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Osaka University

## PERTURBATION AND DEGENERATION OF EVOLUTIONAL EQUATIONS IN BANACH SPACES

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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### § 1. Completely well posed evolutional equations

Let  $E$  be a Banach space and  $E_1$  be another Banach space such that  $E \subset E_1$  and the embedding of  $E$  into  $E_1$  is continuous. Let  $A(t)$  be a continuous linear mapping of  $E$  into  $E_1$  for every fixed  $t$  in the real interval  $[a, b]$  such that  $A(t)u$  is a continuous function on  $[a, b]$  into  $E_1$  for every fixed  $u \in E$ . Then we can easily see,  $A(t)u(t)$  is continuous on  $[a, b]$  into  $E_1$ , if  $u(t)$  is continuous on  $[a, b]$  into  $E$ .

As an  $E$ -solution in  $[a, b]$  of the evolutional equation

$$(0) \quad \partial_t u = A(t)u + f(t) \quad \left( \partial_t = \frac{d}{dt} \right),$$

where  $f(t)$  is an  $E$ -continuous function on  $[a, b]$ <sup>1)</sup>, we understand an  $E$ -continuous function  $u = u(t)$  on  $[a, b]$  such that the strong derivative  $\partial_t u = \lim_{h \rightarrow 0} h^{-1} \{u(t+h) - u(t)\}$  exists in  $E_1$  for  $t \in [a, b]$  and the equation (0) is fulfilled in  $E_1$  for  $t \in [a, b]$ .

The equation (0) is said to be  $E$ -well posed (or simply well posed) in  $[a, b]$  when for any  $\varphi \in E$  there exists one and only one  $E$ -solution  $u = u(t)$  of (0) with the initial value  $u(a) = \varphi$ . We say that the equation (0) is *completely  $E$ -well posed* in  $[a, b]$  when (0) is  $E$ -well posed for any closed subinterval of  $[a, b]$  and the solution  $u = u(t, s, \varphi)$  of (0) with the initial value  $u(s) = \varphi$  ( $a \leq s \leq b$ ) is a continuous function of  $(t, s, \varphi)$  for  $a \leq s \leq t \leq b$ ,  $\varphi \in E$ . If (0) is (completely)  $E$ -well posed in  $[a, b]$  then the associated homogeneous equation

$$(1) \quad \partial_t u = A(t)u$$

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1)  $f(t)$  is said to be  $E$ -continuous on  $[a, b]$  when  $f(t)$  is continuous on  $[a, b]$  into  $E$ .

is also (completely)  $E$ -well posed in  $[a, b]$ . When (1) is completely well posed in  $[a, b]$  then the solution of (1) with the initial condition  $u(s) = \varphi$  ( $s \in [a, b]$ ) can be written in the form

$$(2) \quad u = U(t, s)\varphi,$$

where  $U(t, s)$  is a continuous linear operator on  $E$  into  $E$  for  $a \leq s \leq t \leq b$  with the following properties :

- 1)  $U(t, s)\varphi$  is continuous on  $a \leq s \leq t \leq b$ ,  $\varphi \in E$  into  $E$ ,
- 2)  $U(s, s) = 1$  (identity) for  $s \in [a, b]$
- 3)  $U(t, \sigma)U(\sigma, s) = U(t, s)$  for  $a \leq s \leq \sigma \leq t \leq b$ ,
- 4)  $\partial_t U(t, s)\varphi = A(t)U(t, s)\varphi$  in  $E_1$  for  $a \leq s \leq t \leq b$ ,  $\varphi \in E$ .

Such an operator  $U(t, s)$  is called the *fundamental solution* of (1).

Especially when  $A(t)$  does not depend on  $t$ :  $A(t) = A$ , (1) is completely  $E$ -well posed in any finite interval  $[a, b]$ , if and only if (1) is simply  $E$ -well posed in some finite interval. For, the fundamental solution of (1) has the form  $U = U(t-s)$ . In this case, restricting the domain of  $A$  to such a set of  $u$  that  $Au \in E$ ,  $A$  is the infinitesimal generator of the one-parameter semi-group  $\{U(t)\}_{t \geq 0}$ , since  $U(t+s) = U(s)U(t)$  for  $s, t \geq 0$ . Conversely, if  $A$  is the infinitesimal generator of a one-parameter semi-group  $\{U(t)\}_{t \geq 0}$ , then extending the domain of  $A$  on  $E$  in such a way that the range of  $A$  will be contained in  $E_1$  as given in Remark 1, we obtain a completely  $E$ -well posed equation (1) with  $A(t) = A$ , for any finite interval, with the fundamental solution  $U(t-s) = \exp((t-s)A)$ .

We can easily obtain the following :

**Theorem 1.** *If the homogeneous equation (1) is completely  $E$ -well posed in  $[a, b]$ , and  $f(t)$  is  $E$ -continuous on  $[a, b]$ , then the inhomogenous equation (0) is also completely  $E$ -well posed in  $[a, b]$  and any solution of (0) satisfies*

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma)d\sigma \quad \text{for } a \leq s \leq t \leq b$$

with the fundamental solution  $U(t, s)$  of (1).

REMARK 1. Let  $A(t)$  be a pre-closed linear operator in  $E$  for every  $t$ , and let there exists a closed operator  $A_0$  with a domain in  $E$  such that for the adjoint operators  $A^*(t)$  and  $A_0^*$  of  $A(t)$  and  $A_0$  resp. we have

$$\|A^*(t)u'\| \leq \|A_0^*u'\| + \|u'\| \quad \text{for } u' \in \mathcal{D}^* = \mathcal{D}(A_0^*).$$

Then, defining a new norm of  $u \in E$  by

$$|||u||| = \sup_{u' \in \mathcal{D}^*} |\langle u, u' \rangle| (\|A_0^* u'\| + \|u'\|)^{-1},$$

we get  $|||u||| \leq \|u\|$ . Hence, denoting by  $E_1$  the completion of  $E$  with respect to the new norm, we obtain that the injection of  $E$  into  $E_1$  is continuous and the extension of  $A(t)$  on  $E$  is continuous on  $E$  into  $E_1$ . Cf [1].

On the other hand, if there exists a closed operator  $A_0$  with a domain  $\mathcal{D}$  dense in  $E$  such that

$$\|A(t)u\| \leq \|A_0 u\| + \|u\| \quad \text{for } u \in \mathcal{D},$$

then defining a new norm of  $u \in \mathcal{D}$  by

$$|||u||| = \|A_0 u\| + \|u\|$$

the vector space  $\mathcal{D}$  becomes a Banach space  $E_0$  with the new norm, such that the injection of  $E_0$  into  $E$  is continuous and  $A(t)$  is continuous on  $E_0$  into  $E$  for every  $t \in [a, b]$ .

**§ 2. Stability of solutions of evolutional equations containing a parameter**

Now we consider an evolutional equation containing a parameter  $\varepsilon \geq 0$ :

$$(1)_\varepsilon \quad \partial_t u = A_\varepsilon(t)u + f_\varepsilon(t).$$

Let  $u = u_0(t)$  be an  $E$ -solution of  $(1)_0$  in  $[a, b]$  for  $\varepsilon = 0$ .  $u = u_0(t)$  is said to be *completely E-stable* in  $[a, b]$  with respect to the equation  $(1)_\varepsilon$  for  $\varepsilon \rightarrow 0$ , when the following condition is fulfilled: For any  $\delta > 0$  there exists  $\eta(\delta) > 0$  such that, if  $0 < \varepsilon < \eta(\delta)$ , any  $E$ -solution  $u = u_\varepsilon(t)$  of  $(1)_\varepsilon$  on  $[s, b]$  for any  $s \in [a, b]$  with  $\|u_\varepsilon(s) - u_0(s)\| < \eta(\delta)$  satisfies the inequality

$$\|u_\varepsilon(t) - u_0(t)\| < \delta \quad \text{for } s \leq t \leq b.$$

**Lemma 1.** *Let the equation  $(1)_\varepsilon$  be completely E-well posed in  $[a, b]$  for  $\varepsilon > 0$ . If an  $E$ -solution  $u = u_0(t)$  of  $(1)_0$  is completely E-stable in  $[a, b]$  for  $\varepsilon \rightarrow 0$  with respect to  $(1)_\varepsilon$ , then the fundamental solution  $U_\varepsilon(t, s)$  of the associated homogeneous equation of  $(1)_\varepsilon$ , for sufficiently small  $\varepsilon > 0$ , with some constant  $C$  satisfies the inequality*

$$(2) \quad \|U_\varepsilon(t, s)\| \leq C \quad \text{for } a \leq s \leq t \leq b.$$

Proof. Let  $u = u_0(t)$  be completely stable in  $[a, b]$  for  $\varepsilon \rightarrow 0$  with respect to  $(1)_\varepsilon$ , and  $u = u_\varepsilon(t)$  and  $u = v_\varepsilon(t)$  be solutions of  $(1)_\varepsilon$  such that

$u_\varepsilon(s) = v_0(s)$  and  $\|v_\varepsilon(s) - u_\varepsilon(s)\| < \eta(\delta)$  resp. Then, if  $0 < \varepsilon < \eta(\delta)$  we must have

$$\|u_\varepsilon(t) - v_\varepsilon(t)\| \leq \|u_\varepsilon(t) - u_0(t)\| + \|v_\varepsilon(t) - u_0(t)\| < 2\delta \quad \text{for } s \leq t \leq b.$$

Hence for any  $w \in E$  with  $\|w\| < \eta(\delta)$  holds the inequality  $\|U_\varepsilon(t, s)w\| < 2\delta$  for  $a \leq s \leq t \leq b$ . This asserts Lemma 1.

In order to give our sufficient conditions for the complete stability of a solution, we shall prepare a definition of quasi-regularity of solutions. An  $E$ -solution  $u = u_0(t)$  of

$$(0) \quad \partial_t u = A_0(t)u + f(t)$$

is said to be *quasi-regular* in  $[a, b]$  with respect to an operator  $A_1(t)$ , when for any  $\delta > 0$  there exists an  $E$ -continuous  $v_\delta(t)$  on  $[a, b]$  such that  $\partial_t v_\delta(t) - A_0(t)v_\delta(t) - f(t)$  and  $A_1(t)v_\delta(t)$  are bounded and  $E$ -continuous on  $[a, b]$  and the inequalities

$$\|v_\delta(t) - u_0(t)\| < \delta \quad \text{and} \quad \|\partial_t v_\delta(t) - A_0(t)v_\delta(t) - f(t)\| < \delta$$

hold for  $a \leq t \leq b$ .

Especially if  $A_0(t) = A_1(t) = A$  and  $A$  is the infinitesimal generator of a 1-parameter semi-group and  $f(t)$  is  $E$ -continuous on  $[a, b]$ , then an  $E$ -solution of (0) is quasi-regular in  $[a, b]$  with respect to  $A$ . Indeed in this case we have to set  $v_\delta(t) = (1 - \lambda_\delta^{-1}A)^{-1}u_0(t)$  with sufficiently large  $\lambda_\delta > 0$ .

Now we assume that  $(1)_\varepsilon$  is completely  $E$ -well posed in  $[a, b]$  for  $\varepsilon \geq 0$  and the operator  $A_\varepsilon(t)$  have the form:

$$(3) \quad A_\varepsilon(t) = A_0(t) + \varepsilon A_1(t) \quad (\varepsilon \geq 0).$$

Further let  $f_\varepsilon(t)$  be  $E$ -continuous on  $[a, b]$  and converge to  $f_0(t)$  uniformly on  $[a, b]$  as  $\varepsilon \rightarrow 0$ . Then we have:

**Theorem 2.** *Let  $u = u_0(t)$  be an  $E$ -solution of  $(1)_0$  for  $\varepsilon = 0$  in a finite closed interval  $[a, b]$  and be quasi-regular with respect to  $A_1(t)$  in  $[a, b]$ . In order that  $u = u_0(t)$  be completely  $E$ -stable in  $[a, b]$  with respect to  $(1)_\varepsilon$ , it is necessary and sufficient that for sufficiently small  $\varepsilon > 0$ , the fundamental solution  $U_\varepsilon(t, s)$  of the associated homogeneous equation of  $(1)_\varepsilon$  is uniformly bounded for  $a \leq s \leq t \leq b$ .*

**Proof.** As the necessity of the condition is already given by Lemma 1, we have only to prove the sufficiency.

For any  $\delta > 0$  there exists an  $E$ -continuous  $v_\delta(t)$  on  $[a, b]$  such that

$h_\delta(t) = \partial_t v_\delta(t) - A_0(t)v_\delta(t) - f_0(t)$  and  $A_1(t)v_\delta(t)$  are  $E$ -continuous on  $[a, b]$  with the conditions

$$(4) \quad \|v_\delta(t) - u_0(t)\| < \delta \quad \text{and} \quad \|h_\delta(t)\| < \delta \quad \text{for} \quad a \leq t \leq b.$$

Then we get

$$\partial_t(u_\varepsilon - v_\delta) = A_\varepsilon(t)(u_\varepsilon - v_\delta) + \varepsilon A_1(t)v_\delta + g_{\varepsilon, \delta}(t),$$

where  $g_{\varepsilon, \delta}(t) = f_\varepsilon(t) - f_0(t) + h_\delta(t)$ . Hence, by Theorem 1,

$$\begin{aligned} u_\varepsilon(t) - v_\delta(t) &= U_\varepsilon(t, s)\{u_\varepsilon(s) - v_\delta(s)\} \\ &+ \int_s^t U_\varepsilon(t, \sigma)\{\varepsilon A_1(\sigma)v_\delta(\sigma) + g_{\varepsilon, \delta}(\sigma)\} d\sigma \quad \text{for} \quad a \leq s \leq t \leq b. \end{aligned}$$

Thus by (2) we have

$$\|u_\varepsilon(t) - v_\delta(t)\| \leq C\{\|u_\varepsilon(s) - v_\delta(s)\| + \int_s^t (\varepsilon \|A_1(\sigma)v_\delta(\sigma)\| + \|g_{\varepsilon, \delta}(\sigma)\|) d\sigma\}.$$

There exist positive constants  $\zeta(\varepsilon)$  and  $B_\delta$  such that  $\zeta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $\|f_\varepsilon(t) - f_0(t)\| < \zeta(\varepsilon)$  for  $a \leq t \leq b$  and  $\|A_1(t)v_\delta(t)\| \leq B_\delta$  for  $a \leq t \leq b$ . Thus, by (4), we get

$$\begin{aligned} \|u_\varepsilon(t) - u_0(t)\| &\leq C\{\|u_\varepsilon(s) - u_0(s)\| + \delta + (b-a)(\varepsilon B_\delta + \zeta(\varepsilon) + \delta)\}, \\ &\quad \text{for} \quad a \leq s \leq t \leq b. \end{aligned}$$

First taking  $\delta > 0$  sufficiently small and then letting  $\varepsilon \rightarrow 0$ , we complete the proof.

REMARK 2. The sufficiency of the condition in Theorem 2 remains valid even for the case of infinite interval  $(a, \infty)$  ( $b = \infty$ ), if we add to it the condition

$$\int_a^t \|U_\varepsilon(t, s)\| ds \leq C \quad \text{for} \quad a \leq t < \infty \quad \text{with some constant } C.$$

### § 3. Degeneration of evolutional equations

Let us consider the evolutional equation of the singular form in the parameter  $\varepsilon$ :

$$(1)_\varepsilon \quad \varepsilon \partial_t u = A_\varepsilon(t)u + f_\varepsilon(t) \quad \text{with} \quad \varepsilon > 0$$

and the degenerated equation

$$(1)_0 \quad A_0(t)u + f_0(t) = 0$$

We assume that  $A_\varepsilon(t)$  and  $f_\varepsilon(t)$  have the forms

$$(2) \quad \begin{aligned} A_\varepsilon(t) &= A_0(t) + \varepsilon A_1(t), \\ f_\varepsilon(t) &= f_0(t) + \varepsilon f_1(t) + \varepsilon h_\varepsilon(t), \end{aligned}$$

where  $f_0(t)$ ,  $f_1(t)$  and  $h_\varepsilon(t)$  are  $E$ -continuous on  $[a, b]$  and  $h_\varepsilon(t) \rightarrow 0$  uniformly on  $[a, b]$  as  $\varepsilon \rightarrow 0$ .

A solution  $u = u_0(t)$  of the degenerated equation  $(1)_0$  is said to be *completely  $E$ -stable* in  $[a, b]$  with respect to  $(1)_\varepsilon$  for  $\varepsilon \rightarrow 0$ , when the following condition is fulfilled: For any  $\varepsilon > 0$  there exists some  $\eta(\delta) > 0$  such that, if  $0 < \varepsilon < \eta(\delta)$ , any  $E$ -solution  $u_\varepsilon(t)$  of  $(1)_\varepsilon$  in  $[s, b]$  for any  $s \in [a, b]$  with

$$\|u_\varepsilon(s) - u_0(s)\| < \eta(\delta)$$

satisfies the inequality

$$\|u_\varepsilon(t) - u_0(t)\| < \delta \quad \text{for } s \leq t \leq b.$$

**Theorem 3.** *Assume that  $(1)_\varepsilon$  is completely  $E$ -well posed in a finite closed interval  $[a, b]$  for  $\varepsilon > 0$  and  $A_\varepsilon(t)$  and  $f_\varepsilon(t)$  have the forms (2). Let  $u = u_0(t)$  be a  $E$ -solution of  $(1)_0$  on  $[a, b]$  such that  $u_0(t)$ ,  $\partial_t u_0(t)$  and  $A_1(t)u_0(t)$  are  $E$ -continuous on  $[a, b]$ . In order that  $u = u_0(t)$  is completely stable in  $[a, b]$  for  $\varepsilon \rightarrow 0$  with respect to  $(1)_\varepsilon$  with any  $E$ -continuous  $f_1(t)$  on  $[a, b]$ , it is necessary and sufficient that the fundamental solution  $U_\varepsilon(t, s)$  of  $\partial_t u = \varepsilon^{-1} A_\varepsilon(t)u$  satisfies the following conditions:*

1) *There exists a constant  $C$  such that*

$$\|U_\varepsilon(t, s)\| \leq C \quad \text{for } a \leq s \leq t \leq b \text{ and sufficiently small } \varepsilon > 0.$$

2) *For any  $\alpha, \beta, t$ , and  $v \in E$  such that  $a \leq \alpha < \beta \leq t \leq b$ ,*

$$\int_\alpha^\beta U_\varepsilon(t, s) v ds \rightarrow 0 \quad \text{uniformly on } t \in [\beta, b] \text{ as } \varepsilon \rightarrow 0.$$

*Proof.* The necessity of 1) is obtained in the same way as in the proof of Lemma 1.

To prove the necessity of 2), setting  $f_1(t) = \varphi(t) - A_1(t)u_0(t)$  with an arbitrary  $E$ -continuous  $\varphi(t)$  on  $[a, b]$ , we get from  $(1)_\varepsilon$ ,  $(1)_0$  and (2)

$$\partial_t(u_\varepsilon - u_0) = \varepsilon^{-1} A_\varepsilon(t)(u_\varepsilon - u_0) + \varphi(t) + h_\varepsilon(t).$$

Hence for  $a \leq s \leq t \leq b$

$$\begin{aligned} u_\varepsilon(t) - u_0(t) &= U_\varepsilon(t, s) \{u_\varepsilon(s) - u_0(s)\} \\ &\quad + \int_s^t U_\varepsilon(t, \sigma) \{\varphi(\sigma) + h_\varepsilon(\sigma)\} d\sigma. \end{aligned}$$

By 1) we have if  $0 < \varepsilon < \eta(\delta)$ , as  $\eta(\delta) \leq \delta$ ,

$$\|U_\varepsilon(t, s)\{u_\varepsilon(s) - u_0(s)\}\| \leq C\delta$$

and

$$\left\| \int_s^t U_\varepsilon(t, \sigma) h_\varepsilon(\sigma) d\sigma \right\| \leq (b-a)C\zeta(\varepsilon)$$

for  $a \leq s \leq t \leq b$ , where  $\zeta(\delta) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Hence we have only to show that

$$\left\| \int_s^t U_\varepsilon(t, \sigma) \varphi(\sigma) d\sigma \right\| \rightarrow 0 \text{ uniformly on } a \leq s \leq t \leq b$$

as  $\varepsilon \rightarrow 0$  for any  $E$ -continuous  $\varphi(t)$  on  $[a, b]$ ” implies 2). Put  $\varphi(\sigma) = \psi(\sigma)v$  with any  $v \in E$  and a continuous real valued function  $\psi(\sigma)$  on  $[a, b]$  such that  $0 \leq \psi(\sigma) \leq 1$ ,  $\psi(\sigma) = 1$  for  $\alpha + \delta \leq \sigma \leq \beta$ ,  $\psi(\sigma) = 0$  for  $a \leq \sigma \leq \alpha$  and for  $\beta + \delta \leq \sigma \leq b$ . Then by 1)

$$\left\| \int_s^t U_\varepsilon(t, \sigma) \varphi(\sigma) d\sigma - \int_\alpha^\beta U_\varepsilon(t, \sigma) v d\sigma \right\| \leq 2\delta C \|v\|.$$

Therefore we obtain 2), as  $\delta$  can be taken arbitrarily small.

To prove the sufficiency of 2) with 1), we have only to show that by these conditions

$$\int_s^t U_\varepsilon(t, \sigma) \varphi(\sigma) d\sigma \rightarrow 0 \text{ uniformly for } a \leq s \leq t \leq b \text{ as } \varepsilon \rightarrow 0.$$

Divide the interval  $[a, b]$  into a finite number of consecutive intervals  $[\tau_{\nu-1}, \tau_\nu]$  ( $\nu = 1, \dots, N$ ) in such a way that  $\tau_\nu - \tau_{\nu-1} < \delta$  and  $\|\varphi(\sigma) - \varphi(\tau_\nu)\| < \delta$  for  $\sigma \in [\tau_{\nu-1}, \tau_\nu]$ . Then

$$\left\| \int_s^t U_\varepsilon(t, \sigma) \varphi(\sigma) d\sigma - \sum_{s < \tau_\nu \leq t} \int_{\tau_{\nu-1}}^{\tau_\nu} U_\varepsilon(t, \sigma) \varphi(\tau_\nu) d\sigma \right\| < C(b-a)\delta + 2C\delta \text{ Max } \|\varphi(\sigma)\|.$$

Thus by 2), taking  $\delta$  sufficiently small and letting  $\varepsilon \rightarrow 0$ , we attain to the desired conclusion. Q. E. D.

When  $f_\varepsilon(t)$  has the form, instead of (2),

$$(3) \quad f_\varepsilon(t) = f_0(t) + h_\varepsilon(t),$$

where  $f_0(t)$  and  $h_\varepsilon(t)$  have the same meanings as before, we cannot easily have necessary and sufficient conditions for the complete stability of  $u_0(t)$ , but only sufficient conditions, while we can relax the conditions on  $u_0(t)$  somewhat.

**Theorem 4.** *Let  $u = u_0(t)$  be an  $E$ -continuous solution of (1)<sub>0</sub> on  $[a, b]$  such that for any  $\delta > 0$  there exists an  $E$ -continuous  $v_\delta(t)$  on  $[a, b]$  with*



bounded and  $E$ -continuous  $\partial_t v_\delta(t)$ ,  $A_0(t)v_\delta(t)$  and  $A_1(t)v_\delta(t)$  on  $[a, b]$  satisfying the conditions

$$\|v_\delta(t) - u_0(t)\| < \delta \quad \text{and} \quad \|A_0 v_\delta(t) + f_0(t)\| < \delta \quad \text{for } t \in [a, b].$$

Assume that the fundamental solution  $U_\varepsilon(t, s)$  of  $\partial_t u = \varepsilon^{-1} A_\varepsilon(t)u$ , satisfies for sufficiently small  $\varepsilon > 0$ , the conditions with some constant  $C$ :

- 1)  $\|U_\varepsilon(t, s)\| \leq C$  for  $a \leq s \leq t \leq b$ ,
- 2)  $\int_a^t \|U_\varepsilon(t, s)\| ds \leq \varepsilon C$  for  $a \leq t \leq b$ .

Then  $u = u_0(t)$  is completely  $E$ -stable in  $[a, b]$  with respect to  $(1)_\varepsilon$  for  $\varepsilon \rightarrow 0$ .

Proof will be left to the reader.

REMARK 3. The sufficiency of the conditions 1) and 2) in Theorem 4 remains valid even for the case of infinite interval  $(a, \infty)$  ( $b = \infty$ ).

#### § 4. Degeneration of evolutionary equation when $A_\varepsilon(t) = A$

Consider the evolutionary equation of singular form in  $\varepsilon$ :

$$(1)_\varepsilon \quad \varepsilon \partial_t u = Au + f_\varepsilon(t) \quad (\varepsilon > 0),$$

where  $A$  is the infinitesimal generator of a one parameter semi-group in a reflexive Banach space  $E$ . As it has been stated in § 1, the operator  $A$  can be extended to a continuous linear operator on  $E$  into  $E_1$ , and the equation  $(1)_\varepsilon$  becomes completely  $E$ -well posed in any finite interval. The fundamental solution of the associated homogeneous equation has the form

$$U_\varepsilon(t, s) = \exp(\varepsilon^{-1}(t-s)A),$$

as  $\exp(tA)$  ( $t \geq 0$ ) is the transformation generated by  $A$ .

**Theorem 5.** Let  $u = u_0(t)$  be an  $E$ -solution in a finite closed interval  $[a, b]$  of the degenerated equation

$$Au + f_0(t) = 0$$

with  $E$ -continuous  $\partial_t u_0(t)$  on  $[a, b]$ .

In order that  $u_0(t)$  be completely  $E$ -stable in  $[a, b]$  with respect to  $(1)_\varepsilon$  for  $\varepsilon \rightarrow 0$ , where  $f_\varepsilon$  has the form (2) with any  $E$ -continuous  $f_1$ , it is necessary and sufficient that the following conditions are fulfilled:

- 1) With some constant  $C$ ,  $\|\exp(tA)\| \leq C$  for  $0 \leq t < \infty$ .
- 2)  $Av = 0$  with  $v \in \mathcal{D}(A)$  implies  $v = 0$ ,

where  $\mathcal{D}(A)$  denotes the proper domain of  $A$  before the extension of  $A$  on  $E$ .

Proof. From Theorem 3 we get easily the necessity of 1), as  $U_\varepsilon(t, s) = \exp(\varepsilon^{-1}(t-s)A)$ .

By the mean ergodic theorem in a reflexive Banach space, we obtain from 1) a projective operator  $P$  on  $E$  into  $E$  such that :

$$(3) \quad \lim_{\tau \rightarrow \infty} \tau^{-1} \int_0^\tau \exp(tA)v dt = Pv \quad \text{for any } v \in E, \text{ } ^{2)}$$

$$(4) \quad P \exp(tA) = \exp(tA)P = P^2 = P,$$

and

$$(5) \quad P(E) = \{u \in \mathcal{D}(A); Au = 0\}$$

Thus, for  $a \leq \alpha < \beta \leq t \leq b$  and any  $v \in E$ , setting

$$\begin{aligned} \tau(\varepsilon) &= \varepsilon^{-1}(\beta - \alpha) \text{ we have by (4)} \\ &\int_\alpha^\beta U_\varepsilon(t, s)v ds - (\beta - \alpha)Pv \\ &= \int_\alpha^\beta \exp(\varepsilon^{-1}(t-s)A)v ds - (\beta - \alpha)Pv \\ &= (\beta - \alpha) \exp(\varepsilon^{-1}(t - \beta)A) \left\{ \tau(\varepsilon)^{-1} \int_0^\tau \exp(\sigma A)v d\sigma - Pv \right\}, \end{aligned}$$

hence by (3) and 1), we have

$$\left\| \int_\alpha^\beta U_\varepsilon(t, s)v ds - (\beta - \alpha)Pv \right\| \rightarrow 0 \text{ uniformly for } \beta \leq t \leq b,$$

as  $\tau(\varepsilon) \rightarrow \infty$  for  $\varepsilon \rightarrow 0$ .

Hence the condition 2) in Theorem 3 with 1) is equivalent to :

$$Pv = 0 \quad \text{for every } v \in E.$$

Therefore, by (5) the condition 2) with 1) is equivalent to the condition 2) with 1) in Theorem 3. Q. E. D.

REMARK 4. The sufficiency of the conditions in Theorem 5 remains valid even for the case  $b = \infty$ , if we replace 2) by

$$\int_0^\infty \|\exp(tA)\| dt < \infty.$$

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2) Here the lim means the strong limit in  $E$ .

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