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Author(s)	Sakan, Ken-ichi
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ON QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

KEN-ICHI SAKAN

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1. Introduction. Let U be the upper half-plane and let $\hat{\boldsymbol{R}}=\boldsymbol{R}\cup\{\infty\}$ be the extended real line. The conformal automorphisms of U form the real Möbius group $SL'(2, \boldsymbol{R})$, that is, $SL(2, \boldsymbol{R})$ modulo its center. A discrete subgroup G of $SL'(2, \boldsymbol{R})$ is called a Fuchsian group; the trivial group $G=\{1\}$ is included, where 1 means the identity transformation of $SL'(2, \boldsymbol{R})$. It is well-known that a Fuchsian group G acts discontinuously on U and that the limit set $\Lambda(G)$ is a closed subset of $\hat{\boldsymbol{R}}$, which is invariant under G.

We denote by $L_\infty(U)$ the Banach space consisting of measurable functions μ on U with finite L_∞ norm $||\mu||$. Let f be a sense-preserving homeomorphism of U. Such a mapping f is said to be quasiconformal if it has generalized derivatives $f_{\bar z}$ and f_z which satisfy the Beltrami equation

$$f_{\bar z}=\mu f_z \tag{1.1}$$

for some $\mu\in L_\infty(U)$ with $||\mu||<1$. As is known, a quasiconformal automorphism f of U is extensible to a homeomorphism of $U\cup\hat{\boldsymbol{R}}$. By abuse of language, we introduce no new name for the extended mapping. For every $\mu\in L_\infty(U)$ with $||\mu||<1$, we denote by f_μ the homeomorphic solution of the equation (1.1) which leaves the points 0, 1, ∞ fixed; it is well-known that f_μ exists and that f_μ is uniquely determined by μ (see [9]).

Let h: $\hat{\boldsymbol{R}}\to\hat{\boldsymbol{R}}$ be a sense-preserving homeomorphism such that $h(\infty)=\infty$. Following [9], we say that h is quasisymmetric if there exists a constant $\rho\geqq 1$ such that

$$1/\rho\leqq(h(x+t)-h(x))/(h(x)-h(x-t))\leqq\rho \tag{1.2}$$

for $x\in\boldsymbol{R}$ and $t>0$. In this case, we denote by ρ_h the infimum of the constants ρ which satisfy (1.2).

Let G be a Fuchsian group. We define $H(G)$ as the set of all the quasisymmetric functions h which satisfy the following conditions: for every $\gamma\in G$, there exists $\gamma^*\in SL'(2, \boldsymbol{R})$ such that

$$h\circ\gamma\circ h^{-1}=\gamma^*{}_{|\hat{\boldsymbol{R}}}, \tag{1.3}$$

and

(1.4) h leaves the points 0, 1, ∞ fixed,

where $\gamma^*{}_{|\hat{\boldsymbol{R}}}$ means the restriction of γ^* to $\hat{\boldsymbol{R}}$. We define $\Sigma(G)$ as the family of all the closed subsets σ of $\hat{\boldsymbol{R}}$, which are invariant under G and satisfy

$$\sigma\supset\Lambda(G)\cup\{0, 1, \infty\}\,. \tag{1.5}$$

For $h\in H(G)$ and $\sigma\in\Sigma(G)$, we define $F(G, h, \sigma)$ as the set of all the quasiconformal automorphisms f of U satisfying

$$fGf^{-1}\subset SL'(2, \boldsymbol{R})\,, \tag{1.6}$$

and

$$f_{|\sigma}=h_{|\sigma}\,. \tag{1.7}$$

In view of (1.4), (1.5) and (1.7), we see that each $f\in F(G, h, \sigma)$ is of the form $f=f_\mu$ for some $\mu\in L_\infty(U)$ with $||\mu||<1$. Moreover, for $\mu\in L_\infty(U)$ with $||\mu||<1$, we can easily check that $f=f_\mu$ satisfies (1.6) if and only if μ satisfies the condition

$$\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z)=\mu(z) \qquad \text{for every } \gamma\in G\,.$$

In the case where $F(G, h, \sigma)$ is not empty, we put

$$k(G, h, \sigma)=\inf ||\mu||\,, \tag{1.8}$$

where the infimum is taken over all $f_\mu\in F(G, h, \sigma)$. By means of a normal family argument of quasiconformal mappings, we can check that there exists some $f_\nu\in F(G, h, \sigma)$ with $||\nu||=k(G, h, \sigma)$ (see [9]). Such a mapping f_ν is said to be extremal in the class $F(G, h, \sigma)$.

Let G_1 be a Fuchsian group and let G_2 be a subgroup of G_1. Then G_2 is, of course, a Fuchsian group. It follows from the definitions that $H(G_1)\subset H(G_2)$, $\Sigma(G_1)\subset\Sigma(G_2)$ and that

$$F(G_1, h, \sigma)\subset F(G_2, h, \sigma) \tag{1.9}$$

for $h\in H(G_1)$ and $\sigma\in\Sigma(G_1)$. It is quite interesting to know how far informations about $F(G_i, h, \sigma)$ for $i=1, 2$, have an effect on each other. In this paper we shall be concerned with this kind of problems and prove some theorems.

2. Teichmüller space theory and $F(G, h, \sigma)$. In this section we translate some known results about Teichmüller space theory into the language of our formulation.

At first, the following result follows easily from the definitions.

Lemma 1. *Let G be a Fuchsian group, $h\in H(G)$ and $\sigma\in\Sigma(G)$. Suppose that $F(G, h, \sigma)$ is not empty. Let γ be an arbitrarily chosen and fixed element of G. Then $f_\mu\circ\gamma\circ f_\mu^{-1}$ is identical with γ^* for every $f_\mu\in F(G, h, \sigma)$, where $\gamma^*\in SL'(2, \boldsymbol{R})$ is determined by the condition (1.3) and does not depend on the choice of f_μ. Further, $k(G, h, \sigma)=0$ if and only if $1\in F(G, h, \sigma)$.*

Let L be the lower half-plane. For every Fuchsian group G, we denote by $B(L,G)$ the Banach space consisting of all the holomorphic functions ϕ in L, which satisfy

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \qquad \text{for every } \gamma\in G, \text{ and } \sup|(z-\bar{z})^2\phi(z)|<\infty.$$

If we define $T(G)$ as the set of those $h\in H(G)$ such that $F(G, h, \hat{\boldsymbol{R}})$ is not empty, then $T(G)$ is the so-called (unreduced) Teichmüller space for the Fuchsian group G. From now on, the trivial group $\{1\}$ is simply denoted by 1, provided that there is no confusion. It is well-known in Teichmüller space theory that there exists a canonical injection $\Psi: H(1)\to B(L, 1)$. Earle [4] includes the result:

$$\Psi(H(G)) = \Psi(H(1))\cap B(L, G) \tag{2.1}$$

for every Fuchsian group G. Kra [6] proved that

$$\Psi(T(G)) = \Psi(H(1))\cap B(L, G) \tag{2.2}$$

for every finitely generated Fuchsian group G. Furthermore, it is obvious that

$$F(G, h, \hat{\boldsymbol{R}})\subset F(G, h, \sigma) \qquad \text{for every } \sigma\in\Sigma(G)\,. \tag{2.3}$$

According to (2.1), (2.2) and (2.3), we have the following lemma.

Lemma 2. *Let G be a finitely generated Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $h\in H(G)$ and every $\sigma\in\Sigma(G)$.*

REMARK 1. In the case where $G=1$ and $\sigma=\hat{\boldsymbol{R}}$, it is well-known that Ahlfors and Beurling [1] includes the following result: Suppose that $h\in H(1)$. Then $F(1, h, \hat{\boldsymbol{R}})$ is not empty, and $k(1, h, \hat{\boldsymbol{R}})\leqq\rho_h^2$.

Now we state the following Lemma 3, of which we shall not make use in this paper. It is, however, a noteworthy information with respect to arbitrary, not necessarily finitely generated, Fuchsian groups. The argument developed in the proof of Theorem 2 in Lehto [8] and (2.3) ensure the validity of our Lemma 3.

Lemma 3. *Let G be a Fuchsian group. Then $F(G, h, \sigma)$ is not empty for every $\sigma\in\Sigma(G)$ and every $h\in H(G)$ such that $\rho_h<\sqrt{2}$.*

3. Extremal quasiconformal mappings. Let G be a Fuchsian group and $\sigma\in\Sigma(G)$. Let ϕ be a holomorphic quadratic differential for the configuration (U, G), that is, a holomorphic function in U satisfying

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \qquad \text{for every } \gamma\in G\,.$$

We denote by $A(G, \sigma)$ the space consisting of all the holomorphic quadratic differentials ϕ for (U, G), which are continuously extensible to $\boldsymbol{R}\backslash\sigma$ and real on $\boldsymbol{R}\backslash\sigma$, and satisfy

$$||\phi||_G\equiv\iint_{U/G}|\phi(z)|\,dxdy<\infty.$$

The space $A(G, \sigma)$ is a real Banach space with norm $||\ ||_G$. We denote by $A(G, \sigma)_1$ the set of those $\phi\in A(G, \sigma)$ with $||\phi||_G=1$. As is well-known, $\dim_{\boldsymbol{R}} A(G, \sigma)$ is finite if and only if G is finitely generated and the set $(\sigma\backslash\Lambda(G))/G$ is finite.

The fundamental inequality, referred to in the title of Bers [3], plays an important role to characterize extremal mappings. In his paper [3], Bers proved the most generalized form of the inequality (see Theorem 2 in [3]). It has several very important applications (see [5] and [11]). Here we state slightly modified forms of some results in Gardiner [5] as the following Lemmas 4 and 5. The corresponding results in [5] say that the following Lemmas 4 and 5 hold whenever G has no elliptic elements. But the proofs of them in [5] are based on Theorem 2 in [3] and are still effectively valid, even if G may have elliptic elements.

Lemma 4. *Suppose that* $f_\mu\in F(G, h, \sigma)$. *Put* $K=(1+k)/(1-k)$, *where* $k=k(G,h,\sigma)$. *Then, for every* ϕ *in* $A(G, \sigma)_1$,

$$1/K\leqq\iint_{U/G}|\phi|\frac{|1-\mu\phi/|\phi||^2}{1-|\mu|^2}\,dxdy\,. \tag{3.1}$$

Lemma 5. *Suppose that* $f_\mu\in F(G, h, \sigma)$, *where*

$$\mu = k|\phi|/\phi \tag{3.2}$$

for a constant k *such that* $0\leqq k<1$ *and some* $\phi\in A(G, \sigma)_1$. *Then* f_μ *is unique extremal in the class* $F(G, h, \sigma)$.

A quasiconformal mapping $f_\mu\in F(G, h, \sigma)$, with μ of the above form (3.2), is said to be a Teichmüller mapping with finite norm in the class $F(G, h, \sigma)$.

Lemma 6. *Suppose that* $f_\mu\in F(G, h, \sigma)$. *Then* f_μ *is extremal in the class* $F(G, h, \sigma)$ *if and only if*

$$||\mu|| = \sup\left|\operatorname{Re}\iint_{U/G}\mu\phi\,dxdy\right|, \tag{3.3}$$

where the supremum is taken over all $\phi\in A(G, \sigma)_1$, *and* Re A *denotes the real part of* A.

Proof. The necessity is due to Bers [2]. The sufficiency follows as a by-product of the fundamental inequality. In fact, we can prove the sufficiency by making use of Lemma 4 and, for the sake of completeness, we shall give the proof.

We assume that μ satisfies (3.3). For ϕ in $A(G, \sigma)_1$, we have $-\phi\in A(G, \sigma)_1$. Thus we see that there exists a sequence $\phi_n\in A(G, \sigma)_1$, $n=1, 2, \cdots$, such that

$$\|\mu\| = \lim_{n\to\infty} \mathrm{Re} \iint_{U/G} \mu\phi_n dxdy \,. \tag{3.4}$$

Put $K=(1+k)/(1-k)$, where $k=k(G, h, \sigma)$. Then, by Lemma 4,

$$\begin{aligned} 1/K &\leqq \iint_{U/G} \frac{|\phi_n|\,|1-\mu\phi_n/|\phi_n|\,|^2}{1-|\mu|^2} dxdy \\ &\leqq ((1+\|\mu\|^2)/(1-\|\mu\|^2)) - (2\,\mathrm{Re} \iint_{U/G} \mu\phi_n dxdy)/(1-\|\mu\|^2)\,. \end{aligned}$$

By letting n tend to ∞ in the above inequality, in view of (3.4), we have

$$1/K \leqq (1-\|\mu\|)/(1+\|\mu\|)\,. \tag{3.5}$$

It follows from (1.8) and (3.5) that $\|\mu\|=k$. This means that f_μ is extremal in the class $F(G, h, \sigma)$.

Let $f_\mu\in F(G ,h, \sigma)$ be extremal. Then we have seen that μ satisfies (3.3). In this case, we can easily check that f_μ is a Teichmüller mapping with finite norm if and only if the supremum in the right-hand side of (3.3) is attained for some $\phi_0\in A(G, \sigma)_1$ (see [10]). In particular, this happens, provided that $\dim_{\mathbf{R}} A(G, \sigma)$ is finite. Thus, noting Lemma 2, we have the following result.

Lemma 7. *Let G be a finitely generated Fuchsian group and $\sigma\in\Sigma(G)$. Suppose that the set $(\sigma\backslash\Lambda(G))/G$ is finite. Then, for every $h\in H(G)$, $F(G, h, \sigma)$ is not empty and contains a Teichmüller mapping with finite norm.*

Let G be a Fuchsian group and let Φ be a holomorphic function in U. The Poincaré series $\Theta_G\Phi$ of Φ is defined by

$$(\Theta_G\Phi)(z) = \sum_{\gamma\in G} \Phi(\gamma(z))\gamma'(z)^2\,. \tag{3.6}$$

Bers [3] proved the fundamental inequality by an application of Poincaré series. The following lemma is implicitly established in [3] on the way of the proof of the inequality.

Lemma 8. *Let G be a Fuchsian group and $\sigma\in\Sigma(G)$. Then, for every Φ in*

$A(1, \sigma)$, the series $\Theta_G\Phi$, defined by (3.6), *converges absolutely and uniformly on compact subsets of U. And, further, the mapping Θ_G gives a real linear continuous mapping of $A(1, \sigma)$ onto $A(G, \sigma)$, and the operator norm of Θ_G is not greater than* 1.

REMARK 2. In the case $\sigma=\hat{\boldsymbol{R}}$, the assertion is well-known (see the presentation in Kra [7]).

4. A Fuchsian group G_1 and its subgroup G_2. Let G_1 be a Fuchsian group and let G_2 be a subgroup of G_1. Let $h\in H(G_1)$ and $\sigma\in\Sigma(G_1)$ be given. Suppose that $F(G_1, h, \sigma)$ is not empty. Then, owing to (1.9), we see that $k(G_1, h, \sigma)\geqq k(G_2, h, \sigma)$. Now we prove the following theorem.

Theorem 1. *Let $G_1\supsetneqq G_2$ be Fuchsian groups such that the index $[G_1: G_2]$ of G_2 in G_1 is finite. Let $h\in H(G_1)$ and $\sigma\in\Sigma(G_1)$ be given. Suppose that $F(G_2, h, \sigma)$ is not empty. Then*

(1) *$F(G_1, h, \sigma)$ is not empty,*

and

(2) *$k(G_1, h, \sigma)=k(G_2, h, \sigma)$.*

Proof. By our assumption, the index $n\equiv[G_1: G_2]$ is finite. In this case, it clearly follows that $A(G_1, \sigma)\subset A(G_2, \sigma)$ and that, for every ϕ in $A(G_1, \sigma)$,

$$(4.1) \qquad ||\phi||_{G_1} = ||\phi||_{G_2}/n\,.$$

First we show that, if $F(G_1, h, \sigma)$ is not empty, then $k(G_1, h, \sigma)=k(G_2, h, \sigma)$. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. Then, by Lemma 6, there exists a sequence $\phi_j\in A(G_1, \sigma)_1, j=1, 2,\cdots$, such that

$$(4.2) \qquad k(G_1, h, \sigma) = ||\mu|| = \lim_{j\to\infty}|\operatorname{Re}\iint_{U/G_1}\mu\phi_j\,dxdy|\,.$$

But we can rewrite (4.2) as follows:

$$(4.3) \qquad ||\mu|| = \lim_{j\to\infty}|\operatorname{Re}\iint_{U/G_2}\mu\phi_j/n\,dxdy|\,.$$

Since $\phi_j/n\in A(G_2, \sigma)_1$, we see by (1.9), (4.3) and Lemma 6 that f_μ is extremal in the class $F(G_2, h, \sigma)$. This means $k(G_1, h, \sigma)=k(G_2, h, \sigma)$.

Now we show (1). Since the index $n=[G_1: G_2]$ is finite, if G_2 is a finite group, then so is G_1. In this case, (1) clearly follows by Lemma 2. Thus it remains to prove (1) in the case where G_2 is an infinite group. Since every Fuchsian group consists of a countable number of elements, we may assume that $G_2=\{1, \gamma_1, \gamma_2, \cdots\}$. For every $j=1, 2, \cdots$, let $G_{2,j}$ be the group generated by $\{\gamma_1, \gamma_2, \cdots, \gamma_j\}$. Then

$$G_{2,1} \subset G_{2,2} \subset \cdots, \text{ and } G_2 = G_{2,1} \cup G_{2,2} \cup \cdots .$$

Choose coset representatives $\delta_0 = 1, \delta_1, \cdots, \delta_{n-1}$ in G_1 such that

$$G_1 = G_2 + G_2\delta_1 + \cdots + G_2\delta_{n-1} .$$

For every $j = 1, 2, \cdots$, let $G_{1,j}$ be the group generated by $G_{2,j}$ and $\{\delta_1, \delta_2, \cdots, \delta_{n-1}\}$, and put $G^*_{2,j} = G_{1,j} \cap G_2$. Then, as is well-known in group theory, it follows that

$$[G_{1,j} : G^*_{2,j}] \leqq [G_1 : G_2] < \infty \qquad \text{for } j = 1, 2, \cdots . \tag{4.4}$$

Since every $G_{1,j}$ is finitely generated, every $F(G_{1,j}, h, \sigma)$ is not empty by Lemma 2. Let f_{μ_j} be extremal in the class $F(G_{1,j}, h, \sigma)$ for $j = 1, 2, \cdots$. Then, as was proved in the first half of the proof, we see by (4.4) that

$$\|\mu_j\| = k(G^*_{2,j}, h, \sigma) \qquad \text{for } j = 1, 2, \cdots . \tag{4.5}$$

On the other hand, because $F(G_2, h, \sigma) \subset F(G^*_{2,j}, h, \sigma)$ for $j = 1, 2, \cdots$, we have

$$k \equiv k(G_2, h, \sigma) \geqq k(G^*_{2,j}, h, \sigma) \qquad \text{for } j = 1, 2, \cdots . \tag{4.6}$$

According to (4.5) and (4.6), we see that the family $\{f_{\mu_j}\}$, $j = 1, 2, \cdots$, forms a subfamily of a closed normal family of K-quasiconformal automorphisms of U, where $K = (1+k)/(1-k)$ (see [9]). Taking a subsequence if necessary, we may assume that the sequence f_{μ_j}, $j = 1, 2, \cdots$, converges to a quasiconformal automorphism f_μ of U. Noting Lemma 1, we can easily check that f_μ belongs to $F(G_1, h, \sigma)$. This completes the proof of the theorem.

REMARK 3. Let G be a Fuchsian group, $h \in H(G)$ and $\sigma \in \Sigma(G)$. If G is an infinite group, then we have an enumeration $1, \gamma_1, \gamma_2, \cdots$, of all the elements of G. In this case, for every $j = 1, 2, \cdots$, let G_j be the group generated by $\{\gamma_1, \gamma_2, \cdots, \gamma_j\}$. Since every topological subspace of $\hat{\boldsymbol{R}}$ is separable, if the set $\sigma \backslash (\Lambda(G) \cup \{0, 1, \infty\})$ is infinite, then there exists a countable dense subset $\{a_1, a_2, \cdots, a_j, \cdots\}$ of $\sigma \backslash (\Lambda(G) \cup \{0, 1, \infty\})$. In this case, for every $j = 1, 2, \cdots$, let σ_j be the smallest subset of σ, which belongs to $\Sigma(G)$ and which contains $\{a_1, a_2, \cdots, a_j\}$. Then, for every $j = 1, 2, \cdots$, we can easily check that an arbitrary element of $\sigma_j \backslash \Lambda(G)$ is of the form $\gamma(x)$ for some $\gamma \in G$, where $x = 0, 1, \infty$ or a_i, $1 \leqq i \leqq j$. In particular, for every $j = 1, 2, \cdots$, the set $(\sigma_j \backslash \Lambda(G))/G$ is finite. Under the above assumptions and notations, suppose that $F(G, h, \sigma)$ is not empty. Then, as in the latter part of the proof of Theorem 1, we can check the following results by a normal family argument and Lemma 1:

$$k(G, h, \sigma) = \lim_{j \to \infty} k(G_j, h, \sigma) \tag{a}$$

whenever G is an infinite group, and

(b) $$k(G, h, \sigma) = \lim_{j\to\infty} k(G, h, \sigma_j)$$

whenever the set $\sigma\backslash(\Lambda(G)\cup\{0, 1, \infty\})$ is infinite.

Now we shall state one more lemma and prove some theorems.

Lemma 9. *Let $G_1 \supsetneqq G_2$ be Fuchsian groups and $\sigma\in\Sigma(G_1)$. Let Φ be an arbitrarily chosen and fixed element of $A(1, \sigma)$. Put $\phi_1=\Theta_{G_1}\Phi$ and $\phi_2=\Theta_{G_2}\Phi$. Suppose that $\phi_2\neq 0$. Then*

$$||\phi_1||_{G_1} \leqq ||\phi_2||_{G_2} \leqq ||\Phi||_1 .$$

Furthermore, the three following conditions are equivalent to each other:

(1) $$||\phi_1||_{G_1} = ||\phi_2||_{G_2} ,$$

(2) $$\phi_2\in A(G_1, \sigma) ,$$

and

(3) $$n\equiv [G_1\colon G_2]<\infty, \quad \textit{and} \quad \phi_1 = n\phi_2 .$$

Proof. Choose coset representatives $\delta_0=1, \delta_1, \delta_2, \cdots$, in G_1, which end with δ_{n-1} if $n=[G_1\colon G_2]<\infty$, such that

$$G_1 = G_2+G_2\delta_1+G_2\delta_2+\cdots .$$

Let ω be a fundamental region in U with respect to G_1, that is, ω is an open subset of U such that the measure of the boundary $\partial\omega$ of ω in U vanishes, $\omega\cap\gamma(\omega)$ is empty for $1\neq\gamma\in G_1$, and such that each $z\in U$ is contained in some $\gamma(\omega\cup\partial\omega)$, $\gamma\in G_1$. Then $\omega\cup\delta_1(\omega)\cup\delta_2(\omega)\cup\cdots$ is a fundamental region in U with respect to G_2. We put $m=n-1(\geqq 1)$ if $n=[G_1\colon G_2]<\infty$, otherwise $m=\infty$. Since the series $\Theta_{G_1}\Phi$ converges absolutely and uniformly on compact subsets of U, we see that

(4.7) $$\begin{aligned}\phi_1(z) &= \sum_{i=0}^{m}\Big(\sum_{\gamma\in G_2}\Phi(\gamma\circ\delta_i(z))\gamma'(\delta_i(z))^2\Big)\delta_i'(z)^2 \\ &= \sum_{i=0}^{m}\phi_2(\delta_i(z))\delta_i'(z)^2 .\end{aligned}$$

Thus

(4.8) $$\begin{aligned}\iint_{\omega}|\phi_1(z)|\,dxdy &\leqq \sum_{i=0}^{m}\iint_{\omega}|\phi_2(\delta_i(z))\delta_i'(z)^2|\,dxdy \\ &= \sum_{i=0}^{m}\iint_{\delta_i(\omega)}|\phi_2(z)|\,dxdy \\ &= \iint_{\omega\cup\delta_1(\omega)\cup\cdots}|\phi_2(z)|\,dxdy \\ &= \iint_{U/G_2}|\phi_2(z)|\,dxdy .\end{aligned}$$

Similarly

$$\iint_{U/G_2} |\phi_2| \, dxdy \leqq \iint_U |\Phi| \, dxdy .$$

It remains to prove the latter part of the lemma. Assume that (4.8) holds with equality. Then, for every $i=1, 2, \cdots$,

$$(4.9) \qquad \iint_\omega |\phi_2(z)+\phi_2(\delta_i(z))\delta_i'(z)^2| \, dxdy = \iint_\omega (|\phi_2(z)| + |\phi_2(\delta_i(z))\delta_i'(z)^2|) dxdy .$$

By (4.9), we see that, for every $i=1, 2, \cdots$,

$$(4.10) \qquad \phi_2(z)\overline{\phi_2(\delta_i(z))}\overline{\delta_i'(z)^2} \geqq 0 \qquad \text{for } z \in \omega .$$

We can easily check that, for every $i=1, 2, \cdots$, (4.10) happens if and only if there exists a constant $c_i \geqq 0$ such that

$$\phi_2(\delta_i(z))\delta_i'(z)^2 = c_i\phi_2(z) \qquad \text{for } z \in \omega, \text{ hence for } z \in U .$$

As $\phi_1 \not\equiv \infty$, we see by (4.7) that $c \equiv 1+\sum_{i=1}^{m} c_i$ is finite and that $\phi_1(z)=c\phi_2(z)$. Thus (1) implies (2).

Now we note that, if the index $[G_1\colon G_2]$ is infinite, then $A(G_1, \sigma) \cap A(G_2, \sigma) =\{0\}$. Noting this, we see by (4.7) that (2) implies (3). Finally, by (4.1), (3) clearly implies (1). The proof of our lemma is complete.

Theorem 2. *Let $G_1 \supsetneqq G_2$ be Fuchsian groups and $\sigma \in \Sigma(G_1)$. Let ν be a measurable function in U such that $\nu=k|\phi_2|/\phi_2$, where k is a constant such that $0<k<1$ and $\phi_2 \in A(G_2, \sigma)_1$. Then ϕ_2 belongs to $A(G_1, \sigma)$ if and only if $f_\nu G_1 f_\nu^{-1} \subset SL'(2, \boldsymbol{R})$. Furthermore, in the case where $[G_1\colon G_2]<\infty$ and $\phi_2 \notin A(G_1, \sigma)$, $f_{\nu|\hat{R}}$ does not belong to $H(G_1)$.*

Proof. If ϕ_2 belongs to $A(G_1, \sigma)$, then

$$\nu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \nu(z) \qquad \text{for every } \gamma \in G_1 .$$

So f_ν is compatible with G_1, that is, $f_\nu G_1 f_\nu^{-1} \subset SL'(2, \boldsymbol{R})$.

Now we show the converse. Assume that f_ν is compatible with G_1. By Lemma 8, there exists some $\Phi \in A(1, \sigma)$ such that $\phi_2=\Theta_{G_2}\Phi$. Put $\phi_1=\Theta_{G_1}\Phi$. Then we can easily check that

$$(4.11) \qquad \|\nu\| = \iint_{U/G_2} \nu\phi_2 \, dxdy = \iint_{U/G_1} \nu\phi_1 \, dxdy .$$

By Lemma 9, we have $\|\phi_1\|_{G_1} \leqq 1$. But (4.11) does not hold whenever $\|\phi_1\|_{G_1}<1$. This implies $\|\phi_1\|_{G_1}=1$. Hence we see by Lemma 9 that $\phi_2 \in A(G_1, \sigma)$.

It remains to prove the latter part of the theorem. Assume that $[G_1: G_2]<\infty$ and that $\phi_2 \notin A(G_1, \sigma)$. Put $h=f_{\nu|\hat{R}}$. Clearly f_ν is contained in $F(G_2, h, \sigma)$. Now we shall prove that $h \notin H(G_1)$. Assume the contrary. Then h belongs to $H(G_1)$, and thus $F(G_1, h, \sigma)$ is not empty by (1) in Theorem 1. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. By (2) in Theorem 1, we see that f_μ is extremal in the class $F(G_2, h, \sigma)$, too. On the other hand, by Lemma 5, f_ν is unique extremal in the class $F(G_2, h, \sigma)$. Thus f_μ is identical with f_ν. Therefore we see that f_ν is compatible with G_1, and that $\phi_2 \in A(G_1, \sigma)$. This contradicts our assumption. This shows that $h \notin H(G_1)$.

As immediate corollaries, we deduce the following results.

Corollary 2.1. *Let G_1 be a Fuchsian group which has a subgroup G_2 such that $[G_1: G_2]=\infty$ and let $\sigma \in \Sigma(G_1)$. Suppose that $\nu=k|\phi_2|/\phi_2$, where k is a constant such that $0<k<1$ and $\phi_2 \in A(G_2, \sigma)_1$. Then f_ν is not compatible with G_1.*

Corollary 2.2. *Let G_1 be a Fuchsian group which has a finitely generated subgroup G_2 such that $[G_1: G_2]=\infty$. For a given $\sigma \in \Sigma(G_1)$ and a given $h \in H(G_1)$ with $h_{|\sigma} \neq 1_{|\sigma}$, suppose that $F(G_1, h, \sigma)$ is not empty. Then*

$$k(G_1, h, \sigma)>k(G_2, h, \sigma^*)$$

for every $\sigma^* \in \Sigma(G_2)$ with $\sigma^* \subset \sigma$, such that the set $(\sigma^* \setminus \Lambda(G_2))/G_2$ is finite.

Proof. Assume the contrary. Then $k(G_1, h, \sigma)=k(G_2, h, \sigma^*)$ for some $\sigma^* \in \Sigma(G_2)$, where $\sigma^* \subset \sigma$ and the set $(\sigma^* \setminus \Lambda(G_2))/G_2$ is finite. Let f_μ be extremal in the class $F(G_1, h, \sigma)$. As $h_{|\sigma} \neq 1_{|\sigma}$, f_μ is not conformal by Lemma 1. By our assumption, f_μ is extremal in the class $F(G_2, h, \sigma^*)$, too. Thus, owing to Lemma 5 and Lemma 7, we see that f_μ is a Teichmüller mapping with finite norm in the class $F(G_2, h, \sigma^*)$, and that f_μ is not conformal. Hence, by Corollary 2.1, f_μ does not belong to $F(G_1, h, \sigma)$. This is absurd. This contradiction proves Corollary 2.2.

Let G be a Fuchsian group and $\sigma \in \Sigma(G)$ and let f be a quasiconformal automorphism of U. We say that f is compatible with G on σ if f satisfies the following property: for every $\gamma \in G$, there exists $\gamma^* \in SL'(2, \boldsymbol{R})$ such that

$$f \circ \gamma(x) = \gamma^* \circ f(x) \qquad \text{for all } x \in \sigma . \tag{4.12}$$

Theorem 3. *Let G_1 be a Fuchsian group which has a normal subgroup G_2 such that $[G_1: G_2]=\infty$. For a given $\sigma \in \Sigma(G_1)$ and a given $h \in H(G_1)$ with $h_{|\sigma} \neq 1_{|\sigma}$, suppose that $F(G_2, h, \sigma)$ is not empty. Then $F(G_2, h, \sigma)$ admits no Teichmüller mappings with finite norm.*

We shall prove the following slightly stronger form of Theorem 3.

Theorem 3′. *Let G_1 be a Fuchsian group which has a normal subgroup G_2 such that $[G_1: G_2]=\infty$. Suppose that $\sigma\in\Sigma(G_1)$ and that $\nu=k|\phi|/\phi$, where k is a constant such that $0<k<1$ and $\phi\in A(G_2, \sigma)_1$. Then f_ν is not compatible with G_1 on σ. In particular, $f_{\nu|\hat{R}}$ does not belong to $H(G_1)$.*

Proof. Assume the contrary. Then f_ν is compatible with G_1 on σ. Let γ be an arbitrary element of G_1. Then there exists $\gamma^*\in SL'(2, \boldsymbol{R})$ which satisfies (4.12) for $f=f_\nu$. We put $g=\gamma^{*-1}\circ f_\nu\circ\gamma$ and $h=g_{|\hat{R}}$. Since G_2 is a normal subgroup of G_1, we can easily check that h belongs to $H(G_2)$ and that both g and f_ν belong to $F(G_2, h, \sigma)$. By Lemma 5, we see that f_ν is unique extremal in the class $F(G_2, h, \sigma)$. On the other hand, the norm of the complex dilatation of g is equal to $||\nu||$, and thus g is also extremal in the class $F(G_2, h, \sigma)$. Therefore g is identical with f_ν. In other words, it holds that

$$f_\nu\circ\gamma(z)=\gamma^*\circ f_\nu(z) \qquad \text{for } z\in U. \tag{4.13}$$

Because the choice of $\gamma\in G_1$ is arbitrary, we see by (4.13) that f_ν is compatible with G_1. But it contradicts Corollary 2.1. This contradiction proves the theorem.

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyosi-ku
Osaka 558, Japan