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ON THE BALAYAGE CONSTANT FOR LOGARITHMIC POTENTIALS

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In the paper dedicated to Professor Kiyoshi Noshiro ([2]), we studied on the balayage for logarithmic potentials. On the plane, consider the logarithmic potential

$$U^{\mu}(P)=\int \log \frac{1}{PQ}\, d\mu(Q)\,,$$

μ being a positive measure, P and Q any points and PQ the distance between P and Q. The measure μ is not always assumed with compact support, but will be bounded to positive measures whose logarithmic potentials are never $-\infty$. The total mass of such measures is naturally finite, and the logarithmic potential of such measures is superharmonic in the whole plane and harmonic outside the support of the measure. Remember the definition of the logarithmic capacity of a compact set F. Putting

$$V=\inf_{\mu}\sup_{P} U^{\mu}(P) \quad\text{and}\quad W=\inf_{\mu}\iint \log\frac{1}{PQ}\, d\mu(Q)d\mu(P)$$

for any positive measure μ supported by F with total mass 1, we have always $V=W$. The logarithmic capacity is given by

$$C(F)=e^{-V}=e^{-W}$$

if $V=W<\infty$ and by $C(F)=0$ if $V=W=\infty$. A Borelian set E is said of logarithmic capacity positive when it contains a compact set of logarithmic capacity positive. The results published in the paper ([2]):

Theorem. *Let F be any closed set (compact or non) of logarithmic capacity positive and μ be any positive measure with total mass* 1. *There exist a positive measure μ' supported by F with total mass* 1 *and a non-negative constant γ_{μ} such that*

(1) $U^{\mu'}(P)=U^{\mu}(P)+\gamma_{\mu}$ *on F with a possible exception of a set of logarithmic capacity zero, and*

(2) $U^{\mu'}(P)\leqq U^{\mu}(P)+\gamma_{\mu}$ *everywhere.*

We shall call μ' a balayaged measure of μ onto F and γ_μ a balayage constant. We can construct a balayaged measure such that the reciprocal relation always holds:

(3) $\int(U^{\mu'}-\gamma_\mu)d\nu=\int(U^{\nu'}-\gamma_\nu)d\mu$ *for any positive measure* μ *with total mass* 1, *any positive measure* ν *of finite logarithmic energy with total mass* 1, *their balayaged measures* μ' *and* ν' *and their balayage constants* γ_μ *and* γ_ν.

When the balayage is done so as to hold the reciprocal relation, a balayaged measure is always unique.

DEFINITION. Let F be any closed set. A point P is called a regular point of F if the balayaged measure ε' of the Dirac measure ε at P onto F (keeping the reciprocal relation) coincides with ε and the balayage constant γ_ε reduces to zero.

Theorem. *Two following expressions are equivalant.*

[A] *A point P is a regular point of F.*

[B] *Let μ be any positive measure with total mass* 1, *μ' the balayaged measure of μ onto F and γ_μ the balayage constant. Then, it holds that*

$$U^{\mu'}(P)=U^{\mu}(P)+\gamma_\mu\,.$$

The paper is devoted itself to answer to the question:

"Is there a case when the balayage constant γ_μ always vanishes?" It is easily seen that, if the complement of F is a bounded open set, the balayage constant γ_μ always reduces to zero. The problem consists in the case F is not so. We shall insist that the balayage constant γ_μ vanishes whenever F has a little expanse at the infinity.

DEFINITION. Let E be a set and P_0 any point. E is said *thin at a point* P_0 if P_0 is an outer point of E or if there exists a positive measure μ such that

$$U^{\mu}(P_0)<\varliminf_{P\to P_0}U^{\mu}(P)\qquad(P\in E)\,.$$

Theorem 1. *Let E be a closed set and P_0 any point. Two following statements are equivalent:*

[1] *P_0 is a regular point of F.*

[2] *F is not thin at P_0.*

Proof. First, let us prove that P_0 is a regular point of F if F is not thin at P_0.

Lemma. *Let E be any set not thin at a point P_0 and μ any positive measure. When e is a countable union of compact sets of logarithmic capacity zero, we have*

$$U^{\mu}(P_0)=\varliminf_{P\to P_0}U^{\mu}(P)\qquad(P\in E)$$

$$= \varliminf_{P \to P_0} U^{\mu}(P) \qquad (P \in E-e).$$

The result is obvious when $U^{\mu}(P_0)=\infty$. Suppose that $U^{\mu}(P_0)<\infty$. Let

$$e_n = \left\{P;\ P\in e,\ PP_0 \geqq \frac{1}{n}\right\}$$

and ν_n a positive measure supported by e_n such that $U^{\nu_n}(P)\equiv\infty$ on e_n and $U^{\nu_n}(P_0)<\infty$. Taking the total mass of ν_n sufficiently small beforehand, we can find a positive measure ν supported by $e-\{P_0\}$ such that $U^{\nu}(P)\equiv\infty$ on e and $U^{\nu}(P_0)<\infty$. As the equality

$$U^{\nu}(P_0) = \varliminf_{P\to P_0} U^{\nu}(P)$$

holds for points P of E, we have the same for points P of $E-e$ also. Then, we have

$$\begin{aligned} U^{\mu}(P_0)+U^{\nu}(P_0) &= U^{\mu+\nu}(P_0) = \varliminf_{P\to P_0} U^{\mu+\nu}(P) && (P\in E)\\ &= \varliminf_{P\to P_0} U^{\mu+\nu}(P) && (P\in E-e)\\ &\geqq \varliminf_{P\to P_0} U^{\mu}(P)+\varliminf_{P\to P_0} U^{\nu}(P) && (P\in E-e)\\ &\geqq U^{\mu}(P_0)+U^{\nu}(P_0), \end{aligned}$$

thus the result.

Thereupon, let λ' be the balayaged measure of any circular measure λ (with total mass 1) onto F and γ_λ the balayage constant. As the equality

$$U^{\lambda'}(P) = U^{\lambda}(P)+\gamma_\lambda$$

holds on F with a possible exception of a set e (a F_σ) of logarithmic capacity zero, we have

$$\begin{aligned} U^{\lambda'}(P_0) &= \varliminf_{P\to P_0} U^{\lambda'}(P) \qquad (P\in F, \text{ therefore } \in F-e)\\ &= U^{\lambda}(P_0)+\gamma_\lambda . \end{aligned}$$

Hence, λ_1 and λ_2 being any concentric circular measure (with total mass 1), λ_1', λ_2', ε' the balayaged measures of λ_1, λ_2 and the Dirac measure ε at P_0 onto F respectively and γ_{λ_1}, γ_{λ_2}, γ_ε their balayage constants respectively, we have

$$\begin{aligned} \int U^{\lambda_1-\lambda_2}d\varepsilon &= \int U^{\lambda_1}d\varepsilon - \int U^{\lambda_2}d\varepsilon\\ &= \int (U^{\lambda_1'}-\gamma_{\lambda_1})d\varepsilon - \int (U^{\lambda_2'}-\gamma_{\lambda_2})d\varepsilon\\ &= \int (U^{\varepsilon'}-\gamma_\varepsilon)d\lambda_1 - \int (U^{\varepsilon'}-\gamma_\varepsilon)d\lambda_2 \end{aligned}$$

$$= \int U^{\varepsilon'} d\lambda_1 - \int U^{\varepsilon'} d\lambda_2 = \int U^{\lambda_1 - \lambda_2} d\varepsilon',$$

which induces the equality

$$\int f\, d\varepsilon = \int f\, d\varepsilon'$$

for any non-negative continuous function f with compact support. Accordingly, we have $\varepsilon = \varepsilon'$ and $\gamma_\varepsilon = 0$, thus the result.

Next, let us prove that P_0 is an irregular point of F if F is thin at P_0.

Lemma 1. *Let both μ_1 and μ_2 be positive measures with compact support and with total mass* 1. *For any positive number δ, the function*

$$H(P) = \inf \{U^{\mu_1}(P),\ U^{\mu_2}(P) + \delta\}$$

is the logarithmic potential of a positive measure with compact support and with total mass 1.

In fact, as there holds

$$\begin{aligned} &\lim_{P\to\infty} \{U^{\mu_2}(P) - U^{\mu_1}(P)\} \\ &= \lim_{P\to\infty} \left\{\int \log \frac{PO}{PQ} d\mu_2(Q) - \int \log \frac{PO}{PQ} d\mu_1(Q)\right\} \\ &= \log 1 - \log 1 = 0, \end{aligned}$$

we have

$$H(P) = U^{\mu_1}(P)$$

outside an enough large disc D_0 centered at the origine. D_1 and D_2 being enough large concentric discs and $D_0 \subset D_1 \subset D_2$, take the Riesz decompositions

$$H(P) = U^{\nu_1}(P) + h_1(P) \qquad \text{in } D_1$$

and

$$H(P) = U^{\nu_2}(P) + h_2(P) \qquad \text{in } D_2.$$

both ν_1 and ν_2 are positive measures with compact support ($\subset D_0$) and with total mass 1, and $h_1(P)$ and $h_2(P)$ are harmonic in D_1 and D_2 respectively. Suppose that a disc centered at a point P_0 contains D_0 and is contained in D_1. At such points P_0 we have $h_1(P_0) = h_2(P_0)$ as is easily seen. Hence, $h_1(P)$ is able to provide the harmonic continuation outward D_1. Therefore, $h_1(P)$ (naturally ν_1 also) is independent upon D_1. Accordingly, we have

$$H(P) = U^{\nu}(P) + h(P)$$

in the whole plane, where ν is a positive measure with compact support and with total mass 1 and $h(P)$ is harmonic in the whole plane. Then, we have

$$\lim_{P\to\infty}\{H(P)-U^{\nu}(P)\}=\lim_{P\to\infty}\{U^{\mu_1}(P)-U^{\nu}(P)\}=0,$$

so $h(P)\equiv 0$ in the whole plane.

Lemma 2. *Let F be any closed set of logarithmic capacity positive, μ_n and μ positive measures with total mass 1, μ'_n and μ' their balayaged measures onto F and γ_{μ_n} and $\gamma_{\mu}(\geqq 0)$ their balayage constants. If*

$$U^{\mu_n}(P)\uparrow U^{\mu}(P)$$

everywhere, there holds

$$U^{\mu'_n}(P)-\gamma_{\mu_n}\uparrow U^{\mu'}(P)-\gamma_{\mu}$$

everywhere.

It is since, for any circular measure λ with total mass 1, its balayaged measure λ' onto F and its balayage constant $\gamma_{\lambda}(\geqq 0)$, we have

$$\begin{aligned}\int(U^{\mu'_n}-\gamma_{\mu_n})d\lambda&=\int(U^{\lambda'}-\gamma_{\lambda})d\mu_n\\&=\int U^{\mu_n}d\lambda'-\gamma_{\lambda}\uparrow\int U^{\mu}d\lambda'-\gamma_{\lambda}\\&=\int(U^{\lambda'}-\gamma_{\lambda})d\mu=\int(U^{\mu'}-\gamma_{\mu})d\lambda\,.\end{aligned}$$

Lemma 3. *Let F be any closed set of logarithmic capacity positive, P_0 a regular point of F, ε the Dirac measure at P_0, λ_n the circular measure with total mass 1 centered at P_0 with radius $1/n$, λ'_n the balayaged measure of λ_n onto F and $\gamma_{\lambda_n}(\geqq 0)$ the balayage constant. When $n\to\infty$, we have $\lambda'_n\to\varepsilon$ and $\gamma_{\lambda_n}\to 0$.*

In fact, as

$$U^{\lambda_n}(P)\uparrow\log\frac{1}{PP_0}$$

everywhere, we have

$$U^{\lambda'_n}(P)-\gamma_{\lambda_n}\uparrow\log\frac{1}{PP_0}$$

everywhere. For any concentric circular measures λ_1 and λ_2 with total mass 1, we have

$$\int U^{\lambda_1}d\lambda'_n-\gamma_{\lambda_n}\uparrow U^{\lambda_1}(P_0)$$

and

$$\int U^{\lambda_2}d\lambda'_n-\gamma_{\lambda_n}\uparrow U^{\lambda_2}(P_0)\,,$$

hence we have

$$\int f\, d\lambda'_n \rightarrow f(P_0)$$

for any non-negative continuous function f with compact support. So, we have $\lambda'_n \rightarrow \varepsilon$ and for a point $P(\neq P_0)$ of F

$$\gamma_{\lambda_n} = U^{\lambda'_n}(P) - U^{\lambda_n}(P) \rightarrow \log \frac{1}{PP_0} - \log \frac{1}{PP_0} = 0 .$$

Now, let F be a closed set thin at a point P_0. We are going to prove that P_0 is an irregular point of F. Unless it is an outer point of F, there exists a positive measure μ such that

$$U^{\mu}(P_0) < \varliminf_{P \rightarrow P_0} U^{\mu}(P) \qquad (P \in F) .$$

We may suppose that μ is with compact support and with total mass 1, $U^{\mu}(P_0) < a$ and $U^{\mu}(P) \geqq a$ at each point P of the set $\{P;\, P \in F, PP_0 < r, P \neq P_0\}$. Take a positive number δ such that

$$\log \frac{1}{PP_0} \leqq U^{\mu}(P) + \delta$$

in the set $\{P; P \in F, PP_0 \geqq r\}$. That is possible since the support of μ is compact, $U^{\mu}(P)$ is lower semi-continuous and

$$\lim_{P \rightarrow \infty} \left(U^{\mu}(P) - \log \frac{1}{PP_0} \right) = 0 .$$

The function

$$\inf \left(a + \delta,\ \log \frac{1}{PP_0} \right)$$

is the logarithmic potential of a circular measure ν_1 with total mass 1 centered at P_0, and the function

$$\inf \left(\log \frac{1}{PP_0},\ U^{\mu}(P) + \delta \right)$$

is the logarithmic potential of a positive measure ν_2 with compact support and with total mass 1. Let us turn attention to

$$U^{\nu_1}(P) \leqq U^{\nu_2}(P) \qquad \text{in } F - \{P_0\}$$

including

$$U^{\nu_1}(P_0) > U^{\nu_2}(P_0) .$$

We insist that

$$U^{\nu_1}(P) \leqq U^{\nu_2}(P)$$

at all regular point P of F, therefore P_0 is an irregular point of F. On the contrary, suppose that P_0 is a regular point of F. λ_n being a circular measure with total mass 1 centered at P_0 with radius $1/n$, λ'_n the balayaged measure of λ_n onto F and $\gamma_{\lambda_n}(\geqq 0)$ the balayage constant, there holds

$$\begin{aligned}
U^{\nu_1}(P_0) &= \int \log \frac{1}{P_0Q}\, d\nu_1(Q) = \lim_{n\to\infty} \int (U^{\lambda'_n} - \gamma_{\lambda_n}) d\nu_1 \\
&= \lim_{n\to\infty} \left(\int U^{\nu_1} d\lambda'_n - \gamma_{\lambda_n}\right) \leqq \lim_{n\to\infty} \left(\int U^{\nu_2} d\lambda'_n - \gamma_{\lambda_n}\right) \\
&= \lim_{n\to\infty} \left(\int U^{\lambda'_n} - \gamma_{\lambda_n}\right) d\nu_2 = \int \log \frac{1}{P_0Q}\, d\nu_2(Q) \\
&= U^{\nu_2}(P_0)\,,
\end{aligned}$$

which is a contradiction.

DEFINITION. A set E is said thin at the infinity when E is bounded or when there exists a positive measure μ with total mass 1 such that for a point P_0

$$\varliminf_{P\to\infty}\left(U^{\mu}(P) - \log \frac{1}{PP_0}\right) > 0 \qquad (P \in E)\,.$$

In this time, we have

$$\varliminf_{P\to\infty}\left(U^{\mu}(P) - \log \frac{1}{PP'}\right) > 0 \qquad (P \in E)$$

for any point P', since

$$\lim_{P\to\infty}\left(\log \frac{1}{PP_0} - \log \frac{1}{PP'}\right) = 0\,.$$

Theorem 2. *Let E be any set, P_0 any point and E' the inversion of E with respect to the circle centered at P_0 with radius R. E is thin at the infinity, which is the same to say that E' is thin at P_0.*

Proof. We have

$$E' = \{P';\, P_0P \cdot P_0P' = R^2, \quad \arg \overrightarrow{P_0P} = \arg \overrightarrow{P_0P'}, \quad P \in E\}$$

and

$$\frac{P_0Q'}{P_0P} = \frac{P_0P'}{P_0Q} = \frac{P'Q'}{PQ} \qquad (P,\, Q \in E \text{ and } P',\, Q' \in E')\,.$$

Let μ be any positive measure with total mass 1 which charges no positive mass at P_0 and μ' the positive measure (with total mass 1) defined by $d\mu'(Q') = d\mu(Q)$. Then, there holds

$$U^{\mu}(P)-\log\frac{1}{PP_0}=U^{\mu'}(P')-U^{\mu'}(P_0) \qquad (P\in E \text{ and } P'\in E'),$$

which naturally induces the result.

Theorem 3. *Whenever a closed set F is not thin at the infinity, the balayage constant of the Dirac measure at any point onto F always reduces to zero.*

Proof. Let ε be the Dirac measure at a point P_0, ε' the balayaged measure of ε onto F and $\gamma_\varepsilon(\geqq 0)$ the balayage constant. If P_0 is a regular point of F, we have $\varepsilon'=\varepsilon$ and $\gamma_\varepsilon=0$. If P_0 is an irregular point of F, ε' is a positive measure supported by $F-\{P_0\}$ with total mass 1 and there holds

$$U^{\varepsilon'}(P)=\log\frac{1}{PP_0}+\gamma_\varepsilon$$

on F with a possible exception of the countable union e of compact sets of logarithmic capacity zero. We are going to prove $\gamma_\varepsilon=0$. Let F' and e' be the inversion of F and e with respect to the circle centered at P_0 with radius R respectively. We have

$$\frac{P_0Q'}{P_0P}=\frac{P_0P'}{P_0Q}=\frac{P'Q'}{PQ} \qquad (P,\, Q\in F,\, e \text{ and } P',\, Q'\in F',\, e')$$

and e' is the countable union of compact sets of logarithmic capacity zero. Denoting by ε^* the positive measure defined by $d\varepsilon^*(Q')=d\varepsilon'(Q)$ supported by F' with total mass 1, we have

$$\begin{aligned}&\varliminf_{P\to\infty}\left(U^{\varepsilon'}(P)-\log\frac{1}{PP_0}-\gamma_\varepsilon\right) \qquad (P\in F)\\ &=\varliminf_{P\to\infty}\left(U^{\varepsilon^*}(P')-U^{\varepsilon^*}(P_0)-\gamma_\varepsilon\right) \qquad (P'\in F'),\end{aligned}$$

F' being not thin at P_0,

$$\begin{aligned}&=\varliminf_{P'\to P_0}\left(U^{\varepsilon^*}(P')-U^{\varepsilon^*}(P_0)-\gamma_\varepsilon\right) \qquad (P'\in F'-e')\\ &=0,\end{aligned}$$

while

$$U^{\varepsilon^*}(P_0)=\varliminf_{P'\to P_0}U^{\varepsilon^*}(P') \qquad (P'\in F' \text{ or } P'\in F'-e').$$

Thus, we have $-\gamma_\varepsilon=0$.

Corollary. *Let F be a closed set not thin at the infinity. Suppose that μ is a positive measure with total mass* 1 *whose logarithmic potential is finite and continuous on F, or is the increasing limit of its restrictions μ_n whose logarithmic potentials are finite and continuous on F. Then, the balayage constant of such* *measures μ onto F is always equal to zero.*

For instance, suppose that $U^{\mu}(P)$ is finite and continuous on F. Take a point P_0 where μ charges no positive mass. Denote by μ' the balayaged measure of μ onto F, by γ_{μ} the balayage constant and by F' the inversion of F with respect to the circle centered at P_0 with R. Then, we have

$$\begin{aligned} & U^{\mu'}(P)-U^{\mu}(P)-\gamma_{\mu} \qquad\qquad (P\in F) \\ &= \{U^{\mu^{*\prime}}(P')-U^{\mu^{*\prime}}(P_0)\}-\{U^{\mu^*}(P')-U^{\mu^*}(P_0)\}-\gamma_{\mu} \\ &\qquad\qquad\qquad\qquad (P'\in F'), \end{aligned}$$

where $\mu^{*'}$ and μ^* are the measures defined by the inversion of μ' and μ respectively, and their total mass both are equal to 1. Observing that $U^{\mu}(P)$ on F, $U^{\mu^*}(P')$ on F', both are finite and continuous, the proof is gone forward alike to the theorem.

REMARK. The condition of Corollary is satisfied in case a positive measure μ with total mass 1 is such that
(1) μ is of finite energy:

$$\int U^{\mu} d\mu = \iint \log \frac{1}{PQ}\, d\mu(Q)\, d\mu(P) < \infty ,$$

more generally,
(2) μ charges no positive mass on the set $\{P;\ U^{\mu}(P)=\infty\}$.

Finally, we should like to terminate the paper by giving a few words on closed sets that support the equilibrium measure.

DEFINITION. Let F be a closed set. A positive measure λ supported by F with total mass 1 is called *the equilibrium measure on* F when $U^{\lambda}(P)=V$ (a constant) on F with a possible exception of a set of logarithmic capacity zero and $\leqq V$ in the whole plane. As is well-known, every compact set F of logarithmic capacity positive supports the equilibrium measure, which is unique. In that case, the constant value V of the equilibrium potential is equal to

$$V_F = \inf_{\mu} \sup_{P} \int \log \frac{1}{PQ}\, d\mu(Q)$$

and

$$W_F = \inf_{\mu} \iint \log \frac{1}{PQ}\, d\mu(Q)\, d\mu(P) ,$$

sup taken in the whole plane and inf taken with respect to positive measures μ supported by F with total mass 1.

DEFINITION. A Borelian set E is said *of logarithmic capacity positive* when it contains a compact set of logarithmic capacity positive, otherwise is said *of logarithmic capacity zero*. Further, E is said *of logarithmic capacity finite* when the logarithmic capacity of all the compact sets contained in E is bounded from above.

Then, we have

Theorem 4. *Any closed set F which supports the equilibrium measure is thin at the infinity.*

Proof. Let λ be any positive measure with total mass 1, P_0 a point where λ charges no positive mass and F' the inversion of F with respect to a circle centered at P_0 with radius R. Taking a positive measure λ' defined by $d\lambda'(Q') = d\lambda(Q)$, we have

$$U^{\lambda}(P) = U^{\lambda'}(P') - \log\frac{1}{P'P_0} - U^{\lambda'}(P_0) - 2\log R$$

for P of F and P' of F'. Here, $U^{\lambda'}(P_0)$ is necessarily finite since $U^{\lambda}(P) \not\equiv \infty$ and $> -\infty$. Now, for the equilibrium measure λ on F and the constant value V of the equilibrium potential, λ' is a positive measure supported by $F' - \{P_0\}$ with total mass 1. We have

$$U^{\lambda}(P) - V = 0$$

on F with a possible exception of a set of logarithmic capacity zero, which induces

$$U^{\lambda'}(P') - \log\frac{1}{P'P_0} - (U^{\lambda'}(P_0) + V + 2\log R) = 0$$

on F' with a possible exception of a set of logarithmic capacity zero. Therefore, λ' is the balayaged measure of the Dirac measure ε at P_0 onto F'. As $\lambda' \neq \varepsilon$, P_0 is an irregular point of F' and the balayage constant is given by

$$\gamma = U^{\lambda'}(P_0) + V + 2\log R\,.$$

Theorem 5. *A necessary and sufficient condition in order that a closed set F supports the equilibrium measure is that F is of logarithmic capacity finite.*

Proof. Let λ be the equilibrium measure on F. Take any compact set F_1 contained in F, the balayaged measure λ_1' of λ onto F_1 and the balayage constant $\gamma_1(\geqq 0)$. Then, λ_1' is the equilibrium measure on F_1 and there holds

$$U^{\lambda_1'}(P) = U^{\lambda}(P) + \gamma_1 \geqq U^{\lambda}(P)$$

on F_1 with a possible exception of a set of logarithmic capacity zero. So, denoting by V_F the constant value of $U^{\lambda}(P)$, we have

$$V_{F_1} = \int U^{\lambda_1'} d\lambda_1' \geqq \int U^{\lambda} d\lambda_1' = V_F ,$$

hence

$$C(F_1) = e^{-V}F_1 \leqq e^{-V}F .$$

Conversely, suppose that F is a closed set of logarithmic capacity finite. We are going to construct the equilibrium measure λ on F. Let F_n ($n=1, 2, 3, \cdots$) be compact sets monotone increasing toward F, P_0 an outer point of F, S a circle centered at P_0 with radius R which contains no point of F, F_n' and F' compact sets which are the inversion of F_n and F with respect to S. λ_n being the equilibrium measure on F_n, denote by λ_n' the inverse measure of λ_n with respect to S:

$$d\lambda_n'(Q') = d\lambda_n(Q) \qquad \text{for } Q \text{ of } F_n \text{ and } Q' \text{ of } F_n' .$$

Then, we have

$$U^{\lambda_n}(P) = U^{\lambda_n'}(P') - \log \frac{1}{P'P_0} - U^{\lambda_n'}(P_0) - 2 \log R ,$$

P and P_0 being inverse each other with respect to S. The constant values V_n of the equilibrium potential on F_n produce a monotone decreasing sequence bounded from below. Let V be the limiting number. V is finite. Put

$$\gamma_n = V_n + U^{\lambda_n'}(P_0) + 2 \log R .$$

Then, the measure λ_n' supported by F_n' with total mass 1 is the balayaged measure of the Dirac measure at P_0 onto F_n and γ_n is the balayage constant. The functions

$$U^{\lambda_n'}(P') - \gamma_n \qquad (n = 1, 2, 3, \cdots)$$

produce a monotone increasing sequence at each point P' and the non-negative numbers γ_n a monotone decreasing sequence, therefore the sequence

$$\{U^{\lambda_n'}(P')\} \qquad (n = 1, 2, 3, \cdots)$$

converges at each point P' ([2], see p. 236). We may suppose that the sequence of positive measures λ_n' with total mass 1 supported by the compact sets F_n' ($\subset F'$) converges vaguely. Then, the limiting measure λ' is a positive measure supported by F' with total mass 1, and there holds the ineguality

$$U^{\lambda'}(P') \leqq \lim_{n \to \infty} U^{\lambda_n'}(P')$$

at each point P' and the equality with a possible exception of a set of logarithmic capacity zero. Let us remark that $U^{\lambda'}(P_0)$ is finite. It is since we have

$$U^{\lambda'}(P_0) \leqq \lim_{n\to\infty} U^{\lambda'_n}(P_0)$$
$$\leqq \lim_{n\to\infty} \{U^{\lambda'_n}(P') - \log\frac{1}{P'P_0} - 2\log R - V_n\}$$

for any point P', therefore

$$U^{\lambda'}(P_0) \leqq U^{\lambda'}(P') - \log\frac{1}{P'P_0} - 2\log R - V$$

for a point P' which does not belong to the exceptional set and such that $U^{\lambda'}(P') < \infty$. Making $\gamma_n \downarrow \gamma (\geqq 0)$ and putting

$$V = \gamma - 2\log R - U^{\lambda'}(P_0)\,,$$

we have

$$U^{\lambda'}(P') - \log\frac{1}{P'P_0} - U^{\lambda'}(P_0) - 2\log R = V$$

on F' with a possible exception of a set of logarithmic capacity zero and $\leqq V$ everywhere. Then, the inverse measure λ of λ' with respect to S is a positive measure supported by F with total mass 1, and we have

$$U^{\lambda}(P) = V$$

on F with a possible exception of a set of logarithmic capacity zero and $\leqq V$ everywhere. Thus, λ is the equilibrium measure on F.

QUESTION. Is the converse of Theorem 4 correct? That is, does any closed set F thin at the infinity always support the equilibrium measure? If the question should be affirmative, for any closed set, three expressions — the existence of the equilibrium measure, the finiteness of the logarithmic capacity and the thinness at the infinity — are all equivalent. In the Newtonian case, these expressions are equivalent ([1], see n°14 and n°29), but how about the case of the logarithmic potential?

References

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