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Author(s)	Sun, Hongbin; Wang, Shicheng; Wu, Jianchun
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SELF-MAPPING DEGREES OF TORUS BUNDLES AND TORUS SEMI-BUNDLES

HONGBIN SUN, SHICHENG WANG and JIANCHUN WU

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Abstract

Each closed oriented 3-manifold M is naturally associated with a set of integers $D(M)$, the degrees of all self-maps on M . $D(M)$ is determined for each torus bundle and semi-bundle M . The structure of torus semi-bundle is studied in detail. The paper is a part of a project to determine $D(M)$ for all 3-manifolds in Thurston's picture.

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1. Introduction

1.1. Background. Each closed oriented n -manifold M is naturally associated with a set of integers, the degrees of all self-maps on M , denoted as $D(M) = \{\deg(f) \mid f: M \rightarrow M\}$.

Indeed the calculation of $D(M)$ is a classical topic appeared in many literatures. The result is simple and well-known for dimension $n = 1, 2$, and for dimension $n > 3$, there are many interesting special results (see [2] and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes attractive in the topic and it is possible to calculate $D(M)$ for any closed oriented 3-manifold M . Since Thurston's geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in reasonable sense.

Thurston's geometrization conjecture claims that the each Jaco–Shalen–Johanson decomposition piece of a prime 3-manifold supports one of the eight geometries which are H^3 , $\widetilde{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3 , S^3 and $S^2 \times E^1$ (for details see [11] and [10]). Call a closed orientable 3-manifold M is *geometrizable* if each prime factor of M meets Thurston's geometrization conjecture.

A known rather general fact about $D(M)$ for geometrizable 3-manifolds is the following:

Theorem 1.1 ([12], Corollary 4.3). *Suppose M is a geometrizable 3-manifold. Then M admits a self-map of degree larger than 1 if and only if M is either*

- (1) *covered by a torus bundle over the circle, or*
- (2) *covered by an $F \times S^1$ for some compact surface F with $\chi(F) < 0$, or*
- (3) *each prime factor of M is covered by S^3 or $S^2 \times E^1$.*

The proof of the “only if” part in Theorem 1.1 is based on the theory of simplicial volume, and various results on 3-manifold topology and group theory. The proof of “if” part in Theorem 1.1 is a sequence of elementary constructions, which were essentially known before.

Hence for any M not listed in Theorem 1.1, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether M admits a self map of degree -1 or not. To determine $D(M)$ for geometrizable 3-manifolds listed in Theorem 1.1, let's have a close look of those 3-manifolds from geometric and topological aspects.

Among Thurston's eight geometries, six of them belong to the list in Theorem 1.1. 3-manifolds in (1) are exactly those supporting either E^3 , or Sol or Nil geometries. E^3 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundles or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert spaces having Euclidean orbifolds with three singular points. 3-manifolds in (2) are exactly those support $H^2 \times E^1$ geometry; 3-manifolds supporting S^3 or $S^2 \times E^1$ geometries form a proper subset of (3).

For 3-manifold M with S^3 -geometry, $D(M)$ has been presented recently in [1] in term of the orders of $\pi_1(M)$ and its elements (and determined earlier in [5] when the maps induce automorphisms on π_1). Note an algorithm is given to calculate the degree set of maps between S^3 -manifolds in term of their Seifert invariants [8].

To determine $D(M)$ for the remaining geometrizable 3-manifolds M , the main task is to solve the question for the following three groups ($D(M)$ is rather easy to determine for Seifert manifold M supporting $H^2 \times E^1$ or $S^2 \times E^1$ geometry):

- (a) torus bundles and semi-bundles;
- (b) Nil Seifert manifolds not in (a);

(c) connected sums of 3-manifolds in (3) do not supporting S^3 or $S^2 \times E^1$ geometries. Indeed $D(M)$ for M in (a) will be determined in this paper (hopefully all the remaining cases will be solved in a forthcoming paper by the authors and Hao Zheng).

1.2. Main result. In this paper we calculate $D(M)$ for 3-manifold M which is either a torus bundle or semi-bundle. To do this, we need first to coordinate torus bundles and semi-bundles by integer matrices in Propositions 1.3 and 1.5, then state the results of $D(M)$ in term of those matrices in Theorems 1.6 and 1.7.

CONVENTION. (1) To simplify notions, for a diffeomorphism ϕ on torus T , we also use ϕ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

(2) Each 3-manifold M is oriented, and each 3-submanifold of M and its boundary have induced orientations.

(3) Suppose S (resp. P) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold M . We use $M \setminus S$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along S (resp. removing int P , the interior of P).

DEFINITION 1.2. A torus bundle is $M_\phi = T \times I / (x, 1) \sim (\phi(x), 0)$ where ϕ is a self-diffeomorphism of the torus T and I is the interval $[0, 1]$.

For a torus bundle M_ϕ , we can isotopic ϕ to be a linear diffeomorphism, which means $\phi \in GL_2(\mathbb{Z})$ while not changing M_ϕ . Since we consider the orientable case only, ϕ must be in the special linear group $SL_2(\mathbb{Z})$.

Proposition 1.3. (1) M_ϕ admits E^3 geometry if and only if ϕ is periodical, or equivalently ϕ is conjugate to one of the following matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ of finite order 1, 2, 3, 4 and 6 respectively;

(2) M_ϕ admits Nil geometry if and only if ϕ is reducible, or equivalently ϕ is conjugate to $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \neq 0$;

(3) M_ϕ admits Sol geometry if and only if ϕ is Anosov or equivalently ϕ is conjugate to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $|a + d| > 2$, $ad - bc = 1$.

Proof. See [4]. □

DEFINITION 1.4. Let K be the Klein bottle and $N = K \tilde{\times} I$ be the twisted I -bundle over K . A torus semi-bundle $N_\phi = N \cup_\phi N$ is obtained by gluing two copies along their torus boundary ∂N via a diffeomorphism ϕ . Note N_ϕ is foliated by tori parallel to ∂N with a Klein bottle at the core of each copy of N .

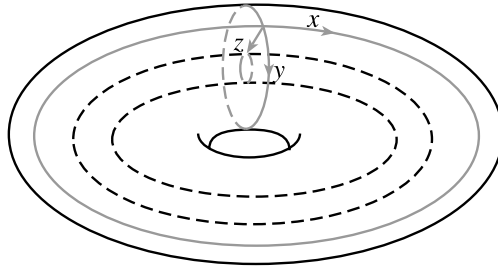


Fig. 1. Coordinates of $S^1 \times S^1 \times I$.

Let (x, y, z) be the coordinate of $S^1 \times S^1 \times I$. Then $N = S^1 \times S^1 \times I/\tau$, where τ is an orientation preserving involution such that $\tau(x, y, z) = (x + \pi, -y, 1 - z)$, and we have the double covering $p: S^1 \times S^1 \times I \rightarrow N$. Let C_x and C_y be the two circles on $S^1 \times S^1 \times \{1\}$ defined by y to be constant and x to be constant, see Fig. 1. Denote by $l_0 = p(C_x)$ (0 slope) and $l_\infty = p(C_y)$ (∞ slope) on ∂N . A *canonical coordinate* is an orientation of $l_0 \cup l_\infty$, hence there are four choices of canonical coordinate on ∂N . Once canonical coordinates on each ∂N are chosen, ϕ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi(l_0, l_\infty) = (l_0, l_\infty) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposition 1.5. *With suitable choice of canonical coordinates of ∂N , we have:*

- (1) N_ϕ admits E^3 geometry if and only if $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- (2) N_ϕ admits Nil geometry if and only if $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ where $z \neq 0$;
- (3) N_ϕ admits Sol geometry if and only if $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0$, $ad - bc = 1$.

Moreover a torus semi-bundle N_ϕ is also a torus bundle if and only if $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ under suitable choice of canonical coordinates.

We will prove Proposition 1.5 in Section 2.

Theorem 1.6. *Using matrix coordinates given by Proposition 1.3, $D(M_\phi)$ is listed in Table 1 for torus bundle M_ϕ , where $\delta(3) = \delta(6) = 1$, $\delta(4) = 0$.*

Theorem 1.7. *Using matrix coordinates given by Proposition 1.5, $D(N_\phi)$ is listed in Table 2 for torus semi-bundle N_ϕ , where $\delta(a, d) = ad/\gcd(a, d)^2$.*

Table 1. Degrees of self maps of orientable torus bundles.

M_ϕ	ϕ	$D(M_\phi)$
E^3	finite order $k = 1, 2$	\mathbb{Z}
E^3	finite order $k = 3, 4, 6$	$\{(kt + 1)(p^2 - \delta(k)pq + q^2) \mid t, p, q \in \mathbb{Z}\}$
Nil	$\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, n \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a + d > 2$	$\{p^2 + (d - a)pr/c - br^2/c \mid p, r \in \mathbb{Z}, \text{ either } br/c, (d - a)r/c \in \mathbb{Z} \text{ or } (p(d - a) - br)/c \in \mathbb{Z}\}$

Table 2. Degrees of self maps of torus semi-bundles.

N_ϕ	ϕ	$D(N_\phi)$
E^3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	\mathbb{Z}
E^3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\{2l + 1 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \neq 0$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, abcd \neq 0, ad - bc = 1$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\}, \text{ if } \delta(a, d) \text{ is even or } \{(2l + 1)^2 \mid l \in \mathbb{Z}\} \cup \{(2l + 1)^2 \cdot \delta(a, d) \mid l \in \mathbb{Z}\}, \text{ if } \delta(a, d) \text{ is odd}$

1.3. Remark on orientation reversing homeomorphisms. Suppose M is a torus bundle or semi-bundle. Then any non-zero degree map is homotopic to a covering ([12] Corollary 0.4). Hence if $-1 \in D(M)$ (which is computable by Theorems 1.6 and 1.7), then M admits an orientation reversing self homeomorphism.

If M is a torus semi-bundle, or M supports the geometry of either E^3 or Nil, then when M admits an orientation reversing self homeomorphism is explicitly presented in the following:

- Corollary 1.8.** (1) *A torus semi-bundle N_ϕ admits an orientation reversing homeomorphism if and only if ϕ is either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ where $abc \neq 0$.*
 (2) *A torus bundle M_ϕ supporting E^3 geometry admits an orientation reversing homeomorphism if and only if ϕ is either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.*
 (3) *If M supports Nil geometry, then M admits no orientation reversing homeomorphism.*

For torus bundle with given Anosov monodromy, even we can calculate whether $-1 \in D(M_\phi)$, but there seems no simple description as in Corollary 1.8. (The referee

informed us that there is a convenient description of when $-1 \in D(M_\phi)$, see Lemma 1.7, [9].)

EXAMPLE 1.9. For the torus bundle M_ϕ , $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $-1 \in D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $|a+d|=3$, then $-1 \in D(M_\phi)$. Since $p^2 + (d-a)pr/c - br^2/c = -1$ has solution $p = 1-d$, $r = c$ when $a+d=3$, and solution $p = -1-d$, $r = c$ when $a+d=-3$.

EXAMPLE 1.10. For the torus bundle M_ϕ , $\phi = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $-1 \notin D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $a+d \pm 2$ has prime decomposition $p_1^{e_1} \cdots p_n^{e_n}$ such that $p_i = 4l+3$ and $e_i = 2m+1$ for some i , then $-1 \notin D(M_\phi)$. Since if the equation $p^2 + (d-a)pr/c - br^2/c = -1$ has integer solution, $((a+d)^2 - 4)r^2 - 4c^2/c^2$ should be a square of rational number. That is $((a+d)^2 - 4)r^2 - 4c^2 = s^2$ for some integer s . Therefore $(a+d+2)(a+d-2)r^2$ is a sum of two squares. By a fact in elementary number theory, neither $a+d+2$ nor $a+d-2$ has $4k+3$ type prime factor with odd power (see p.279, [7]).

EXAMPLE 1.11. Note if $-1 \in D(M)$, then $k \in D(M)$ implies $-k \in D(M)$. For the torus bundle M_ϕ , $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers > 0 , exactly 1, 4, 5, 9, 11, 16, 19, 20 $\in D(M_\phi)$.

1.4. Organization of the paper. Theorems 1.6 and 1.7 will be proved in Sections 3 and 4 respectively. To prove these theorems, we need have a careful look of the structures of torus bundle and semi-bundles. This is carried out in Section 2.

We explain more about Section 2. The most convenient and useful reference for us is “Notes on basic 3-manifold topology” by Hatcher [4], which is not formally published, but widely circulated (see <http://www.math.cornell.edu/~hatcher/>). In particular Chapter 2 of [4] is devoted to the study of torus bundles and semi-bundles. Theorems 2.3 and 2.4 about classifications of torus bundles and semi-bundles are quoted from [4] directly. It seems that the proof of Theorem 2.4 in [4] missed an existed and rather complicated case, so we rewrite a proof for it (most parts still follow that in [4]). Lemma 2.6 studies incompressible surfaces in torus semi-bundle, which relies on the proof of Theorem 2.4. Then Proposition 1.5 is proved by using Theorem 2.4, Lemma 2.6, and Lemma 2.8 which presents the relation between gluing maps of a torus semi-bundles and its torus bundle double covers. Finally, Theorem 2.9 studies lifting of maps between torus semi-bundles to their torus bundle double covers.

2. Structures of orientable torus bundles and semi-bundles

2.1. Some elementary facts. All facts in this sub-section are known, and one can find them in [6], or more directly in [4].

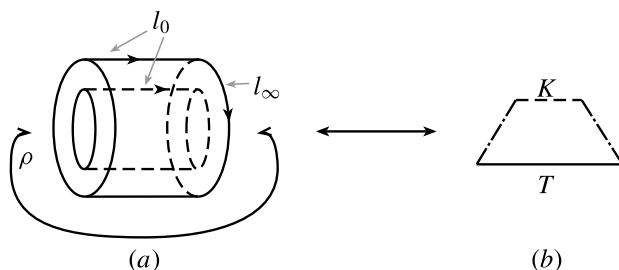


Fig. 2. Coordinates of ∂N .

DEFINITION 2.1. Suppose an oriented 3-manifold M' is a circle bundle with a given section F , where F is a compact surface with boundary components $c_1, \dots, c_n, \dots, c_{n+b}$ with $n > 0$. On each boundary component of M' , orient c_i and the circle fiber l_i so that the product of their orientations match with the induced orientation of M' . Now attaching n solid tori S_i to the first n boundary tori of M' so that the meridian of S_i is identified with slope $r_i = a_i c_i + b_i l_i$ with $a_i > 0$. Denote the resulting manifold by M which has the Seifert fiber structure extended from the circle bundle structure of M' .

We will denote this Seifert fibering of M by $M(\pm g, b; r_1, \dots, r_s)$ where g is the genus of the section F of M , with the sign $+$ if F is orientable and $-$ if F is non-orientable, here ‘genus’ of nonorientable surfaces means the number of RP^2 connected summands. When $b = 0$, call $e(M) = \sum_1^s r_i$ the Euler number of the Seifert fibration.

Another view of N described in Fig. 2 (a): N is obtained from $S^1 \times I \times I$ by identifying $S^1 \times I \times \{0\}$ with $S^1 \times I \times \{1\}$ via a diffeomorphism ρ which reflects both the S^1 and I factors. Fig. 2 (b) is a schematic picture of N which will be used in the paper.

We list some properties of N as:

Lemma 2.2. (1) N has two types of Seifert fiber structures:

- I: $M(0, 1; 1/2, -1/2)$ in which l_0 on ∂N is a regular fiber and l_∞ is the boundary of the section defining the Seifert invariant.
- II: $M(-1, 1; \)$ in which l_∞ on ∂N is a regular fiber and l_0 is the boundary of the section defining the Seifert invariant.

(2) N has three types of essential (orientable, incompressible, ∂ -incompressible) surfaces:

- I. A torus parallel to ∂N .
- II. An annulus whose boundary is l_∞ in ∂N (Fig. 3 (a)) which does not separate N .
- III. An annulus whose boundary is l_0 in ∂N (Fig. 3 (b)) which separates N .

(3) Suppose M is a torus bundle or semi-bundle and F is a closed incompressible surface in M , then F is union of parallel tori.

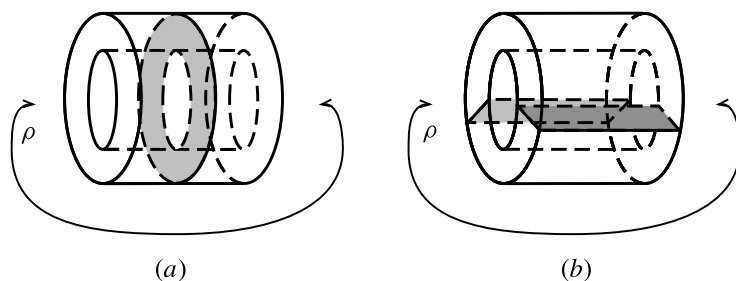


Fig. 3. Essential surface in N .

2.2. Classifications of torus bundles and semi-bundles. Orientable torus bundles and semi-bundles are classified by two theorems below.

Theorem 2.3 ([3]; [4], Theorem 2.6). *An orientable torus bundle M_ϕ is diffeomorphic to M_ψ if and only if ϕ conjugates to $\psi^{\pm 1}$ in $GL_2(\mathbb{Z})$.*

Theorem 2.4 ([4], Theorem 2.8). *The torus semi-bundle N_ϕ is diffeomorphic to N_ψ if and only if $\phi = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \psi^{\pm 1} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in $GL_2(\mathbb{Z})$, with independent choices of signs understood.*

Proof. (We start the proof as that in [4].) Suppose $f: N_\phi \rightarrow N_\psi$ is a diffeomorphism and T, T' are the torus fibers of N_ϕ, N_ψ respectively. $N_\psi \setminus T' = N_1 \cup N_2$ where N_1, N_2 are homeomorphic to N .

Since f is a diffeomorphism, two components of $N_\psi \setminus f(T)$ are both homeomorphic to N . We can isotope f , such that every component of $f(T) \cap N_i$ is an essential surface in $N_i, i = 1, 2$. So $f(T) \cap N_i$ is in the three types listed in Lemma 2.2 (2). Thus either $f(T)$ is parallel to T' , or ψ takes l_0 or l_∞ to l_0 or l_∞ .

Suppose $f(T)$ is parallel to T' . We can assume $f(T) = T'$. Then ϕ must be obtained from ψ by composing on the left and right homeomorphisms of ∂N which extend to homeomorphisms of N . Such homeomorphisms must preserve both l_0 and l_∞ (may reverse the directions), since l_0 is the unique slopes of the boundaries of essential separating annulus and l_∞ is the unique slopes of the boundaries of essential non-separating annulus in N . Theorem 2.4 is proved in this situation.

Suppose ψ takes l_0 or l_∞ to l_0 or l_∞ . Then there are three cases as below:

CASE (1) ψ takes l_∞ to l_0 (if ψ takes l_0 to l_∞ , then we consider ψ^{-1}).

CASE (2) ψ takes l_∞ to l_∞ .

CASE (3) ψ takes l_0 to l_0 .

(The proof in [4] claims that only Case (3) is possible, while we show below that only Case (2) is impossible.)

Case (1). Now $\psi = \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}$, and $N_\psi = M(-1, 0; 1/2, -1/2, z)$, and $e(M) = z$. Note:

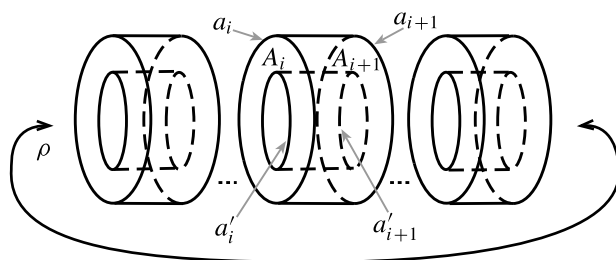


Fig. 4. Cut N_i through type (II) surfaces.

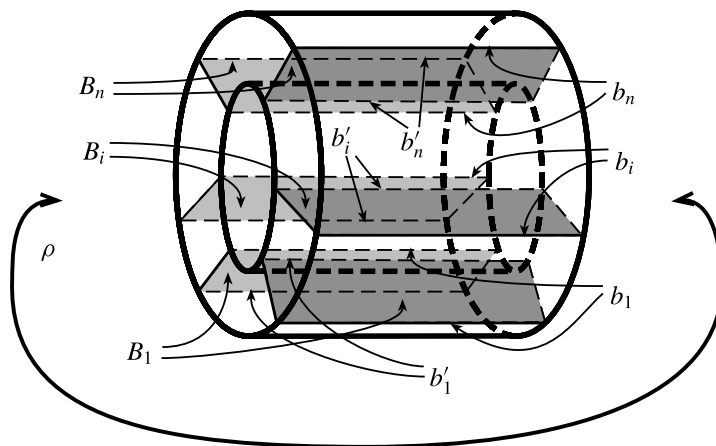


Fig. 5. Cut N_i through type (III) surfaces.

(i) $f(T) \cap N_1$ are n parallel annuli A_1, \dots, A_n of type (II) (see Fig. 4), which are located in a cyclic order in N . Set $\partial A_i = a_i \cup a'_i$, then $2n$ circles $a_1, \dots, a_n, a'_1, \dots, a'_n$ are located in cyclic order in ∂N_1 .

(ii) $f(T) \cap N_2$ are annuli B_1, \dots, B_n of type (III) (see Fig. 5), where B_{i+1} is next to B_i , $i = 1, \dots, n - 1$ in N_2 . Set $\partial B_i = b_i \cup b'_i$ then $2n$ circles $b_1, \dots, b_n, b'_1, \dots, b'_n$ are located in cyclic order in ∂N_2 .

If $n = 1$, we can check that ψ pastes A_1 and B_1 to a Klein bottle, which contradicts the fact that $f(T)$ is torus. When $n > 1$, we can assume ψ pastes a_1 to b_1 and pastes a_2 to b_2 , after reindexing A_i if necessary. By the orders of sequences of $a_1, \dots, a_n, a'_1, \dots, a'_n$ and $b_1, \dots, b_n, b'_1, \dots, b'_n$ on ∂N_1 and ∂N_2 , we have a_i is pasted to b_i , and a'_i pasted to b'_{n-i} , $i = 1, \dots, n$. So $A_i, A_{n-i}, B_i, B_{n-i}$ are pasted to one component of $f(T)$ in N_ψ , and $f(T)$ has $[(n + 1)/2]$ components. Since $f(T)$ is connected, we have $n = 2$.

Now $N_1 \setminus f(T)$ can be presented as two I -bundles over annulus: $I \times A_1$ and $I \times A_2$, where $f(T) \cap N_1 = A_1 \cup A_2$, as in Fig. 4. $N_2 \setminus f(T)$ can be presented as an I -bundle

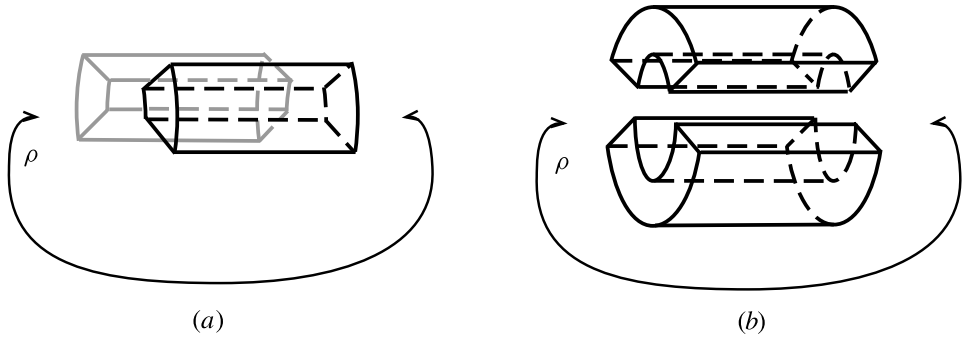


Fig. 6.

over annulus $I \times B$ as in Fig. 6 (a) and two solid tori P_1 and P_2 with the core of $P_i \cap \partial N_2$ to be the $(2, 1)$ curve of ∂P_i as in Fig. 6 (b).

If we glue those five pieces along ∂N , we get two components of $N_\psi \setminus f(T)$ which are $N'_1 = P_1 \cup_{\partial N} I \times A_1 \cup_{\partial N} P_2$ and $N'_2 = I \times A_2 \cup_{\partial N} I \times B$ (re-index A_i if needed), each of them is a copy of N . Moreover under the inherited Seifert structure of N_ψ , $N'_1 = M(0, 1; 1/2, -1/2)$ and $N'_2 = M(-1, 1;)$.

If we consider that $M(-1, 0; 1/2, -1/2, z)$ is obtained by identifying N'_1 and N'_2 along $f(T)$, we get a new semi-bundle structure so that $f(T)$ become a fiber torus. Since the Euler number of the Seifert structure is z , the new gluing map must be $\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}^{\pm 1}$. This reduces us to the situation that $f(T)$ is parallel to T' .

Case (2). Both $f(T) \cap N_i$ are type (II) surfaces, for $i = 1, 2$ (Fig. 4). Hence $f(T) \cap N_1$ is exactly as that in Case (1) (i). Similarly, $f(T) \cap N_2$ are n parallel annulus B_1, \dots, B_n located in a cyclic order in N . Set $\partial B_i = b_i \cup b'_i$, then $2n$ circles $b_1, \dots, b_n, b'_1, \dots, b'_n$ are located in cyclic order in ∂N_2 .

We can assume ψ paste a_1 to b_1 and paste a_2 to b_2 (re-index $\{B_i\}$ if needed). Then we have a_i is pasted to b_i , and a'_i pasted to b'_i , $i = 1, \dots, n$. So A_i and B_i are pasted to one component of $f(T)$ in N_ψ . Since $f(T)$ is connected, $n = 1$. But here $f(T)$ does not separate N_ψ , it is impossible.

Case (3). (We copy the proof of [4] for this case.) Now $\psi = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, and $N_\psi = M(0, 0; 1/2, 1/2, -1/2, -1/2, z)$, $e(N_\psi) = z$. (Both $f(T) \cap N_i$ are type (III).)

We may assume that $f(T)$ has been isotoped to be either vertical or horizontal in this Seifert fibering. Since a connected horizontal essential surface is not separating, $f(T)$ must be vertical. Then $f(T)$ must separate $M(0, 0; 1/2, 1/2, -1/2, -1/2, z)$ into two copies of N both having the inherited Seifert structure $M(0, 1; 1/2, -1/2)$. We can rechoose the semi-bundle structure so that $f(T)$ become a fiber torus. Then for the new torus semi-bundle structure the gluing map must also be $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. This reduces us to the situation that $f(T)$ is parallel to T' . □

2.3. Incompressible surfaces.

Lemma 2.5 ([4], Lemma 2.7). *For a torus bundle M_ϕ , if ϕ is not conjugate to $\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, then any essential closed surface in M_ϕ is isotopic to a union of torus fibers.*

Lemma 2.6. *If a torus semi-bundle N_ϕ has no torus bundle structure, then any essential closed surface in N_ϕ is isotopic to copies of torus fibers of a torus semi-bundle structure on N_ϕ , which is isomorphic to N_ϕ .*

Proof. Let F be an essential close surface in $N_\phi = N_1 \cup N_2$. By Lemma 2.2 (3), F is a union of parallel tori. For our purpose we may assume that F is a torus. Isotope F so that $F \cap N_i$ is essential in N_i . Then each component of $F \cap N_i$ must be in one of the three types listed in Lemma 2.2.

If $F \cap N_i$ is of type (I), then the proof is finished.

There are two cases remaining:

- (a) Both $F \cap N_i$ are of type (II) for $i = 1, 2$ (Fig. 4). Then $N_i \setminus F$ are I-bundles over $N_i \cap F$. Gluing those two I-bundles along ∂N will get an I-bundle over F and N_ϕ is obtained from this I-bundle by identifying its top and bottom, which provides a torus bundle structure of N_ϕ .
- (b) Some $F \cap N_i$ is of type (III), say $i = 2$ (Fig. 5). Then F is the same as $f(T)$ either in Case (1) or Case (3) of the proof of Theorem 2.4, depends on $F \cap N_1$ is of type (III) or type (II).

As indicated in the proof of Theorem 2.4, we can rechoose the new torus semi-bundle structure N_ψ so that F become a fiber torus; moreover if choosing suitable coordinates, we can make ψ to be ϕ . □

2.4. Coordinates of torus semi-bundles. Call a map $g: (M, \partial M) \rightarrow (M', \partial M')$ is *proper* if $g^{-1}(\partial M') \subset \partial M$.

Lemma 2.7. *If $V = T \times I$ with the two boundaries T^+, T^- and $g: (V, T^+, T^-) \rightarrow (N, \partial N)$ is a proper map, then $(g|_{T^+})_* = \tau_* \cdot (g|_{T^-})_*$, where $\tau_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Proof. Let $p: T \times I \rightarrow N$ be the double covering and τ be the deck transformation map.

Since $g_*(\pi_1(V)) = (g|_{T^+})_*(\pi_1(T^+)) \subset \pi_1(\partial N) \subset \pi_1(N)$, thus g can be lifted to a map $\tilde{g}: V \rightarrow T \times I$.

$$\begin{array}{ccc}
 (T \times I, T \times \{0\}, T \times \{1\}) & \xrightarrow{\tau} & (T \times I, T \times \{1\}, T \times \{0\}) \\
 \tilde{g} \nearrow & \downarrow p & \nwarrow p \\
 (V, T^+, T^-) & \xrightarrow{g} & (N, \partial N)
 \end{array}$$

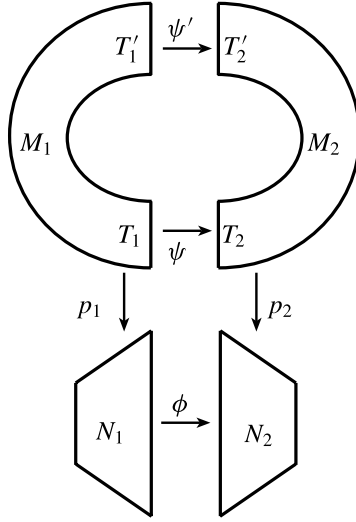


Fig. 7. N_ϕ is double covered by $M_{\tau\phi\tau^{-1}}$.

From the commuted diagram above, we have:

$$\begin{cases} g|_{T^-} = p|_{T \times \{1\}} \circ \tilde{g}|_{T^-}, \\ g|_{T^+} = p|_{T \times \{1\}} \circ \tau|_{T \times \{0\}} \circ \tilde{g}|_{T^+}. \end{cases}$$

We can choose coordinate on $(T \times I, T \times \{0\}, T \times \{1\})$, such that $p|_{T \times \{1\}} = id$.

When considering fundamental group, we have $(\tilde{g}|_{T^-})_* = (\tilde{g}|_{T^+})_*$. Thus by the above equation:

$$(g|_{T^+})_* = \tau_* \cdot (g|_{T^-})_*$$

where $\tau_* = (\gamma|_{T \times \{0\}})_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. □

Lemma 2.8. *A torus semi-bundle N_ϕ is doubly covered by a torus bundle $M_{\tau\phi\tau^{-1}}$ where $\tau(x, y) = (x + \pi, -y)$ with suitable choice of coordinate (x, y) on the torus.*

Proof. Let $N_\phi = N_1 \cup_\phi N_2$ with $\partial N_1 = \partial N_2 = T$. Let $p: M \rightarrow N_\phi$ be the double cover, where M is a torus bundle, $p^{-1}(N_i) = M_i$ is homeomorphic to $T \times I$, $p^{-1}(T) = T_1 \cup T_2$. Cut M along T_1, T_2 , get $M \setminus T_1 \cup T_2$. The two boundaries of M_i are denoted by T_i and T'_i , T_1 is pasted to T_2 by ψ , T'_1 is pasted to T'_2 by ψ' . Let $p_i = p|_{M_i}$. All of these are shown in Fig. 7.

We can choose coordinate on T_1, T_2 , such that $(p_i|_{T_i})_* = id$. Since T'_i is parallel to T_i , we can identify $\pi_1(T'_i)$ with $\pi_1(T_i)$. By Lemma 2.7, we have $(p_i|_{T'_i})_* = \tau_* \cdot (p_i|_{T_i})_*$.

From Fig. 7, we know that

$$\begin{cases} (p_2|_{T_2})_* \circ \psi = \phi \circ (p_1|_{T_1})_*, \\ (p_2|_{T'_2})_* \circ \psi' = \phi \circ (p_1|_{T'_1})_*. \end{cases}$$

Then we get

$$\begin{cases} \psi = \phi, \\ \psi' = \tau \circ \phi \circ \tau. \end{cases}$$

Thus M has the torus bundle structure $M_{\psi'/\psi^{-1}} = M_{\tau\phi\tau\phi^{-1}}$. □

By Theorem 2.4, and the fact that $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}^{-1} = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix}$, with suitable choice of canonical coordinates of ∂N , we can set ϕ is one of the four matrices: $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$, $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0, ad - bc = 1$.

When ϕ is in the first three matrices, N_ϕ is a Seifert manifold with Euler number z . N_ϕ is E^3 manifold if $z = 0$ and is Nil manifold if $z \neq 0$. Now suppose $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0, ad - bc = 1$. Then by Lemma 2.8, N_ϕ is double covered by $M_{\tau\phi\tau\phi^{-1}}$. Since

$$(\tau\phi\tau\phi^{-1})_* = \tau_* \cdot \phi_* \cdot \tau_* \cdot \phi_*^{-1} = \begin{pmatrix} ad + bc & -2ab \\ -2cd & ad + bc \end{pmatrix},$$

we have

$$|\text{Trace}((\tau\phi\tau\phi^{-1})_*)| = 2|ad + bc| = 2|ad - bc + 2bc| = 2|2bc + 1| > 2.$$

By Proposition 1.3, $M_{\tau\phi\tau\phi^{-1}}$ admits Sol geometry, thus N_ϕ admits Sol geometry. The first part of Proposition 1.5 is proved.

If N_ϕ also has torus bundle structure, it must have non-separating essential torus. Recall the proof of Lemma 2.6, an essential torus in N_ϕ can be non-separating only if case (a) is happened, and in this case $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ under suitable choice of canonical coordinates, and N_ϕ does have torus bundle structure. This finishes the ‘‘moreover’’ part of Proposition 1.5.

2.5. Lifting automorphism from semi-bundle to bundle.

Theorem 2.9. *Suppose $f: N_\phi \rightarrow N_\psi$ is a non-zero degree map and $f^{-1}(T')$ is a union of copies of T , where T, T' are the torus fiber of N_ϕ, N_ψ respectively. Then we*

have commute diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\tilde{f}} & M' \\
 p \downarrow & & \downarrow p' \\
 N_\phi & \xrightarrow{f} & N_\psi
 \end{array}$$

where M, M' are the torus bundle which are double covers of N_ϕ, N_ψ respectively and $\tilde{f}: M \rightarrow M'$ is a lift of f .

Proof. We only have to check $f_*(p_*(\pi_1(M))) \subset p'_*(\pi_1(M'))$.

Let \tilde{T}, \tilde{T}' be one of the lifting of T, T' in M, M' respectively. In torus bundle M , we have the exact sequence:

$$1 \rightarrow \pi_1(\tilde{T}) \rightarrow \pi_1(M) \rightarrow \pi_1(S^1) \rightarrow 1.$$

In torus semi-bundle N_ϕ , we have another exact sequence:

$$1 \rightarrow \pi_1(T) \rightarrow \pi_1(N_\phi) \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow 1.$$

Since $f^{-1}(T')$ is a union of copies of T , we can assume $f(T) = T'$. Then we have the commuted diagram (every row is exact):

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \pi_1(\tilde{T}) & \xrightarrow{\tilde{i}_1} & \pi_1(M) & \xrightarrow{\tilde{j}_1} & \pi_1(S^1) & \longrightarrow & 1 \\
 & & (p|_*) \downarrow & & p_* \downarrow & & \bar{p}_* \downarrow & & \\
 1 & \longrightarrow & \pi_1(T) & \xrightarrow{i_1} & \pi_1(N_\phi) & \xrightarrow{j_1} & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & (f|_*) \downarrow & & f_* \downarrow & & \tilde{f}_* \downarrow & & \\
 1 & \longrightarrow & \pi_1(T') & \xrightarrow{i_2} & \pi_1(N_\psi) & \xrightarrow{j_2} & \mathbb{Z}_2 * \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & (p'|_*) \downarrow & & p'_* \downarrow & & \bar{p}'_* \downarrow & & \\
 1 & \longrightarrow & \pi_1(\tilde{T}') & \xrightarrow{\tilde{i}_2} & \pi_1(M') & \xrightarrow{\tilde{j}_2} & \pi_1(S^1) & \longrightarrow & 1,
 \end{array}$$

here $\bar{p}_*, \bar{p}'_*, \tilde{f}_*$ are the maps among the fundamental groups of the base spaces of fiber bundles induced by the maps among the fundamental groups of the total spaces.

We present the group $\mathbb{Z}_2 * \mathbb{Z}_2$ by $\langle a, b \mid a^2 = b^2 = 1 \rangle$ and choose the generator a, b such that $\bar{p}_*(1) = ab, \bar{p}'_*(1) = ab$ (here 1 is the generator of $\pi_1(S^1)$).

Since $a^2 = b^2 = 1$, so $\tilde{f}_*(a)^2 = \tilde{f}_*(b)^2 = 1$, then $\tilde{f}_*(a), \tilde{f}_*(b)$ must be of the form $ab \cdots ba$ or $ba \cdots ab$, and $\tilde{f}_*(ab) = (ab)^k$ or $(ba)^k = (ab)^{-k}$. So $\tilde{f}_*(\bar{p}_*(\pi_1(S^1))) \subset \bar{p}'_*(\pi_1(S^1))$.

For any $\alpha \in \pi_1(M)$, let $\beta = f_*(p_*(\alpha))$. Since $j_2(\beta) = \tilde{f}_*(\tilde{p}_*(\tilde{j}_1(\alpha))) \in \tilde{p}'_*(\pi_1(S^1))$, and there is $\gamma \in \pi_1(M')$ such that $\tilde{p}'_*(\tilde{j}_2(\gamma)) = j_2(\beta)$, so

$$j_2(p'_*(\gamma) \cdot \beta^{-1}) = \tilde{p}'_*(\tilde{j}_2(\gamma)) \cdot j_2(\beta^{-1}) = j_2(\beta) \cdot j_2(\beta^{-1}) = 1.$$

Since $(p'|)_*$ is an isomorphism, there is $\delta \in \pi_1(\tilde{T}')$ such that $i_2((p'|)_*(\delta)) = p'_*(\gamma) \cdot \beta^{-1}$. We have

$$p'_*(\tilde{i}_2(\delta^{-1}) \cdot \gamma) = i_2((p'|)_*(\delta^{-1})) \cdot p'_*(\gamma) = (p'_*(\gamma) \cdot \beta^{-1})^{-1} \cdot p'_*(\gamma) = \beta.$$

So $f_*(p_*(\pi_1(M))) \subset p'_*(\pi_1(M'))$, thus \tilde{f} exists. □

3. The degrees of self maps of torus bundles

We are going to prove Theorem 1.6 (ref. Proposition 1.3). There are two cases to consider:

CASE 1: ϕ is conjugated to $\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$. Now M_ϕ is a Seifert manifold whose Euler number of Seifert fibering $e(M_\phi)$ is equal to n .

(1.I) If $n = 0$, M_ϕ is T^3 or $S^1 \tilde{\times} S^1 \tilde{\times} S^1$. Here $\phi = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, any 2×2 integer matrix A commutes with ϕ , so M_ϕ admits self maps of any degrees.

(1.II) If $n \neq 0$, for a none zero degree map $f: M_\phi \rightarrow M_\phi$, by [12, Corollary 0.4], f is homotopic to a covering map $g: M_\phi \rightarrow M_\phi$. We can choose a suitable Seifert fibering of M_ϕ such that g is a fiber preserving map. Denote the orbifold of M_ϕ by $O(M_\phi)$. By [10, Lemma 3.5], we have:

$$(3.1) \quad \begin{cases} e(M_\phi) = e(M_\phi) \cdot \frac{l}{m}, \\ \deg(g) = l \cdot m, \end{cases}$$

where l is the covering degree of $O(M_\phi) \rightarrow O(M_\phi)$ and m is the fiber degree.

Since $e(M_\phi) \neq 0$, from equation (3.1) we get $l = m$. Thus $\deg(f) = \deg(g)$ is a square number. Conversely, given a square number l^2 , it is easy to construct a covering map $f: M_\phi \rightarrow M_\phi$ of degree l^2 .

CASE 2: $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not conjugated to $\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

Theorem 3.1. *Suppose ϕ is not conjugated to $\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ M_ϕ admits a self map of degree $l \neq 0$ if and only if there exist a 2×2 nondegenerate integer matrix A and a positive integer k such that $l = k \cdot \epsilon \cdot \det(A)$ and $A \cdot \phi_* = (\phi^\epsilon)_*^k \cdot A$ where $\epsilon = \pm 1$.*

Proof. For a torus fiber $T \in M_\phi$, T is incompressible. Suppose $f: M_\phi \rightarrow M_\phi$ is a self-map of degree $l \neq 0$. By [6, Lemma 6.5], f is homotopic to $g: M_\phi \rightarrow M_\phi$

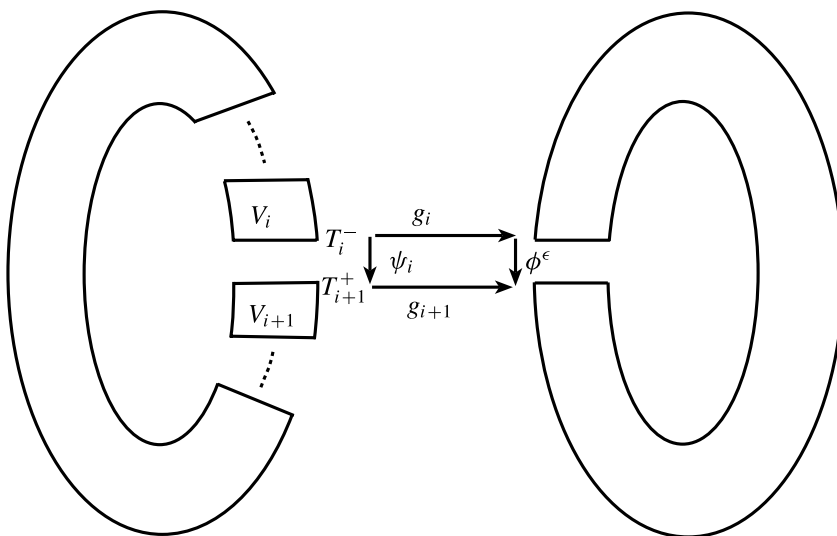


Fig. 8. Non-zero degree self-map of M_ϕ .

such that $g^{-1}(T)$ is an incompressible surface of M_ϕ . Thus by Lemma 2.5, $g^{-1}(T)$ is isotopic to a union of torus fibers.

Suppose $M_\phi \setminus g^{-1}(T)$ has k components V_1, \dots, V_k . Each V_i is a $T \times I$. Denote two torus boundary components of V_i by T_i^+ and T_i^- , and the homeomorphism gluing T_i^- to T_{i+1}^+ by ψ_i see Fig. 8. Then $M_{\psi_k \circ \dots \circ \psi_1} = M_\phi$. By choosing suitable coordinate on the torus fiber, we have $\psi_k \circ \dots \circ \psi_0 = \phi^\epsilon$, $\epsilon = \pm 1$ according to Theorem 2.3. Below we assume $\psi_k \circ \dots \circ \psi_0 = \phi$ (replace ϕ by ϕ^{-1} if needed). Let $\tilde{g}: M_\phi \setminus g^{-1}(T) \rightarrow M_\phi \setminus T$ be the map induced by g . We have the following commuted diagram:

$$(3.2) \quad \begin{array}{ccc} M_\phi \setminus g^{-1}(T) & \xrightarrow{\tilde{g}} & M_\phi \setminus T \\ \cup \psi_i \downarrow & & \downarrow \phi^\epsilon \\ M_\phi & \xrightarrow{g} & M_\phi. \end{array}$$

Denote the restriction of \tilde{g} to V_i by g_i . From the commuted diagram in Fig. 8, we have:

$$(3.3) \quad g_{i+1}|_{T_{i+1}^+} \circ \psi_i = \phi^\epsilon \circ g_i|_{T_i^-},$$

where $\epsilon = \pm 1$, $i = 1, \dots, k$ and if $i = k$ then $i + 1$ is 1.

Since T_i^- is parallel to T_i^+ , we can identify $\pi_1(T_i^-)$ with $\pi_1(T_i^+)$. Thus $(g_i|_{T_i^-})_* = (g_i|_{T_i^+})_*$ and $(\psi_k)_* \cdots (\psi_1)_* = \phi_*$ on fundamental group. The identity (3.3) deduces that:

$$\begin{aligned} (g_1|_{T_1^+})_* \cdot \phi_* &= (g_1|_{T_1^+})_* \cdot (\psi_k)_* \cdots (\psi_1)_* \\ &= (g_{k+1}|_{T_{k+1}^+})_* \cdot (\psi_k)_* \cdots (\psi_1)_* \\ &= \phi_*^\epsilon \cdot (g_k|_{T_k^-})_* \cdot (\psi_{k-1})_* \cdots (\psi_1)_* \\ &= \phi_*^\epsilon \cdot (g_k|_{T_k^+})_* \cdot (\psi_{k-1})_* \cdots (\psi_1)_* \\ &= \cdots \\ &= (\phi_*^\epsilon)^k \cdot (g_1|_{T_1^+})_*. \end{aligned}$$

Set $A = (g_1|_{T_1^+})_*$ and get:

$$(3.4) \quad A \cdot \phi_* = (\phi_*^\epsilon)^k \cdot A.$$

Clearly $|\deg(g)| = k|\det(A)|$. The sign of $\deg(g)$ is decided by ϵ and the sign of $\det(A)$. Thus $l = \deg(f) = \deg(g) = k \cdot \epsilon \cdot \det(A)$.

Conversely, we set $\psi_1 = \cdots = \psi_{k-1} = id$, $\psi_k = \phi$ and construct the map $\tilde{g}: M_\phi \setminus g^{-1}(T) \rightarrow M_\phi \setminus T$ such that $\tilde{g}|_{V_i} = (\phi^{\epsilon \cdot (i-1)} \circ A) \times id: T \times I \rightarrow T \times I$ for $i = 1, \dots, k$. This construction fits the commuted diagram (3.2). Thus we get the quotient $g: M_\phi \rightarrow M_\phi$ whose degree is equal to $k \cdot \epsilon \cdot \det(A)$. \square

Suppose $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ where $p, q, r, s \in \mathbb{Z}$. We use equation (3.4) to solve p, q, r, s and then can determine l by Theorem 3.1.

(2.I) If ϕ is Anosov which means the absolute value of one eigenvalue of ϕ is larger than 1 while the other is less than 1. In this case, the k in the equation (3.4) must be equal to 1. We have:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\epsilon \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Solve this matrix equation and get:

$$A = \begin{cases} \begin{pmatrix} p & \frac{br}{c} \\ r & \frac{cp + (d-a)r}{c} \end{pmatrix} & (\epsilon = 1), \\ \begin{pmatrix} p & \frac{p(d-a) - br}{c} \\ r & -p \end{pmatrix} & (\epsilon = -1) \end{cases}$$

where $br/c, (d-a)r/c, (p(d-a) - br)/c \in \mathbb{Z}$.

By Theorem 3.1, we have:

$$l = p^2 + \frac{(d-a)}{c} \cdot pr - \frac{b}{c} \cdot r^2.$$

(2.II) If ϕ is periodic, may assume ϕ is either $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

(A) If $\phi = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ (ϕ has order 3), the equation (3.4) means:

$$A \cdot \phi_* = \begin{cases} A & (k \equiv 0 \pmod{3}), \\ \phi_*^\epsilon \cdot A & (k \equiv 1 \pmod{3}), \\ \phi_*^{2\epsilon} \cdot A & (k \equiv 2 \pmod{3}). \end{cases}$$

After solving all the above possible cases, we get:

$$A = \begin{cases} \begin{pmatrix} p & q \\ -q & p-q \end{pmatrix} & (k \equiv 1 \pmod{3}, \epsilon = 1), \\ \begin{pmatrix} p & q \\ q-p & -p \end{pmatrix} & (k \equiv 1 \pmod{3}, \epsilon = -1), \\ \begin{pmatrix} p & q \\ q-p & -p \end{pmatrix} & (k \equiv 2 \pmod{3}, \epsilon = 1), \\ \begin{pmatrix} p & q \\ -q & p-q \end{pmatrix} & (k \equiv 2 \pmod{3}, \epsilon = -1). \end{cases}$$

If $k \equiv 0 \pmod{3}$, we have $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which induces degree 0 map.

By Theorem 3.1:

$$l = \begin{cases} k \cdot (p^2 - pq + q^2) & (k \equiv 1 \pmod{3}), \\ k \cdot (-p^2 + pq - q^2) & (k \equiv 2 \pmod{3}). \end{cases}$$

It's easy to deduce that:

$$l = (3t+1)(p^2 - pq + q^2), \quad t, p, q \in \mathbb{Z}.$$

The same method is applied to the other two cases and we get:

(B) If $\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then:

$$l = (4t+1)(p^2 + q^2), \quad t, p, q \in \mathbb{Z}.$$

(C) If $\phi = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, then:

$$l = (6t + 1)(p^2 - pq + q^2), \quad t, p, q \in \mathbb{Z}.$$

4. The degrees of self maps of torus semi-bundles

We are going to prove Theorem 1.7 (ref. Proposition 1.5). We will assume that torus semi-bundle N_ϕ considered in this section has no torus bundle structure, otherwise $D(N_\phi)$ is determined in Section 3.

Suppose the degree of $f: N_\phi \rightarrow N_\phi$ is $l \neq 0$ and T is a torus fiber of N_ϕ . By [6, Lemma 6.5], f is homotopic to $g: N_\phi \rightarrow N_\phi$ such that $g^{-1}(T)$ is incompressible in N_ϕ . Thus by Lemma 2.6 and its proof (also ref. the proof of Theorem 2.4), we have $g^{-1}(T)$ is isotopic to either a union of torus fibers, or a union of torus fibers of another semi-bundle structure which is isomorphic to the original one. Also the later case happen only if N_ψ is a Nil manifold. Note by Theorem 2.9 and the proof in Section 3 (1.II), Nil 3-manifolds admits no orientation reversing homeomorphism.

Suppose now $g^{-1}(T)$ has k connected components, then $N_\phi \setminus g^{-1}(T)$ has two copies of N , denoted by V_0 and V_k , and $k - 1$ copies of $T \times I$, denoted by $V_i, i = 1, \dots, k - 1$. Denote the boundaries of V_0 and V_k by T_0^- and T_k^+ , the boundaries of V_i by T_i^+ and $T_i^-, i = 1, \dots, k - 1$, and the gluing map from T_i^- to T_{i+1}^+ by $\psi_i (i = 0, \dots, k - 1)$ see Fig. 9.

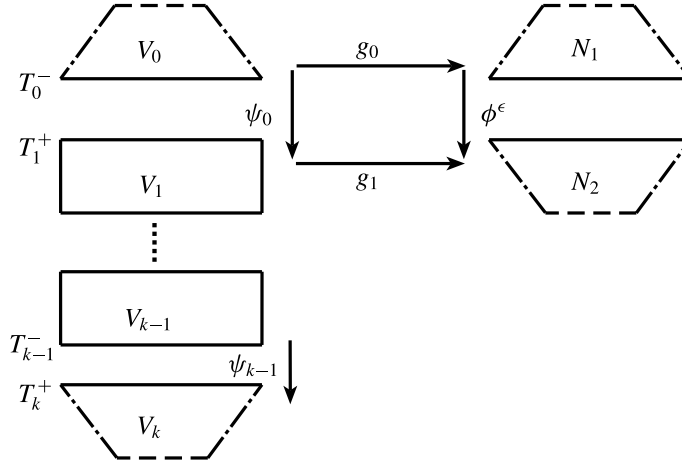
Then $N_{\psi_{k-1} \circ \dots \circ \psi_0} = N_\phi$, and $\psi_{k-1} \circ \dots \circ \psi_0 = \phi^\epsilon, \epsilon = \pm 1$ by Theorem 2.4 (with a suitable orientation of the canonical coordinate). Below we assume $\psi_{k-1} \circ \dots \circ \psi_0 = \phi$ (replace ϕ by ϕ^{-1} if needed). Let $\tilde{g}: N_\phi \setminus g^{-1}(T) \rightarrow N_\phi \setminus T$ be the map induced by g , and we have commuted diagram:

$$(4.1) \quad \begin{array}{ccc} N_\phi \setminus g^{-1}(T) & \xrightarrow{\tilde{g}} & N_\phi \setminus T \\ \cup \psi_i \downarrow & & \downarrow \phi^\epsilon \\ N_\phi & \xrightarrow{g} & N_\phi. \end{array}$$

Since T_i^+ is parallel to T_i^- , we can identify $\pi_1(T_i^+)$ with $\pi_1(T_i^-) (i = 0, \dots, k - 1)$. Thus $(\psi_{k-1})_* \dots (\psi_0)_* = \phi_*$ on fundamental group. Denote the restriction of \tilde{g} on V_i by g_i . Then $g: V_i \rightarrow N_1$ if i even, and $g: V_i \rightarrow N_2$ if i odd.

Lemma 4.1. *Under the canonical basis (l_0, l_∞) , $(g_0|_{T_0^-})_*$ is of the form $\begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}$ where $n \neq 0, m, n \in \mathbb{Z}$, and so is $(g_k|_{T_k^+})_*$.*

Proof. Let $g: N \rightarrow N$ be a proper map, we argue that under the basis (l_0, l_∞) , $(g|_{\partial N})_*$ is of the form $\begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}$ where $n \neq 0, m, n \in \mathbb{Z}$.

Fig. 9. Non-zero degree self-map of N_ϕ .

Choose a presentation $\pi_1(N) = \langle a, b \mid a = bab \rangle$ with $l_0 = a^2$ and $l_\infty = b$. Suppose $g_*(a) = a^{m'}b^q$, $g_*(b) = a^pb^n$. Since $g_*(a) = g_*(b)g_*(a)g_*(b)$, we get:

$$a^{m'}b^q = a^pb^na^{m'}b^qa^pb^n = a^{m'+2p}b^{(-1)^{m'+p}n+(-1)^p \cdot q+n}.$$

Thus:

$$\begin{cases} m' = m' + 2p, \\ q = (-1)^{m'+p} \cdot n + (-1)^p \cdot q + n, \end{cases} \implies \begin{cases} p = 0, \\ m' \text{ odd}, \end{cases} \quad \text{or} \quad \begin{cases} p = 0, \\ n = 0. \end{cases}$$

Abandon the case that $p = n = 0$ for g_0 is non-zero degree map and let $m' = 2m + 1$, we get: $g_*(a) = a^{2m+1}b^q$, $g_*(b) = b^n$.

Since $\pi_1(\partial N) = \langle a^2, b \mid [a^2, b] = 1 \rangle$ and $g_*(a^2) = a^{2m+1}b^qa^{2m+1}b^q = a^{4m+2}$, we have

$$(g|_{\partial N})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}. \quad \square$$

Theorem 4.2. *If N_ϕ has no torus bundle structure, then N_ϕ admits a self map of degree $l \neq 0$ if and only if there exist a positive integer k and two integer matrices A_1, A_2 of form $\begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$, $m, n \in \mathbb{Z}$, $n \neq 0$, satisfying the following equation:*

$$A_2 \cdot \phi_* = \begin{cases} (\phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^\epsilon \cdot \tau_*)^{s-1} \cdot \phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^\epsilon \cdot A_1 & (k = 2s), \\ (\phi_*^\epsilon \cdot \tau_* \cdot \phi_*^{-\epsilon} \cdot \tau_*)^s \cdot \phi_*^\epsilon \cdot A_1 & (k = 2s + 1), \end{cases}$$

such that $l = k \cdot \epsilon \cdot \det(A_1)$ where $\epsilon = \pm 1$.

Proof. From Fig. 9, we know that:

$$(4.2) \quad g_{i+1}|_{T_{i+1}^+} \circ \psi_i = \begin{cases} \phi^\epsilon \circ g_i|_{T_i^-} & (i \equiv 0 \pmod{2}), \\ \phi^{-\epsilon} \circ g_i|_{T_i^-} & (i \equiv 1 \pmod{2}), \end{cases}$$

where $\epsilon = \pm 1$, $i = 0, \dots, k-1$.

Thus if $k = 2s$ is even, then:

$$(4.3) \quad \begin{aligned} (g_k|_{T_k^+})_* \cdot \phi_* &= (g_k|_{T_k^+})_* \cdot (\psi_{k-1})_* \cdots (\psi_0)_* && \text{by Fig. 9} \\ &= \phi_*^{-\epsilon} \cdot (g_{k-1}|_{T_{k-1}^-})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* && \text{by (4.2)} \\ &= \phi_*^{-\epsilon} \cdot \tau_* \cdot (g_{k-1}|_{T_{k-1}^+})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* && \text{by Lemma 2.8} \\ &= \dots \\ &= (\phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^\epsilon \cdot \tau_*)^{s-1} \cdot \phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^\epsilon \cdot (g_0|_{T_0^-})_* \end{aligned}$$

If $k = 2s + 1$ is odd, then:

$$(4.4) \quad \begin{aligned} (g_k|_{T_k^+})_* \cdot \phi_* &= (g_k|_{T_k^+})_* \cdot (\psi_{k-1})_* \cdots (\psi_0)_* \\ &= \phi_*^\epsilon \cdot (g_{k-1}|_{T_{k-1}^-})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* \\ &= \phi_*^\epsilon \cdot \tau_* \cdot (g_{k-1}|_{T_{k-1}^+})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* \\ &= \dots \\ &= (\phi_*^\epsilon \cdot \tau_* \cdot \phi_*^{-\epsilon} \cdot \tau_*)^s \cdot \phi_*^\epsilon \cdot (g_0|_{T_0^-})_* \end{aligned}$$

It is easy to see that $|\deg(g)| = k|\det(g_0|_{T_0^-})_*|$. The sign of $\deg(g)$ is decided by both ϵ and the sign of $\det(g_0|_{T_0^-})_*$. Thus $l = \deg(f) = \deg(g) = k \cdot \epsilon \cdot \det(g_0|_{T_0^-})_*$. Finally by applying Lemma 4.1, we finish the proof of one direction of Theorem 4.2.

Conversely, if given k, A_1, A_2 , then we can easily construct the maps $g_0, g_k: N \rightarrow N$ such that $(g_0|_{T_0^-})_* = A_1$, $(g_k|_{T_k^+})_* = A_2$. Set $\psi_0 = \dots = \psi_{k-2} = id$, $\psi_{k-1} = \phi$ and $g_i: T \times I \rightarrow N$ ($i = 1, \dots, k-1$) is a map such that:

$$g_i|_{T_i^+} = \begin{cases} \phi^\epsilon \circ g_{i-1}|_{T_{i-1}^-} & (i \equiv 1 \pmod{2}), \\ \phi^{-\epsilon} \circ g_{i-1}|_{T_{i-1}^-} & (i \equiv 0 \pmod{2}). \end{cases}$$

Then $\tilde{g} = \bigcup g_i$ fits the commutative diagram (4.1). Thus we get the quotient map $g: N_\phi \rightarrow N_\phi$ of degree $k \cdot \epsilon \cdot \det(A_1)$. \square

Given $\phi_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ and suppose $(g_0|_{T_0^-})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$, $(g_k|_{T_k^+})_* = \begin{pmatrix} 2m'+1 & 0 \\ 0 & n' \end{pmatrix}$ where $m, n, m', n' \in \mathbb{Z}$.

CASE 1: $abcd \neq 0$, $ad - bc = 1$. (It should be noted that $(\tau\phi\tau\phi^{-1})_*$ is Anosov.)

Since $g: N_\phi \rightarrow N_\phi$ satisfies $g^{-1}(T)$ is copies of torus fiber, by Theorem 2.9 g can be lift to $g': M_{\tau\phi\tau\phi^{-1}} \rightarrow M_{\tau\phi\tau\phi^{-1}}$. By the argument of Anosov monodromy case in Section 3, the degree of g' in the S^1 direction is 1. So we have $k = 1$.

By equation (4.4), we have:

$$(g_1|_{T_1^+})_* \cdot \phi_* = \phi_*^\epsilon \cdot (g_0|_{T_0^-})_*.$$

If $\epsilon = 1$, then:

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation we have:

$$\begin{cases} n = 2m + 1, \\ m' = m, \\ n' = 2m + 1. \end{cases}$$

Thus $(g_0|_{T_0^-})_* = \begin{pmatrix} 2m + 1 & 0 \\ 0 & 2m + 1 \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = (2m + 1)^2.$$

If $\epsilon = -1$, then:

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation we have:

$$\begin{cases} n = -(2m' + 1), \\ (2m' + 1) \cdot a = (2m + 1) \cdot d, \\ n' = -(2m + 1). \end{cases}$$

Suppose $(2m + 1) = u \cdot a/\gcd(a, d)$, then both u and $a/\gcd(a, d)$ must be odd. Similarly, since $n = 2m' + 1 = -u \cdot d/\gcd(a, d)$ is odd, then $d/\gcd(a, d)$ is odd also.

Thus $(g_0|_{T_0^-})_* = \begin{pmatrix} u \cdot a/\gcd(a, d) & 0 \\ 0 & -u \cdot d/\gcd(a, d) \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = u^2 \cdot \frac{ad}{\gcd(a, d)^2}.$$

This degree can be realized here if and only if $ad/\gcd(a, d)^2$ is odd.

CASE 2: $abcd = 0$. Then there are three subcases.

$$(2.I) \quad \phi_* = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

In this case N_ϕ is a torus bundle which has been discussed in Section 3.

$$(2.II) \quad \phi_* = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}, \text{ or equivalently } \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}.$$

When $z \neq 0$, we discuss the following four possible cases:

(A) If $\epsilon = 1$ and $k = 2s$ is even, then by equation (4.3), we have the following equation:

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & zk \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution.

(B) If $\epsilon = -1$ and $k = 2s$ is even, then by equation (4.3):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & 0 \\ zk & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution either.

(C) If $\epsilon = 1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 0 & 1 \\ 1 & kz \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation:

$$\begin{cases} n = (-1)^s(2m' + 1), \\ n' = (-1)^s(2m + 1), \\ n' = (-1)^s kn. \end{cases}$$

So $2m + 1 = kn$, thus k is odd, if k exists.

Then $(g_k|_{T_k^+})_* = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & k(2m' + 1) \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = k \cdot \epsilon \cdot \det((g_k|_{T_k^+})_*) = k^2 \cdot (2m' + 1)^2.$$

This degree is an odd square number. In another hand, when $k = 1$, all odd square number can be realized as a degree: $(g_k|_{T_k^+})_* = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & 2m' + 1 \end{pmatrix}$.

(D) If $\epsilon = -1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} -zk & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution.

When $z = 0$, the same method will show that $\deg(g)$ is odd, and all odd numbers can be realized.

$$(2.III) \quad \phi_* = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

In this case, $\deg(g)$ can be determined as in case (2.II).

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Hongbin Sun
School of Mathematical Sciences
Peking University
Beijing 100871
China
e-mail: hongbin.sun2331@gmail.com

Shicheng Wang
School of Mathematical Sciences
Peking University
Beijing 100871
China
e-mail: wangsc@math.pku.edu.cn

Jianchun Wu
School of Mathematical Sciences
Peking University
Beijing 100871
China
e-mail: wujianchun@math.pku.edu.cn