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Author(s)	Zhang, Wenpeng
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Osaka University

ON A HYBRID MEAN VALUE OF CERTAIN HARDY SUMS AND RAMANUJAN SUM

WENPENG ZHANG

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1. Introduction

Let c be a natural number and d an integer prime to c . The classical Dedekind sum

$$(1) \quad S(d, c) = \sum_{j=1}^c \left(\left(\frac{j}{c} \right) \right) \left(\left(\frac{dj}{c} \right) \right),$$

with

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer,} \end{cases}$$

describes the behaviour of the logarithm of eta-function (cf. [7]) under modular transformations. B.C. Berndt [3] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 z), \quad \text{Im } z > 0.$$

Put $Vz = (az + b)(cz + d)$ with $a, b, c, d \in \mathbb{Z}$, $c > 0$, and $ad - bc = 1$. Then

$$(2) \quad \log \theta(Vz) = \log \theta(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i + \frac{1}{4} \pi i S_1(d, c),$$

where

$$S_1(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

The sums $S_1(d, c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums (cf., e.g., Lemma 2). Some arithmetical

properties of $S_1(d, c)$ can be found in R. Sitaramachandrarao [10]. In [13], the author studied the $2m$ -th power mean of $S_1(d, c)$, and proved that the asymptotic formula

$$\sum_{h=1}^{p-1} |S_1(h, p)|^{2m} = p^{2m} \frac{\zeta^2(2m) (1 - 1/4^m)}{\zeta(4m) (1 + 1/4^m)} + O\left(p^{2m-1} \exp\left(\frac{6 \ln p}{\ln \ln p}\right)\right)$$

holds for all odd prime p and positive integer m , where $\zeta(s)$ is the Riemann zeta-function and $\exp(y) = e^y$. In this paper, we shall study the distribution problem of the hybrid mean value involving $S_1(d, c)$ and Ramanujan’s sum

$$R_c(d) = \sum'_{b=1}^c e\left(\frac{db}{c}\right),$$

where $e(y) = e^{2\pi iy}$, \sum'_b denotes the summation over all b such that $(b, c) = 1$. In fact, we use the estimates for character sums and the mean value theorem of Dirichlet L -functions to obtain an interesting hybrid mean value formula involving $S_1(d, c)$ and $R_c(d)$. That is, we shall prove the following:

Theorem. *Let $c \geq 3$ be an odd number. Then we have the asymptotic formula*

$$\sum'_{h=1}^c R_c(2h+1)S_1(2h, c) = \phi^2(c) \prod_{p|c} \left(1 + \frac{1}{p}\right) \prod_{p|c} \left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O(c^{1+\epsilon}),$$

where ϵ be any fixed positive number, $\phi(d)$ be the Euler function and $\prod_{p|c}$ denotes the product over prime divisor p of c such that $p | c$ and $p^2 \nmid c$.

From this Theorem we may immediately deduce the following:

Corollary. *If $c \geq 3$ be a square-full odd number, then we have*

$$\sum'_{h=1}^c R_c(2h+1)S_1(2h, c) = \phi^2(c) \prod_{p|c} \left(1 + \frac{1}{p}\right) + O(c^{1+\epsilon}).$$

2. Some Lemmas

To complete the proof of Theorem, we need the following Lemmas.

Lemma 1. *Let integer $q \geq 3$ and $(h, q) = 1$. Then we have the identity*

$$S(h, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi)|^2,$$

where χ denotes a Dirichlet character modulo d with $\chi(-1) = -1$, and $L(s, \chi)$ denotes the Dirichlet L -function corresponding to χ .

Proof (see reference [11]). □

Lemma 2. *Let integer $q \geq 2$ and $(h, q) = 1$. Then we have the identity*

$$S_1(h, q) = -8S(h + q, 2q) + 4S(h, q).$$

Proof. This formula is an immediate consequence of (5.9) and (5.10) in [10]. □

Lemma 3. *Let χ be a character modulo q , generated by the primitive character χ_m modulo m . Then we have the identity*

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right) \mu\left(\frac{q}{m}\right) \tau(\chi_m),$$

where $\mu(n)$ be the Möbius function.

Proof (see Lemma 1.3 of reference [2]). □

Lemma 4. *Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, χ be a Dirichlet character modulo q . Then we have the identities*

$$\sum_{\chi \bmod q}^* \chi(r) = \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right),$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters modulo q , and $J(q)$ denotes the number of primitive characters modulo q .

Proof. From the properties of characters we know that for any character χ modulo q , there exists one and only one $d | q$ and primitive character χ_d^* modulo d such that $\chi = \chi_d^* \chi_q^0$, where χ_q^0 denotes the principal character modulo q . So we have

$$\sum_{\chi \bmod q} \chi(r) = \sum_{d|q} \sum_{\chi \bmod d}^* \chi(r) \chi_q^0(r) = \sum_{d|q} \sum_{\chi \bmod d}^* \chi(r).$$

Combining this formula and Möbius inversion, and noting the identity

$$\sum_{\chi \bmod q} \chi(r) = \begin{cases} \phi(q), & \text{if } r \equiv 1 \pmod{q}; \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{\chi \bmod q}^* \chi(r) = \sum_{d|q} \mu(d) \sum_{\chi \bmod (q/d)} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right) \phi(d).$$

Taking $r = 1$, we immediately get

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right).$$

This proves Lemma 4. □

Lemma 5. *Let $q > 1$ be an odd number and $(h, q) = 1$. Then we have*

$$S_1(h, q) = \begin{cases} -\frac{16}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi \chi_2^0)|^2, & \text{if } 2 \mid h; \\ 0, & \text{if } 2 \nmid h, \end{cases}$$

where χ_d^0 denotes the principal character mod d .

Proof. Note that $\sum_{d|2q} f(d) = \sum_{d|q} f(d) + \sum_{d|q} f(2d)$. So from Lemma 1 and Lemma 2 we get

$$\begin{aligned} S_1(h, q) &= -8S(h+q, 2q) + 4S(h, q) \\ &= -\frac{4}{\pi^2 q} \sum_{d|2q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h+q) |L(1, \chi)|^2 \\ &\quad + \frac{4}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi)|^2 \\ (3) \quad &= -\frac{4}{\pi^2 q} \sum_{d|q} \frac{(2d)^2}{\phi(2d)} \sum_{\substack{\chi \bmod 2d \\ \chi(-1)=-1}} \chi(h+q) |L(1, \chi)|^2. \end{aligned}$$

It is clear that 2 has only one principal character χ_2^0 , so for any odd number q , from (3) we obtain

$$S_1(h, q) = -\frac{16}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h+q) \chi_2^0(h+q) |L(1, \chi \chi_2^0)|^2$$

$$= \begin{cases} -\frac{16}{\pi^2 q} \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi \chi_2^0)|^2, & \text{if } 2 \mid h ; \\ 0, & \text{if } 2 \nmid h . \end{cases}$$

This completes the proof of Lemma 5. □

Lemma 6. *Let q be any integer with $q \geq 3$. Then for any integer $k \mid q$ with $k \geq 2$, we have the asymptotic formula*

$$\sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}}^* |L(1, \chi \chi_q^0)|^2 = \frac{\pi^2}{12} J(k) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O(q^\epsilon),$$

where χ_q^0 be the principal character.

Proof. First for any parameter $q \leq N \leq q^2$ and any non-principal character χ modulo k , applying Abel’s identity (see reference [1]) we have

$$(4) \quad L(1, \chi \chi_q^0) = \sum_{n=1}^{\infty} \frac{\chi(n) \chi_q^0(n)}{n} = \sum_{1 \leq n \leq N} \frac{\chi(n) \chi_q^0(n)}{n} + \int_N^{\infty} \frac{A(y, \chi \chi_q^0)}{y^2} dy,$$

where $A(y, \chi \chi_q^0) = \sum_{N < n \leq y} \chi(n) \chi_q^0(n)$. Applying Pólya-Vinogradov inequality we have

$$\begin{aligned} |A(y, \chi \chi_q^0)| &= \left| \sum_{N < n \leq y} \chi(n) \chi_q^0(n) \right| = \left| \sum_{d|q} \mu(d) \chi(d) \sum_{N/d < n \leq y/d} \chi(n) \right| \\ &\ll \left(\sum_{d|q} |\mu(d)| \right) \sqrt{k} \ln k \ll 2^{\omega(q)} \sqrt{k} \ln k \ll k^{1/2} q^\epsilon, \end{aligned}$$

where $\omega(q)$ denotes the number of all different prime divisors of q . From this estimate and (4) we get

$$L(1, \chi \chi_q^0) = \sum_{1 \leq n \leq N} \frac{\chi(n) \chi_q^0(n)}{n} + O\left(\frac{k^{1/2} q^\epsilon}{N}\right)$$

and

$$\sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}}^* |L(1, \chi \chi_q^0)|^2 = \sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq n \leq N} \frac{\chi(n) \chi_q^0(n)}{n} \right|^2$$

$$\begin{aligned}
 & + O\left(\frac{k^{1/2}q^\epsilon}{N} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq n \leq N} \frac{\chi(n)\chi_q^0(n)}{n} \right|\right) + O\left(\frac{k^2q^\epsilon}{N^2}\right) \\
 (5) \quad & = \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq n \leq N} \frac{\chi(n)\chi_q^0(n)}{n} \right|^2 + O\left(\frac{k^{3/2}q^\epsilon}{N}\right).
 \end{aligned}$$

Note that for $(a, k) = 1$, from Lemma 4 we have

$$\begin{aligned}
 \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}}^* \chi(a) &= \frac{1}{2} \sum_{\chi \bmod k}^* (1 - \chi(-1))\chi(a) \\
 &= \frac{1}{2} \sum_{\chi \bmod k}^* \chi(a) - \frac{1}{2} \sum_{\chi \bmod k}^* \chi(-a) \\
 &= \frac{1}{2} \sum_{u|(k,a-1)} \mu\left(\frac{k}{u}\right) \phi(u) - \frac{1}{2} \sum_{u|(k,a+1)} \mu\left(\frac{k}{u}\right) \phi(u).
 \end{aligned}$$

So that we have

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}}^* \left| \sum_{1 \leq n \leq N} \frac{\chi(n)\chi_q^0(n)}{n} \right|^2 \\
 &= \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{\chi_q^0(mn)}{mn} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}}^* \chi(m)\overline{\chi}(n) \\
 &= \frac{1}{2} \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{\chi_q^0(mn)}{mn} \sum_{u|(k,m-n)} \mu\left(\frac{k}{u}\right) \phi(u) \\
 &\quad - \frac{1}{2} \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{\chi_q^0(mn)}{mn} \sum_{u|(k,m+n)} \mu\left(\frac{k}{u}\right) \phi(u) \\
 &= \frac{1}{2} J(k) \sum_{\substack{1 \leq n \leq N \\ (n,q)=1}} \frac{1}{n^2} + O\left(\sum_{u|k} \phi(u) \sum_{\substack{1 \leq n \leq N \\ (n,q)=1}} \sum_{1 \leq h \leq N/u} \frac{1}{n(hu+n)}\right) \\
 &\quad + O\left(\sum_{u|k} \phi(u) \sum_{1 \leq n < u} \frac{1}{n(u-n)}\right) + O\left(\sum_{u|k} \sum_{\substack{1 \leq n \leq N \\ (n,q)=1}} \sum_{1+(n/u) \leq h \leq N/u} \frac{\phi(u)}{n(hu-n)}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} J(k) \zeta(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O\left(\sum_{u|k} \frac{\phi(u)}{u} \ln^2 N\right) \\
 (6) \quad &= \frac{\pi^2}{12} J(k) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O(q^\epsilon),
 \end{aligned}$$

where $\zeta(n)$ be the Riemann zeta-function and $\zeta(2) = \pi^2/6$.

Taking $N = q^2$, combining (4), (5) and (6) we may immediately obtain the asymptotic formula

$$\sum_{\substack{\chi \pmod k \\ \chi(-1)=-1}}^* |L(1, \chi \chi_q^0)|^2 = \frac{\pi^2}{12} J(k) \prod_{p|q} \left(1 - \frac{1}{p^2}\right) + O(q^\epsilon).$$

This completes the proof of Lemma 6. □

3. Proof of the Theorem

In this section, we complete the proof of Theorem. First for any primitive character χ_m modulo m , from the properties of Gauss sums we have

$$\tau(\chi_m) \tau(\overline{\chi_m}) = -m \text{ if } \chi(-1) = -1$$

and for any odd number c and $d | c$ with any character χ modulo d , note that

$$\begin{aligned}
 \sum_{h=1}^{c'} \chi(2h) R_c(2h+1) &= \sum_{b=1}^c \sum_{h=1}^c \chi(2h) \chi_c^0(2h) e\left(\frac{(2h+1)b}{c}\right) \\
 (7) \quad &= \sum_{b=1}^c \overline{\chi}(b) \chi_c^0(b) e\left(\frac{b}{c}\right) \sum_{h=1}^c \chi(h) \chi_c^0(h) e\left(\frac{h}{c}\right) = \tau(\overline{\chi} \chi_c^0) \tau(\chi \chi_c^0).
 \end{aligned}$$

From (7), Lemma 3 and Lemma 5 we have

$$\begin{aligned}
 &\sum_{h=1}^c S_1(2h, c) R_c(2h+1) \\
 &= -\frac{16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod d \\ \chi(-1)=-1}} \left(\sum_{h=1}^{c'} \chi(2h) R_c(2h+1) \right) |L(1, \chi \chi_c^0)|^2 \\
 &= -\frac{16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \pmod d \\ \chi(-1)=-1}} \tau(\overline{\chi} \chi_c^0) \tau(\chi \chi_c^0) |L(1, \chi \chi_c^0)|^2
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{m|d} \sum_{\substack{\chi \pmod m \\ \chi(-1)=-1}}^* \tau(\overline{\chi}\chi_c^0)\tau(\chi\chi_c^0)|L(1, \chi\chi_d^0\chi_2^0)|^2 \\
 &= \frac{-16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{m|d} \sum_{\substack{\chi \pmod m \\ \chi(-1)=-1}}^* \left| \chi\left(\frac{c}{m}\right) \right|^2 \mu^2\left(\frac{c}{m}\right) \tau(\overline{\chi})\tau(\chi)|L(1, \chi\chi_{2d}^0)|^2 \\
 (8) \quad &= \frac{16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{m|d} \sum_{\substack{\chi \pmod m \\ \chi(-1)=-1}}^* m \left| \chi\left(\frac{c}{m}\right) \right|^2 \mu^2\left(\frac{c}{m}\right) |L(1, \chi\chi_{2d}^0)|^2.
 \end{aligned}$$

Let $c = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Now for any $m \mid c$ and χ modulo m , it is clear that $\chi(c/m) = 0$, if $(c/m, m) > 1$. $\chi(c/m)\mu(c/m) \neq 0$ if and only if $m = ut$, where $t \mid v$. From these properties, (8) and Lemma 6 we have

$$\begin{aligned}
 &\sum_{h=1}^c R_c(2h+1)S_1(2h, q) \\
 &= \frac{16}{\pi^2 c} \sum_{d|c} \frac{d^2}{\phi(d)} \sum_{m|d} \sum_{\substack{\chi \pmod m \\ \chi(-1)=-1}}^* m \left| \chi\left(\frac{c}{m}\right) \right|^2 \mu^2\left(\frac{c}{m}\right) |L(1, \chi\chi_{2d}^0)|^2 \\
 &= \frac{16}{\pi^2 c} \sum_{d|v} \frac{u^2 d^2}{\phi(ud)} \sum_{m|d} \sum_{\substack{\chi \pmod{um} \\ \chi(-1)=-1}}^* mu |L(1, \chi\chi_{2ud}^0)|^2 \\
 &= \frac{16u^2}{\pi^2 v \phi(u)} \sum_{d|v} \frac{d^2}{\phi(d)} \sum_{m|d} m \sum_{\substack{\chi \pmod{um} \\ \chi(-1)=-1}}^* |L(1, \chi\chi_{2ud}^0)|^2 \\
 &= \frac{16u^2}{\pi^2 v \phi(u)} \sum_{d|v} \frac{d^2}{\phi(d)} \sum_{m|d} m \left(\frac{\pi^2}{12} J(um) \prod_{p|2ud} \left(1 - \frac{1}{p^2}\right) + O((ud)^\epsilon) \right) \\
 &= \frac{16u^2 J(u)}{12v \phi(u)} \sum_{d|v} \frac{d^2}{\phi(d)} \sum_{m|d} m J(m) \prod_{p|2ud} \left(1 - \frac{1}{p^2}\right) + O(c^{1+\epsilon}) \\
 &= \frac{u^2 \phi^2(u)}{u \phi(u)} \frac{1}{v} \prod_{p|u} \left(1 - \frac{1}{p^2}\right) \sum_{d|v} \frac{d^2}{\phi(d)} \prod_{p|d} \left(1 - \frac{1}{p^2}\right) \sum_{m|d} m J(m) + O(c^{1+\epsilon}) \\
 &= \phi^2(u) \prod_{p|u} \left(1 + \frac{1}{p}\right) \cdot \frac{1}{v} \sum_{d|v} d \prod_{p|d} \left(1 + \frac{1}{p}\right) (1 + p(p-2)) + O(c^{1+\epsilon}) \\
 &= \phi^2(u) \prod_{p|u} \left(1 + \frac{1}{p}\right) \cdot \frac{1}{v} \prod_{p|v} (1 + (p+1)(p-1)^2) + O(c^{1+\epsilon})
 \end{aligned}$$

$$\begin{aligned}
&= \phi^2(uv) \prod_{p|vu} \left(1 + \frac{1}{p}\right) \cdot \prod_{p|v} \left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O(c^{1+\epsilon}) \\
&= \phi^2(c) \prod_{p|c} \left(1 + \frac{1}{p}\right) \prod_{p \nmid c} \left(1 + \frac{1}{(p+1)(p-1)^2}\right) + O(c^{1+\epsilon}),
\end{aligned}$$

where we have used the identity $J(u) = \phi^2(u)/u$, if u be a square-full number. This completes the proof of Theorem.

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Department of Mathematics
 Northwest University
 Xi'an, Shaanxi
 P. R. China
 e-mail: wpzhang@nwu.edu.cn