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COMBINATORIAL PREBUNDLES PART II*

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1. Introduction

It is well known that smooth m spheres embedded in smooth $m+2$ manifolds have the trivial normal bundles, provided $m \geq 3$. This is a direct consequence from the fact that SO_2 has the same homotopy type as the circle. The purpose of the paper is to show the analogous Theorem for locally flatly embedded *PL m* spheres of codimension two except for the case $m=4$.

Theorem A. *Let f: $S \rightarrow W$ be a locally flat PL embedding of the m sphere S into a PL $m+2$ manifold W. Suppose that W is orientable and $m \neq 2,4$. Then f has the trivial normal 2 cell bundle: That is to say, the embedding f is collared.*

The assumption that W is orientable may be weakened by saying that a regular neighborhood of $f(S)$ in W is orientable. If $m \geq 2$, then normal prebundles for f are clearly orientable. We have, therefore,

Addendum. *Every locally flat PL embedding of the m sphere of codimension two is collared, provided that $m \geq 5$ or $m=3$.*

REMARK. The case $m=4$ is unknown for the author.

From Theorem A, we shall deduce:

Theorem B. *The $k(\neq 3)$-th homotopy group $\pi_k(PR_2)$ of the structural group PR_2 of 2 prebundles is isomorphic to $\pi_k(O_2)$.*

The following was proven in [3]:

Proposition 1.1. *The structural group ΠL_2 of PL 2 cell bundles has the homotopy type of the orthogonal group O_2.*

So we have:

Corollary to Theorem B. $\pi_k(PR_2, \Pi L_2) \cong 0$ *for* $k \neq 3$ *and* $\pi_3(PR_2, \Pi L_2) \cong \pi_3(PR_2)$.

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2. Applications

Let M and W be PL manifolds. Recall that a PL embedding $f: M \rightarrow W$ is *oriented*, if M and W are oriented. Two oriented PL embeddings $f: M \rightarrow W$ and $g: M \rightarrow W'$ are *equivalent* if there is an orientation preserving PL homeomorphism $h: W \rightarrow W'$ such that $hf=g$. The equivalence of oriented PL embeddings is clearly a proper equivalence relation. Let S_k denote the standard oriented PL k sphere. A *PL m knot* means a locally flat PL embedding $f: S_m \rightarrow S_{m+2}$. Then the PL homeomorphism class of $S_{m+2}-f(S_m)$, called the *complement*, is an invariant of the equivalence class of the knot.

By Theorem A we may sharpen Levine's unknotting theorem in codimension two as follows.

Theorem C. (J. Levine, [5]) *Suppose that $m \geq 5$. A PL m knot is trivial if the complement is a homotopy circle.*

By Theorem 4.4 in [6] and by Corollary to Theorem B the following existence theorem of a normal PL 2 cell bundle is derived from the obstruction theory for reducing combinatorial prebundles into PL cell bundles. (In Part III, we shall give the precise description of the obstruction theory.)

Theorem D. *Let M be a PL m manifold. Suppose that M is compact and $H^4(M, \pi_3(PR_2)) \cong 0$. Then any locally flat embedding f of M into a PL $m+2$ manifold W has a normal PL 2 cell bundle. More precisely, if K and L are partitions of M and W such that $f: K \rightarrow L$ is simplicial and that $F(K)$ is full in L, then there is a normal PL cell bundle $v(f)$ for f which is compatible with the dual cell structures of K and L, for compatibility see* [6].

By the universal coefficient theorem the assumption $H^4(M, \pi_3(PR_2)) \cong 0$ is always satisfied by such a manifold M that $H_3(M)$ is torsion free and $H_4(M) \cong 0$.

A PL manifold pair (W, M) is *smoothable* if W and M are smoothable so that there are a smooth manifold pair $(\boldsymbol{W}, \boldsymbol{M})$ and a smooth triangulation $h: W \rightarrow \boldsymbol{W}$ such that $h(M)=\boldsymbol{M}$. Suppose that M admits a normal PL 2 cell bundle v in W. Since by Proposition 1.1 ΠL_2 has the same homotopy type as 0_2, the normal bundle v triangulates a vector bundle. Therefore M has a normal PL microbundle in W which triangulates a vector bundle. Thus, applying Theorem D and Theorem 7.3 in [4] we have the following:

Corollary D.1 *Let (W, M) be a locally flat PL $(m+2, m)$ manifold pair. Suppose that M is closed and $H^4(M, \pi_3(PR_2)) \cong 0$. If W is smoothable then the pair (W, M) is smoothable.*

It is well known that there is a non smoothable 5 connected PL 12 manifold M which is piecewise linearly embeddable into the euclidean 14 space R^{14}.

Hence we have the following example.

EXAMPLE. There is an example of a closed 5 connected PL 12 manifold having a PL embedding into R^{14}, but having no locally flat PL embedding.

Recall that two oriented PL embeddings $f: M \to W$ and $g: M \to W'$ are *microequivalent* if there exist neighborhoods U and U' of $f(M)$ and $g(M)$ in W and W' respectively and a PL homeomorphism $h: U \to U'$ preserving orientations induced from those of W and W' so that $hf=g$.

Let M be a closed oriented PL manifold. For any oriented proper embedding f of M of codimension 2, H. Noguchi has defined an invariant $\chi(f) \in H^2(M)$ under the microequivalence class of f, which is called the *Euler class* of f.

REMARK. In his paper [7], p. 120, the class $\chi(f)$ is denoted by ω and called the Stiefel-Whitney class.

Finally we shall prove the following.

Theorem E. *Let M be a closed oriented PL manifold. Suppose that $H^4(M, \pi_3(PR_2)) \cong 0$. Then two oriented locally flat PL embeddings $f: M \to W$ and $g: M \to W'$ of M of codimension two are microequivalent if and only if $\chi(f) = \chi(g)$.*

3. Definitions and Lemmas

In the following we restrict ourselves in the PL category.

To prove Theorem A we need the following definition. Let $\{E, K, \Sigma\}$ be an n prebundle. A *collared non zero section* of E is a pair (G, g) consisting of an embedding $G: |K| \times J^{n-1} \to \partial E$ and a non zero section $g: K \to \partial E$ such that $G(x, 0) = g(x)$ for all x in $|K|$, and $G(A \times J^{n-1}) \subset h(A \times \partial J^n)$ for all pair (A, h) in Σ.

Lemma 3.1 (k) *Let K be a k dimensional complex and let $\{E, K, \Sigma\}$ be an n prebundle.*

Suppose that E has a collared non zero section (G, g).

Then E collapses to $G(|K| \times J^{n-1})$.

Proof. We prove Lemma 3.1 (k) by induction on the dimension k.

(0): Trivial.

$(k) \Rightarrow (k+1)$: Assuming inductively that (k) is proven, we prove $(k+1)$. Let A be an arbitrary $k+1$ simplex of K. Since $(G/\partial A \times J^{n-1}, g/\partial A)$ is a collared non zero section of $E/\partial A$, it follows from (k) that $E/\partial A$ collapses to $G(\partial A \times J^{n-1})$. Hence $E/\partial A \cup G(A \times J^{n-1})$ is an $n+k$ cell on the boundary of the $n+1+k$ cell E/A. Therefore E/A collapses to $E/\partial A \cup G(A \times J^{n-1})$. Let K^k denote the k skeleton of K. By the above argument, E collapses to $E/K^k \cup G(|K| \times J^{n-1})$. By (k) E/K^k collapses to $G(|K^k| \times J^{n-1})$. It follows that E collapses to $G(|K| \times J^{n-1})$, completing the induction.

Lemma 3.2 *Let M be a closed m manifold and let N be a normal n prebundle of an embedding $f: M \to W$ over a partition K of M such that $N \subset \operatorname{Int} W$.*

Suppose that N has a collared non zero section (G, g).

Then the following three statements hold;

(1) *There is an embedding* $F: M\times J^n \to W$ *such that* $F(M\times J^{n-1}\times I)=N$ *and* $F(x, 0)=G(x)$ *for all* x *in* $M\times J^{n-1}$, *and*

(2) *any regular neighborhood of* $f(M)$ *in* W *is homeomorphic to the product space* $M\times J^n$, *and*

(3) W-$f(M)$ *and* W-$g(M)$ *are homeomorphic.*

Proof. By the existence of a collar of ∂N in W, see Corollary to Lemma 24 in [11], there is an embedding $F_1: M\times J^n \to W$ such that $F_1(M\times J^{n-1}\times I)\subset N$ and $F_1(x, 0)=G(x)$ for all x in $M\times J^{n-1}$. Since $F_1(M\times J^{n-1}\times I)$ collapses to $G(M\times J^{n-1})$, and since by Lemma 3.1 N also collapses to $G(M\times J^{n-1})$, it follows that they are regular neighborhoods of $G(M\times J^{n-1})$ mod ∂V-Int $G(M\times J^{n-1})$ in W-Int V, where V denotes the submanifold $F_1(M\times J^{n-1}\times[-1, 0]$. By the uniqueness of relative regular neighborhoods there is a homeomorphism $F_2: W\to W$ such that $F_2/V=$ id., and $F_2F_1(M\times J^{n-1}\times I)=N$. Then $F=F_2F_1$ is the required embedding in (1).

Let U denote the image $F(M\times J^n)$. Then U is obviously a regular neighborhood of $g(M)$ in W. Since U collapses to $F(M\times J^{n-1}\times I)=N$, it follows that U is a regular neighborhood of $f(M)$ in W. By the uniqueness of regular neighborhoods we have (2). To prove (3) we choose partitions K_1, K_2 and L of $f(M)$, $g(M)$ and W respectively such that K_1, K_2 are full subcomplexes of L and that $N(K_1', L')$ and $N(K_2', L')$ are contained in Int $F(M\times J^n)$, where $N(K_i', L')$, $i=1, 2$ stand for derived neighborhoods of K_i, $i=1, 2$ in L. Thus we have infinite sequences of derived neighborhoods.

$U\supset N(K_i', L')\supset\cdots\supset N(K_i^{(p)}, L^{(p)})\supset\cdots$ $i=1, 2$ such that for any neighborhoods V_1, V_2 of $f(M)$, $g(M)$ in W respectively there is an integer p so that $N(K_i^{(p)}, L^{(p)})\subset V_i$ for $i=1, 2$.

By virtue of the regular neighborhood annulus theorem in [1], p. 725, there are homeomorphisms

$h_1: U\text{-}f(M)\to\partial U\times[0, \infty)$ and $h_2: U\text{-}g(M)\to\partial U\times[0, \infty)$ such that $h_i(x)=(x, 0)$ for all x in ∂U and for $i=1, 2$.

Thus we have the required homeomorphism $h: W\text{-}f(M)\to W\text{-}g(M)$ by setting h/W-Int $U=$id. and $h/U\text{-}f(M)=h_2^{-1}h_1$.

This completes the proof of Lemma 3.2.

4. The proof of Theorems

In the section, we shall prove Theorems A, B and E.

Proof of Theorem A. Since W is orientable, f has an oriented normal prebundle N over $S=\partial\Delta_{m+1}$. Let A be an m simplex of S and let B denote both the complex $S-A$ and the cell S-Int A. By Corollary 4.2 in [6], N/B and N/A are trivial prebundles. Hence we have trivializations $h_1: B\times(J^2, 0)\to N/B$ and

h_2: $A\times(J^2, 0)\to N/A$ so that $h_2^{-1}h_1/\partial A\times J^2$: $\partial A\times(J^2, 0)\to\partial A\times(J^2, 0)$ is an orientation preserving 2 prebundle isomorphism.

In case $m=1$; Since $\pi_0(PR_n)\cong\pi_0(0_n)\cong Z_2$, for all n and the non trivial element is the class of orientation reversing homeomorphisms of $(J^n, 0)$ onto itself, it follows that $h_2^{-1}h_1/\partial A\times J^2$ is extendable to an isomorphism h_3: $A\times(J^2, 0)\to A\times(J^2, 0)$.

Hence the required isomorphism h: $S\times(J^2, 0)\to N$ is obtained by setting $h/B\times(J^2, 0)=h_1$ and $h/A\times(J^2, 0)=h_2h_3$, completing the proof in case $m=1$.

In case $m=3$,; (The proof is essentially given in [7], p. 124.)

We consider the restriction $h'=h_2^{-1}h_1/\partial A\times\partial J^2$.

Since h' induces the identity map of $H_2(\partial A\times\partial J^2)+H_1(\partial A\times\partial J^2)=Z+Z$, it follows from the Theorem 13.2 in [0] that h' is isotopic to the identity or T. But T may not be extended to a homeomorphism of $\partial A\times J^2$ fixing $\partial A\times O$. Therefore h' is isotopic to the identity.

So we may extend $h_2^{-1}h_1/\partial A\times J^2$ to a homeomorphism of $\partial(A\times J^2)$ fixing $\partial A\times O$.

By the join extension, we have a homeomorphism h_3 of $A\times J^2$ fixing $A\times O$ such that $h_3/\partial A\times J^2=h_2^{-1}h_1/\partial A\times J^2$.

Thus we have the required isomorphism

$$h\colon S\times(J^2, 0)\to N$$

by setting $h/B\times J^2=h_1$ and $h/A\times J^2=h_2h_3$, completing the proof in case $m=3$.

In case $m\geq 5$; Firstly we show that N has a collared non zero section. For N/B we have a collared non zero section (G_1, g_1) by setting $g_1(x)=h_1(x, 0, 1)$ for all x in B and $G_1(x, u)=h_1(x, u, 1)$ for all (x, u) in $B\times J$. Let X and Y denote the $m+1$ sphere $\partial(A\times J^2)$ and the $m-1$ sphere ∂A. Then the embedding $\times 0^2$: $Y\to X$ has a trivial normal prebundle $Y\times J^2$. Put $g'=h_2^{-1}g_1/Y$ and $G'=h_2^{-1}G_1/Y\times J$. Then (G', g') is a collared non zero section of $Y\times J^2$. By Lemma 3.2 there is an embedding f: $Y\times J^2\to X$ such that X-$g'(Y)$ is homeomorphic to X-$Y\times 0$, and that $f(Y\times J\times I)=Y\times J^2$ and $f(x, 0)=G'(x)$ for all x in $Y\times J$. Since X-$Y\times 0$ is a homotopy circle and since m-$1\geq 4$, applying the argument due to J. Levine in [5], and then using the existence theorem of a compatible collar, see Lemma 24 in [11], we have a collared non zero section (G_2, g_2) of N/A such that $G_2/Y\times J=G'$. Thus the required collared non zero section (G, g) of N is well defined by setting $(G, g)/(A\times J, A)=(h_2G_2, h_2g_2)$ and $(G, g)/(B\times J, B)=(G_1, g_1)$.

Secondly we prove that N is actually trivial.

Again by Lemma 3.2 there is an embedding F: $S\times J^2\to W$ such that $F(S\times J^2)=N$ and $F(x, 0)=G(x)$ for all x in $S\times J$. We will change the homeomorphism into an isomorphism. Let a and b denote interior points of A and B respectively. Then $h_1(b\times\partial J^2)\cap F(S\times J\times 1)=F(b\times J\times 1)$. Consider the intersection of $h_1(b\times\partial J^2)$ and $F(a\times\partial J^2)$. Since $1+1-(m+1)=1-m<0$, by the general position argument, see Chapter 6 of [11], we may assume that $h(b\times\partial J^2)\cap F(S\times J\times 1\cup a\times\partial J^2)=F(b\times J\times 1)$, and moreover for a sufficiently small regular neighborhood C of a in Int A, $h_1(b\times\partial J^2)\cap F(S\times J\times 1\cup C\times\partial J^2)=F(b\times J\times 1)$.

Let D denote the $k+1$ cell $S\times\partial J^2$-Int $(S\times J\times 1\cup C\times\partial J^2)$, and let L denote the 1 cell $b\times(\partial J^2\text{-Int } J\times 1)$.

Then $F^{-1}h_1(L)$ and L are two arcs in D, and $F^{-1}h\,|\,\partial L=\text{id}$..

Since $m+1-1=m>2$, by Corollary 1 to Lemma 9 in [11] we may also assume that $F^{-1}h_1/b\times\partial J^2=\text{id}$.. Moreover by the uniqueness of regular neighborhoods of $b\times\partial J^2$ in $S\times\partial J^2$, we may assume that $F^{-1}h_1(B\times\partial J^2)=B\times\partial J^2$. Then the homeomorphism $F^{-1}h_1/B\times\partial J^2$ is clearly extendable to a homeomorphism H: $S\times\partial J^2\rightarrow S\times\partial J^2$. Thus the homeomorphism FH: $S\times\partial J^2\rightarrow\partial N$ is an isomorphism of the associated 1 sphere prebundle ∂N of N. Therefore by 3.1 in [6], N is trivial, completing the proof.

Proof of Theorem B. Combining Addendum to Theorem A and the Theorem 4.6 in [6], we conclude that $\pi_{m-1}(PR_2)$ consists of only one element for $m\geq 5$ and $m=3$.

Hence $\pi_m(PR_2)\cong 0\cong\pi_m(0_2)$ for $m\geq 4$ and $m=2$. Since $\pi_0(PR_2)\cong Z_2\cong\pi_0(0_2)$, it remains to prove that $\pi_1(PR_2)\cong\pi_1(0_2)$.

By Proposition 1.1 and by the Proposition 3.1, (ii) in [6], we have $\pi_1(\Pi L_2)\cong\pi_1(0_2)$ and $\pi_1(PR_2)\cong\pi_1(\partial PR_2)$, where ∂PR_2 stands for the structural group of 1 sphere $\partial J^2(=S^1)$ prebundles.

Since each element of $\pi_1(\partial PR_2)$ is represented by a homeomorphism h of $I\times S^1$ onto itself fixing $\partial I\times S^1$, we associate to each element $\{h\}$ of $\pi_1(\partial PR_2)$ the homotopy class $w\{h\}$ of the map

$p_2h(\times e)$: $(I,\partial I)\rightarrow(S^1,e)$, where $e=(0,1)$, p_2: $I\times S^1\rightarrow S^1$ and $(\times e)$: $I\rightarrow I\times S^1$ stand for the maps $(x,y)\rightarrow y$ and $x\rightarrow(x,e)$

for x in I and y in S^1, respectively. Then the function w: $\pi_1(\partial PR_2)\rightarrow\pi_1(S^1)$ is clearly a well defined homomorphism such that a diagram

$$\begin{array}{ccc} \pi_1(PR_2) & \cong & \pi_1(\partial PR_2) \\ i\big\uparrow & & \big\downarrow w \\ \pi_1(\Pi L_2)\cong\pi_1(0_2) & \cong & \pi_1(S^1) \end{array}$$

commutes, where i stands for the homomorphism induced from the inclusion map. Hence w is surjective. It remains to prove that w is injective. Notice that for $\{h\}$ in $\pi_1(\partial PR_2)$, $w\{h\}$ coincides with the winding number of h which is defined in [0], p. 313. Therefore by the Theorem 7.2 in [0], if $w\{h\}=w\{g\}$, then homeomorphisms h and g of IxS^1 fixing ∂IxS^1 are isotopic keeping ∂IxS^1 fixed.

Hence $\{h\}=\{g\}$, completing the proof.

Proof of Theorem E.

Suppose that $\chi(f)=\chi(g)$. Since ΠL_2 is homotopy equivalent to 0_2, it should be noted that the isomorphism class of every orientable 2 cell bundle x is completely determined by the Euler class $\chi(x)$.

Let K, L and L' denote partitions of M, W and W' respectively such that $f\colon K\to L$ and $g\colon K\to L'$ are simplicial and $f(K)$ and $g(K)$ are full in L and L' respectively. By Theorem D, there are normal cell bundles $v(f)$ and $v(g)$ for f and g which are compatible with the dual cell structures of K, L and K, L' respectively. It follows from the definitions of $\chi(f)$ and $\chi(g)$ see [7], p. 120, that $\chi(f)=\chi(v(f))$ and $\chi(g)=\chi(v(g))$. Hence $\chi(v(f))=\chi(v(g)$.

Therefore $v(f)$ and $v(g)$ are isomorphic. Thus f and g are microequivalent. This completes the proof of Theorem E.

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