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ON SEMI-FIELD PLANES OF EVEN ORDER

YUTAKA HIRAMINE

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1. Introduction

Let $\pi$ be a non-Desarguesian semi-field plane with an autotopism group $G$ and let $u(\pi)$ denote the number of the orbits of $G$ on the points not incident with any side of the autotopism triangles.

In their paper [9], M.J. Kallaher and R.A. Liebler have conjectured that $u(\pi) \geq 5$ and they have proved that the conjecture is true if $G$ is solvable and the order of $\pi$ is not $2^6$.

In this paper we treat semi-field planes of even order whose autotopism groups are not necessarily solvable and prove the following.

**Theorem 1.** Let $\pi$ be a non-Desarguesian semi-field plane of order $2^r$. If $r$ is not divisible by 4, then $u(\pi) \geq 5$.

The proof requires the use of the Kallaher-Liebler's theorem mentioned above and the following lemma which we prove in section 3.

**Lemma 2.** Let $\pi$ be a non-Desarguesian semi-field plane of order $2^6$ with a solvable autotopism group. Then $u(\pi) \geq 5$.

2. Notations and preliminaries

Our notation is largely standard and taken from [3] and [6]. Let $G$ be a permutation group on $\Omega$. For $X \leq G$ and $\Delta \subset \Omega$, we define $F(X) = \{ \alpha \in \Omega | \alpha^x = \alpha \text{ for all } x \in X \}$, $X(\Delta) = \{ x \in X | x^\Delta = \Delta \}$, $X_\Delta = \{ x \in X | x^\Delta = \alpha \text{ for all } \alpha \in \Delta \}$ and $X^\Delta = X(\Delta)/X_\Delta$, the restriction of $X$ on $\Delta$. When $X$ is a collineation group of a projective plane, we denote by $F(X)$ the set of fixed points and fixed lines of $X$.

**Lemma 2.1.** Let $G$ be a transitive permutation group on a finite set $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then $|F(M)| = |N_G(M)| \times |ccl_G(M) \cap H||H|$. Here $ccl_G(M) \cap H = \{ g^{-1}Mg | g^{-1}Mg \subset H, g \in G \}$.

Proof. Set $W = \{(L, \alpha) | L \in ccl_G(M), \alpha \in F(L)\}$ and $W_a = \{ L | L \in ccl_G(M), \alpha \in F(L) \}$.
\( \alpha \in F(L) \). By the transitivity of \( G \), \( |W_\alpha| = |W_\beta| \) holds for every \( \alpha, \beta \in \Omega \). Counting the number of elements of \( W \) in two ways, we obtain \( |G; N_\alpha(M)| \times |F(M)| = |G; H| \times |\text{cl}_\alpha(M) \cap H| \). Thus we have the lemma.

**Lemma 2.2.** Let \( PG(2, q) \) denote the Desarguesian projective plane of order \( q \) where \( q = 2^n \) and \( n \equiv 1 \pmod{2} \). Set \( Y = PSL(3, q) \) and \( X = \langle f \rangle Y \), where \( f \) is a field automorphism of \( Y \) of order \( n \). Set \( G = X_{P, G, R} \) and \( N = G \cap Y \), where \( P = [1, 0, 0], Q = [0, 1, 0] \) and \( R = [0, 0, 1] \).

(i) Let \( A \) be a noncyclic abelian \( p \)-subgroup of \( G \) of order \( p^2 \) for a prime \( p \).

Then \( A \) is not semi-regular on the set of points contained in \( PG(2, q) - F(A) \).

(ii) Let \( C \) be a cyclic subgroup of \( G \) of order \( q-1 \). Then \( C \subseteq N \).

**Proof.** Since \( A \cap N = \pm 1 \) and \( N = Z_{q-1} \times Z_{q-1} \), \( p \) is an odd prime. Let \( T \) be the translation group with respect to the line \( g \) joining \([1, 0, 0] \) and \([0, 1, 0] \). Deny (i) and let \( \Omega \) denote the set of points in \( F(A) \). Then, by Theorem 5.3.6 of [3], \( T = \langle C_T(x) | 1 \leq x \in A \rangle \). By the semi-regularity of \( A \), \( C_T(x) \) acts on \( \Omega \) for each \( x \in A - \{1\} \). Hence \( T \) acts on \( \Omega \).

Let \( \Delta \) denote the set of points not incident with the line \( g \). Clearly \([0, 0, 1] \in \Delta \cap \Omega \). Since \( T \) is transitive on \( \Delta \), we have (i).

Set \( D = C \cap N \) and let \( \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \) be a generator of \( D \). Then \( C < D \) and \( D < C^N \leq G \leq N = Z_n \) and so \( |D| \geq (q-1)/n \). Set \( \langle h \rangle = C_{<f, g}(D) \) and \( s = |\langle h \rangle| \). Then \( n = r \times s \) for an integer \( r \). It follows that \( ba^{-1}, ca^{-1} \in GF(2^r)^* \). Hence \( |D| \leq 2^r - 1 \). From this, \( 2^r - 1 \geq |D| \geq (q-1)/n \). We can easily verify that \( s = 1 \). Therefore \( C_{<f, g}(D) = 1 \), whence \( C \leq C_G(D) = NC_{<f, g}(D) = N \). Thus \( C \subseteq N \).

In the rest of the paper we assume the following.

**Hypothesis 2.3.** Let \( \pi \) be a non-Desarguesian semi-field plane of order \( 2^r \) coordinatized by a semi-field \( D \) with respect to the points \( U_1 = (0, 0), U_2 = (0), U_3 = (\infty) \) and let \( G \) be the automorphism group of \( \pi \) with respect to \( U_1, U_2, U_3 \). Let \( l_i \) be the line joining \( U_j \) and \( U_k \) for \( i, j, k \) with \( \{i, j, k\} = \{1, 2, 3\} \) and let \( \Phi(\pi) \) be the set of points of \( \pi \) not incident with \( l_1, l_2 \) or \( l_3 \). Let \( u(\pi) \) denote the number of \( G \)-orbits on \( \Phi(\pi) \). Set \( K_i = G_{U_i, l_i} \) for \( 1 \leq i \leq 3 \) and let \( N_1, N_2 \) or \( N_3 \) be the right, middle or left nucleus, respectively.

\( D \) may be considered as a right vector space over \( N_1 \) or \( N_2 \) and as a left vector space over \( N_2 \) or \( N_3 \). The multiplicative group \( N_i^* \) is isomorphic to \( K_i \) for each \( i \) with \( 1 \leq i \leq 3 \) (Chapter 8 of [6]). Set \( l_i = l_i - \{U_j, U_k\} \) for \( i, j, k \) with \( \{i, j, k\} = \{1, 2, 3\} \).
3. The proof of Lemma 2.
Throughout this section \( \pi \) is a projective plane satisfying the hypothesis 2.3 and the following.

**Hypothesis 3.1.**
(i) The order of \( \pi \) is \( 2^6 \).
(ii) Set \( u=u(\pi) \). Then \( u \leq 4 \).
(iii) The automorphism group \( G \) is solvable.

**Lemma 3.2.** \( |K_t|=1, 3 \) or 7 for every \( t \in \{1, 2, 3\} \) and \( u=3 \) or 4.

Proof. Since \( \pi \) is non-Desarguesian, \( D \) is not a field. Hence, \( N_t \) is isomorphic to \( GF(2) \), \( GF(4) \) or \( GF(8) \) for \( t \in \{1, 2, 3\} \). By Theorem 8.2 of [6], \( |K_t|=1, 3 \) or 7.

By Corollary 4.1.1 of [9] and Hypothesis 3.1 (ii), \( u=3 \) or 4.

**Lemma 3.3.** If \( G \) is transitive on \( \mathcal{I}_t \) for some \( t \in \{1, 2, 3\} \), then the following hold.

(i) \( G/K_t \leq \Gamma L(1, 2^6) \) and \( G/K_t \) contains an element of order 9.

(ii) Let \( m \) be an arbitrary line through \( U_t \) such that \( m \neq l_i, l_k \) for \( \{t, j, k\} = \{1, 2, 3\} \). Set \( A=m \cap l_j \). Then \( G_m=G_A \), \( |G: G_A|=3^3 \cdot 7 \) and the number of \( G_A \)-orbits on \( m-\{U_t, A\} \) is equal to \( u \).

(iii) Let \( \Delta_1, \Delta_2, \ldots, \Delta_u \) be the orbits stated in (ii). Set \( x_s=|\Delta_s|, 1 \leq s \leq u \), and assume that \( x_1 \leq x_2 \leq \cdots \leq x_u \). Then \( |G_A| \) is divisible by \( x_s \) for every \( s \) and \( 6 \times |K_t| \) is divisible by \( |G_A| \). Furthermore \( \sum_{s=1}^{u} x_s = 63 \).

Proof. By Lemma 2.1 of [9], \( G \) is a transitive linear group on \( D \). Hence it follows from a Huppert's theorem ([7]) that \( G/K_t \leq \Gamma L(1, 2^6) \). If \( G/K_t \) contains no element of order 9, then its Sylow 3-subgroup is an elementary abelian 3-subgroup of order at most 9. By the structure of \( \Gamma L(1, 2^6) \), \( G/K_t \) is not a transitive linear group, a contradiction. Thus \( G/K_t \) contains an element of order 9 and (i) holds.

Let \( m, A \) be as in (ii). Since \( G \) fixes \( U_t \) and \( l_j, l_k \), we have \( G_m=G_A \). Clearly \( |G: G_A|=|A^c|=|I_t|=2^{2^t}-1=3^3 \cdot 7 \). As any point of \( \Phi(\pi) \) lies on a line of \( [U_t]-\{l_j, l_k\} \), \( \Phi(\pi) \cap m (m=U_t, A) \) is a union of \( u \) \( G_A \)-orbits, hence (ii) holds.

Since \( G/K_t \leq \Gamma L(1, 2^6) \), \( G_A/K_t \leq Z_6 \). Hence \( 6 \times |K_t| \) is divisible by \( |G_A| \). Clearly \( x_s=|\Delta_s| \) divides \( |G_A| \) and \( \sum_{s=1}^{u} x_s=|\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_u|=|I_t|=2^6-1=3^3 \cdot 7 \). Thus (iii) holds.

**Lemma 3.4.** Suppose \( u=4 \). Then there exists \( i \in \{1, 2, 3\} \) having the following properties:
(i) $G$ is transitive on $I_i$.
(ii) $K_i$ is isomorphic to $Z_7$ and $G$ has a normal Sylow 7-subgroup and
(iii) $|G: G_A|=63$, $G_A|K_i$ is isomorphic to $Z_6$ and $C_{G_A}(K_i)=K_i$ for each $A \in I_i$.

Proof. By Lemma 6.1 of [9], there exists $i \in \{1, 2, 3\}$ such that $G$ is transitive on $I_i$. Assume that $K_i=Z_7$. Then $K_i \leq Z_3$ by Lemma 3.2. Let $m, A, x_i$ be as in Lemma 3.3. We have $x_i 6|K_i|=6$ or 18 and $x_1+x_2+x_3+x_4=63$, hence $|K_i|=3$, $|G_A|=18$ and $(x_1, x_2, x_3, x_4)=(9, 18, 18, 18)$.

Let $z$ be an involution in $G_A$. Then $z$ is a Baer involution and so $|F(z) \cap (m-\{U_i, A\})|=7$ because $m \in F(z)$. If $F(z) \cap \Delta_s \neq \phi$, then $|\Delta_s| \leq \frac{1}{7}|G_A|$. In particular $F(z) \cap \Delta_s=\phi$ for $s \geq 2$ and so $|F(z) \cap \Delta_1|=7$. Since $G_A|K_i=Z_6$ and $z \in K_i$, $C_{G_A}(z)\neq \langle z \rangle$. Hence an element of $C_{G_A}(z)$ of order 3 acts on $F(z)$ and fixes at least one point on it. It follows that $|\Delta_1| \leq \frac{1}{3}|G_A|=3$, a contradiction. Therefore we have $K_i=Z_7$ and so $G$ has a normal Sylow 7-subgroup by Lemma 3.3. Thus (ii) holds.

Let $m=(U_i A)$, $\Delta_s$, $x_i$ for $t=i$ be as in Lemma 3.3 (ii). Since $G_A \geq K_i \simeq Z_7$ and $K_i$ acts semi-regularly on $m-\{U_i, A\}$, $7 \mid |\Delta_s|=x_i$ for all $s \in \{1, 2, 3, 4\}$. Moreover, by Lemma 3.3, $x_1+x_2+x_3+x_4=63$. Hence $(x_1, x_2, x_3, x_4)=(7, 7, 7, 42)$ or $(14, 14, 14, 21)$ and so $|G_A|=42$. Thus $G_A/K_i \simeq Z_6$ by the similar argument as in the proof of Lemma 3.3 (iii). Let $y$ be an element of $C_{G_A}(K_i)$ and assume that $y$ is 2 or 3. Since $G_A/K_i \simeq Z_6$ and $K_i \simeq Z_7$, $y$ is contained in the center of $G_A$. Hence $G_A$ acts on $F(y)$ and therefore $\Delta_s$ is contained in $F(y)$ for each $s$ with $|\Delta_s| \neq x_i$. As above, $(x_1, x_2, x_3, x_4)=(7, 7, 7, 42)$ or $(14, 14, 14, 21)$ and hence $|F(y) \cap m|=21+2=23$. Since $F(y) \cap \Phi(y)=\phi$, $y$ is a planar collineation. Therefore $y=1$, a contradiction. Thus (ii) holds.

Lemma 3.5. Suppose $u=4$ and let notations be as in Lemma 3.4. Then, for some $s \in \{1, 2, 3\}-\{i\}$ $O(G)$ has no orbit of length 7 on $l_s$.

Proof. Suppose false. Let $P$ be a Sylow 7-subgroup of $G$. By Lemma 3.4 (ii), $|P|=7^2$ and $P$ is a normal subgroup of $G$. Let $s \in \{1, 2, 3\}-\{i\}$ and let $\Omega_i$ be a $P$-orbit of length 7 on $l_s$. Then there exists another $P$-orbit of length 7, say $\Omega_2$, on $l_s$ because $T_7^7|l_{\Omega_2}|=|\Omega_2|-|\Omega_2|$.

Let $Q$ be a Sylow 3-subgroup of $O(G)$. By Lemmas 3.3 and 3.4, $K_i \simeq Z_7$ and a Sylow 3-subgroup of $G/K_i$ is isomorphic to that of a Sylow 3-subgroup of $\Gamma L(1, 2^9)$. Hence $Q=\langle a, b \mid a^7=b^2=1, b^{-1}ab=a^b \rangle$ for suitable $a, b$ in $Q$. We note that $Q'=\{[Q, Q]=\langle a^2 \rangle$.

Since $|\Omega_1|=|\Omega_2|=7<9$, $a^2$ acts trivially on $\Omega_1 \cup \Omega_2$, hence $|F(a^2) \cap l_s|=2+|\Omega_1|+|\Omega_2|=16$. As $s \in \{1, 2, 3\}-\{i\}$ is arbitrary, $a^2$ is planar and moreover we have $F(a^2)=\pi$, by Theorem 3.7 of [6], which implies that $a^2=1$. This
is a contradiction. Thus we have the lemma.

Lemma 3.6. \( u=3 \).

Proof. Assume that \( u \neq 3 \). Then, by Lemma 3.2, \( u=4 \) and we can apply Lemmas 3.4 and 3.5. Let notations be as in them.

Let \( P \) be a Sylow 7-subgroup of \( G \) and \( \Gamma \) the set of \( P \)-orbits on \( \overline{I} \). Set \( H=\Phi(G) \). Since \( P \) is a normal subgroup of \( H \) by Lemma 3.4 (ii), \( H \) induces a permutation group on \( \Gamma \). Since \( P \supseteq K \) and \( K \) is semi-regular on \( \overline{I} \), every \( P \)-orbit in \( \Gamma \) has length 7 or \( 7^2 \). If an orbit in \( \Gamma \) has length \( 7^2 \), \( \Gamma \) contains exactly two \( P \)-orbits of length 7, which are also \( H \)-orbits of length 7, contrary to Lemma 3.5. Therefore each \( P \)-orbit in \( \Gamma \) has length 7 and so \( |\Gamma|=7 \).

If \( H \) acts transitively on \( \Gamma \), \( G \) is transitive on \( \overline{I} \) and therefore \( G/K_s \leq \Gamma L(1, 2^6) \) by Lemma 3.3 (i). It follows that \( \Gamma L(1, 2^6) \supseteq \langle \Phi \rangle K_s \supseteq \langle G \rangle \langle G \rangle K_s \cap K_s=\langle G \rangle K_s \). Therefore an involution in \( \langle G \rangle \) centralizes a Sylow 7-subgroup of \( G \), which is central to \( \langle G \rangle \), contrary to Lemma 3.4 (iii). Hence \( H \) is not transitive on \( \Gamma \).

Let \( Q \) be a Sylow 3-subgroup of \( H \). Then \( |Q|=27 \) and \( [Q, Q]=Q' \supseteq Z_3 \) as in the proof of Lemma 3.5. Since \( H=\Phi Q, \Gamma^u=\Gamma^e \). On the other hand \( H \) is not transitive on \( \Gamma \). Hence \( Q' \) is abelian and therefore \( Q' \) acts trivially on \( \Gamma \). We note that \( G/\Phi \leq Z_6 \) or \( G/\Phi \leq GL(2, 7) \) according as \( P \leq Z_4 \) or \( P \leq Z_7 \times Z_7 \), respectively. Hence \( Q' \) is contained in \( \Phi \). Since \( Q' \) acts trivially on \( \Gamma \) and each orbit \( \Delta \in \Gamma \) is of length 7, \( F(Q') \cap \Delta \neq \phi \). Therefore \( Q' \leq K_s \) because \( [P, Q']=1 \). In particular \( Q' \) is semi-regular on \( I \), where \( \{j\}=\{1, 2, 3\} \). Hence \( QK_j \) is transitive on \( I \). By Lemma 3.3 (i), \( G/K_j \supseteq \Gamma L(1, 2^6) \) and \( K_j=Z_7 \). Let \( z \) be an involution in \( G \). Then \( [z, P] \leq K_j \cap K_j=1 \) and so \( z \in C_{\Phi}(K_j) \), contrary to Lemma 3.4 (iii). This we have \( u=3 \).

Lemma 3.7. Assume that there exists a line \( I \) through \( U \) with \( I \Phi / \Phi \) such that \( G \) acts transitively on \( \overline{I} \) on \( \overline{I} \), where \( \{i, j, k\}=\{1, 2, 3\} \), such that \( G_i \) acts transitively on \( I \) on \( \{U_i, l \cap I\} \). Then the following hold.

(i) \( G_i \) is transitive on \( I \) for \( t=j, k \).

(ii) \( G \) has two or three orbits on \( I \).

Proof. Let \( A_1 \), \( A_2 \in \overline{I} \) and set \( B_i=U_l A_i \cap l \) and \( B_2=U_j A_2 \cap l \). By assumption, there exists an element \( x \in G_1 \) such that \( B_2^x=B_2 \). Since \( U_l A_2 \cap l=B_2 \Rightarrow B_2^x=U_l A_2 \cap l \) and \( A_2, A_1 \in \overline{I} \), it follows that \( A_1^x=A_2 \). Hence \( G_i \) is transitive on \( I \). Similarly \( G_i \) is transitive on \( I \). Thus (i) holds.

Let \( d \) be the number of \( G \)-orbits on \( I \). Clearly \( d \) is at most 3. If \( d=1 \), \( G \) acts transitively on \( \Phi(\pi) \), contrary to \( u=3 \). Thus (ii) holds.

Lemma 3.8. Let \( I \) be the line satisfying the assumption in Lemma 3.7. If \( 7^2 \) \mid \mid G \) and \( 7^2 \not\mid \mid G \), then \( K_s=Z_3 \) and \( |G|=2 \cdot 3^2 \cdot 7^2 \).
Proof. By Lemmas 3.2, 3.3 (i) and 3.7 (i), $K_j$ and $K_k$ are isomorphic to $Z_7$; otherwise $7^2 
mid |G|$. Set $A=\cap l_i$. Then $G_t=G_{A}$ and so $G_t/K_t=G_{A}/K_t$. Since $G/K_t \leq GL(6, 2)$, $G_t/K_t$ is isomorphic to a subgroup of $L$, where

$$L = \begin{bmatrix}
1 & a_2 & \cdots & a_6 \\
0 & M \\
0 & & & & \\
& & & & 
\end{bmatrix}, a_2, \cdots, a_6 \in GF(2), M \in GL(5, 2).$$

Since $L/O_2(L)=GL(5, 2)$, a Sylow 3-subgroup of $L$ is an elementary abelian group of order 9. On the other hand, by Lemmas 3.3 and 3.7 (i), $G_t$ contains an element of order 9. Therefore $K_t \cong Z_3$.

For a subgroup $X$ of $G$, $X$ denotes the homomorphic image of $X$ in $G/K_t$. Since $K_t \cong K_j \cap K_k$ and $G \leq GL(6, 2)$, $K_j \times K_k$ is a Sylow 7-subgroup of $G$ and so $K_j \times K_k$ has two subgroups $\langle a \rangle$ and $\langle b \rangle$ of order 7 which fix nonzero vectors on $I_i$. Set $H=O(G)$. By Lemmas 3.3 (i) and 3.7 (i), $G/K_t \leq \Gamma L(1, 2^t)$ for $t \in \{j, k\}$, so that $|G:H| \leq 2$. Since $G \cong K_t$ for $t \in \{j, k\}$, $H$ normalizes $K_j$, $K_k$, $\langle a \rangle$ and $\langle b \rangle$. As $K_t$ acts semi-regularly on $I_i$, we have $K_t=\langle a \rangle \times \langle b \rangle$ for $t \in \{j, k\}$. Without loss of generality, we can assume that $\langle ab \rangle=K_j$. Let $g \in H$. Then $g^{-1}ag=a^p$ and $g^{-1}bg=b^q$ for some $p, q$ with $1 \leq p, q \leq 6$, so we have $g^{-1}abg=a^pb^q \in K_j=\langle ab \rangle$. Hence $p=q$. From this, $H/\langle a \rangle \times \langle b \rangle \leq O(\text{Aut}(Z_7))=Z_3$. Since $C_{GL(6, 2)}(\langle a \rangle \times \langle b \rangle)=\langle a \rangle \times \langle b \rangle$, we have $|H||3^2\langle a \rangle \times \langle b \rangle|=3\cdot 7^2$ and therefore $|H|\mid 3^2\cdot 7^2$. Thus we obtain $|G|=2\cdot 3^2\cdot 7^2$.

**Lemma 3.9.** Let $i \in \{1, 2, 3\}$ and set $\{i, j, k\}=\{1, 2, 3\}$. Then the following hold.

(i) For every line $m \in [U_i]-\{l_i, l_i\}$, $G_m$ has three orbits on $m-\{U_i, m \cap l_i\}$.

(ii) $G$ acts transitively on $I_i$ and $G/K_t \leq \Gamma L(1, 2^t)$.

Proof. Deny (i). Then, since $u=\omega(\pi)=3$, there exists a line $l \in [U_i]$ satisfying the assumption of Lemma 3.7. Let $\{\Omega_1, \Omega_2, \ldots, \Omega_p\}$ be the set of $G$-orbits on $I_i$ and set $b_i=|\Omega_i|$ for $1 \leq i \leq p$. By Lemma 3.7 (ii), $p=2$ or 3.

Assume $p=3$. Set $b=\max\{b_1, b_2, b_3\}$, $b=|\Omega_v|$ and let $A \in \Omega_v$. Since $u=3$, $G_A$ is transitive on $m-\{U_i, A\}$, where $m=AU_i$. Therefore $63 \mid |G_A|$. Hence $3b \mid |G|$ because $|G|=b|G_A|$. By Lemmas 3.2, 3.3 (i) and 3.7 (i), we have $|G|=2\cdot 3^4\cdot 7^2$ and so $b \mid 2\cdot 3^2\cdot 7$. Since $3b \geq b_1+b_2+b_3=63$, it follows that $21 \leq b \leq 63$, hence $b=21$ or 42 and $3\cdot 3^2\cdot 7^2 \mid |G|$, contrary to Lemma 3.8. Thus $p \neq 3$.

Assume $p=2$. Let $A \in \Omega_1$, $B \cup \Omega_2$ and set $g=AU_i$, $h=BU_i$. Since $u=3$, without loss of generality we may assume that $G_A$ is transitive on $g-\{U_i, A\}$ and that $G_h$ has two orbits on $h-\{U_i, B\}$, say $\Gamma_1, \Gamma_2$. Similarly as in the last paragraph we obtain the following:

$$b_1, b_2 \mid |G|, |G| \mid 2\cdot 3^4\cdot 7^2, b_1+b_2=63.$$
Hence \( \{b_1, b_2\} = \{21, 42\}, \{14, 49\} \) or \( \{9, 54\} \). We note that \( |G:G_A|=|G:G_b|\) = \( b_1\), \( |G:G_b| = |G:G_A| = b_2\) and \( 63 \mid |G_A| \).

If \( \{b_1, b_2\} = \{21, 42\} \), \( |G| = |G_A|b_1 \) and \( 21 \mid b_1 \). Hence \( 3^3 \cdot 7^2 \mid |G| \), contrary to Lemma 3.8.

If \( \{b_1, b_2\} = \{14, 49\} \), \( |G:G_A|=14 \) because \( 7^2 \not| |G| \). Hence \( |G:G_b|=49 \). By Lemma 3.8, \( |G| \mid 2 \cdot 3^3 \cdot 7^2 \). Therefore \( |G_A| \mid 18 \). Since \( \{U_1, B\} \) is a union of \( G_A \)-orbits \( \Gamma_1, \Gamma_2 \), we have \( |\Gamma_1| + |\Gamma_2| = 63 \) and \( |\Gamma_1|, |\Gamma_2| \mid 18 \). This is a contradiction.

If \( \{b_1, b_2\} = \{9, 54\} \), we have \( |G:G_A|=9 \) as \( 3^5 \not| |G| \). Hence \( 3^4 \mid |G| \) and so \( 7^2 \not| |G| \) by Lemma 3.8. Therefore \( |G| = 2 \cdot 3^4 \cdot 7 \). From this, \( |G_b|=21 \). Hence \( |\Gamma_1|, |\Gamma_2| \mid 21 \). However, \( |\Gamma_1| + |\Gamma_2| = 63 \), a contradiction. Thus we have (i), and (ii) follows immediately from (i).

By Lemma 3.9, we can apply Lemma 3.3 for every \( t \in \{1, 2, 3\} \) and obtain the following.

**Lemma 3.10.** Let notations be as in Lemma 3.3. Then the following hold.

(i) \( 3^3 \cdot 7 \mid |G|, |G_b| \mid 2 \cdot 3^3 \cdot 7^2 \) and \( 3^4 \not| |G| \).

(ii) \( 3^3, 7^2 \not| G_s \) for all \( s \in \{1, 2, 3\} \).

Proof. By Lemmas 3.2 and 3.3 (i) (ii), we have (i). By Lemma 3.3 (ii) (iii), \( |G_A| = |G|/63 \) and \( x_s \mid |G_A| \). Hence \( x_s \mid 2 \cdot 3^2 \cdot 7 \). Thus we have (ii).

**Lemma 3.11.** Let notations be as in Lemma 3.3 and assume that \( 21 \mid x_2 \). Then the following hold.

(i) \( K_1=K_2=K_3=Z_7 \) and \( G/K_i \) is isomorphic to a subgroup of \( \Gamma L(1, 2^6) \) of index at most \( 2 \) for each \( t \in \{1, 2, 3\} \).

(ii) Let \( Q \) be a Sylow 3-subgroup of \( G \). Then \( |Q|=3^3 \) and \( Q=\langle a, b| a^3=b_7=1, b^{-1}ab=a^3 \rangle \) for suitable \( a, b \) in \( Q \). Moreover, for any element \( v \) of order 3 in \( Q-Z(Q) \), \( F(v) \) is a subplane of order 4.

Proof. By Lemma 3.3, \( x_3 \) divides \( |G_A| \) and \( |G:G_A|=3^3 \cdot 7 \), so that \( 3^3 \cdot 7^2 \mid |G| \). It follows from Lemma 3.10 (i) that \( |G| = 3^3 \cdot 7^2 \) or \( 2 \cdot 3^3 \cdot 7^2 \). Therefore (i) holds.

By (i), the order of a Sylow 3-subgroup \( Q \) of \( G \) is \( 3^3 \). Hence \( Q \) is of the form stated in (ii) by the structure of \( \Gamma L(1, 2^6) \). We note that \( Q \) has exactly two conjugacy classes of subgroups of order 3. Let \( v \in Q-Z(Q) \) such that \( \langle v \rangle \cong Z_3 \). Then, as an element in \( \Gamma L(1, 2^6) \), \( v \) fixes three nonzero elements, that is, \( |F(v) \cap \mathcal{L}_1| = 3 \) for all \( t \in \{1, 2, 3\} \). Hence \( F(v) \) is a subplane of order 4.

**Lemma 3.12.** Let notations be as in Lemma 3.3. Then \( (x_1, x_2, x_3) = (7, 14, 42), K_i=Z_7 \) and \( G/K_i=\Gamma L(1, 2^6) \) for each \( t \in \{1, 2, 3\} \).

Proof. By Lemmas 3.3 (iii) and 3.10, we have \( x_1 \leq x_2 \leq x_3, x_1+x_2+x_3=63 \)
and $3^3, 7^2 \not\mid x_n$, $x_i \mid |G| 2\cdot 3^4 \cdot 7^2$ for $s \in \{1, 2, 3\}$. Hence $(x_1, x_2, x_3) = (21, 21, 21)$ or $(7, 14, 42)$. On the other hand $K_1 \lhd K_2 \lhd K_3 \cong Z_7$ by Lemma 3.11 (i).

Assume that $(x_1, x_2, x_3) = (21, 21, 21)$. Let $\Delta_i$ be the $G$-orbit containing $P_s$ for $s \in \{-1, 2, 3\}$. Hence $(x_1, x_2, x_3) = (21, 21, 21)$ or $(7, 14, 42)$. On the other hand $K_1 \lhd K_2 \lhd K_3$ by Lemma 3.11 (i).

Lemma 3.13. Let $\Delta_i$ be as defined in Lemma 3.3. Then the following hold.

(i) Let $P \in \Delta_i$. Then $G = \langle x \rangle = Z_6$ and a Sylow 7-subgroup of $G$ acts on $F(x^3) \cap I_t$ for all $t \in \{1, 2, 3\}$.

(ii) $F(x^3)$ is a subplane of $\pi$ of order $2^e$ for $e = 2, 3$.

Proof. Similarly as in the proof of Lemma 3.12, we obtain $|G_p| = |G|/|P_p| = 2 \cdot 3^3 \cdot 7^1 / 63 \cdot 7 = 6$. Since $G_p \leq G_A$, $G_p \cap K_i = 1$ and $G_A/K_i = Z_6$, we have $G_p = G_p K_i / K_i \cong Z_6$. Hence $G_p = Z_6$. Set $\langle x \rangle = G_p$. Clearly $x^3$ is an involution in $G_p$ and so by the property of $\Gamma L(1, 2^e)$, $x^3$ centralizes the Sylow 7-subgroup of $G/K_i$ for all $t \in \{1, 2, 3\}$. Let $S$ be the Sylow 7-subgroup of $G$. Then $[x^3, S] \leq K_i = 1$ and therefore $S$ centralizes $x^3$. Hence $S$ acts on $F(x^3) \cap I_t$ for all $t \in \{1, 2, 3\}$. Thus (i) holds.

By Theorem 4.3 of [6], $F(x^3)$ is a subplane of $\pi$ of order $2^3$ and by Lemmas 3.11 (ii) and 3.12, $F(x^3)$ is a subplane of order $2^2$.

If we coordinatize $\pi$ by choosing $(0, 0)$ as $U_1$, $(0)$ as $U_2$, $(\infty)$ as $U_3$, $(1, 1)$ as $P$ which was defined in Lemma 3.13, then we get a semi-field $F$. In general, $F$ is not always isomorphic to $D$ and since $\pi$ is non-Desarguesian, $F$ is not a field. Thus $\pi$ is a semi-field plane coordinatized by $F$ and it also satisfies Hypothesis 2.3.

Lemma 3.14. Set $F_1 = \{d \mid d \in F, (d, 0) \in F(x^3)\}$, $F_2 = \{d \mid d \in F, (0, d) \in F(x^3)\}$ and $F_3 = \{d \mid d \in F, (d, 0) \in F(x^3)\}$. Then $F_1 = F_2 = GF(8)$ and $F_3 = GF(4)$.

Proof. Since $F(x)$ contains $(0, 0), (0), (\infty)$ and $(1, 1)$, it also contains $(1)$. By Lemma 3.13 and the definition of the coordinatization of $\pi$, we have the lemma.

Lemma 3.15. Let $N_1$, $N_2$ or $N_3$ be the right, middle or left nucleus, respectively. Then $N_1 = N_2 = N_3 = GF(8)$.

Proof. By Lemmas 3.12, we have $N_t = GF(8)$ for all $t \in \{1, 2, 3\}$. Furthermore, the multiplicative group $N_t^* = \{d \mid (d, 0) \in (1, 0)^{k_t}\}$ for $t = 1, 2$ and $N_3^* = \{d \mid (0, d) \in (0, 1)^{k_3}\}$ by the proof of Theorems 7.9 and 8.2 of [6]. Since
$K_1$ and $K_2$ are semi-regular on $I_3$ and $(1, 0)\in F(x^i)$, it follows from Lemma 3.13 that $N_1=N_2=F_1$. Similarly $N_3=F_2$. By Lemma 3.14, we have $N_1=N_2=N_3=GF(8)$.

**Lemma 3.16.** Set $N=N_1=N_2=N_3$ and $F_3^*=\langle \theta \rangle$.

(i) $N$ does not contain $\theta$ and $F$ is a right and left vector space over $N$ with a basis $\{1, \theta\}$.

(ii) For any $\xi \in F$, $(\xi \theta)\theta=\xi(\theta^2)$.

**Proof.** (i) follows immediately from Lemmas 3.14 and 3.15. Set $\xi=a+b\theta$ for $a, b \in N$. Then $(\xi \theta)\theta=((a+b\theta)\theta)\theta=(a\theta+(b\theta)\theta)\theta=(a\theta)\theta+(b\theta)\theta\theta=a\theta^2+b\theta^3$ because $a, b \in N=N_3$ and $\langle \theta \rangle=F_3^*$. Hence $(\xi \theta)\theta=a\theta^2+(b\theta)\theta^2=\xi(\theta^2)$. Thus (ii) holds.

**Lemma 3.17.** $\theta \in N$.

**Proof.** Let $\xi, \eta \in F$ and set $\xi=a+b\theta$, $\eta=c+d\theta$ for $a, b, c, d \in N$. Then, $(\xi \eta)\theta=((a+b\theta)(c+d\theta))\theta=(ac)\theta+((b\theta)c)\theta+(a(d\theta))\theta+((b\theta)(d\theta))\theta$. Similarly $\xi(\eta \theta)=a(c\theta)+(b\theta)(c\theta)+a((d\theta)\theta)+(b\theta)((d\theta)\theta)$. Since $a \in N=N_3$ and $c \in N=N_2$, we have $(a\theta)\theta=a(\theta^2)$, $(b\theta)c\theta=(b\theta)(c\theta)$ and $(a(d\theta))\theta=a((d\theta)\theta)$. Since $d \in N=N_2$, $((b\theta)(d\theta))\theta=(((b\theta)d)\theta)d\theta$ and by Lemma 3.16, $((b\theta)d)\theta^2=(b\theta)(d\theta)^2$, so that $((b\theta)(d\theta))\theta=((b\theta)d)\theta^2=\theta((d\theta)d\theta)=\theta((d\theta)\theta)$ as $d \in N=N_2=N_3$. Hence $(\xi \eta)\theta=\xi(\eta \theta)$ and so $\theta \in N_3=N$.

Proof of Lemma 2.

By Lemmas 3.16 (i) and 3.17, we obtain a contradiction and so the lemma holds.

### 4. The proof of Theorem 1

Throughout this section $\pi$ is a semi-field plane satisfying Hypothesis 2.3 and the following.

**Hypothesis 4.1.** $r \equiv 0 \pmod{4}$ and $u(\pi)\leq 4$.

**Lemma 4.2.** (i) $G$ is not solvable.

(ii) $u(\pi)=2, 3$ or 4.

(iii) There exists $i \in \{1, 2, 3\}$ such that $G$ is transitive on $I_i$.

**Proof.** By Theorem of [8], Theorem 6.3 of [9] and the lemma proved in §3, we have (i).

It follows from Kallaher's theorem [8] that $u(\pi)\neq 1$ and so (ii) holds.

If $u(\pi)=2$ or 3, we have (iii) by a similar argument as in the proof of Lemma 3.7. If $u=4$, we can apply Lemma 6.1 of [9] and (iii) follows.
Lemma 4.3. Let $S$ be a Sylow 2-subgroup of $G$ and set $\pi_0=F(S)$, $H=G(\pi_0)$. Then the following hold.

(i) $S\not\approx 1$ and $S$ is semi-regular on $\pi_0$.

(ii) $\pi_0$ is a Baer subplane of $\pi$.

(iii) $G'\cong PSL(2, q)$ for some even $q$. Moreover $H=O(G)N_G(S)$ and $|G:H|=q+1$.

Proof. By the Feit-Thompson theorem [2] and Lemma 4.1 (i), the order of $G$ is even and so $S\not\approx 1$. Let $z$ be an involution in the center of $S$. Then $F(z)$ is a Baer subplane of order $2^{r/2}$ and $S^{F(z)}$ is a collineation group. By Hypothesis 4.1, $2^{r/4}$ is not an integer. Therefore $S^{F(z)}=1$. Hence (i) and (ii) hold.

By Lemma 4.2 (ii), $G\not\approx H$ and clearly $H\not\supset S$. Hence $H$ is a strongly embedded subgroup of $G$. By a Bender's theorem [1] and by Corollary 3.2 of [4], (iii) holds.

Lemma 4.4. Set $\Delta=\pi_0\cap I_i$ and $\Gamma=\{\Delta^g|g\in G\}$. Then the following hold.

(i) $I_i=\cup_{\Delta^g\in \Gamma} \Delta^g$ and $\Delta^g\cap \Delta^h=\phi$ for distinct $\Delta^g$ and $\Delta^h$ in $\Gamma$.

(ii) Set $N=O(G)$. Then $G(\Delta)=H\supset N=G\Gamma$ and $G$ is doubly transitive on $\Gamma$.

Proof. By Lemma 4.3 (iii), $H=N\cdot N_G(S)$. Since $G(\pi_0)\leq G(\Delta)$ and $H$ is a maximal subgroup of $G$, we have $H=G(\Delta)$. Hence $G$ is doubly transitive on $\Gamma$ (See [1] §3). Since $N$ is a normal subgroup of $G$ and $N\leq G(\Delta)$, $N$ is contained in $G\Gamma$ and so $N=G\Gamma$ by Lemma 4.4 (iv). Thus (ii) holds.

Clearly $\Delta^g\subset I_i$ for all $g\in G$, hence $I_i=\cup_{\Delta^g\in \Gamma} \Delta^g$ by Lemma 4.2 (iii). Suppose $\Delta^g\not=\Delta^h$ and $\Delta^g\cap \Delta^h=\phi$ and set $g=xy^{-1}$. Then $\Delta^g=\Delta$ and $\Delta^g\cap \Delta=\phi$. By Lemma 4.3 (i), $S$ and $S^{g}$ fix $\Delta^g\cap \Delta$ pointwise. By (ii), $G=\langle N, S, S^g, G(\Delta)\cap G(\Delta^g)\rangle$. Hence $G$ fixes $\Delta^g\cap \Delta$ as a set, contrary to Lemma 4.2 (iii). Thus (i) holds.

Lemma 4.5. $q^2=2^r$ and $|\Delta|=q-1$, $|\Gamma|=q+1$.

Proof. By Lemmas 4.3 (iii) and 4.4 (ii), $|\Gamma|=|G: H|=q+1$ and by Lemma 4.4 (i) $|\Gamma|=|I_i|/|\Delta|=(2^r-1)/|\Delta|$. On the other hand $|\Delta|=2^{r/2}-1$ since $\pi_0$ is a Baer subplane of $\pi$. Hence $q^2=2^r$ and $|\Delta|=q-1$.

Lemma 4.6. $\pi_0(=F(S))$ is a Desarguesian projective plane of order $q$ and the number of $N_G(S)$-orbits on $\Phi(\pi)\cap \pi_0$ is one or three.

Proof. Let $\Lambda$ be a $G$-orbit on $\Phi(\pi)$ and supose $\Lambda\cap \pi_0=\phi$. Let $P\in \Lambda\cap \pi_0$. Then $G_P\geq S$. Hence $|\Lambda|=|G: G_P|=1$ (mod 2) and moreover $N_G(S)$ is transitive on $\Lambda\cap \pi_0$ by Theorem 3.5 of [11]. Since $|\Phi|=1$ (mod 2)
and \( u = u(\pi) \leq 4 \), the number of \( G \)-orbits \( \Lambda \) on \( \Phi \) such that \( \Lambda \cap \pi_0 \neq \phi \) is one or three. Hence the number of \( N_G(S) \)-orbits on \( \pi_0 \cap \Phi \) is one or three.

Since the order of \( \pi_0 \) is \( 2^{r/2} \) and \( 2^{r/4} \) is not an integer, the autotopism group of \( \pi_0 \) is of odd order. By Theorem 6.3 of [9] and Theorem of [8], \( \pi_0 \) is a Desarguesian plane of order \( q \).

By Lemma 4.3, \(|G:G(\pi_0)|=q+1\). We set \( \{\pi_0^s\mid g \in G\} = \{\pi_0, \pi_1, \ldots, \pi_q\} \). Then the following lemma holds.

**Lemma 4.7.** Set \( N = \mathcal{O}(G) \). Then

(i) \( N_{\pi_s} \) acts faithfully on \( \pi_t \) and \( |N_{\pi_s}| \mid (q-1)^2(r/2) \) for all \( s, t \) \((s \neq t)\) and

(ii) \( N_{\pi_s} \) is a normal subgroup of \( N \) and \( [N_{\pi_s}, N_{\pi_t}] = 1 \) for all \( s, t \) \((s \neq t)\).

**Proof.** By Lemma 4.4 (ii), \( N \) acts on \( \pi_t \) and so \( N_{\pi_t} \) is a normal subgroup of \( N \). By Lemma 4.3 (ii), \( \pi_s \) and \( \pi_t \) are Baer subplanes of \( \pi \), so that \( N_{\pi_s} \cap N_{\pi_t} = 1 \). Hence \( N_{\pi_s} \) acts faithfully on \( \pi_t \) and \( [N_{\pi_s}, N_{\pi_t}] \leq N_{\pi_s} \cap N_{\pi_t} = 1 \). Moreover \( |N_{\pi_s}| \mid (q-1)^2(r/2) \) since \( \pi_t \) is a Desarguesian plane of order \( q \).

**Lemma 4.8.** Assume \( N_{\pi_0} \neq 1 \) and let \( P \) be a minimal normal subgroup of \( N_{\pi_0} \) and let \( p \) be a prime dividing the order of \( P \). Then a Sylow \( p \)-subgroup of \( N_{\pi_0} \) is cyclic and \( P \) is a normal subgroup of \( N \). Moreover \( P \) is isomorphic to \( Z_p \).

**Proof.** Let \( P \) be a Sylow \( p \)-subgroup of \( N_{\pi_0} \). Since \( F(Q) = \pi_0 \), \( Q \) is semi-regular on \( \pi_t - \pi_0 \) for \( t \neq 0 \). By Lemma 2.2 (i) and Theorem 5.4.10 of [3], \( Q \) is cyclic. Hence, by Lemma 4.7 (ii), we have the lemma.

**Lemma 4.9.** Let \( P \) be as in Lemma 4.8. Then the following hold.

(i) Set \( L = \langle P^t \mid g \in G \rangle \). Then \( L \) is a normal subgroup of \( G \) and is an elementary abelian \( p \)-group.

(ii) \( p \not| r \) and \( |L| \leq p^3 \).

**Proof.** (i) follows immediately from Lemma 4.8. Clearly \( L \leq N \). Set \( X = N_{\pi_0} \). Since \( X \cap L = P \) and \( L/P = LX/X \leq N/X = N^{r_0} \), \( |L/P| \) is at most \( p^3 \). Moreover \( |L/P| \leq p^3 \) if \( p \not| r \). Therefore it suffices to show \( p \not| r \). Assume \( p \mid r \). Since \( H \) normalizes \( X \), \( P \) is a normal subgroup of \( H \) and so \( L \) contains at least \( q+1 \) subgroups of order \( p \). Hence \( q+1 \leq (p^4-1)/(p-1) = p^3 + p^2 + p + 1 \). On the other hand \( p \mid r/2 \) and \( q = 2^{r/2} \), so that \( (r/2)^3 + (r/2)^2 + r/2 + 1 \geq 2^{r/2} + 1 \). From this \( r = 6 \) or \( 10 \) and \( p = r/2 \). But \( p \not| q-1 \) for \( r = 6 \) or \( 10 \). Therefore, \( |L/P| \leq p \) and so \( q+1 \leq (p^2-1)/(p-1) = p+1 \leq 6 \), a contradiction. Thus \( p \not| r/2 \).

**Lemma 4.10.** \( N_{\pi_0} = 1 \).

**Proof.** Assume \( N_{\pi_0} \neq 1 \) and let \( P \), \( L \) be as in Lemma 4.8, 4.9, respectively.
If \(|C_6(L)|\) is even, all Sylow 2-subgroups of \(G\) are contained in \(C_6(L)\) by Lemma 4.3 (iii). Hence \(<S^y_1|y \in G>\) acts on \(F(P) (=\pi_0)\), which is contrary to \(G(\pi_0)=H\). Therefore \(|C_6(L)|\) is odd. In particular \(S\) is isomorphic to a subgroup of \(G/C_6(L)\).

By Lemmas 4.3 and 4.9, \((G/C_6(L))' \leq SL(3, p)\) and \(|G/C_6(L):(G/C_6(L))'\) is odd. Hence \(S\) is isomorphic to a subgroup of \(SL(3, p)\). Since a Sylow 2-subgroup of \(SL(3, p)\) is semi-dihedral or wreathed, \(S\) is an elementary abelian group of order 4 and so \(q=2^r\). Hence \(r=4 \equiv 0 \pmod{4}\), a contradiction.

**Lemma 4.11.** Let \(G^{(\omega)}\) denote the last term of the derived series of \(G\). Set \(M=G^{(\omega)}\). Then \(M=PSL(2, q)\).

Proof. Let \(X\) be a subgroup of \(G\) generated by all Sylow 2-subgroups of \(G\). By Lemma 4.3 (iii), \(X \leq M\) and \(|M/X|\) is odd. It follows from the Feit-Thompson theorem that \(M=M^{(\omega)} \leq X\) and hence \(X=M\). By Lemmas 4.4 (ii) and 4.10, \([S, N] \leq N \cap G_\chi=N_\chi=1\), so that \(N\) centralizes \(X (=M)\) and \(M \cap N = Z(M)\), \(M/Z(M)=PSL(2, q)\). By a property of \(PSL(2, q)\), \(M \cong PSL(2, q)\).

**Lemma 4.12.** (i) Let \(t \in \{1, 2, 3\}, P \in I_t\) and let \(X\) be a subgroup of \(G\). Then \(|F(X) \cap I_t|=2^a+1\) for an integer \(a \geq 1\).

(ii) \(M\) is transitive on \(I_t\) and \(|M_P|=q\) for \(P \in I_t\). Here \(i\) is the integer defined in Lemma 4.2 (iii).

Proof. Let \(A\) be the full collineation group of \(\pi\) and set \(T_1=A_{(U_3, l_3)}, T_2=A_{(U_2, l_2)}, T_3=A_{(U_1, U_2)}\). Since \(U_3\) is a translation point and \(l_1\) is a translation line, \(T_1,T_2,T_3 \cong \mathbb{Z}_2\) and \(XT_1\) is a transitive linear group on \(I_1\). Since \((XT_1)_P=\mathcal{X}\), we have (i) by Lemma 2.1.

Let \(\{\Delta_t, \ldots, \Delta_u\}\) be the set of \(M\)-orbits on \(I_t\). Since \(G\) is transitive on \(I_t\) and \(G \not\subset M\), \(|\Delta_t|=\cdots=|\Delta_u|=1 \pmod{2}\). Let \(P \in \Delta_i\) and set \(M_P=CS\) with \(C \leq Z_\chi\) and \(|N_\chi(S): M_P|=k\). As \(M=PSL(2, q)\), \(2q-1\) and \(F(M_P) \cap \Delta_v=\phi\) for each \(v \in \{1, \ldots, m\}\).

Assume \(C \neq 1\). Then \(|N_\chi(C)|=2(q-1)\) as \(M=PSL(2, q)\). By Lemma 2.1, \(|F(C) \cap I_t|=m \times 2(q-1)\times |S|=2mk\) and applying (i), we have \(2mk=2^a-1\) for an integer \(a \geq 1\), a contradiction. Thus \(C=1\) and \(|M_P|=q\). Therefore \(|P^M|=|M|: M_P|=q^2-1\) and (ii) follows.

**Lemma 4.13.** Let \(j \in \{1, 2, 3\} \setminus \{i\}\) and \(P \in I_j\). Then \(q \mid |M_P|\).

Proof. By Lemma 4.3 (i), it suffices to consider the case that \(|M_P|=1 \pmod{2}\). As \(M=PSL(2, q)\), \(M \leq Z_\chi\). Since \(|I_j|=q^2-1 \geq |P^M|=|M:M_P|\) and \(P^M \cap F(S)=\phi\), we have \(M_P=Z_\chi+1\) and \(|l_j-P^M|=|F(S) \cap I_j|=q+1\). Hence \(F(S) \cap I_j=F(M) \cap I_j\). Therefore \(|F(M_P)|=q+1+\frac{2(q+1) \times 1}{|M_P|}=q+3\) by Lem-
Lemma 4.14. \( M \) is transitive on \( \bar{l}_j \) and \( M_P \) is a Sylow 2-subgroup of \( M \) for each \( j \in \{1, 2, 3\} \) and \( P \in \bar{l}_j \).

Proof. By Lemma 4.12 (ii), we may assume \( j \in \{1, 2, 3\} - \{i\} \). First we argue that \( F(M) \cap \bar{l}_j = \phi \). Set \( \Delta = F(M) \cap \bar{l}_j \) and assume \( \Delta \neq \phi \). Let \( \pi_0 \) be as defined in Lemma 4.6 and set \( N_M(S) = DS \) with \( D = Z_{q-1} \). By Lemma 4.12 (ii), \( D^o \cong D \) and \( \pi_0 \cap F(D) \supset \Delta \). Since \( \pi_0 \) is a Desarguesian plane of order \( q \), \( F(D) \supseteq \pi_0 \cap I_j \) by Lemma 2.2 (ii). Therefore, by Lemmas 2.1 and 4.13, \( |F(D) \cap \bar{l}_j| = |\Delta| + 2(q-1 - |\Delta|) = 2(q-1) - |\Delta| \). Applying Lemma 4.12 (i), \( |\Delta| = 2^{a-1} \) and \( 2(q-1) - |\Delta| = 2^b - 1 \) for integers \( a, b \geq 1 \), hence \( 2q = 2^a + 2^b \). However, as \( |\Delta| < |\pi_0 \cap I_j| = q-1 < |F(D) \cap I_j| = 2^b - 1 \), we have \( 2^a < q < 2^b \). This is a contradiction. Thus \( F(M) \cap I_j = \phi \).

Let \( \{\Delta_1, \ldots, \Delta_m\} \) be the set of \( M \)-orbits on \( I_j \). By Lemma 4.3, \( |\Delta_t| \big| q^2 - 1 \) for each \( t \). Assume \( |M_p| = q \) for some \( P \in \bar{l}_j \). We may assume \( P \in \Delta_1 \) and \( M_P \supseteq S \). Set \( M_P = CS \) with \( 1 \neq C \leq Z_{q-1} \). By a similar argument as in the last paragraph \( F(S) \cap I_j \subset F(C) \cap I_j \) and so \( F(C) \cap \Delta_t = \phi \) for each \( t \). Hence \( |F(C) \cap \Delta_t| = 2 \times |F(S) \cap \Delta_t| = 2(q-1) - |\Delta_t| \) and so \( |F(C) \cap I_j| = 2q \), contrary to Lemma 4.12 (i). Thus \( |M_P| = q \) and \( M \) is transitive on \( I_j \).

Let \( X \) be the full collineation group of \( \pi \) and set \( A = X(t_1, t_2) \), \( B = X(u_3, u_4) \) and \( T = AB \). Since \( U_2 \) is a translation point and \( I_1 \) is a translation line, \( A \) and \( B \) are elementary abelian normal 2-subgroups of \( X \) of order \( q^4 \). Hence \( T \) is a normal 2-subgroup of \( X \).

Lemma 4.15. (i) \( T \) is a nonabelian normal 2-subgroup of \( X \).
(ii) \( C_T(x) = 1 \) for any element \( x \neq 1 \) of \( M \) of odd order.

Proof. If \( T \) is abelian, \( T_P = 1 \) for \( P \in I_1 \) because \( A \) acts transitively on the set of points not incident with the line \( I_1 \). Hence \( |T| = |T: T_P| = q^4 + q^2 + 1 - (q^2 + 1) = q^4 \) and so \( T = A = B \), a contradiction. Thus (i) holds.

Assume \( C_T(x) \neq 1 \) and let \( t \) be an involution in \( C_T(x) \). Then, there exist element \( t_1 \in A \) and \( t_2 \in B \) such that \( t = t_1t_2 \). By Lemma 4.14, \( F(X) = \{U_1, U_2, U_3, l_1, l_2, l_3\} \) and so \( t \) acts on \( \{U_1, U_2, U_3\} \). Since \( F(X) = \{U_3, l_1\} \), it follows that \( (U_2)^t \in I_1 \) and \( (U_3)^t \in U_1 \). Hence we have \( (U_1)^t = U_1 \) and \( (U_2)^t = U_2 \) and so \( F(t_2) = F(t_3) \supseteq \{U_2, U_3\} \). Therefore \( t_2 \in X(u_3, t_1) \leq X(t_4, t_5) \), which implies \( t \in X(t_4, t_5) \). However, as \( (U_3)^t = U_1 \), this is a contradiction. Thus \( C_T(x) = 1 \).

Proof of Theorem 1.

Since \( 3 \big| |M| = |PSL(2, q)| \), there exists an element \( x \in M \) of order 3.
By Lemma 4.15 (ii), $C_{\tau}(x) = 1$. Applying Theorem 8.2 of [5] to the group $MT$, $T$ is an abelian 2-group, which is contrary to Lemma 4.15 (i). Thus we have the theorem.

References