Burgess, W.D. Fuller, K.R. Green, E.L. and Zacharia, D.
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LEFT MONOMIAL RINGS-A GENERALIZATION OF MONOMIAL ALGEBRAS

W.D. BURGESS[1], K.R. FULLER, E.L. GREEN[2] AND D. ZACHARIA

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0. Introduction

A particularly tractable class of finite dimensional algebras defined by quivers and relations is that of monomial algebras, i.e., those for which the ideal of relations is generated by a collection of paths. The homological structure of these algebras is very well understood and some constructions for them are even algorithmic. There is, for example, an algorithm due to Green, Happel and Zacharia (see [11], where the algebras are called 0-relations algebras) for constructing the projective resloutions of the simple modules which determines their projective dimensions in a predictable number of steps. The Cartan determinant conjecture is known to be true for these algebras since they are positively graded ([18]) and the finitistic dimensions are finite ([12] and [13]) and are thoroughly understood due to the recent work of B.Z. Huisgen ([13] and [14]). Other properties of monomial algebras will be cited below.

Here we introduce a class of left artinian rings which includes that of monomial algebras and we show that many of the above results remain valid within it. The proposed rings, called *left monomial rings* (see Definition 2.2) will include monomial algebras and the more general 0-relations algebras given by species and 0-relations, as well as left (almost) serial rings, right serial rings, hereditary artinian rings and more. To each such ring R is associated a monomial algebra A so that, in many ways, R and A have the "same" homological properties (see Theorem 2.3); enough so that, for example, the projective dimensions of the corresponding simple modules are the same. (See Theorem 2.3 and its corollary.)

1. Tree modules

We fix throughout a basic left artinian ring R with radical J. In the sequel

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$\{e_1, \cdots, e_n\}$ will denote a complete set of primitive orthogonal idempotents and for $i=1, \cdots, n$, $S_i=Re_i/Je_i$ will be the simple left module corresponding to e_i. Modules will always be left R-modules. The composition length of a module M is denoted $c(M)$. An element r of some Re_i will be called *normed*, if for some j, $e_j r=r$.

We begin by looking at a special class of modules before presenting a definition of left monomial rings.

Let M be an R-module. A subset $\mathcal{X}$ of $M\setminus\{0\}$ is said to be *normed* in case

(1) $\mathcal{X}=\cup_{i=1}^n e_i\mathcal{X}$, and

(2) if $x, y\in\mathcal{X}$, $x\neq y$ then $Rx\neq Ry$.

Then $\mathcal{X}\cup\{0\}$ becomes the set of non-zero nodes of a *module diagram* (in the sense of [3] or [5]), which is also denoted $\mathcal{X}$, with arrows $x\rightarrow y$ when $Ry\subset Rx$ and if for $z\in\mathcal{X}$, $Ry\subseteq Rz\subseteq Rx$ implies $z=y$ or $z=x$. Since the least element 0 of a module diagram behaves in an entirely predictable way, we only talk about the non-zero nodes of these diagrams in the sequel. Implicitly, however, 0 belongs to every (sub)diagram and $\{0\}$ is a subdiagram.

In such a module diagram $\mathcal{X}$, $\mathcal{V}\subseteq\mathcal{X}$ is a *subdiagram*, written $\mathcal{V}\leq\mathcal{X}$, in case $x\in\mathcal{V}$ and $x\rightarrow y$ imply $y\in\mathcal{V}$, and the *radical* of $\mathcal{V}$, denoted $\mathcal{J}\mathcal{V}$, is $\{x\in\mathcal{V}\mid v\rightarrow x \text{ for some } v\in\mathcal{V}\}$. The *top* of $\mathcal{V}$ is defined as $\mathcal{V}\setminus\mathcal{J}\mathcal{V}$ and will be written $\mathcal{V}^T$. If $x\in\mathcal{X}$ then $\mathcal{U}(x)$ stands for the smallest subdiagram containing x; $\mathcal{U}(x)$ is *local* in the sense that $\mathcal{J}\mathcal{U}(x)=\mathcal{U}(x)\setminus\{x\}$ is the unique maximal proper subdiagram of $\mathcal{U}(x)$. The lattice of subdiagrams (under $\cup$ and $\cap$) of $\mathcal{X}$ is denoted $\mathcal{L}(\mathcal{X})$, and the lattice of submodules of a module M is written $\mathcal{L}(M)$.

If $\mathcal{X}$ is a normed subset of M then $\mathcal{X}$ together with the functions $\delta\colon \mathcal{L}(\mathcal{X})\rightarrow\mathcal{L}(M)$ via $\delta\colon \mathcal{U}\mapsto R\mathcal{U}$ and $\lambda\colon \mathcal{X}\rightarrow\{1, \cdots, n\}$ via $\lambda(x)=i$ if $x=e_ix$ becomes a *diagram for M* in case.

(M0) $\delta\colon \mathcal{L}(\mathcal{X})\rightarrow\mathcal{L}(M)$ is a lattice monomorphism;

(M1) card $\mathcal{X}=c(M)$;

(M2) $\delta(\mathcal{J}\mathcal{V})=J\delta(\mathcal{V})$ for all $\mathcal{V}\in\mathcal{L}(\mathcal{X})$;

(M3) $\delta(\mathcal{V})/\delta(\mathcal{U})\cong S_{\lambda(x)}$ if $\mathcal{U}<\mathcal{V}$ in $\mathcal{L}(\mathcal{X})$ and $\mathcal{V}=\mathcal{U}\cup\{x\}$.

We say that M is a *tree module* with *tree subset* $\mathcal{X}$ or that $(M, \mathcal{X})$ is a tree module if the above conditions hold and the diagram $\mathcal{X}$ is a disjoint union of local subdiagrams, $\mathcal{X}=\mathcal{U}(x_1)\dot{\cup}\cdots\dot{\cup}\,\mathcal{U}(x_m)$, which are trees in the sense that for each $y\in\mathcal{U}(x_j)$, $j=1, \cdots, m$, there is a unique path from x_j to y.

If $\mathcal{U}\leq\mathcal{V}\leq\mathcal{X}$ are module diagrams, $\mathcal{V}/\mathcal{U}$ is the module diagram obtained from $\mathcal{V}$ by identifying all the nodes of $\mathcal{U}$ with the node 0 (see [5, page 24]).

Before characterizing tree modules we list some properties of them which will be of use in the sequel.

Proposition 1.1. *Let M be a tree module with tree subset $\mathcal{X}$. Then the following hold.*

(a) *If $\mathcal{U}<\mathcal{V}$ are subdiagrams of $\mathcal{X}$ then $\mathcal{V}\backslash\mathcal{U}+R\mathcal{U}$ is a tree subset for the module $R\mathcal{V}/R\mathcal{U}$. As diagrams, $\mathcal{V}\backslash\mathcal{U}+R\mathcal{U}$ may be identified with $\mathcal{V}/\mathcal{U}$.*

(b) *For any subdiagram $\mathcal{V}$ of $\mathcal{X}$, $R\mathcal{V}=\bigoplus_{v\in\mathcal{V}^\tau} Rv$.*

Proof. Part (a) is by [5, Proposition 1.1] and the observation that since δ is an injection, $v\neq w$ in $\mathcal{V}\backslash\mathcal{U}$ implies $v+R\mathcal{U}\neq w+R\mathcal{U}$. Indeed, if $v-w\in R\mathcal{U}$ then $R(\mathcal{U}(v)\cup\mathcal{U})=R(\mathcal{U}(w)\cup\mathcal{U})$ implies $\mathcal{U}(v)\cup\mathcal{U}=\mathcal{U}(w)\cup\mathcal{U}$ and thus $v\in\mathcal{U}(w)$ and $w\in\mathcal{U}(v)$, giving $v=w$.

Part (b) also follows from the injectivity of δ and the fact that $\mathcal{V}$ is the disjoint union $\bigcup_{v\in\mathcal{V}^\tau}\mathcal{U}(v)$, which is since $\mathcal{X}$ is a disjoint union of trees. □

The next step is to characterize tree modules in a way which will be easier to use. It connects the tree structure with the radical layers.

Proposition 1.2. *Let $\mathcal{X}$ be a normed subset of a module M of Loewy length m. Then $(M,\mathcal{X})$ is a tree module if and only if*

(*) *$\mathcal{X}$ can be written $\mathcal{X}=\mathcal{Y}_0\cup\cdots\cup\mathcal{Y}_{m-1}$ so that $M=\bigoplus_{y\in\mathcal{Y}_0} Ry$; and for each l, $1\leq l\leq m-1$, and $x\in\mathcal{Y}_{l-1}$, there are subsets $\mathcal{Y}_{lx}\subseteq\mathcal{Y}_l$ so that $\mathcal{Y}_l=\bigcup_{x\in\mathcal{Y}_{l-1}}\mathcal{Y}_{lx}$ and $Jx=\bigoplus_{y\in\mathcal{Y}_{lx}} Ry$.*

Moreover under these conditions, $J^lM=\bigoplus_{y\in\mathcal{Y}_l} Ry$, for $l=1,\cdots,m-1$.

Proof. ($\Rightarrow$). By Proposition 1.1(b) we set $\mathcal{Y}_0=\mathcal{X}^\tau$. If we assume that subsets $\mathcal{Y}_0,\cdots,\mathcal{Y}_{l-1}$ have been chosen which satisfy the condition (*) then for each $x\in\mathcal{Y}_{l-1}$, $Jx=R\mathcal{I}\mathcal{U}(x)$ and $\mathcal{Y}_{lx}$ is chosen to be $(\mathcal{I}\mathcal{U}(x))^\tau$. Condition (*) is then satisfied for $\mathcal{Y}_l=\bigcup_{x\in\mathcal{Y}_{l-1}}\mathcal{Y}_{lx}$.

($\Leftarrow$). The module diagram made from $\mathcal{X}$ satisfying (*) is easily seen to be a disjoint union of trees (more formally, this is done by induction on the Loewy length).

In order to make $\mathcal{X}$ into a tree subset, functions δ and λ are defined by $\delta\colon\mathcal{U}\mapsto R\mathcal{U}$ for $\mathcal{U}\leq\mathcal{X}$ and for $x\in\mathcal{X}$, $Rx/Jx\cong S_{\lambda(x)}$.

If $\mathcal{U}\leq\mathcal{X}$ then it will follow from (*) that $R\mathcal{U}=\bigoplus_{x\in\mathcal{U}^\tau} Rx$. To see this, note that there is $k\geq 0$ such that $\mathcal{U}^\tau=\mathcal{U}_1\cup\mathcal{U}_2$ with $\emptyset\neq\mathcal{U}_1\subseteq\mathcal{Y}_k$ and $\mathcal{U}_2\subseteq\bigcup_{l>k}\mathcal{Y}_l$. Now

$$R\mathcal{U}=R\mathcal{U}^\tau\subseteq\bigoplus_{u\in\mathcal{U}_1} Ru\oplus\left(\bigoplus_{x\in\mathcal{Y}_k\backslash\mathcal{U}_1} Jx\right)$$

and each $v\in\mathcal{U}_2$ belongs to some Jx with $x\in\mathcal{Y}_k$. Since $v\notin Ru$, for $u\in\mathcal{U}_1$, we must have

$$R\mathcal{U}_2\subseteq\bigoplus_{x\in\mathcal{Y}_k\backslash\mathcal{U}_1} Jx\subseteq J^{k+1}M\,.$$

Thus

$$R\mathcal{U} = \left(\bigoplus_{u\in\mathcal{U}_1} Ru\right)\oplus R\mathcal{U}_2$$

and, inductively, we also see that $R\mathcal{U}_2=\oplus_{u\in\mathcal{U}_2^\tau} Ru$, so the assertion follows.

The verification that δ is an injective lattice homomorphism proceeds exactly as in Part 6 of the proof of [9, Theorem 1]. For $\mathcal{U}\leqq\mathcal{X}$ we need to show $R\mathcal{J}\mathcal{U}=J\mathcal{U}$. The direct sum $R\mathcal{U}=\oplus_{x\in\mathcal{U}^\tau} Rx$ permits us to restrict attention to subdiagrams of the form $\mathcal{U}(x)$, $x\in\mathcal{Y}_{l-1}$. But then

$$J\mathcal{U}(x) = Jx = \bigoplus_{y\in\mathcal{Y}_{lx}} Ry = R\left(\bigcup_{y\in\mathcal{Y}_{lx}}\mathcal{U}(y)\right) = R\mathcal{J}\mathcal{U}(x)\,.$$

The second equality is from (*). The last follows since $\mathcal{X}$ is, as already noted, a disjoint union of trees. Hence $\cup_{y\in\mathcal{Y}_{lx}}\mathcal{U}(y)=\mathcal{J}\mathcal{U}(x)$.

Finally we see that if $\mathcal{U}\leq\mathcal{X}$ then card $\mathcal{U}=c(R\mathcal{U})$ (again there is a reduction to $\mathcal{U}=\mathcal{U}(x)$, $x\in\mathcal{Y}_{l-1}$, and induction on l). From this, if $\mathcal{U}<\mathcal{V}\leq\mathcal{X}$ and $\mathcal{V}=\mathcal{U}\cup\{x\}$, $x=e_jx$, then $R\mathcal{V}/R\mathcal{U}\cong Re_jx/Je_jx=S_{\lambda(x)}$.

The last statement is easily proved by induction on l. □

Corollary 1.3. *A module M of Loewy length m contains a normed subset $\mathcal{X}$ such that $(M,\mathcal{X})$ is a tree module if and only if J^kM is a direct sum of local submodules, for $k=0,1,\cdots,m-1$.*

Proof. ($\Rightarrow$). This follows from (*) of Proposition 1.2.

($\Leftarrow$). A subset $\mathcal{X}$ of M is constructed which satisfies (*) of Proposition 1.2. To start, we write $M=J^0M=\oplus_{\alpha=1}^{h(0)} N_{0,\alpha}$, a direct sum of local submodules. Normed generators $x_{0,\alpha}$ are chose for each $N_{0,\alpha}$ and we set $\mathcal{Y}_0=\{x_{0,1},\cdots,x_{0,h(0)}\}$. Now $JM=\oplus_{\alpha=1}^{h(0)} JN_{0,\alpha}$ is a direct sum of local submodules and so, by the Krull-Schmidt Theorem, each $JN_{0,\alpha}$ is itself a direct sum of local submodules. We write $JM=\oplus_{\alpha=1}^{h(1)} N_{1,\alpha}$, a direct sum of local submodules chosen so that for each α, there is β so that $N_{1,\alpha}\subset N_{0,\beta}$. A choice of normed generators $x_{1,\alpha}$ for the $N_{1,\alpha}$ gives the set $\mathcal{Y}_1$. This process is now repeated for JM and the other layers. The condition (*) is clearly satisfied. □

A *diagram isomorphism* between module diagrams $\mathcal{X}$ and $\mathcal{Y}$ is a bijection which is compatible with the arrows and the functions λ. The next proposition will be used in later sections.

Proposition 1.4. *Let $(M,\mathcal{X})$ and $(N,\mathcal{Y})$ be tree modules with $M\cong N$ as R-modules. Then there is a diagram isomorphism $\varphi\colon\mathcal{X}\to\mathcal{Y}$.*

Proof. This is clear for the case $c(M)=1$. Suppose now that M is local, $c(M)=m>1$ and that the theorem is true for (local) modules of composition length $<m$. We have that $R(\mathcal{J}\mathcal{X})=JM\cong JN=R(\mathcal{J}\mathcal{Y})$. Write $JM=\oplus_{x\in(\mathcal{J}\mathcal{X})^\tau} Rx$ and $JN=\oplus_{y\in(\mathcal{J}\mathcal{Y})^\tau} Ry$. Since $JM\cong JN$, the Krull-Schmidt Theorem gives a bijiection $\theta\colon(\mathcal{J}\mathcal{X})^\tau\to(\mathcal{J}\mathcal{Y})^\tau$ so that $Rx\cong R\theta(x)$, $x\in(\mathcal{J}\mathcal{X})^\tau$. The induction hypothesis yields tree isomorphisms $\rho_x\colon\mathcal{U}(x)\to\mathcal{U}(\theta(x))$, for $x\in(\mathcal{J}\mathcal{X})^\tau$. Suppose $\mathcal{X}^\tau=\{x_0\}$ and $\mathcal{Y}^\tau=\{y_0\}$. Then a tree isomorphism $\varphi\colon\mathcal{X}\to\mathcal{Y}$ is defined as follows: $\varphi(x_0)=y_0$ and if $x\neq x_0$ there is a unique $u\in(\mathcal{J}\mathcal{X})^\tau$ so that $Rx\subseteq Ru$; then $\varphi(x)=\rho_u(x)$.

Using this and the Krull-Schmidt Theorem once more, the assumption that M be local may be dropped. $\square$

2. Left monomial rings

Before coming to our proposed generalization of monomial algebras to left artinian rings, we look at somewhat milder conditions to be imposed on our ring R. Recall ([5]) that a homomorphism of module diagrams φ is *quotient monic* if for $x\neq y$ in the domain of φ, $\varphi(x)\neq\varphi(y)$ unless both of the latter are 0.

Proposition 2.1. *Suppose $({}_RR,\mathcal{X})$ is a tree module with $\mathcal{X}=\bigcup_{i,j=1}^n e_j\mathcal{X}e_i$, a disjoint union, and $\{e_1,\cdots,e_n\}\subseteq\mathcal{X}$. Suppose further that for $i,j=1,\cdots,n$ and $a\in e_j(\mathcal{J}\mathcal{X})^\tau e_i$, there is a subdiagram $\mathcal{A}(a)\leq\mathcal{X}e_j$ so that* $\operatorname{lann}_{Re_j}(a)=R\mathcal{A}(a)$. *Then there are quotient monic maps $\varphi_a\colon\mathcal{X}e_j\to\mathcal{U}(a)$ for each $a\in e_j(\mathcal{J}\mathcal{X})^\tau e_i$ so that $(R,\mathcal{X},\{\varphi_a\})$ is a left diagram for R and the diagram $(\mathcal{X},\{\varphi_a\})$ is an algebra diagram in which each $\mathcal{P}_i=\mathcal{X}e_i$ is a tree. In particular if K is a field, the semigroup algebra $K\mathcal{X}$ is a monomial algebra.*

Proof. By Proposition 1.1, $(Re_i,\mathcal{X}e_i)$ is a tree module. For $a\in e_j(\mathcal{J}\mathcal{X})^\tau e_i$, the projective cover $\varphi\colon Re_j\to Ra$, defined by $\varphi(e_j)=a$, has kernel $R\,\mathcal{A}(a)$. Now Ra has a tree subset $\mathcal{U}(a)$ and $Re_j/R\mathcal{A}(a)$ also has one, $(\mathcal{X}e_j\backslash\mathcal{A}(a))+R\mathcal{A}(a)$ (which is isomorphic to $\mathcal{X}e_i/\mathcal{A}(a)$, as diagrams by Proposition 1.1). But then $\mathcal{X}e_j/\mathcal{A}(a)\cong\mathcal{U}(a)$ by Proposition 1.4. Let ψ be a tree isomorphism $\psi\colon\mathcal{X}e_j/\mathcal{A}(a)\to\mathcal{U}(a)$. Define φ_a by $\varphi_a(x)=\psi(x)$ for $x\in\mathcal{X}e_i/\mathcal{A}(a)$ and $\varphi_a(x)=0$ otherwise. Since ψ is a bijection, φ_a is quotient monic.

By [5, Theorem 4.2], the facts that the $\mathcal{X}e_i$ are trees and the φ_a quotient monic show that $(\mathcal{X},\{\varphi_a\})$ is an algebra diagram and that $K\mathcal{X}$ is a monomial algebra. $\square$

At this stage it is possible to define a class of left artinian rings which, as will be seen, behave, in many important respects, like monomial algebras. The conditions on R need to be somewhat stronger than in Proposition 2.1. It should be noticed that while the existence of a tree subset for ${}_RR$ is independent of the choice of primitive idempotents (by Corollary 1.3), the stronger requirements of the following definition may depend on this choice.

DEFINITION 2.2. *Suppose ${}_RR$ has a tree subset $\mathcal{X}$ with $\{e_1, \cdots, e_n\} \subseteq \mathcal{X}$ and $\mathcal{X} = \bigcup_{i,j=1}^{n} e_j \mathcal{X} e_i$. Suppose further that if $x \in e_j \mathcal{X} e_i$ then there is an $\mathcal{A}(x) \leq \mathcal{X} e_j$ so that* $\mathrm{lann}_{Re_j}(x) = R\mathcal{A}(x)$. *Then $(R, \mathcal{X})$ (or simply R) is called a left monomial ring.*

In a left monomial ring it will be convenient to write $\mathcal{N}$ for $\mathcal{J}\mathcal{X} = \mathcal{X} \setminus \{e_1, \cdots, e_n\}$ and for $k > 1$, $\mathcal{N}^k$ is defined as $\mathcal{J}\mathcal{N}^{k-1}$.

Families of examples of left monomial rings are discussed in detail in the next section. The following list mentions some of them.

(i) all left almost serial rings ([4], see Proposition 3.2).

(ii) all rings given by species and 0-relations ([3], see Proposition 3.3). (Monomial algebras are, of course, among these.)

(iii) all hereditary left artinian rings ([9, Theorem 1 and its proof]).

(iv) all left artinian rings with $J^2 = 0$ and those with $J^3 = 0$ such that J is a direct sum of local left idelas and each Je_i/J^2e_i is "square free". (See Proposition 3.4.)

(v) all left locally distributive left artinian rings whose radical is a direct sum of local left ideals ([10, Theorem 3] and [6, Proposition 3.8]).

(vi) all left and right artinian rings which are *right* serial (Proposition 3.5).

It follows from [5, Theorem 4.2 and Propostion 2.7] that if A is a monomial K-algebra there is a subset $\mathcal{Y}$ of A such that $(A, \mathcal{Y})$ is a monomial ring and $A = K\mathcal{Y}$; moreover, $\mathcal{Y} = \mathcal{S}/\mathcal{I}$, where $\mathcal{S}$ is the semigroup of paths in the quiver of A and $\mathcal{I}$ is the semigroup ideal of monomial relations. We have seen that if $(R, \mathcal{X})$ is a monomial ring (or even with the less stringent conditions of Proposition 2.1), the set $\mathcal{X}$, along with its diagram maps φ_a, has a natural diagram structure. For any field K, there is a diagram algebra $K\mathcal{X}$, which is, in fact, a monomial algebra ([5, Theorem 4.4]). Such a monomial algebra, for any field K, is called an *associated monomial algebra for R*. The next result shows that R and an associated monomial algebra $K\mathcal{X}$ have many properties in common. The elements of $\mathcal{X}$ will retain their names whether they are found in R or in $K\mathcal{X}$.

Theorem 2.3. *Let R be a basic left artinian ring. If $(R, \mathcal{X})$ is a left monomial ring and K a field, let $K\mathcal{X}$ be an associated monomial algebra with simple modules $\hat{S}_i \cong K\mathcal{X}e_i/K\mathcal{N}e_i$. Then for each $i = 1, \cdots, n$, the simple R-module S_i and the simple $K\mathcal{X}$-module $\hat{S}_i$ have minimal projective resolutions*

$$\cdots \to Q_m \to \cdots \to Q_1 \xrightarrow{\theta_1} Q_0 \xrightarrow{\theta_0} S_i \to 0$$

and

$$\cdots \to \hat{Q}_m \to \cdots \to \hat{Q}_1 \xrightarrow{\hat{\theta}_1} \hat{Q}_0 \xrightarrow{\hat{\theta}_0} \hat{S}_1 \to 0$$

over R and $K\mathcal{X}$, respectively, such that Q_m and $\hat{Q}_m$ are tree modules with isomorphic diagrams $\mathcal{Q}_m$ and $\hat{\mathcal{Q}}_m$ respectively. Further, $\mathrm{Im}\,\theta_m$ *and* $\mathrm{Im}\,\hat{\theta}_m$ *are tree modules with isomorphic diagrams $\mathcal{I}_m$ and $\hat{\mathcal{I}}_m$, respectively; and if $x \in \mathcal{X}$, there are analogous statements above minimal projective resolutions of $Rx = R\mathcal{U}(x)$ and of $K\mathcal{X}x = K\mathcal{U}(x)$.*

Proof. A minimal projective resolution of S_i begins with $Je_i = \bigoplus_{x \in (\mathcal{N}e_i)^\tau} Rx$, and, similarly, one for $\hat{S}_i$ with $K\mathcal{N}e_i = \bigoplus_{x \in (\mathcal{N}e_i)^\tau} K\mathcal{X}x$. Hence it suffices to consider the last statement of the theorem.

Let $x \in e_i \mathcal{X} e_j$. By Definition 2.2, $\mathrm{lann}_{Re_k}(x) = R\mathcal{A}(x) = \bigoplus_{y \in \mathcal{A}(x)^\tau} Ry$ and so there is a projective cover $Re_j \to Rx$ with kernel $\bigoplus_{y \in \mathcal{A}(x)^\tau} Ry$, which means that the next step of the resolution is made up of the proejctive cover of a direct sum of local modules of the same sort. By Propositions 1.1 and 1.4 there is a tree isomorphism $\mathcal{X}e_j/\mathcal{A}(x) \cong \mathcal{X}x = \mathcal{U}(x)$ which yields a $K\mathcal{X}$ homomorphism $K\mathcal{X}e_j \to K\mathcal{X}x$ with kernel $K\mathcal{A}(x) = \bigoplus_{y \in \mathcal{A}(x)^\tau} K\mathcal{X}y$ (see [5, Theorem 2.5]). Thus an identical statement can be made about the projective cover of the $K\mathcal{X}$-module $K\mathcal{X}x$. $\square$

As a consequence, all the facts known about monomial algebras which derive from the projective resolutions of the local left ideals generated by paths can be carried over to the left monomial ring case. The following corollary lists various of these consequences. Some of them are "corollaries" in the sense that they have been proved earlier for monomial algebras and the theorem can be applied to carry the results over to the more general setting.

Corollary 2.4. *Assume $(R, \mathcal{X})$ is as in Theorem 2.3. Then the following statements hold.*

(i) *For any simple S_i of R,* $\mathrm{pr.dim.}\, S_i = \mathrm{pr.dim.}\, \hat{S}_i$ *for any associated monomial algebra $K\mathcal{X}$.*

(ii) *For any $x \in \mathcal{X}$,* $\mathrm{pr.dim.}\, Rx = \mathrm{pr.dim.}\, K\mathcal{X}x$ *for any associated monomial algebra $K\mathcal{X}$.*

(iii) *For any associated monomial algebra $K\mathcal{X}$,* $\mathrm{l.gl.dim.}\, R = \mathrm{gl.dim.}\, K\mathcal{X}$.

(iv) *The Cartan determinant conjecture is true for R (i.e., if* $\mathrm{l.gl.dim.}\, R < \infty$ *then the determinant of the left Cartan matrix for R is* 1).

(v) *The strong "no loops" conjecture is true for R (i.e., if* $\mathrm{pr.dim.}\, S_i < \infty$ *then the quiver for ${}_R R$ has no loops at i).*

(vi) *The injectively defined finitistic dimension of R is finite.*

Proof. Parts (i), (ii) and (iii) follow immediately from the theorem.

(iv): This follows since R and $K\mathscr{X}$ have the same left Cartan matrices. Now $K\mathscr{X}$ is positively graded (by radical layers) and so Wilson's result [18, Corollary 2.3] and (iii) give the conclusion. (It may also be remarked that the radical layering in R-mod gives a Cartan filtration in the sense of [7], as may readily be verified as in the proof of [6, Corollary 3.9].

(v): The corresponding result for $K\mathscr{X}$ is found in [15, Corollary 6.2] and it carries over to R.

(vi) : As mentioned in [8, Proposition 4.3], the reasoning of [16, Proposition 1.8] shows that this follows from the fact that the syzygies in a minimal projective resolution of R/J have only a finite set of indecomposable components-they are, up to isomorphism, from among the Rx, $x\in\mathscr{X}$. Hence [8, Proposition 4.3] applies. □

3. Examples

Several classes of examples of left monomial rings will be presented. The first of these generalizes that of left almost serial rings studied in [4] (a class which in turn includes left serial rings) and also hereditary artinian rings.

Definition 3.1. *A left artinian ring R is called left uniformly monomial if*

$$J\cong\bigoplus_{l=1}^{m}\frac{Re_{i_l}}{J^{u_l}e_{i_l}},$$

for various $1\leq i_l\leq n$ and $u_l\geq 1$. When this occurs, the following notation is used. For each $1\leq i\leq n$,

$$Je_i\cong\bigoplus_{l=1}^{m(i)}\frac{Re_{k(i,l)}}{J^{u(i,l)}e_{k(i,l)}}.$$

Proposition 3.2. *Let R be a left uniformly monomial ring, then R is a left monomial ring.*

Proof. By definition and the Krull-Schmidt Theorem, each Je_i is a direct sum of local left ideals of the form $Re_j/J^{u(i,l)}e_j$. Its radical is isomorphic to $Je_j/J^{u(i,l)}e_j$, itself either 0 or a direct sum of local left ideals of the same type. Condition (*) of Proposition 1.2 can be seen to be verified. In order to do so it is convenient to fix epimorphisms $\theta_{i,l}\colon Re_{k(i,l)}\to L_{i,l}\subseteq Je_i$ with kernels $J^{u(i,l)}e_{k(i,l)}$. Then $\mathscr{X}$ is defined as follows. First $\mathscr{X}_0=\{e_1,\cdots,e_n\}$ and $\mathscr{X}_1=\{\theta_{i,l}(e_{k(i,l)})\mid 1\leq i\leq n;\ 1\leq l\leq m(i)\}$. The remaining layers of $\mathscr{X}$ are defined inductively by setting $\mathscr{X}_r$ to be the set of non-zero elements of the form $\theta_{i_1,l_1}\circ\theta_{i_2,l_2}\circ\cdots\circ\theta_{i_r,l_r}(e_{k(i_r,l_r)})\in J^re_{i_1}$, which is what was required. It remains to check the annihilator condition of Definition 2.2.

Abbreviate $e_{k(i_j,l_j)}$ by $e_{k(j)}$, θ_{i_j,l_j} by θ_j and $u(i_j,l_j)$ by $u(j)$. Let $0\neq x=$

$\theta_1\circ\theta_2\circ\cdots\circ\theta_r(e_{k(r)})$. Then there is a $t\geq 0$, depending on $u(1), \cdots, u(r)$, such that

$$\mathrm{lann}_{Re_{k(r)}}(x) = \ker(\theta_1\circ\theta_2\circ\cdots\circ\theta_r) = J^t e_{k(r)} .$$

But this annihilator is $R\mathcal{X}_t e_{k(r)}$. □

We will need the following additional remark about uniformly monomial rings in Proposition 4.3. If $x=\theta_{i_1,l_1}\circ\theta_{i_2,l_2}\circ\cdots\circ\theta_{i_r,l_r}(e_{k(i_r,l_r)})$ and $y=\theta_{j_1,p_1}\circ\theta_{i_2,l_2}\circ\cdots\circ\theta_{j_t,p_t}(e_{k(j_t,p_t)})$, then $xy=0$ if $k(j_t,p_t)\neq i_1$. But if $k(j_t,p_t)=i_1$, then $xy=\theta_{i_1,l_1}\circ\theta_{i_2,l_2}\circ\cdots\circ\theta_{i_r,l_r}\circ\theta_{j_1,p_1}\circ\theta_{i_2,l_2}\circ\cdots\circ\theta_{j_t,p_t}(e_{k(j_t,p_t)})$, which is either 0 or in $\mathcal{X}$.

Another class of rings to be studied is that of split left artinian rings given by a species with 0-relations ([3, Definition 2.1]). Such a ring R is constructed with the following data:

(i) a directed graph Γ.

(ii) to each vertex v_i of Γ is attached a division ring D_i.

(iii) to each arrow $v_i \xrightarrow{a} v_j$ is attached a D_j—D_i-bivector space $M(a)$ which is left finite dimensional. The set of paths in Γ is denoted Π.

(iv) the *tensor ring* $\mathcal{T}$ given by these data is $D_1\times\cdots\times D_n\oplus \bigoplus_{p\in\Pi} t(p)$, where for $p\in\Pi$, $p=v_{i_1}\xrightarrow{a_1} v_{i_2}\xrightarrow{a_2}\cdots\xrightarrow{a_{m-1}} v_{i_m}$, $t(p)=M(a_{m-1})\otimes M(a_{m-2})\otimes\cdots\otimes M(a_1)$. The vertices v_i are identified with orthogonal idempotents e_i.

(v) a 0-relation is a subgroup of $\mathcal{T}$ of the form $r=t(p)$ for some path of length ≥ 2.

(vi) there is a set ρ of 0-relations such that for some m, every $p\in\Pi$ of length $\geq$m contains a subpath giving rise to an element of ρ.

Then R is defined as $R=\mathcal{T}/(\rho)$.

Such a ring may be thought of as those elements of the tensor ring whose terms are from those paths not in (ρ). Call this set of paths $\mathcal{P}$. (This means that every path *not* in $\mathcal{P}$ has a subpath giving rise to one of the elements of ρ.) The paths in $\mathcal{P}$ are called *non-zero*. Then $R=D_1\times\cdots\times D_n\oplus \bigoplus_{p\in\mathcal{P}} t(p)$. For each arrow $v_i\to v_j$, fix a left a D_j-basis for $M(a)$, say $\{x_{a(1)}, \cdots, x_{a(m(a))}\}$. Set $\mathcal{X}$ to be the set of all simple non-zero tensors whose factors are basis elements, along with the idempotents $e_1, \cdots, e_n$ which come from the the vertices. That is, if $a_1, \cdots, a_r$ gives a non-zero path p from v_j to v_i, then it would yield elements of $e_j\mathcal{X}e_i$ of the form $x_{a_r(i_r)}\otimes\cdots\otimes x_{a_1(i_1)}=x$. For a given path p, there is a set $\mathcal{X}(p)$ of such elements and $t(p)=D_j\mathcal{X}(p)$.

If we fix p as above, let $\Pi(p)$ be the set of all paths q such that $qp\neq 0$. With this notation

$$Rx = D_j x \oplus \bigoplus_{q \in \Pi(p)} t(q) \otimes x ,$$

where the sums are as abelian groups. Now for $x \in \mathcal{X}(p)$ and $y \in \mathcal{Y}(q)$, when is $Rx \subseteq Ry$? For $x \in \mathcal{X}(p)$ to be in Ry, there must be a path r so that $p = rq$ and x must have the form $z \otimes y$ for some $z \in \mathcal{X}(r)$. Hence if $\{x_\alpha\}_{\alpha \in A}$ is a set elements of $\mathcal{X}$ with the property that $Rx_\alpha \nsubseteq Rx_\beta$ if $\alpha \neq \beta$, then the sum $\sum_A Rx_\alpha$ is direct. Indeed, if $\sum_A r_\alpha x_\alpha = 0$, we may assume (by multiplication on the left and right by suitable idempotents) that each $r_\alpha x_\alpha$ is a D_k-linear combination of elements of $e_k \mathcal{X} e_i$, say $r_\alpha x_\alpha = \sum d_{\alpha\beta} y_{\alpha\beta} \otimes x_\alpha$, for some suitable collection of $y_{\alpha\beta} \in \mathcal{X}$ and $d_{\alpha\beta} \in D_k$. Then $0 = \sum_\alpha r_\alpha x_\alpha = \sum_\alpha \sum_\beta d_{\alpha\beta} y_{\alpha\beta} \otimes x_\alpha$. Hence if we fix some $0 \neq z = y_{\alpha\beta} \otimes x_\alpha$, then $\sum_{y_{\gamma\delta} \otimes x_\gamma = z} d_{\gamma\delta} = 0$. Look at such a sum for a fixed z. If $y_{\gamma\delta} \otimes x_\gamma = y \otimes_{\mu\nu} x_\nu$ then either $x_\gamma = t \otimes x_\mu$ for some $t \in \mathcal{X}$ or $x_\mu = t' \otimes x_\gamma$ for some $t' \in \mathcal{X}$. By assumption on our set $\{x_\alpha\}$, neither is possible except for trivial factorizations. Hence for any one z there is at most one $r_\alpha x_\alpha$ with non-zero term in z. This makes it impossible for any $r_\alpha x_\alpha$ to be non-zero.

Proposition 3.3. *A left artinian* 0*-relations ring given by a species with* 0*-relations is a left monomial ring.*

Proof. We have just seen that $\mathcal{X}$ is a tree set for ${}_RR$ and it is clear that $\operatorname{card}\mathcal{X} = c({}_RR)$. To complete the verification that $(R, \mathcal{X})$ is a left monomial ring, consider $x \in e_j \mathcal{X} e_i$. Then

$$\operatorname{lann}_{Re_j}(x) = \sum_{y \in \mathcal{X} e_j, y \otimes x = 0} Ry .$$

It should be noted that $\mathcal{X} \cup \{0\}$ is a multiplicative semigroup in this case. (The question of semigroups will be examined in Section 4.) □

If R is such that $J^3 = 0$ and J is a direct sum of local left ideals then ${}_RR$ automatically has a tree module structure. More is needed to get the left monomial ring structure.

Proposition 3.4. *Let R be a left artinian ring such that $J^3 = 0$, J is a direct sum of local left ideals and each Je_i/J^2e_i is a direct sum of pairwise non-isomorphic simple modules. Then R is left monomial.*

Proof. Write $Je_i = L_{i1} \oplus \cdots \oplus L_{im(i)}$, a direct sum of local left ideals. For each $j = 1, \cdots, m(i)$, choose a projective cover $\theta_{ij}\colon Re_{k(i,j)} \to L_{ij}$ and let $x_{ij} = \theta_{ij}(e_{\beta(i,j)})$. These are the elements of $\mathcal{M}^\tau$. Set $k = \beta(i,j)$, $\theta = \theta_{ij}$ and $Je_k = Rx_{k1} \oplus \cdots \oplus Rx_{km(k)}$, where the x_{kl} are the elements chosen by the process just described. Now $JL_{ij} = \theta(Je_k) = \sum_{l=1}^{m(k)} R\theta(x_{kl})$. We also have $J^2 e_k \subseteq \ker\theta$. If $\sum_{l=1}^{m(k)} r_l \theta(x_{kl}) = 0$ is a non-trivial expression, then for some α, $\sum e_\alpha r_l \theta(x_{kl}) x_{kl} = 0$ is non-trivial. It follows that $\sum e_\alpha r_l x_{kl} = \sum e_\alpha r_l e_{\beta(k,l)} \in \ker\theta$. By hypothesis there is at most one term where some $\beta(k, l) = \alpha$ and r_l is a unit in $e_\alpha R e_\alpha$, which is not in J^2e_k. But then $\sum e_\alpha r_l x_{kl}=0$ has at most one non-zero term. This is absurd. Hence $\ker\theta$ is a direct sum of those L_{kl} for which $\theta(L_{kl})=0$. This means that $\mathrm{lann}_{Re_k}(x_{ij})$ is generated by elements of $\mathcal{N}^\tau$. The remaining layer, $\mathcal{N}^2$, of $\mathcal{X}$ consists of the non-zero images under the various θ of the elements of $\mathcal{N}^\tau$. These are in the socle and so for $x=e_jx\in\mathcal{N}^2$, $\mathrm{lann}_{Re_i}(x)=R\mathcal{N}e_j$. □

Proposition 3.5. *Let R be a left and right artinian ring which is right serial. Then R is a left monomial ring.*

Proof. According to Sumioka ([17, Lemma 2.3] with the sides interchanged), each radical layer is a direct sum of local left ideals. By Corollary 1.3 we have a tree structure for ${}_RR$. It remains to check the annihilator condition. But if $a\in(e_j\mathcal{N}^r)^\tau$ is a generator of one of the local left ideals making up J^re_i, then [17, Lemma 2.7] shows that $e_k\mathrm{lann}_{Re_j}(a)=e_kJ^{s_k}e_j$, which is generated by $e_k\mathcal{N}^{s_k}e_j$. □

EXAMPLE. A simple example of a left monomial ring which does not fit into any of the classes listed after Definition 2.2 follows. Let R be the ring of lower triangular matrices where $S=\boldsymbol{Z}/(16)$ and $N=J(S)$

$$R=\begin{bmatrix} S & 0 \\ S/N\oplus S/N & S \end{bmatrix}$$

4. Diagram semigroups

An associated monomial algebra $K\mathcal{X}$ of a left monomial ring $(R,\mathcal{X})$ is always a semigroup algebra ([5, Theorem 4.2]). In fact the tree isomorphism constructed in Proposition 1.4 shows that the elements of $\mathcal{X}$ in $K\mathcal{X}$ may be identified with compositions of the quotient monic maps φ_α of Proposition 2.1. It is not known if a left monomial ring $(R,\mathcal{X})$ always has a semigroup internal to it (i.e., if there is a tree subset $\mathcal{Y}$ so that $(R,\mathcal{Y})$ is a left monomial ring and $\mathcal{Y}\cup\{0\}$ is a multiplicative semigroup). In this section a criterion for that will be established and it will be shown that left uniformly monomial rings, right serial rings and left monomial rings with $J^3=0$ all have such semigroups. It was already noted in the proof of Proposition 3.3 that a split ring of the "species with 0-relations" kind had a semigroup.

Proposition 4.1. *Let R be basic left artinian. Define $\mathcal{A}_0=\{e_1,\cdots,e_n\}$ and let $\mathcal{A}_1=\bigcup_{i,j=1}^n e_j\mathcal{A}_1e_i\subseteq J$ and let $\mathcal{A}_l=\mathcal{A}_1^l\setminus\{0\}$, for $l=1,\cdots,m$, where m is the Loewy length of R. Assume the following two conditions:*

1) *if $a_1,\cdots,a_r$ and $b_1,\cdots,b_s$ are from $\mathcal{A}_1$ and $0\neq a_1\cdots a_r=b_1\cdots b_s$ then $r=s$ and for $j=1,\cdots,r$, $a_j=b_j$; and*

2) *$J^k=\bigoplus_{x\in\mathcal{A}_k}Rx$ for $k=1,\cdots,m-1$.*

If $\mathcal{X}=\bigcup_{k=0}^{m-1}\mathcal{A}_k$ then $\mathcal{X}\cup\{0\}$ is a subsemigroup of $(R,\cdot)$ which makes $(R,\mathcal{X})$ into a left monomial ring.

Proof. The second condition shows, by Corollary 1.3, that $({}_RR,\mathcal{X})$ is a tree module. We need to check the annihilator condition of Definition 2.2. If $x\in e_j\mathcal{X}$ and Rx is simple then $\mathrm{lann}_{Re_f}\,x=Je_j=\bigoplus_{y\in\mathcal{A}_1e_j}Ry$. We proceed by induction on the Loewy length, $L(Rx)$, of Rx. If $L(Rx)=k>1$, we suppose that for $y\in e_p\mathcal{X}e_j$ with $L(Ry)<k$, $\mathrm{lann}_{Re_p}(y)$ is a direct sum of left ideals of the form Rz, $z\in\mathcal{Z}\subseteq\mathcal{X}$. Consider $r\in\mathrm{lann}_{Re_j}(x)$. There is some $\mathcal{Y}=\mathcal{Y}^{\tau}\subseteq\mathcal{X}e_j$ so that $r\in\bigoplus_{y\in\mathcal{Y}}Ry$, say $r=\sum_{y\in\mathcal{Y}}r_yy$. The hypotheses show that the sum of the non-zero left ideals Ryx, $y\in\mathcal{Y}$, remains direct. Hence $rx=\sum_{y\in\mathcal{Y}}r_yyx=0$ if and only if each $r_yyx=0$, which occurs if and only if each $r_y\in\mathrm{lann}_{Re_{p(y)}}(yx)=R\mathcal{U}_y$, for some $\mathcal{U}_y\subseteq\mathcal{X}e_{p(y)}$, by the induction hypothesis. But then

$$\sum_{y\in\mathcal{Y}}R\mathcal{U}_xy\subseteq\mathrm{lann}_{Re_j}(x)\subseteq\sum_{y\in\mathcal{Y}}R\mathcal{U}_yy$$

and $\sum_{y\in\mathcal{Y}}R\mathcal{U}_yy=R(\bigcup_{y\in\mathcal{Y}}\mathcal{U}_y)y$. □

Corollary 4.2. *Let K be a field and suppose that A is a split K-algebra with $\mathcal{X}=\bigcup_{k=0}^{m-1}\mathcal{A}_k$ as in Proposition 4.1. Then A can be presented as a monomial algebra. Moreover, there are canonical one-to-one corespondences between the subsets $\mathcal{A}_k$ and the sets of paths of length k in the quiver of A which do not contain those used for relations in the presentation of A as a monomial algebra.*

Proof. It is clear that the tree subset $\mathcal{X}$ is a K-basis for A. Hence $A=K\mathcal{X}$ is a semigroup algebra and the semigroup satisfies the conditions of [5, Theorem 4.2]. This shows that A is a monomial algebra. □

Proposition 4.3. *Let R be a basic left artinian ring. If R is one of the following types of ring, then there is a semigroup $\mathcal{X}\cup\{0\}$ in R so that $(R,\mathcal{X})$ has the structure of a left monomial ring.*

1) *R is a left uniformly monomial ring.*
2) *R is a artinian and right serial.*
3) *R is a left monomial ring with $J^3=0$.*

Proof. 1) The remark after Proposition 3.2 gives this result. (This fact was also obtained independently by Y. Wang.)

2) In this case, choose $\mathcal{N}^{\tau}$ to be a set of normed generators of the local left ideals making up J, as in Proposition 3.5. If $a\in\mathcal{N}^{\tau}$ then [17, Corollary 2.8] says, in particular, that if $J^t=\bigoplus_{u\in\mathcal{U}}Ru$, for some set of normed elements $\mathcal{U}$, then the sum of the non-zero terms of the form Rua, $u\in\mathcal{U}$, remains direct. This means that Proposition 4.1 can be applied using $\mathcal{A}_1=\mathcal{N}^{\tau}$.

3) For the start it suffices to assume that $(R,\mathcal{X})$ is left monomial (with no

hypothesis on the Loewy length). For $x \in e_j \mathcal{M}^r e_i$, there is a projective cover $Re_j \to Jx$ with kernel $R\mathcal{A}(x)$. The restriction of this map to Je_j gives an epimorphism $\theta: Je_j \to Rx$, given by right multiplication by x. Now $\ker(\theta) = Je_j \cap R\mathcal{A}(x)$. Write $Je_j = \oplus_{y \in \mathcal{M}^r e_j} Ry$ and $\mathcal{X}e_j = (\mathcal{X}e_j \cap \mathcal{A}(x)) \cup (\mathcal{X}e_j \setminus \mathcal{A}(x))$. Label $\mathcal{X}e_j \setminus \mathcal{A}(x) = \mathcal{Y}$. Then $\theta|_{R\mathcal{Y}} = \iota$ is still an epimorphism $R\mathcal{Y} \to Jx$. Its kernel is $R(\mathcal{Z} \cap \mathcal{A}(x))$, where $\mathcal{Z} = \{z \in \mathcal{X} \mid Rz \subseteq Ry$ for some $y \in \mathcal{Y}\}$. Then ι (right multiplication by x) induces an isomorphism of $R\overline{\mathcal{Z}} \to Jx$, where $\overline{\mathcal{Z}} = \mathcal{Z} \setminus \mathcal{A}(x) + R\mathcal{A}(x)$ (by Proposition 1.1). But $R\overline{\mathcal{Z}} = \oplus_{y \in \mathcal{Y}} R(y + R\mathcal{A}(x))$ by the definition of $\mathcal{Y}$. Hence $Jx = \oplus_{y \in \mathcal{Y}} Ryx$.

Now if we impose that $J^3 = 0$, the conditions of Proposition 4.1 are satisfied. □

No example is known of a left monomial ring which does not have such a semigroup. However, as is easy to see, if $(R, \mathcal{X})$ is a left monomial ring, $\mathcal{X} \cup \{0\}$ need not be closed under multiplication. We do not know whether a left monomial ring wich is a split algebra over a field necessarily is a monomial algebra (with respect to *some* semigroup) even though we have seen that it behaves much like one.

An alternate approach to left monomial rings, would be to assume directly that R contains an appropriate semigroup. The details of doing this will not be given but the reader will get an idea of how this might be done from the basic definition. Let Γ be the (left) quiver of R with set of vertices $\{e_1, \cdots, e_n\}$ and set of arrows $A = \{a_l \mid l \in I\}$, a set of normed representatives, in $\cup_{i=1}^n Re_i$, of the simple submodules of J/J^2. Let B be the set of paths in Γ. For $w \in B$, $\lambda(w)$ denotes its length. There is a function, called *evaluation*, $\rho: B \to R$ defined as follows. If $w = \bullet \xrightarrow{a_1} \bullet \cdots \bullet \xrightarrow{a_m} \bullet$ then $\rho(w) = a_m \cdots a_1$, this product in R.

DEFINITION 4.4. *The (finite) set T of non-zero elements of $\rho(B)$ is called a monomial system (with respect $\{e_1, \cdots, e_n\}$) if for each $j \geq 0$,*

$$J^j = \bigoplus_{w \in B, \rho(w) \neq 0, \lambda(w) = j} R\rho(w) .$$

If for some choice of the set of arrows A, $\rho(B) \setminus \{0\}$ is a monomial system, then R is said to have a monomial system.

By definition, if R has a monomial system T then T is a tree subset for ${}_R R$. Clearly $T \cup \{0\}$ is a semigroup as well. The fact that the annihilator condition of Definition 2.2 is satisfied requires more effort to prove, but that is indeed the case.

EXAMPLE 4.5. The following example shows that it does not suffice that ${}_R R$ be a tree module in order that R be a left monomial ring-even for split algegebras over a field for which $J^3=0$. Consider the quiver

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \arrow[r, shift left, "b"] \arrow[r, shift right, "c"'] & \overset{3}{\bullet} \arrow[r, "d"] \arrow[dr, "f"'] & \overset{4}{\bullet} \\ & & & \overset{5}{\bullet} \end{tikzcd},$$

subject to the relations $(b-c)a$, fb and dc. These data define an algebra R over any field K. The module ${}_RR$ has the tree diagram

$$\begin{array}{ccccccccc} Re_1 & & Re_2 & & & Re_3 & & Re_4 & Re_5 \\ {}_1e_1 & & e_2 & & & e_3 & & e_4 & e_5 \\ \downarrow & \swarrow & & \searrow & \swarrow & & \searrow & & \\ {}_2a_1 & {}_3b_2 & & {}_3c_2 & {}_4d_3 & & {}_5f_3 & & \\ \downarrow & \downarrow & & \downarrow & & & & & \\ {}_3x_1 & {}_4y_2 & & {}_5z_2 & & & & & \end{array},$$

where $x=ba=ca$, $y=db$, $z=fc$ (the subscripts ${}_ju_i$ indicate that $u=e_jue_i$). Each power of the radical is a direct sum of local left ideals so by Corollary 1.3, $({}_RR, \mathcal{X})$ is a tree module where $\mathcal{X}=\{e_1, e_2, e_3, e_4, e_5, a, b, c, d, f, x, y, z\}$. We shall see that the annihilator condition fails, not only for $\mathcal{X}$ but for any choice of tree subset $\mathcal{Y}$. Let us, for the moment, assume that we are dealing with a tree subset with respect to the given choice of primitive idempotents and that $\mathcal{Y}$ is such a tree subset. Since $e_2Re_1=Ka$, $e_2\mathcal{Y}e_1=\{ka\}$ for some $0\neq k\in K$. Thus $\mathrm{lann}_{Re_2}(ka)=R(b-c)$. Now $e_3\mathcal{Y}e_2\subset Kb+Kc$ and so the subset $\mathcal{A}(a)\subset\mathcal{Y}$ must contain an element of the form $l(b-c)$, $0\neq l\in K$. Now $Rl(b-c)\simeq Re_3$ and $\mathrm{Soc}\,Rl(b-c)=\mathrm{Soc}\,R(b-c)=Ky\oplus Kz=\mathrm{Soc}\,Re_2$. But if R is a left monomial ring Je_2 must be a direct sum of local left ideals generated by elements of $\mathcal{Y}e_2$, one of which is $l(b-c)$. This is impossible. Finally, the reasoning above will work just as well for any other choice of primitive idempotents once we have made the following observation (which requires some computations which will be omitted in the interests of brevity). The only primitive idempotents e_1' for which $Re_1'\simeq Re_1$ have the form $e_1'=e_1+ka+lx$ for some scalars k and l. A primitive idempotent e_2' with $Re_2'\simeq Re_2$ has the form $e_2'=e_2+ka$, k a scalar. Finally, for our computations, a primitive idempotent e_3' such that $Re_3'\simeq Re_3$ has the form $e_3'=e_3+k_1x+k_2d+k_3f$. With this, the above conclusions about $\mathcal{Y}$ hold in general. (It can also be remarked that replacing b by $b-c$ allows us to see that R_R is also a tree module, cf. [5, Theorem 4.2.])

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W.D. Burgess
Department of Mathematics
University of Ottawa
Ottawa, Ontario
Canada K1N 6N5

K.R. Fuller
Department of Mathematics

University of Iowa
Iowa City, Iowa
U.S.A. 52242

E.L. Green
Department of Mathematics
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia
U.S.A. 24061

D. Zacharia
Department of Mathematics
Syracuse University
Syracuse, New York
U.S.A. 13244