Xu, H.W. and Ren, X.
Osaka J. Math.
45 (2008), 747–756

CLOSED HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A SYMMETRIC MANIFOLD

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(Received August 30, 2006, revised July 26, 2007)

Abstract

We prove a rigidity theorem for closed hypersurfaces with constant mean curvature in a symmetric Riemannian manifold, which is a generalization of main results in [3] and [15].

1. Introduction

It seems interesting to generalize the famous optimal rigidity theorem for minimal hypersurfaces in a sphere due to J. Simons, H.B. Lawson Jr., and S.S. Chern, M. do Carmo and S. Kobayashi to general cases (see [4], [8], [12]). Q.M. Cheng and H. Nakagawa [3], and H.W. Xu [15] proved the following optimal rigidity theorem for hypersurfaces of constant mean curvature in a sphere independently.

Theorem A ([3], [15]). *Let M^n be an n-dimensional closed hypersurface with constant mean curvature H in a unit sphere S^{n+1}. If the squared norm of the second fundamental form S satisfies*

$$S \leq \alpha(n, H),$$

then M is congruent to one of the following

(1) *totally umbilic sphere* $S^n(1/\sqrt{1+H^2})$;
(2) *one of the Clifford minimal hypersurface* $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ *in* $S^{n+1}(1)$, *for* $k = 1, 2, \ldots, n-1$;
(3) *the isoparametric hypersurface* $S^{n-1}(1/\sqrt{1+\lambda^2}) \times S^1(\lambda/\sqrt{1+\lambda^2})$ *in* $S^{n+1}(1)$.

2000 Mathematics Subject Classification. Primary 53C40; Secondary 53C42.

Research supported by the NSFC, Grant No. 10771187, 10231010 and Natural Science Foundation of Zhejiang Province.

Here λ *and* $\alpha(n, H)$ *are given by*

$$\lambda = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}$$

and

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Motivated by Theorem A and a theorem due to G. Huisken [7], B. Andrews [2] proposed a following conjecture on mean curvature flow for closed hypersurfaces in a unit sphere.

Conjecture. *Let* $M_0 = F_0(M)$ *be a closed hypersurface in* S^{n+1} *which satisfies*

$$S < \alpha(n, H). \tag{1.1}$$

Then there exists a smooth family of hypersurfaces $\{M_t = F_t(M)\}_{0\le t<T}$ *which satisfy* (1.1) *and move by mean curvature flow with initial data* M_0. *Either* $T < \infty$ *and* M_t *is asymptotic to a family of geodesic spheres shrinking to their common centre, or* $T = \infty$ *and* M_t *approaches to a great sphere.*

The topological sphere theorem due to K. Shiohama and H.W. Xu [11] says that any closed hypersurface in S^{n+1} which satisfies $S < \alpha(n, H)$ must be a topological sphere, which provides an positive evidence to the conjecture above. In this paper, we generalize Theorem A as follows.

Main Theorem. *Let* N^{n+1} *be an* $(n+1)$*-dimensional simply connected symmetric Riemannian manifold with* δ *pinched curvature, i.e.,* $\delta \le K_N \le 1$, *and* M^n *be a closed hypersurface with constant mean curvature* H *in* N^{n+1}. *If*

$$(S - nH^2)[\alpha(n, H) - S - 2n(1-\delta)] - \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2} \ge 0,$$

then M *is congruent to one of the following*
(1) *totally umbilical hypersurface*;
(2) *one of the Clifford minimal hypersurface* $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ *in* $S^{n+1}(1)$, *for* $k = 1, 2, \ldots, n-1$;
(3) *the isoparametric hypersurface* $S^{n-1}(1/\sqrt{1+\lambda^2}) \times S^1(\lambda/\sqrt{1+\lambda^2})$ *in* $S^{n+1}(1)$.
Here $\alpha(n, H)$, λ *are defined as in* Theorem A.

Consequently we have

Corollary. *Let M^n be an n-dimensional closed minimal hypersurface in N^{n+1} with curvature K_N satisfying $\delta \leq K_N \leq 1$. If the squared norm of the second fundamental form S satisfies*

$$S \leq (2\delta - 1)n,$$

then M is congruent to one of the following
(1) *totally geodesic submanifold*;
(2) *one of the Clifford minimal hypersurface $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \ldots, n-1$.*

It should be mentioned that when M^n is a minimal hypersurface in N^{n+1}, then our pinching condition reduces to $S \leq (2\delta - 1)n$, which is weaker than the one in [5], [10] and [14].

Motivated by the main theorem, one can propose an analogue of the conjecture above for closed hypersurfaces in a symmetric Riemannian manifold with δ pinched curvature.

2. Preliminaries

Throughout this paper, let M^n be an n-dimensional closed hypersurface isometrically immersed in an $(n+1)$-dimensional simply connected symmetric Riemannian manifold N^{n+1}. The following convention of indices are used throughout.

$$1 \leq i, j, k, \ldots, \leq n,$$
$$1 \leq A, B, C, \ldots, \leq n+1.$$

Choose an orthonormal frame field $\{e_A\}$ in a neighborhood of $p \in M$ such that the $\{e_i\}$ span the tangent space T_pM to M at p. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ be the connection 1-forms of N. Restricting these forms to M, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

The curvature tensors of N, M are denoted by K_{ABCD}, R_{ijkl} respectively. The second fundamental form of M is denoted by h and the mean curvature normal field by ξ. Denote the mean curvature of M and squared norm of h by $H = \|\xi\|$ and S respectively. We have then

$$h = \sum_{i,j} h_{ij}\omega^i \otimes \omega^j \otimes e_{n+1}, \tag{2.1}$$

$$\xi = \frac{1}{n}\sum_{i=1}^{n} h_{ii} e_{n+1}, \tag{2.2}$$

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.3}$$

$$S = \sum_{i,j}(h_{ij})^2. \tag{2.4}$$

DEFINITION 2.1. M is called a hypersurface with constant mean curvature if H is constant. In particular, M is called minimal hypersurface if $H = 0$.

We denote the first and second covariant derivatives of h_{ij} by h_{ijk} and h_{ijkl} respectively, which are defined as in [4]. Following to [4] and [16], we have

$$h_{ijk} - h_{ikj} = -K_{n+1ijk}, \tag{2.5}$$

and the Ricci formula

$$h_{ijkl} - h_{ijlk} = \sum_s h_{sj} R_{sikl} + \sum_s h_{is} R_{sjkl}. \tag{2.6}$$

Let $K_{n+1ijkl}$ be the covariant derivative of K_{n+1ijk} as the section of $T^{\perp}M \otimes T^*M \otimes T^*M \otimes T^*M$ and $K_{ABCD;E}$ the covariant derivative of K_{ABCD} as curvature tensor of N. Restricted to M we have

$$\sum_l K_{n+1ijkl}\omega_l = dK_{n+1ijk} + \sum_s K_{n+1sjk}\omega_{is} + \sum_s K_{n+1ijs}\omega_{ks}, \tag{2.7}$$

and

$$K_{n+1ijk;l} = K_{n+1ijkl} - K_{n+1in+1k}h_{jl} - K_{n+1ijn+1}h_{kl} + \sum_m K_{mijk}h_{ml}. \tag{2.8}$$

DEFINITION 2.2. N is called a symmetric Riemannian manifold if for every $p \in N$ there exists an isometric $\sigma_p\colon N \to N$ such that $\sigma_p(p) = p$, and the differential of σ_p at p is equal to $-I_p$, where I_p is the identity transformation of T_pN. The Laplacian of the second fundamental form is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$.

The following propositions will be used in the proof of Main Theorem.

Proposition 2.3 ([3], [15]). *If $a_1,\dots,a_n$ are n real numbers with $\sum_{i=1}^n a_i = 0$, then*

$$\left|\sum_{i=1}^n a_i^3\right| \le (n-2)[n(n-1)]^{-1/2}\left(\sum_{i=1}^n a_i^2\right)^{3/2}.$$

Moreover, the equality holds if and only if at least $n-1$ numbers of a_i's are equal.

Proposition 2.4. *If the function* $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^n x_i y_i$ *satisfies*

$$\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n x_i^2 = \Lambda, \quad \delta \le y_i \le 1. \tag{2.9}$$

Then

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) \ge \frac{1}{2}(\delta - 1)(n\Lambda)^{1/2}.$$

Proof. We assume

$$x_1 \le x_2 \le \cdots \le x_k \le 0 \le x_{k+1} \le \cdots \le x_n.$$

Thus

$$\begin{aligned} f(x_1, \ldots, x_n, y_1, \ldots, y_n) &= \sum_{i=1}^k x_i y_i + \sum_{i=k+1}^n x_i y_i \\ &\ge \sum_{i=1}^k x_i + \delta \sum_{i=k+1}^n x_i \\ &= (\delta - 1) \sum_{i=k+1}^n x_i. \end{aligned} \tag{2.10}$$

By (2.9) we have

$$\begin{aligned} k\Lambda &= k \sum_{i=1}^k x_i^2 + k \sum_{i=k+1}^n x_i^2 \\ &\ge \left(\sum_{i=1}^k x_i\right)^2 + \frac{k}{n-k}\left(\sum_{i=k+1}^n x_i\right)^2 \\ &= \frac{n}{n-k}\left(\sum_{i=k+1}^n x_i\right)^2. \end{aligned} \tag{2.11}$$

So by (2.11) we have

$$\left(\sum_{i=k+1}^n x_i\right)^2 \le \frac{k(n-k)}{n}\Lambda \le \frac{n}{4}\Lambda.$$

Thus

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) \ge (\delta - 1) \sum_{i=k+1}^n x_i \ge \frac{1}{2}(\delta - 1)(n\Lambda)^{1/2}.$$

This proves Proposition 2.4. □

From (2.5) and (2.6),

$$\Delta h_{ij} = -\sum_k (K_{n+1kikl} + K_{n+1ijkk}) + \sum_{k,m}(h_{mk}R_{mijk} + h_{im}R_{mkjk}). \tag{2.12}$$

Since N is a symmetric manifold, N is complete and locally symmetric. Thus

$$K_{ABCD;E} \equiv 0$$

for all A, B, C, D, E. This together with (2.3), (2.8) and (2.12) implies

$$\frac{1}{2}\Delta S = \sum_{i,j,k}(h_{ijk})^2 + \sum_{i,j} h_{ij}\Delta h_{ij} = X + Y + Z, \tag{2.13}$$

where

$$\begin{aligned}
X &= nH \operatorname{tr} H_{n+1}^3 - (\operatorname{tr} H_{n+1}^2)^2,\\
Y &= 2\sum_{m,k,i,j}(h_{mj}h_{ij}K_{mkik} + h_{mk}h_{ij}K_{mijk}) + \sum_{i,j,k}(h_{ijk})^2,\\
Z &= -\sum_{i,j,k}(K_{n+1kn+1k}h_{ij}h_{ij} + K_{n+1kin+1}h_{kj}h_{ij} + K_{n+1in+1k}h_{jk}h_{ij} + K_{n+1ijn+1}h_{kk}h_{ij}).
\end{aligned}$$

3. Proof of Main Theorem

The following lemmas are useful in the proof of the main theorem.

Lemma 3.1. $X \geq (S - nH^2)[2nH^2 - S - (n(n-2)/\sqrt{n(n-1)})H(S - nH^2)^{1/2}]$.

Proof. Let $\{e_i\}$ be an orthonormal frame at a point on M such that the matrix $H_{n+1} = (h_{ij})_{n\times n}$ takes the diagonal form and such that $h_{ij} = \lambda_i\delta_{ij}$ for all i, j. Set

$$\begin{aligned}
f_k &= \sum_{i=1}^{n}(\lambda_i)^k,\\
B_k &= \sum_{i=1}^{n}(\mu_i)^k,\\
\mu_i &= H - \lambda_i.
\end{aligned}$$

Then we have

$$B_1 = 0, \quad B_2 = S - nH^2, \tag{3.1}$$

and

$$B_3 = 3HS - 2nH^3 - f_3. \tag{3.2}$$

From (3.1), (3.2) and Proposition 2.3, we get

$$\begin{aligned} X &= nHf_3 - S^2 \\ &\geq nH\left[3HS - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}B_2^{3/2}\right] - S^2 \\ &\geq (S - nH^2)\left[2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S - nH^2)^{1/2}\right]. \end{aligned}$$

This proves Lemma 3.1. □

Lemma 3.2. $Y \geq 2\delta n(S - nH^2)$.

Proof. It follows that

$$\begin{aligned} Y &= 2\sum_{i,k}[K_{ikik}(h_{ii})^2 + K_{kiik}h_{kk}h_{ii}] \\ &= \sum_{i,k} K_{ikik}(\lambda_i - \lambda_k)^2 \\ &\geq \delta \sum_{i,k}(\lambda_i - \lambda_k)^2 \\ &= 2n\delta(S - nH^2). \end{aligned}$$

This proves Lemma 3.2. □

Lemma 3.3. $Z \geq -n(S - nH^2) - (1/2)(1-\delta)n^{3/2}H\sqrt{S - nH^2}$.

Proof.

$$\begin{aligned} Z &= -\sum_{k,i} K_{n+1kn+1k}(\lambda_i)^2 + \sum_{k,i} K_{n+1kn+1k}\lambda_k\lambda_i \\ &\geq -nS + nH\sum_k K_{n+1kn+1k}\lambda_k \\ &= -n(S - nH^2) + nH\sum_k K_{n+1kn+1k}\mu_k, \end{aligned} \tag{3.3}$$

where we set $\mu_k = \lambda_k - H$. Since $\sum_k \mu_k = 0$, $\sum_k \mu_k^2 = S - nH^2$ and $\delta \leq K_{n+1kn+1k} \leq 1$, by Proposition 2.4, we have

$$Z \geq -n(S - nH^2) - \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2}. \tag{3.4}$$

This proves Lemma 3.3. □

Proof of Main Theorem. If $S = nH^2$, then $\lambda_i = H$ for $i = 1, 2, \dots, n$, which means that M is a totally umbilic submanifold.

If $S \neq nH^2$, then $S > nH^2$. By the assumption that

$$(S - nH^2)[\alpha(n, H) - S - 2n(1-\delta)] - \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2} \geq 0,$$

we get

$$S \leq \alpha(n, H) - 2n(1-\delta) \leq \alpha(n, H).$$

Combining this with Lemmas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}
\frac{1}{2}\Delta S \geq{}& (S - nH^2)\left[-n + 2n\delta + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S - nH^2)^{1/2}\right] \\
&- \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2} \\
\geq{}& (S - nH^2)\Bigg[n + 2nH^2 - \alpha(n, H) - \frac{n(n-2)}{\sqrt{n(n-1)}}H(\alpha(n, H) - nH^2)^{1/2} \\
&\qquad + \alpha(n, H) - S - 2n(1-\delta)\Bigg] - \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2} \\
={}& (S - nH^2)[\alpha(n, H) - S - 2n(1-\delta)] - \frac{1}{2}(1-\delta)n^{3/2}H\sqrt{S - nH^2}.
\end{aligned}$$

By the assumption and Hopf's maximum principle, we see that S must be a constant. This implies that the inequalities in (3.3) and (3.4) become equalities. Since $S > nH^2 \geq 0$, it follows from (3.3) that $K_{n+1kn+1k} = 1$ for all $k = 1, \dots, n$. On the other hand, it follows from $S > nH^2$ and $\sum_k \mu_k = 0$ that there exist k and l such that $\mu_k < 0$ and $\mu_l > 0$, where $1 \leq k < l \leq n$. By (3.4) and Proposition 2.4, we have $K_{n+1ln+1l} = \delta$ for some l. Therefore

$$\delta = 1 \quad \text{and} \quad (S - nH^2)[S - \alpha(n, H)] = 0,$$

which implies that $S = \alpha(n, H)$ and N is isometric to a unit sphere. It follows from Theorem A that M must be congruent to either

(i) one of the Clifford minimal hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$ in $S^{n+1}(1)$, for $k = 1, 2, \dots, n-1$; or

(ii) the isoparametric hypersurface $S^{n-1}(1/\sqrt{1+\lambda^2}) \times S^1(\lambda/\sqrt{1+\lambda^2})$ in $S^{n+1}(1)$, where λ is given by

$$\lambda = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}.$$

This proves the theorem. □

ACKNOWLEDGEMENTS. We would like to thank the referee for his/her valuable comments and suggestions.

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