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TORSION FREE EXTENDING MODULES

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1. Introduction

We recall that a module is extending if every complement submodule is a direct summand. In [6] we showed that, over a commutative domain R, a non-torsion module is extending if and only if it is of the form "injective $\oplus$ extending torsion free reduced", and that a torsion free reduced module is extending if and only if it is a finite direct sum of uniform submodules each pair of which is extending.

In Theorem 1, we provide now a characterization of the extending property for such pairs, and thereby complete the description of non-torsion extending R-modules. (For the torsion case, cf. [7]). The drawback of this characterization, viz. that it is formulated in terms of local data, is removed in Theorems 5 and 6, under the assumption that a certain natural overring S of R is noetherian. The subsequent corollaries state what can be said if R itself is noetherian.

The last section presents a number of examples, which demonstrate that our various technical conditions cannot be relaxed.

Throughout this paper R will be a commutative domain with quotient field K.

A submodule N of a module M is a complement submodule, if there is another submodule N' such that N is maximal with respect to $N \cap N' = 0$.

Let T be an overring of R. The conductor of R in T is the largest ideal of R which is also an ideal of T.

2. Direct sums of two uniform modules over commutative domains

In this section we characterize all torsion free (reduced) extending modules which are direct sums of two uniform submodules.

Let M_1 and M_2 be torsion free reduced uniform R-modules. Since the M_i are embeddable into the quotient field K of R, we may assume $M_i \subsetneq K$. Let $A := \{q \in K: qM_1 \subset M_2\}$ and $B := \{q \in K: qM_2 \subset M_1\}$. For any R-submodule X of K, let $O(X) := \{q \in K: qX \subset X\}$. Denote $O(M_1) \cap O(M_2)$ by S. If $M_i \neq 0$, then $A \cong \hom_R(M_1, M_2)$, and $B \cong \hom_R(M_2, M_1)$, and $O(M_i) \cong \mathrm{end}_R(M_i)$.

Theorem 1. *Let $M=M_1\oplus M_2$ be a torsion free reduced R-module, where the M_i are uniform. Then the following statements are equivalent:*
1) *M is extending;*
2) *for every maximal ideal P of S, $O(A_P)$ coincides with $O(B_P)$, and is a valuation ring with maximal ideal $W\subset A_PB_P$. If $A_P\cong W\cong B_P$, then $O(A_P)$ is discrete.*

Proof. 1) $\Rightarrow$ 2): Let M be extending. By [6], Corollary 8, we have that A and B are non-zero. By [6], Theorem 7, we obtain $q^{-1}A\cap S+qB\cap S=S$ for each $0\neq q\in K$. It follows that $q^{-1}A_P\cap S_P+qB_P\cap S_P=S_P$ for every maximal ideal P of S, and hence $q\in A_P$ or $q^{-1}\in B_P$.

By the same argument as in [7], Theorem 20, we can show that $O(A_P)$ coincides with $O(B_P)$ and is a valuation ring with maximal ideal $W\subset A_PB_P$, and that if $A_P\cong W\cong B_P$, then $O(A_P)$ $(=O(B_P))$ is discrete.
2) $\Rightarrow$ 1): The same argument as in [7], Theorem 20, shows that $q\in A_P$ or $q^{-1}\in B_P$ for all $0\neq q\in K$ and every maximal ideal P of S. It follows that $q^{-1}A_P\cap S_P+qB_P\cap S_P=S_P$, and hence $q^{-1}A\cap S+qB\cap S=S$. Therefore M is extending, by [6], Theorem 7.

Corollary 2. *Let $M=M_1\oplus M_2$ be an extending R-module as in Theorem 1. Then $O(A)$ coincides with $O(B)$, and is integrally closed.*

Proof. By Theorem 1, $O(A_P)=O(B_P)$ is a valuation ring for all maximal ideals P of S. Since any valuation ring is integrally closed, we have that $\bigcap_P O(A_P)$ $(=\bigcap_P O(B_P))$ is integrally closed. It is clear that $AO(A_P)\subset A_P$ for all P; hence $A[\bigcap_P O(A_P)]\subset\bigcap_P A_P=A$, i.e. $\bigcap_P O(A_P)\subset O(A)$. Thus $\bigcap_P O(A_P)=O(A)$. Similarly $\bigcap_P O(B_P)=O(B)$.

Corollary 3. *Let N be a uniform torsion free reduced R-module. Then N^n is extending if and only if $O(N)$ is a Prüfer domain.*

Proof. Theorem 1, and [6], Theorem 11.

Corollary 4. *Let P be a maximal ideal of a commutative domain R. Then the following statements are equivalent:*
1) *$P\oplus R$ is extending;*
2) *$O(P)$ is a Prüfer domain and P is a maximal ideal of $O(P)$. R_Q is a valuation ring for all maximal ideals Q different from P.*

Proof. Theorem 1, Corollary 3, and observing that $(R:P)P=R$ or $(R:P)P=P$.

3. Direct sums of uniform modules over noetherian domains

Theorem 5. *Let $M=M_1\oplus M_2$ be a torsion free reduced R-module, where*

the M_i are uniform. Let S be noetherian. Then the following statements are equivalent:
1) *M is extending;*
2) *$O(A)$ coincides with $O(B)$, and is a Dedekind domain. AB is a product of distinct maximal ideals of $O(A)$. There is a one-to-one correspondence between the maximal ideals of $O(A)$ and the maximal ideals of S, via contraction;*
3) *the integral closure S' of S is Dedekind and is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of S and of S'. The conductor D of S in S' is a product of distinct maximal ideals of S' (or S). A and B are S'-modules with $AB=D$.*

Proof. 1) $\Rightarrow$ 2). Let M be extending. By Corollary 2, $O(A)=O(B)$ is integrally closed. By [6], Corollary 8, AB is a non-zero ideal of S which is also an ideal of $O(A)$. Then AB is contained in the conductor D of S in $O(A)$. Since S is noetherian and $D\neq 0$, we have that $O(A)$ is a finitely generated S-module. It follows that $O(A)$ is noetherian and integral over S, and hence $O(A)$ is the integral closure of S.

Since A is a fractional ideal of S and hence finitely generated, $O(A)_P=O(A_P)$ is a rank one discrete valuation ring, for every maximal ideal P of S, by Theorem 1. Since $O(A)_P$ is integral over S_P, it follows that S_P is one dimensional for all P. Hence S is one dimensional, and thus $O(A)$ is Dedekind.

We show that for each maximal ideal P of S there exists a unique maximal ideal $\mathfrak{P}$ of $O(A)$ such that $P=\mathfrak{P}\cap S$. The existence of such $\mathfrak{P}$ is due to $O(A)$ being integral over S. The uniqueness follows from the fact that $O(A)_P=O(A)_{\mathfrak{P}}$ for any maximal ideal $\mathfrak{P}$ of $O(A)$ lying over P. This establishes the one-to-one correspondence, via contraction.

Now we show that AB is a product of distinct maximal ideals of $O(A)$. Since $O(A)_P=O(A)_{\mathfrak{P}}$, for every maximal ideal P of S, where $\mathfrak{P}\cap S=P$, we have $(AB)_{\mathfrak{P}}=(AB)_P$. Hence $(AB)_{\mathfrak{P}}=\mathfrak{P}_{\mathfrak{P}}$ for any maximal ideal $\mathfrak{P}$ of $O(A)$ containing AB, by Theorem 1. On the other hand, AB is an ideal of $O(A)$, $AB=\Pi\mathfrak{P}^{n(\mathfrak{P})}$. It follows that $(AB)_{\mathfrak{P}}=\mathfrak{P}_{\mathfrak{P}}^{n(\mathfrak{P})}$, and by comparison we conclude that $n(\mathfrak{P})=1$.
2) $\Rightarrow$ 3). By 2), $O(A)=O(B)=: S'$ is Dedekind and is the integral closure of S, hence a maximal equivalent order.

Now let D be the conductor of S in S'; it is clear that $AB\subset D$. Since A and B are non zero, M_1S' and M_2S', as S'-modules, can be embedded in each other. By [5], Lemma 12, $M_1S'=M_2I$ where I is a fractional ideal of S'. Since $M_iD\subset M_i$ $(i=1, 2)$, it follows that $M_2ID\subset M_1$ and $M_1I^{-1}D\subset M_2$. Hence $ID\subset B$ and $I^{-1}D\subset A$, and thus $D^2\subset AB\subset D$. Since AB, by 2), is a product of distinct maximal ideals of S', we deduce $AB=D$.

Now we show that for any maximal ideal P of S, the unique maximal ideal of S' lying over P is $\mathfrak{P}=PS'$. This means that the inverse of the one-to-one correspondence via contraction (in Condition 2)) is given by extension. Since

$P=\mathfrak{P}\cap S\subset\mathfrak{P}$, we have $PS'\subset\mathfrak{P}$. Hence $PS'=\mathfrak{P}^n$, $n\geq 1$. If $n>1$, then $\mathfrak{P}D\subset\mathfrak{P}\cap S=P\subset PS'=\mathfrak{P}^n$, and thus $D\subset\mathfrak{P}^{n-1}\subset\mathfrak{P}$. Therefore $D\subset\mathfrak{P}\cap S\subset\mathfrak{P}^n$, which contradicts that D is a product of distinct maximal ideals of S'.
3) $\Rightarrow$ 1): Condition 2) of Theorem 1 can be easily verified, by using that A and B are S'-modules, and $AB=D$.

For more than two uniform modules M_i, we use the notations $A_{ij}:=\{q\in K: qM_i\subset M_j\}$, $A_{ii}=O(M_i)$, and $S_{ij}=O(M_i)\cap O(M_j)$.

We combine Theorem 6 with Theorem 11 from [6] to obtain the following:

Theorem 6. *Let $M=\bigoplus_{i=1}^{n} M_i$ be a torsion free reduced R-module, where the M_i are uniform. Let $S:=\bigcap_{i=1}^{n} O(M_i)$ be noetherian. Then the following statements are equivalent:*
1) *M is extending;*
2) *the integral closure S' of S is Dedekind, and is a (maximal) equivalent order. There is a one-to-one correspondence, via contraction (and extension) between the maximal ideals of S' and of S. The conductor D of S in S' is a product of distinct maximal ideals of S' (or S). For all $i\neq j$, the A_{ij} are S'-modules, and $D\subset A_{ij}A_{ji}\subset S'$;*
3) *there is a Dedekind domain $R\subset L\subset K$, maximal ideals $\mathfrak{P}_k (k=1, 2, \cdots, n)$ of L, and subfields F_k of $L/\mathfrak{P}_k$ such that the A_{ij} are L-modules and $\bigcap_{k=1}^{n}\mathfrak{P}_k\subset A_{ij}A_{ji}\subset L$ $(i\neq j)$, and S is the full inverse image of $\bigoplus_{k=1}^{n} F_k$ under the natural homomorphism $L\rightarrow L/\bigcap_{k=1}^{n}\mathfrak{P}_k\cong\bigoplus_{k=1}^{n} L/\mathfrak{P}_k$.*

Proof. 1) $\Rightarrow$ 2): Let M be extending. Then $M_i\oplus M_j$ is extending for all $i\neq j$. It is clear that $I:=A_{12}A_{23}\cdots A_{n-1n}A_{n1}$ is a non zero ideal of S which is also an ideal of S_{ij}. Since S is noetherian, S_{ij} is noetherian for all $i\neq j$. By Theorem 5, the $O(A_{ij})$ are Dedekind. Since $o\neq I\subset D$ (the conductor of S in $O(A_{ij})$), it follows that $O(A_{ij})$ is the integral closure of S; and hence all $O(A_{ij})$ coincide. We denote this ring by S'.

We show that $(S_{ij}: D)D=D_{ij}$, where D_{ij} is the conductor of S_{ij} in S'. It is clear that $(S_{ij}: D)D\subset D_{ij}$. Now let $x\in D_{ij}$ be arbitrary. For any $y\in(S': D)$, we have $yD\subset S'$ and hence $xyD\subset xS'\subset S_{ij}$. It follows that $xy\in(S_{ij}: D)$, hence $x(S': D)\subset(S_{ij}: D)$. Thus $x\in xS'=x(S': D)D\subset(S_{ij}: D)D$. Therefore $(S_{ij}: D)\,D=D_{ij}$, and hence $O(D)=O(S_{ij}: D)=S'$.

By Theorem 5 and since $O(D)=O(S_{ij}: D)=S'$, we have that $D\oplus S_{ij}$ is extending for all $i\neq j$. 2) follows, by Theorem 5, once we show that $D\oplus S$ is extending. Since S is noetherian, it is enough to show that $D_P\oplus S_P$ is extending, for every maximal ideal P of S. To this end we consider two cases:
Case 1. $P\supset D$. Since $D\oplus S_{ij}$ is extending for all $i\neq j$, we have that $D_P\oplus(S_{ij})_P$

is extending. Since $O(D_P)=O(D)_P=S'_P$ is local, we have, by [6], Theorem 7, that D_P and $q(S_{ij})_P$ are comparable for every $q\in K$. If $q(S_{ij})_P\subset D_P$ for some $i\neq j$, then $qS_P\subset q(S_{ij})_P\subset D_P$. On the other hand, if $D_P\subset q(S_{ij})_P$ for all $i\neq j$, then $D_P\subset \bigcap_{i\neq j} q(S_{ij})_P=qS_P$. It follows that D_P and qS_P are comparable for every $q\in K$, and hence, by [6], Theorem 7, $D_P\oplus S_P$ is extending whenever $P\supset D$.
Case 2. $P\not\supset D$. Then $D_P=S_P=S'_P$ is a rank one discrete valuation ring, and thus $D_P\oplus S_P$ is extending.
2) $\Rightarrow$ 3): From 2), the conductor of S in S' is a product of distinct maximal ideals of S', i.e. $D=\prod_{i=1}^{n}\mathfrak{P}_i=\bigcap_{i=1}^{n}\mathfrak{P}_i$. Let $\mathfrak{P}_i\cap S=:P_i$, it follows that $D=\bigcap_{i=1}^{n}P_i$; and hence $\bigoplus_{i=1}^{n}k_i:=\bigoplus_{i=1}^{n}S/P_i\cong S/\bigcap_{i=1}^{n}P_i=S/D\subset S'/\bigcap_{i=1}^{n}\mathfrak{P}_i\cong\bigoplus_{i=1}^{n}S'/\mathfrak{P}_i$, where $k_i=S/P_i=S/\mathfrak{P}_i\cap S\cong S+\mathfrak{P}_i/\mathfrak{P}_i\subset S'/\mathfrak{P}_i$.

By 2), the A_{ij} are S'-modules and $A_{ij}A_{ji}\supset\bigcap_{i=1}^{n}\mathfrak{P}_i$.
3) $\Rightarrow$ 1) Let $I:=\bigcap_{i=1}^{n}\mathfrak{P}_i$. Since I is a nonzero ideal of S which is also an ideal of L, the conductor D of S in L is nonzero. Since S is noetherian, we obtain that L is the integral closure of S. Since $A_{ij}A_{ji}\supset I$, we get that $A_{ij}A_{ji}$ is a product of distinct maximal ideals of L.

From $S/I=\bigoplus_{i=1}^{n}k_i\subset\bigoplus_{i=1}^{n}L/\mathfrak{P}_i=L/I$, we see that the maximal ideals of S and of L containing I are in one-to-one correspondence. On the other hand, for any maximal ideal P of S not containing I, $S_P=L_P$ is a rank one discrete valuation ring. Therefore we obtain a one-to-one correspondence between all the maximal ideals of L and of S. Since the A_{ij} are L-modules, it follows that $O(A_{ij})=L$ for all $i\neq j$. Since $S_{ij}\supset S$, Condition 2) of Theorem 5 is satisfied for all $i\neq j$, and hence $M_i\oplus M_j$ is extending. Therefore M is extending, by [6] Theorem 11.

REMARKS. (i) We note that Condition 2) of Theorem 6 mainly deals with the relationship between S and its integral closure, and that further data from M enter only in the last sentence. Loosely speaking, this condition says that S is "almost integrally closed" and the M_i are "almost isomorphic".
(ii) Condition 3) is convenient for the construction of examples, starting with an arbitrary Dedekind domain.
(iii) Even if the ring R is noetherian, S need not be, and then Theorem 6 does not apply. However, if R is noetherian of Krull dimension one, then every overring is noetherian of Krull dimension one ([8], Theorem 13), hence in particular S is noetherian and S' is Dedekind ([8], Theorem 96). Thus in this case, the rest of Condition 2) yields a complete characterization of all torsion free reduced extending R-modules.

Conversely, again if R is noetherian, and if one of the M_i is finitely generated (and hence all M_i are isomorphic to ideals of R, cf. [6], Corollary 8), then it follows that the Krull dimension is one.

Corollary 7. *Let R be a noetherian domain. Let M_1, M_2 be finitely generated torsion free reduced uniform R-modules. If $M_1 \oplus M_2$ is extending, then R has Krull dimension one.*

Proof. It is clear that $R \rightarrow S \rightarrow O(M_1) \cong \hom_R(M_1, M_1) \rightarrowtail \hom_R(R^n, M_1) \cong M_1^n$. Since M_1^n is noetherian, it follows that S is noetherian and integral over R. Thus S and R have the same Krull dimension.

Now if $M_1 \oplus M_2$ is extending, then, by Theorem 5, S is one dimensional, and therefore so is R.

We now prove a generalization of Corollary 4, for arbitrary ideals, in case R is noetherian.

Corollary 8. *Let R be a noetherian domain, and I be an ideal of R. Then the following statements are equivalent:*
1) *$R \oplus I$ is extending;*
2) *the integral closure R' of R is Dedekind. There is a one-to-one correspondence, via contraction (and extension), between the maximal ideals of R and of R'. The conductor D of R in R' is a product of distinct maximal ideals of R'. I is an ideal of R';*
3) *$O(I)$ is Dedekind, and $(R: I) I$ is a product of distinct maximal ideals of $O(I)$. $O(I)_P$ is a discrete rank one valuation ring for all maximal ideal P of R containing $(R: I) I$.*

Proof. Corollary 7, and Theorem 5.

4. Examples

The first example shows that the condition "if $A_P \cong W \cong B_P$, then $O(A_P)$ is discrete" in Theorem 1 does not follow from the rest of Condition 2).

EXAMPLE 9. Let V be a valuation ring which is not discrete, with maximal ideal W, and $V/W = \boldsymbol{Q}$ the field of rational numbers. Choose additive subgroups M_1 and M_2, $W \subset M_1, M_2 \subset V$, such that M_1/W, M_2/W are of incomparable types $\mathfrak{T}_1$ and $\mathfrak{T}_2$, and such that $\mathfrak{T}_1(P)$, $\mathfrak{T}_2(P)$ are not both ∞ for any prime number P. Then $O(M_1) \cap O(M_2) = S$ is the full inverse image of $\boldsymbol{Z}$ under the natural homomorphism $V \rightarrow V/W = \boldsymbol{Q}$. One can show that $A_P = A = W = B = B_P$, for every maximal ideal P of S. Consequently one has $O(A_P) = O(B_P) = O(W) = V$ and $W = A_P B_P$.

Our second example shows that, in contrast to Corollary 3, if $M_1 \oplus M_2$ is extending with $M_1 \not\cong M_2$, then neither $O(M_2)$ nor $O(M_1) \cap O(M_2) =: S$ need be Prüfer domains.

EXAMPLE 10. Let $F[[t]]$ be the ring of formal power series over a field F. Let k be a proper subfield of F. Let $M_1:=tF[[t]]$ and $M_2:=k+tF[[t]]$. By Corollary 4, $M_1\oplus M_2$ is extending. $O(M_2)=S=k+tF[[t]]$ is local but not a valuation ring, hence not a Prüfer domain.

The following example refers to Theorem 5 (3). It shows that, if the integral closure S' of S is Dedekind, and there is a one-to-one correspondence between the maximal ideals of S' and of S, via contraction and extension, then the conductor D of S in S' need not be a product of distinct maximal ideals of S'.

EXAMPLE 11. Let $S':=F[t]$ be the polynomial ring over a field F. Let k be a proper subfield of F such that F is a finite dimensional over k. Let $S:=k+kt+t^2S'$. Then the conductor D of S in S' is t^2S', and hence S' is a maximal equivalent order. Since $S/t^2S'\cong k[t]/t^2k[t]$, we see that $P:=kt+t^2S'$ is the only maximal ideal of S containing D. It is easy to show that $PS'=tS'$ and $tS'\cap S=P$. This suffices to establish the one-to-one correspondence, via contraction and extension, between all maximal ideals of S' and of S. But $D=t^2S'$ is not a product of distinct maximal ideals of S'.

The next example shows that the statement "A and B are S'-modules" does not follow from the rest of condition 3) of Theorem 5.

EXAMPLE 12. Let $S':=F[t]$ and let k be as in Example 11. Let $S=k+tS'$. Then $D=tS'$ is the conductor. Let V be a proper k-subspace of F such that $\dim_k V\geq 2$, and let $M_1:=Vt+t^2S'$ and $M_2:=S$.

Then $B=M_1$ and $A=(S:B)$. Since $BS'=(Vt+t^2S')\, S'=tS'\subset S$, we have $S'\subset A$. Now let $a\in A$, hence $aB\subset S$. It follows that $at^2S'\subset S$, and thus $at^2\in D=tS'$. Then $at\in S'$, and therefore $at=x+yt$ with $x\in F$ and $y\in S'$. On the other hand, $atV=(x+yt)\, V\subset S$; it follows that $xV\subset k$. Since $\dim_k V\geq 2$, we obtain that $x=0$ and $at=yt\in tS'$. Therefore $A=S'$ and $AB=tS'=D$. The one-to-one correspondence, via contraction and extension, between the maximal ideals of S' and of S can be easily established. Hence all the conditions of Theorem 5 (3) are satisfied, except that B is not an S'-module.

The last example shows that, in contrast to Theorem 5 and Corollary 7, if S is not noetherian, then S need not be of Krull dimension one, and if R is noetherian but the M_i are infinitely generated then R need not be of Krull dimension one.

EXAMPLE 13. Let R be a commutative noetherian domain with quotient field K, and with Krull $\dim(R)>1$. There exists a valuation ring $R\subset V\subset K$ such that Krull $\dim(V)>1$; hence V is not noetherian (cf. [9] Chapter V Exercise 3). By Corollary 3, $V\oplus V$ is extending as an R-module; and obviously

$S=V$.

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