (C·6), we obtain

$$A_q \left(\frac{3}{2} - q \right) \left((z_+^2 - 1) P_{1/2}^{-q}(z_+) + \zeta \left(\frac{3}{2} + q \right) z_+ P_{3/2}^{-q}(z_+) \right)$$

$$+ B_q \left(\frac{3}{2} + q \right) \left((z_+^2 - 1) P_{1/2}^q(z_+) + \zeta \left(\frac{3}{2} - q \right) z_+ P_{3/2}^q(z_+) \right) = 0,$$

$$A_q \left(\frac{3}{2} - q \right) \left((z_-^2 - 1) P_{1/2}^{-q}(z_-) + \zeta \left(\frac{3}{2} + q \right) z_- P_{3/2}^{-q}(z_-) \right)$$

$$+ B_q \left(\frac{3}{2} + q \right) \left((z_-^2 - 1) P_{1/2}^q(z_-) + \zeta \left(\frac{3}{2} - q \right) z_- P_{3/2}^q(z_-) \right) = 0, \tag{C·10}$$

where $z_\pm = \cosh(y_\pm/\ell)$.

In order for a non-trivial solution for A_q and B_q to exist, the determinant must vanish. Imposing this condition explicitly, we have

$$\left((z_+^2 - 1) P_{1/2}^{-q}(z_+) + \zeta \left(\frac{3}{2} + q \right) z_+ P_{3/2}^{-q}(z_+) \right)$$

$$\times \left((z_-^2 - 1) P_{1/2}^q(z_-) + \zeta \left(\frac{3}{2} - q \right) z_- P_{3/2}^q(z_-) \right)$$

$$-\left((z_-^2 - 1)P_{1/2}^{-q}(z_-) + \zeta\left(\frac{3}{2} + q\right)z_- P_{3/2}^{-q}(z_-)\right)$$

$$\times\left((z_+^2 - 1)P_{1/2}^{q}(z_+) + \zeta\left(\frac{3}{2} - q\right)z_+ P_{3/2}^{q}(z_+)\right) = 0. \qquad (\text{C·11})$$

The pole at $q = 3/2$, which corresponds to the zero mode, is divided out in deriving Eq. (C·11). If there exists a solution of Eq. (C·11) for $q > 3/2$, its existence implies the existence of a tachyonic bound state.

C.3. *Existence of a tachyonic bound state*

From Eq. (C·11), we have

$$\frac{(z_-^2 - 1)P_{1/2}^{q}(z_-) + \zeta\left(\frac{3}{2} - q\right)z_- P_{3/2}^{q}(z_-)}{(z_-^2 - 1)P_{1/2}^{-q}(z_-) + \zeta\left(\frac{3}{2} + q\right)z_- P_{3/2}^{-q}(z_-)}$$

$$= \frac{(z_+^2 - 1)P_{1/2}^{q}(z_+) + \zeta\left(\frac{3}{2} - q\right)z_- P_{3/2}^{q}(z_-)}{(z_+^2 - 1)P_{1/2}^{-q}(z_+) + \zeta\left(\frac{3}{2} + q\right)z_+ P_{3/2}^{-q}(z_+)}. \qquad (\text{C·12})$$

Using the definition of the Legendre functions,[32]

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left[-\nu,\, \nu+1;\, 1-\mu;\, \frac{1-z}{2}\right], \qquad (\text{C·13})$$

we see that the left-hand side of Eq. (C·12) is generally much larger than the right-hand side for $q \gg 1$ for fixed z_+ and z_-. Therefore, in order for this equation to be satisfied, we must have

$$q - \frac{3}{2} \simeq \frac{(z_-^2 - 1)P_{1/2}^{q}(z_-)}{\zeta z_- P_{3/2}^{q}(z_-)} \to \frac{z_-^2 - 1}{\zeta z_-} \quad \text{for } q \to \infty. \qquad (\text{C·14})$$

This is a consistent solution for $\zeta \ll 1$. Thus a tachyonic bound state exists in the de Sitter brane case as well.

The tachyon mass is given by

$$\mu H = \sqrt{q^2 - 9/4}\, H \simeq \frac{(z_-^2 - 1)H\ell}{\zeta z_- \ell}. \qquad (\text{C·15})$$

In the low energy case, we have $z_+ > z_- \gg 1$ and $H\ell \simeq 1/z_+ \ll 1$. Hence, the above reduces to

$$\mu H \simeq \frac{\Omega}{\zeta \ell}, \qquad (\text{C·16})$$

where

$$\Omega := \frac{b(z_-)}{b(z_+)} \simeq \frac{z_-}{z_+} \sim e^{-(y_+ - y_-)/\ell}. \qquad (\text{C·17})$$

This is consistent with the result for the Minkowski brane.[23]

In the high energy case, $H\ell \gg 1$, we have

$$\mu H \simeq \frac{\Omega^2 H}{\zeta (H\ell)^2} \ll \frac{H}{\zeta} \,. \tag{C.18}$$

Thus the high background expansion rate of the brane suppresses the tachyonic mass, resulting in a tendency to stabilize the two-brane system.

References

1) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 4690; hep-th/9906064.
2) J. Garriga and T. Tanaka, Phys. Rev. Lett. **84** (2000), 2778; hep-th/9911055.
3) R. Maartens, gr-qc/0312059.
4) R. C. Myers, Phys. Rev. D **36** (1987), 392.
5) S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D **65** (2002), 023521; hep-th/0108172.
6) N. Deruelle and J. Madore, gr-qc/0305004.
7) J. E. Kim, B. Kyae and H. M. Lee, Nucl. Phys. B **582** (2000), 296 [Errata; **591** (2000), 587]; hep-th/0004005.
8) B. Abdesselam and N. Mohammedi, Phys. Rev. D **65** (2002), 084018; hep-th/0110143.
9) C. Charmousis and J. F. Dufaux, Class. Quantum Grav. **19** (2002), 4671; hep-th/0202107.
10) E. Gravanis and S. Willison, Phys. Lett. B **562** (2003), 118; hep-th/0209076.
11) J. E. Lidsey and N. J. Nunes, Phys. Rev. D **67** (2003), 103510; astro-ph/0303168.
12) M. H. Dehghani, hep-th/0404118.
13) K. i. Maeda and T. Torii, Phys. Rev. D **69** (2004), 024002; hep-th/0309152.
14) J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62** (2000), 084013; hep-th/0003271.
15) R. G. Cai, Phys. Rev. D **65** (2002), 084014; hep-th/0109133.
16) M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628** (2002), 295; hep-th/0112045.
17) M. Cvitan, S. Pallua and P. Prester, Phys. Lett. B **546** (2002), 119; hep-th/0207265.
18) M. Cvitan, S. Pallua and P. Prester, Phys. Lett. B **555** (2003), 248; hep-th/0212029.
19) T. Clunan, S. F. Ross and D. J. Smith, Class. Quantum Grav. **21** (2004), 3447; gr-qc/0402044.
20) N. Deruelle and M. Sasaki, Prog. Theor. Phys. **110** (2003), 441; gr-qc/0306032.
21) C. Brans and R. H. Dicke, Phys. Rev. **124** (1961), 925.
22) S. C. Davis, hep-th/0402151.
23) C. Charmousis and J. F. Dufaux, hep-th/0311267.
24) G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15** (1977), 2738.
25) W. Israel, Nuovo Cim. B **44** (1966), 1 [Errata; **48** (1967), 463].
26) S. C. Davis, Phys. Rev. D **67** (2003), 024030; hep-th/0208205.
27) J. Garriga and M. Sasaki, Phys. Rev. D **62** (2000), 043523; hep-th/9912118.
28) Y. M. Cho, I. P. Neupane and P. S. Wesson, Nucl. Phys. B **621** (2002), 388; hep-th/0104227.
29) U. Gen and M. Sasaki, Prog. Theor. Phys. **108** (2002), 471; gr-qc/0201031.
30) U. Gen and M. Sasaki, Prog. Theor. Phys. **105** (2001), 591; gr-qc/0011078.
31) S. Kobayashi, K. Koyama and J. Soda, Phys. Lett. B **501** (2001), 157; hep-th/0009160.
32) See e.g., H. Bateman, *Higher Transcendental Functions* Vol. I (McGraw-Hill, New York, 1953).
33) Y. Himemoto, T. Tanaka and M. Sasaki, Phys. Rev. D **65** (2002), 104020; gr-qc/0112027.
34) T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D **62** (2000), 024012; gr-qc/9910076.
35) H. Yoshiguchi and K. Koyama, hep-th/0403097.
36) H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. No. 78 (1984), 1.