ON QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

KEN-ICHI SAKAN

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1. Introduction. Let $U$ be the upper half-plane and let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the extended real line. The conformal automorphisms of $U$ form the real Möbius group $SL'(2, \mathbb{R})$, that is, $SL(2, \mathbb{R})$ modulo its center. A discrete subgroup $G$ of $SL'(2, \mathbb{R})$ is called a Fuchsian group; the trivial group $G = \{1\}$ is included, where 1 means the identity transformation of $SL'(2, \mathbb{R})$. It is well-known that a Fuchsian group $G$ acts discontinuously on $U$ and that the limit set $\Lambda(G)$ is a closed subset of $\hat{\mathbb{R}}$, which is invariant under $G$.

We denote by $L_{\infty}(U)$ the Banach space consisting of measurable functions $\mu$ on $U$ with finite $L_{\infty}$ norm $||\mu||$. Let $f$ be a sense-preserving homeomorphism of $U$. Such a mapping $f$ is said to be quasiconformal if it has generalized derivatives $f_{\bar{z}}$ and $f_z$ which satisfy the Beltrami equation

$$f_{\bar{z}} = \mu f_z$$

for some $\mu \in L_{\infty}(U)$ with $||\mu|| < 1$. As is known, a quasiconformal automorphism $f$ of $U$ is extensible to a homeomorphism of $U \cup \hat{\mathbb{R}}$. By abuse of language, we introduce no new name for the extended mapping. For every $\mu \in L_{\infty}(U)$ with $||\mu|| < 1$, we denote by $f_\mu$ the homeomorphic solution of the equation (1.1) which leaves the points 0, 1, $\infty$ fixed; it is well-known that $f_\mu$ exists and that $f_\mu$ is uniquely determined by $\mu$ (see [9]).

Let $h: \hat{\mathbb{R}} \rightarrow \hat{\mathbb{R}}$ be a sense-preserving homeomorphism such that $h(\infty) = \infty$. Following [9], we say that $h$ is quasisymmetric if there exists a constant $\rho \geq 1$ such that

$$1/\rho \leq (h(x+t) - h(x))/(h(x) - h(x-t)) \leq \rho$$

for $x \in \mathbb{R}$ and $t > 0$. In this case, we denote by $\rho_h$ the infimum of the constants $\rho$ which satisfy (1.2).

Let $G$ be a Fuchsian group. We define $H(G)$ as the set of all the quasisymmetric functions $h$ which satisfy the following conditions: for every $\gamma \in G$, there exists $\gamma^* \in SL'(2, \mathbb{R})$ such that

$$h \circ \gamma \circ h^{-1} = \gamma^* \circ h,$$
and

\[(1.4) \quad h \text{ leaves the points } 0, 1, \infty \text{ fixed,} \]

where \(\gamma^*|_h\) means the restriction of \(\gamma^*\) to \(\hat{H}\). We define \(\Sigma(G)\) as the family of all the closed subsets \(\sigma\) of \(\hat{H}\), which are invariant under \(G\) and satisfy

\[(1.5) \quad \sigma \supset \Lambda(G) \cup \{0, 1, \infty\}.\]

For \(h \in H(G)\) and \(\sigma \in \Sigma(G)\), we define \(F(G, h, \sigma)\) as the set of all the quasiconformal automorphisms \(f\) of \(U\) satisfying

\[(1.6) \quad fGf^{-1} \subset SL'(2, \mathbb{R}),\]

and

\[(1.7) \quad f|_{\sigma} = h|_{\sigma}.\]

In view of (1.4), (1.5) and (1.7), we see that each \(f \in F(G, h, \sigma)\) is of the form \(f = f_{\mu}\) for some \(\mu \in L_\infty(U)\) with \(||\mu|| < 1\). Moreover, for \(\mu \in L_\infty(U)\) with \(||\mu|| < 1\), we can easily check that \(f = f_{\mu}\) satisfies (1.6) if and only if \(\mu\) satisfies the condition

\[\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \quad \text{for every } \gamma \in G.\]

In the case where \(F(G, h, \sigma)\) is not empty, we put

\[(1.8) \quad k(G, h, \sigma) = \inf ||\mu||,\]

where the infimum is taken over all \(f_{\mu} \in F(G, h, \sigma)\). By means of a normal family argument of quasiconformal mappings, we can check that there exists some \(f_{\nu} \in F(G, h, \sigma)\) with \(||\nu|| = k(G, h, \sigma)\) (see [9]). Such a mapping \(f_{\nu}\) is said to be extremal in the class \(F(G, h, \sigma)\).

Let \(G_1\) be a Fuchsian group and let \(G_2\) be a subgroup of \(G_1\). Then \(G_2\) is, of course, a Fuchsian group. It follows from the definitions that \(H(G_1) \subset H(G_2)\), \(\Sigma(G_1) \subset \Sigma(G_2)\) and that

\[(1.9) \quad F(G_1, h, \sigma) \subset F(G_2, h, \sigma)\]

for \(h \in H(G_1)\) and \(\sigma \in \Sigma(G_1)\). It is quite interesting to know how far informations about \(F(G_i, h, \sigma)\) for \(i = 1, 2,\) have an effect on each other. In this paper we shall be concerned with this kind of problems and prove some theorems.

2. **Teichmüller space theory and \(F(G, h, \sigma)\).** In this section we translate some known results about Teichmüller space theory into the language of our formulation.

At first, the following result follows easily from the definitions.
Lemma 1. Let $G$ be a Fuchsian group, $h \in H(G)$ and $\sigma \in \Sigma(G)$. Suppose that $F(G, h, \sigma)$ is not empty. Let $\gamma$ be an arbitrarily chosen and fixed element of $G$. Then $f_\gamma \circ \gamma \circ f_\gamma^{-1}$ is identical with $\gamma^*$ for every $f_\gamma \in F(G, h, \sigma)$, where $\gamma^* \in \text{SL}'(2, \mathbb{R})$ is determined by the condition (1.3) and does not depend on the choice of $f_\gamma$. Further, $k(G, h, \sigma) = 0$ if and only if $1 \in F(G, h, \sigma)$.

Let $L$ be the lower half-plane. For every Fuchsian group $G$, we denote by $B(L, G)$ the Banach space consisting of all the holomorphic functions $\phi$ in $L$, which satisfy
\[
\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \quad \text{for every } \gamma \in G, \text{ and } \sup |(z - \bar{z})^2 \phi(z)| < \infty.
\]
If we define $T(G)$ as the set of those $h \in H(G)$ such that $F(G, h, \hat{R})$ is not empty, then $T(G)$ is the so-called (unreduced) Teichmüller space for the Fuchsian group $G$. From now on, the trivial group $\{1\}$ is simply denoted by 1, provided that there is no confusion. It is well-known in Teichmüller space theory that there exists a canonical injection $\Psi : H(1) \to B(L, 1)$. Earle [4] includes the result:

\[
(2.1) \quad \Psi(H(G)) = \Psi(H(1)) \cap B(L, G)
\]
for every Fuchsian group $G$. Kra [6] proved that

\[
(2.2) \quad \Psi(T(G)) = \Psi(H(1)) \cap B(L, G)
\]
for every finitely generated Fuchsian group $G$. Furthermore, it is obvious that

\[
F(G, h, \hat{R}) \subset F(G, h, \sigma) \quad \text{for every } \sigma \in \Sigma(G).
\]
According to (2.1), (2.2) and (2.3), we have the following lemma.

Lemma 2. Let $G$ be a finitely generated Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $h \in H(G)$ and every $\sigma \in \Sigma(G)$.

Remark 1. In the case where $G = 1$ and $\sigma = \hat{R}$, it is well-known that Ahlfors and Beurling [1] includes the following result: Suppose that $h \in H(1)$. Then $F(1, h, \hat{R})$ is not empty, and $k(1, h, \hat{R}) \leq \rho_1^2$.

Now we state the following Lemma 3, of which we shall not make use in this paper. It is, however, a noteworthy information with respect to arbitrary, not necessarily finitely generated, Fuchsian groups. The argument developed in the proof of Theorem 2 in Lehto [8] and (2.3) ensure the validity of our Lemma 3.

Lemma 3. Let $G$ be a Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $\sigma \in \Sigma(G)$ and every $h \in H(G)$ such that $\rho_h < \sqrt{2}$.

3. Extremal quasiconformal mappings. Let $G$ be a Fuchsian group
and $\sigma \subseteq \Sigma(G)$. Let $\phi$ be a holomorphic quadratic differential for the configuration $(U, G)$, that is, a holomorphic function in $U$ satisfying

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \quad \text{for every } \gamma \in G.$$ 

We denote by $A(G, \sigma)$ the space consisting of all the holomorphic quadratic differentials $\phi$ for $(U, G)$, which are continuously extensible to $R\setminus \sigma$ and real on $R\setminus \sigma$, and satisfy

$$||\phi||_g \equiv \iint_{U/G} |\phi(z)| \, dx \, dy < \infty.$$

The space $A(G, \sigma)$ is a real Banach space with norm $|| \cdot ||_g$. We denote by $A(G, \sigma)_1$ the set of those $\phi \in A(G, \sigma)$ with $||\phi||_g = 1$. As is well-known, $\dim_R A(G, \sigma)$ is finite if and only if $G$ is finitely generated and the set $(\sigma \setminus \Lambda(G))/G$ is finite.

The fundamental inequality, referred to in the title of Bers [3], plays an important role to characterize extremal mappings. In his paper [3], Bers proved the most generalized form of the inequality (see Theorem 2 in [3]). It has several very important applications (see [5] and [11]). Here we state slightly modified forms of some results in Gardiner [5] as the following Lemmas 4 and 5.

The corresponding results in [5] say that the following Lemmas 4 and 5 hold whenever $G$ has no elliptic elements. But the proofs of them in [5] are based on Theorem 2 in [3] and are still effectively valid, even if $G$ may have elliptic elements.

**Lemma 4.** Suppose that $f_\mu \in F(G, h, \sigma)$. Put $K = (1 + k)/(1 - k)$, where $k = k(G, h, \sigma)$. Then, for every $\phi$ in $A(G, \sigma)_1$,

$$1/K \leq \iint_{U/G} |\phi| \frac{|1 - \mu \phi|/|\phi|}{1 - |\mu|^2} \, dx \, dy .$$

(3.1)

**Lemma 5.** Suppose that $f_\mu \in F(G, h, \sigma)$, where

$$\mu = k|\phi|/|\phi|$$

for a constant $k$ such that $0 \leq k < 1$ and some $\phi \in A(G, \sigma)_1$. Then $f_\mu$ is unique extremal in the class $F(G, h, \sigma)$.

A quasiconformal mapping $f_\mu \in F(G, h, \sigma)$, with $\mu$ of the above form (3.2), is said to be a Teichmüller mapping with finite norm in the class $F(G, h, \sigma)$.

**Lemma 6.** Suppose that $f_\mu \in F(G, h, \sigma)$. Then $f_\mu$ is extremal in the class $F(G, h, \sigma)$ if and only if

$$||\mu|| = \sup |\text{Re} \iint_{U/G} \mu \phi \, dx \, dy| ,$$

(3.3)
where the supremum is taken over all \( \phi \in A(G, \sigma) \), and \( \text{Re} A \) denotes the real part of \( A \).

Proof. The necessity is due to Bers [2]. The sufficiency follows as a by-product of the fundamental inequality. In fact, we can prove the sufficiency by making use of Lemma 4 and, for the sake of completeness, we shall give the proof.

We assume that \( \mu \) satisfies (3.3). For \( \phi \) in \( A(G, \sigma) \), we have \(-\phi \in A(G, \sigma)\). Thus we see that there exists a sequence \( \phi_n \in A(G, \sigma) \), \( n = 1, 2, \ldots \), such that

\[
\|\mu\| = \lim_{n \to \infty} \text{Re} \int_{U/G} \mu \phi_n \, dx \, dy.
\]

Put \( K = (1+k)/(1-k) \), where \( k = k(G, h, \sigma) \). Then, by Lemma 4,

\[
1/K \leq \int_{U/G} \frac{|\phi_n| (1 - \|\mu\|/|\phi_n|)^2}{1 - |\mu|^2} \, dx \, dy
\]

\[
\leq ((1 + \|\mu\|^2)/(1 - |\mu|^2)) - (2 \text{ Re} \int_{U/G} \mu \phi_n \, dx \, dy)/(1 - |\mu|^2).
\]

By letting \( n \) tend to \( \infty \) in the above inequality, in view of (3.4), we have

\[
1/K \leq (1 - \|\mu\|)/(1 + \|\mu\|).
\]

It follows from (1.8) and (3.5) that \( \|\mu\| = k \). This means that \( f_\mu \) is extremal in the class \( F(G, h, \sigma) \).

Let \( f_\mu \in F(G, h, \sigma) \) be extremal. Then we have seen that \( \mu \) satisfies (3.3). In this case, we can easily check that \( f_\mu \) is a Teichmüller mapping with finite norm if and only if the supremum in the right-hand side of (3.3) is attained for some \( \phi_0 \in A(G, \sigma) \) (see [10]). In particular, this happens, provided that \( \dim_{R} A(G, \sigma) \) is finite. Thus, noting Lemma 2, we have the following result.

**Lemma 7.** Let \( G \) be a finitely generated Fuchsian group and \( \sigma \in \Sigma(G) \). Suppose that the set \( (\sigma \setminus \Lambda(G))/G \) is finite. Then, for every \( h \in H(G) \), \( F(G, h, \sigma) \) is not empty and contains a Teichmüller mapping with finite norm.

Let \( G \) be a Fuchsian group and let \( \Phi \) be a holomorphic function in \( U \). The Poincaré series \( \Theta_\sigma \Phi \) of \( \Phi \) is defined by

\[
(\Theta_\sigma \Phi)(z) = \sum_{\gamma \in \sigma} \Phi(\gamma(z))\gamma'(z)^2.
\]

Bers [3] proved the fundamental inequality by an application of Poincaré series. The following lemma is implicitly established in [3] on the way of the proof of the inequality.

**Lemma 8.** Let \( G \) be a Fuchsian group and \( \sigma \in \Sigma(G) \). Then, for every \( \Phi \) in
A(1, \sigma), the series \( \Theta_0 \Phi \), defined by (3.6), converges absolutely and uniformly on compact subsets of \( U \). And, further, the mapping \( \Theta_0 \) gives a real linear continuous mapping of \( A(1, \sigma) \) onto \( A(G, \sigma) \), and the operator norm of \( \Theta_0 \) is not greater than 1.

Remark 2. In the case \( \sigma = \hat{\mathcal{R}} \), the assertion is well-known (see the presentation in Kra [7]).

4. A Fuchsian group \( G_1 \) and its subgroup \( G_2 \). Let \( G_1 \) be a Fuchsian group and let \( G_2 \) be a subgroup of \( G_1 \). Let \( h \in H(G_1) \) and \( \sigma \in \Sigma(G_1) \) be given. Suppose that \( F(G_1, h, \sigma) \) is not empty. Then, owing to (1.9), we see that \( k(G_1, h, \sigma) \geq k(G_2, h, \sigma) \). Now we prove the following theorem.

Theorem 1. Let \( G_1 \supseteq G_2 \) be Fuchsian groups such that the index \([G_1:G_2]\) of \( G_2 \) in \( G_1 \) is finite. Let \( h \in H(G_1) \) and \( \sigma \in \Sigma(G_1) \) be given. Suppose that \( F(G_2, h, \sigma) \) is not empty. Then

1. \( F(G_1, h, \sigma) \) is not empty,

and

2. \( k(G_1, h, \sigma) = k(G_2, h, \sigma) \).

Proof. By our assumption, the index \( n = [G_1:G_2] \) is finite. In this case, it clearly follows that \( A(G_1, \sigma) \subseteq A(G_2, \sigma) \) and that, for every \( \phi \) in \( A(G_1, \sigma) \),

(4.1) \[ ||\phi||_{G_1} = ||\phi||_{G_2}/n. \]

First we show that, if \( F(G_1, h, \sigma) \) is not empty, then \( k(G_1, h, \sigma) = k(G_2, h, \sigma) \). Let \( f_\mu \) be extremal in the class \( F(G_1, h, \sigma) \). Then, by Lemma 6, there exists a sequence \( \phi_j \in A(G_1, \sigma), j = 1, 2, \ldots \), such that

(4.2) \[ k(G_1, h, \sigma) = ||\mu|| = \lim_{j \to \infty} |\Re \int_{U/G_1} \mu \phi_j \, dxdy|. \]

But we can rewrite (4.2) as follows:

(4.3) \[ ||\mu|| = \lim_{j \to \infty} |\Re \int_{U/G_2} \mu \phi_j/\sqrt{n} \, dxdy|. \]

Since \( \phi_j/\sqrt{n} \in A(G_2, \sigma) \), we see by (1.9), (4.3) and Lemma 6 that \( f_\mu \) is extremal in the class \( F(G_2, h, \sigma) \). This means \( k(G_1, h, \sigma) = k(G_2, h, \sigma) \).

Now we show (1). Since the index \( n = [G_1:G_2] \) is finite, if \( G_2 \) is a finite group, then so is \( G_1 \). In this case, (1) clearly follows by Lemma 2. Thus it remains to prove (1) in the case where \( G_2 \) is an infinite group. Since every Fuchsian group consists of a countable number of elements, we may assume that \( G_2 = \{1, \gamma_1, \gamma_2, \ldots\} \). For every \( j = 1, 2, \ldots \), let \( G_{2,j} \) be the group generated by \( \{\gamma_1, \gamma_2, \ldots, \gamma_j\} \). Then
Choose coset representatives \( \delta_0 = 1, \delta_1, \ldots, \delta_{n-1} \) in \( G \) such that
\[
G_1 = G + G_2 \delta_1 + \cdots + G_2 \delta_{n-1}.
\]

For every \( j = 1, 2, \ldots \), let \( G_1,j \) be the group generated by \( G_2,j \) and \( \{\delta_1, \delta_2, \ldots, \delta_{n-1}\} \), and put \( G_2^* = G_1,j \cap G_2 \). Then, as is well-known in group theory, it follows that

\[
[G_1,j; G_2^*] \subseteq [G_1; G_2] < \infty \quad \text{for} \quad j = 1, 2, \ldots.
\]

Since every \( G_1,j \) is finitely generated, every \( F(G_1,j, h, \sigma) \) is not empty by Lemma 2. Let \( f_{\mu,j} \) be extremal in the class \( F(G_1,j, h, \sigma) \) for \( j = 1, 2, \ldots \). Then, as was proved in the first half of the proof, we see by (4.4) that

\[
||\mu_j|| = k(G_2^*, h, \sigma) \quad \text{for} \quad j = 1, 2, \ldots.
\]

On the other hand, because \( F(G_2, h, \sigma) \subseteq F(G_2^*, h, \sigma) \) for \( j = 1, 2, \ldots \), we have

\[
k = k(G_2, h, \sigma) \leq k(G_2^*, h, \sigma) \quad \text{for} \quad j = 1, 2, \ldots.
\]

According to (4.5) and (4.6), we see that the family \( \{f_{\mu,j}\}, j = 1, 2, \ldots \), forms a subfamily of a closed normal family of \( K \)-quasiconformal automorphisms of \( U \), where \( K = (1+k)/(1-k) \) (see [9]). Taking a subsequence if necessary, we may assume that the sequence \( f_{\mu,j}, j = 1, 2, \ldots \), converges to a quasiconformal automorphism \( f_\mu \) of \( U \). Noting Lemma 1, we can easily check that \( f_\mu \) belongs to \( F(G_1, h, \sigma) \). This completes the proof of the theorem.

**Remark 3.** Let \( G \) be a Fuchsian group, \( h \in H(G) \) and \( \sigma \in \Sigma(G) \). If \( G \) is an infinite group, then we have an enumeration \( 1, \gamma_1, \gamma_2, \ldots \), of all the elements of \( G \). In this case, for every \( j = 1, 2, \ldots \), let \( G_j \) be the group generated by \( \{\gamma_1, \gamma_2, \ldots, \gamma_j\} \). Since every topological subspace of \( \mathbb{R} \) is separable, if the set \( \sigma \setminus (\Lambda(G) \cup \{0, 1, \infty\}) \) is infinite, then there exists a countable dense subset \( \{a_1, a_2, \ldots, a_j, \ldots\} \) of \( \sigma \setminus (\Lambda(G) \cup \{0, 1, \infty\}) \). In this case, for every \( j = 1, 2, \ldots \), let \( \sigma_j \) be the smallest subset of \( \sigma \), which belongs to \( \Sigma(G) \) and which contains \( \{a_1, a_2, \ldots, a_j\} \). Then, for every \( j = 1, 2, \ldots \), we can easily check that an arbitrary element of \( \sigma\setminus \Lambda(G) \) is of the form \( \gamma(x) \) for some \( \gamma \in G \), where \( x = 0, 1, \infty \) or \( a_i, 1 \leq i \leq j \). In particular, for every \( j = 1, 2, \ldots \), the set \( (\sigma \setminus \Lambda(G))/G \) is finite. Under the above assumptions and notations, suppose that \( F(G, h, \sigma) \) is not empty. Then, as in the latter part of the proof of Theorem 1, we can check the following results by a normal family argument and Lemma 1:

\[
(a) \quad k(G, h, \sigma) = \lim_{j \to \infty} k(G_j, h, \sigma)
\]

whenever \( G \) is an infinite group, and
(b) \[ k(G, h, \sigma) = \lim_{j \to \infty} k(G, h, \sigma_j) \]

whenever the set \( \sigma \backslash (\Lambda(G) \cup \{0, 1, \infty\}) \) is infinite.

Now we shall state one more lemma and prove some theorems.

**Lemma 9.** Let \( G_1 \supseteq G_2 \) be Fuchsian groups and \( \sigma \in \Sigma(G_1) \). Let \( \Phi \) be an arbitrarily chosen and fixed element of \( A(1, \sigma) \). Put \( \phi_1 = \Theta_{G_1} \Phi \) and \( \phi_2 = \Theta_{G_2} \Phi \).

Suppose that \( \phi_2 \neq 0 \). Then

\[ \|\phi_1\|_{G_1} \leq \|\phi_2\|_{G_2} \leq \|\Phi\|_1. \]

Furthermore, the three following conditions are equivalent to each other:

1. \( \|\phi_1\|_{G_1} = \|\phi_2\|_{G_2} \),
2. \( \phi_2 \in A(G_1, \sigma) \),
3. \( n = [G_1 : G_2] < \infty \), and \( \phi_1 = n\phi_2 \).

**Proof.** Choose coset representatives \( \delta_0 = 1, \delta_1, \delta_2, \ldots \), in \( G_1 \), which end with \( \delta_n \) if \( n = [G_1 : G_2] < \infty \), such that

\[ G_1 = G_2 + G_2\delta_1 + G_2\delta_2 + \cdots. \]

Let \( \omega \) be a fundamental region in \( U \) with respect to \( G_1 \), that is, \( \omega \) is an open subset of \( U \) such that the measure of the boundary \( \partial \omega \) of \( \omega \) in \( U \) vanishes, \( \omega \cap \gamma(\omega) \) is empty for \( 1 \neq \gamma \in G_1 \), and such that each \( z \in U \) is contained in some \( \gamma(\omega) \cup \delta(\omega) \cup \cdots \) is a fundamental region in \( U \) with respect to \( G_2 \). We put \( m = n - 1(\geq 1) \) if \( n = [G_1 : G_2] < \infty \), otherwise \( m = \infty \). Since the series \( \Theta_{G_1} \Phi \) converges absolutely and uniformly on compact subsets of \( U \), we see that

\[
\phi_i(z) = \sum_{i=0}^{m} \left( \sum_{\gamma \in G_2} \Phi(\gamma \circ \delta_i(z)) \gamma'(\delta_i(z))^2 \delta'(z)^2 \right)
= \sum_{i=0}^{m} \phi_2(\delta_i(z)) \delta'(z)^2.
\]

Thus

\[
\int_\omega |\phi_i(z)| \, dx \, dy \leq \sum_{i=0}^{m} \int_\omega |\phi_2(\delta_i(z)) \delta'(z)^2| \, dx \, dy
= \int_{\omega \cup \delta(\omega) \cup \cdots} |\phi_2(z)| \, dx \, dy
= \int_{U/G_2} |\phi_2(z)| \, dx \, dy.
\]
Similarly
\[ \int_{\mathbb{U}/G_2} |\phi_2| \, dx \, dy \leq \int_{\mathbb{U}} |\Phi| \, dx \, dy. \]

It remains to prove the latter part of the lemma. Assume that (4.8) holds with equality. Then, for every \( i = 1, 2, \ldots \),

\[ \iint_{\mathbb{U}} |\phi_2(x) + \phi_2(\delta_i(x))\delta_i(z)|^2 \, dx \, dy = \iint_{\mathbb{U}} (|\phi_2(x)| + |\phi_2(\delta_i(x))\delta_i(z)|^2) \, dx \, dy. \]

By (4.9), we see that, for every \( i = 1, 2, \ldots \),

\[ \phi_2(x)\overline{\phi_2(\delta_i(z))}\delta_i(z)^2 \geq 0 \quad \text{for } z \in \omega. \]

We can easily check that, for every \( i = 1, 2, \ldots \), (4.10) happens if and only if there exists a constant \( c \geq 0 \) such that

\[ \phi_2(\delta_i(z))\delta_i(z)^2 = c_i \phi_2(z) \quad \text{for } z \in \omega, \text{ hence for } z \in \mathbb{U}. \]

As \( \phi_1 \equiv \infty \), we see by (4.7) that \( c \equiv 1 + \sum_{i=1}^{\lambda} c_i \) is finite and that \( \phi_1(z) = c\phi_2(z) \). Thus (1) implies (2).

Now we note that, if the index \([G_1; G_2]\) is infinite, then \( A(G_1, \sigma) \cap A(G_2, \sigma) = \{0\} \). Noting this, we see by (4.7) that (2) implies (3). Finally, by (4.1), (3) clearly implies (1). The proof of our lemma is complete.

**Theorem 2.** Let \( G_1 \supsetneq G_2 \) be Fuchsian groups and \( \sigma \in \Sigma(G_1) \). Let \( \nu \) be a measurable function in \( \mathbb{U} \) such that \( \nu = k |\phi_2|/|\phi_2| \), where \( k \) is a constant such that \( 0 < k < 1 \) and \( \phi_2 \in A(G_2, \sigma) \). Then \( \phi_2 \) belongs to \( A(G_1, \sigma) \) if and only if \( f_\nu G_1 f_\nu^{-1} \subseteq SL'(2, \mathbb{R}) \). Furthermore, in the case where \([G_1; G_2] < \infty \) and \( \phi_2 \notin A(G_1, \sigma) \), \( f_\nu \) does not belong to \( H(G_1) \).

**Proof.** If \( \phi_2 \) belongs to \( A(G_1, \sigma) \), then

\[ \nu(\gamma(z)\gamma'(z)/\gamma'(z)) = \nu(z) \quad \text{for every } \gamma \in G_1. \]

So \( f_\nu \) is compatible with \( G_1 \), that is, \( f_\nu G_1 f_\nu^{-1} \subseteq SL'(2, \mathbb{R}) \).

Now we show the converse. Assume that \( f_\nu \) is compatible with \( G_1 \). By Lemma 8, there exists some \( \Phi \in \mathcal{A}(1, \sigma) \) such that \( \phi_2 = \Theta_{G_2} \Phi \). Put \( \phi_1 = \Theta_{G_1} \Phi \). Then we can easily check that

\[ ||\nu|| = \iint_{U/G_2} \nu \phi_2 \, dx \, dy = \iint_{U/G_1} \nu \phi_1 \, dx \, dy. \]

By Lemma 9, we have \( ||\phi_1||_{G_1} \leq 1 \). But (4.11) does not hold whenever \( ||\phi_1||_{G_1} < 1 \). This implies \( ||\phi_1||_{G_1} = 1 \). Hence we see by Lemma 9 that \( \phi_2 \in A(G_1, \sigma) \).
It remains to prove the latter part of the theorem. Assume that \([G_1 : G_2] < \infty\) and that \(\phi_2 \in \mathcal{A}(G_1, \sigma)\). Put \(h = f_{1,k}\). Clearly \(f_1\) is contained in \(F(G_2, h, \sigma)\). Now we shall prove that \(h \notin H(G_1)\). Assume the contrary. Then \(h\) belongs to \(H(G_1)\), and thus \(F(G_1, h, \sigma)\) is not empty by (1) in Theorem 1. Let \(f_\mu\) be extremal in the class \(F(G_1, h, \sigma)\). By (2) in Theorem 1, we see that \(f_\mu\) is extremal in the class \(F(G_2, h, \sigma)\), too. On the other hand, by Lemma 5, \(f_\mu\) is unique extremal in the class \(F(G_2, h, \sigma)\). Thus \(f_\mu\) is identical with \(f_1\). Therefore we see that \(f_1\) is compatible with \(G_1\), and that \(\phi_2 \in \mathcal{A}(G_1, \sigma)\). This contradicts our assumption. This shows that \(h \notin H(G_1)\).

As immediate corollaries, we deduce the following results.

**Corollary 2.1.** Let \(G_1\) be a Fuchsian group which has a subgroup \(G_2\) such that \([G_1 : G_2] = \infty\) and let \(\sigma \in \Sigma(G_1)\). Suppose that \(v = k|\phi_2|/\phi_2\), where \(k\) is a constant such that \(0 < k < 1\) and \(\phi_2 \in \mathcal{A}(G_2, \sigma)\). Then \(f_1\) is not compatible with \(G_1\).

**Corollary 2.2.** Let \(G_1\) be a Fuchsian group which has a finitely generated subgroup \(G_2\) such that \([G_1 : G_2] = \infty\). For a given \(\sigma \in \Sigma(G_1)\) and a given \(h \in H(G_1)\) with \(h_{1,\sigma} \neq 1_{1,\sigma}\), suppose that \(F(G_1, h, \sigma)\) is not empty. Then

\[
k(G_1, h, \sigma) > k(G_2, h, \sigma^*)
\]

for every \(\sigma^* \in \Sigma(G_2)\) with \(\sigma^* \subset \sigma\), such that the set \((\sigma^* \setminus \Lambda(G_2))|G_2\) is finite.

**Proof.** Assume the contrary. Then \(k(G_1, h, \sigma) = k(G_2, h, \sigma^*)\) for some \(\sigma^* \in \Sigma(G_2)\), where \(\sigma^* \subset \sigma\) and the set \((\sigma^* \setminus \Lambda(G_2))|G_2\) is finite. Let \(f_\mu\) be extremal in the class \(F(G_1, h, \sigma)\). As \(h_{1,\sigma} \neq 1_{1,\sigma}\), \(f_\mu\) is not conformal by Lemma 1. By our assumption, \(f_\mu\) is extremal in the class \(F(G_2, h, \sigma^*)\), too. Thus, owing to Lemma 5 and Lemma 7, we see that \(f_\mu\) is a Teichmüller mapping with finite norm in the class \(F(G_2, h, \sigma^*)\), and that \(f_\mu\) is not conformal. Hence, by Corollary 2.1, \(f_\mu\) does not belong to \(F(G_1, h, \sigma)\). This is absurd. This contradiction proves Corollary 2.2.

Let \(G\) be a Fuchsian group and \(\sigma \in \Sigma(G)\) and let \(f\) be a quasiconformal automorphism of \(U\). We say that \(f\) is compatible with \(G\) on \(\sigma\) if \(f\) satisfies the following property: for every \(\gamma \in G\), there exists \(\gamma^* \in SL'(2, \mathbb{R})\) such that

\[
f \circ \gamma(x) = \gamma^* \circ f(x) \quad \text{for all } x \in \sigma.
\]

**Theorem 3.** Let \(G_1\) be a Fuchsian group which has a normal subgroup \(G_2\) such that \([G_1 : G_2] = \infty\). For a given \(\sigma \in \Sigma(G_1)\) and a given \(h \in H(G_1)\) with \(h_{1,\sigma} \neq 1_{1,\sigma}\), suppose that \(F(G_2, h, \sigma)\) is not empty. Then \(F(G_2, h, \sigma)\) admits no Teichmüller mappings with finite norm.

We shall prove the following slightly stronger form of Theorem 3.
Theorem 3'. Let $G_1$ be a Fuchsian group which has a normal subgroup $G_2$ such that $[G_1: G_2] = \infty$. Suppose that $\sigma \in \Sigma(G_1)$ and that $\nu = k\phi|/\phi$, where $k$ is a constant such that $0 < k < 1$ and $\phi \in A(G_2, \sigma)$. Then $f_\nu$ is not compatible with $G_1$ on $\sigma$. In particular, $f_{\nu,h}$ does not belong to $H(G_1)$.

Proof. Assume the contrary. Then $f_\nu$ is compatible with $G_1$ on $\sigma$. Let $\gamma$ be an arbitrary element of $G_1$. Then there exists $\gamma^* \in SL'(2, \mathbb{R})$ which satisfies (4.12) for $f = f_\nu$. We put $g = \gamma^* f_{\nu} \gamma$ and $h = g_{\nu,h}$. Since $G_2$ is a normal subgroup of $G_1$, we can easily check that $h$ belongs to $H(G_2)$ and that both $g$ and $f_\nu$ belong to $F(G_2, h, \sigma)$. By Lemma 5, we see that $f_\nu$ is unique extremal in the class $F(G_2, h, \sigma)$. On the other hand, the norm of the complex dilatation of $g$ is equal to $||\nu||$, and thus $g$ is also extremal in the class $F(G_2, h, \sigma)$. Therefore $g$ is identical with $f_\nu$. In other words, it holds that

$$f_\nu \circ \gamma(z) = \gamma^* f_\nu(z) \quad \text{for } z \in U.$$  

Because the choice of $\gamma \in G_1$ is arbitrary, we see by (4.13) that $f_\nu$ is compatible with $G_1$. But it contradicts Corollary 2.1. This contradiction proves the theorem.

References


Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan