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THE PRINCIPLE OF LIMITING ABSORPTION FOR THE NON-SELFADJOINT SCHröDINGER OPERATOR IN $\mathbb{R}^2$

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Introduction

The present paper is a continuation of [3] and is devoted to extending the results obtained in [3] to the non-selfadjoint Schrödinger operator in $\mathbb{R}^2$.

In the paper [3] we considered the non-selfadjoint Schrödinger operator

\[
(0.1) \quad L = -\sum_{j=1}^{\infty} \left( \frac{\partial}{\partial x_j} + ib_j(x) \right)^2 + Q(x)
\]

in $\mathbb{R}^N$, where $N$ is a positive integer such that $N \geq 2$, and the complex-valued function $Q(x)$ and the real-valued functions $b_j(x)$ ($j=1, 2, \ldots, N$) are assumed to satisfy some asymptotic conditions at infinity. Among others we have shown the following: Let us define a Hilbert space $\mathcal{L}_\beta = \mathcal{L}_\beta(\mathbb{R}^N)$ ($\beta \in \mathbb{R}$) by

\[
(0.2) \quad \mathcal{L}_\beta = \{ f(x) | (1 + |x|)^\beta f(x) \in \mathcal{L}_2(\mathbb{R}^N) \}
\]

with its inner product

\[
(0.3) \quad (f, g) = \int_{\mathbb{R}^N} (1 + |x|)^\beta f(x) \overline{g(x)} \, dx
\]

and norm

\[
(0.4) \quad \|f\|_\beta = [(f, f)_\beta]^{1/2}.
\]

If $\kappa \in \mathcal{C}_{\pm} = \{ \kappa \in \mathcal{C} | \kappa \neq 0 \text{ and } \text{Im } \kappa \geq 0 \}$ does not belong to an exceptional set which is called the set of the singular points of $L$, then the operator $(L - \kappa^2)^{-1}$ is well-defined as a bounded linear operator from $L_{2,\kappa}^{(1+\varepsilon)/2}$ into $L_{2,-(1+\varepsilon)/2}$ ($\varepsilon > 0$) with the estimate

\[
(0.5) \quad \|(L - \kappa^2)^{-1}\| = O(|\kappa|^{-1}) \quad (|\kappa| \to \infty).
\]

Here $u = (L - \kappa^2)^{-1} f \in L_{2,-(1+\varepsilon)/2}$ ($f \in L_{2,\kappa}^{(1+\varepsilon)/2}$) is a unique solution of the equation

\[
(0.6) \quad (L - \kappa^2) u = f
\]

with a sort of "radiation condition", and $\|(L - \kappa^2)^{-1}\|$ means the operator norm.
of \((L - \epsilon^2)^{-1}\) from \(L_{2, (\xi+\epsilon)/2}\) into \(L_{2, -\xi (\xi+\epsilon)/2}\).

In this paper, modifying the method of [3], we shall show that the estimate (0.5) holds good for \(L\) defined in \(\mathbb{R}^2\) with \(b_j(x) = 0, j = 1, 2\). In our case \(L\) takes the form

\[(0.7)\quad L = -\Delta + Q(x)\]

At the same time it will be shown that the other results obtained in [3] also hold for \(L\) in \(\mathbb{R}^2\). Throughout this paper we shall use the same notations as in [3]. For example \(\partial_j u = \frac{\partial u}{\partial x_j}, D_j u = D_j \infty u = \partial_j u + (\bar{x}_j/(2r))u - i\epsilon \bar{x}_j u, r = |x|, \bar{x}_j = x_j/r, D_j u = (D_j u) \bar{x}_j + (D_j u) \bar{x}_j\) etc.

1. A priori estimates

Let us define a differential operator \(L\) in \(\mathbb{R}^2\) by (0.7), where \(Q(x)\) is a complex-valued function on \(\mathbb{R}^2\) and \(L\) is regarded as an operator from \(H^2_{loc}\) into \(L^2_{loc}\). We decompose \(Q(x)\) as \(Q(x) = Q_0(x) - iQ(x)\). Throughout this paper the following is assumed: \(V_0(x)\) is a real-valued, measurable function such that the radial derivative exists

\[(1.1)\quad |V_0(x)| \leq C(1 + |x|)^{-\delta}, \quad \frac{\partial V_0}{\partial |x|} \leq C(1 + |x|)^{-1-\delta}\quad (x \in \mathbb{R}^2).
\]

\(V(x)\) is a complex-valued, measurable function which satisfies

\[(1.2)\quad |V(x)| \leq C(1 + |x|)^{-1-\delta}\quad (x \in \mathbb{R}^2).
\]

Here \(C\) and \(\delta\) are positive constants.

Now let us note that with no loss of generality \(V_0(x)\) can be assumed to satisfy

\[(1.3)\quad V_0(x) = 0\quad (|x| \leq R)
\]

by replacing \(V_0\) and \(V\) with \(\alpha V_0\) and \((1-\alpha)V_0 + V\), respectively, \(\alpha(x)\) being a real-valued, \(C^\infty\)-function such that

\[(1.4)\quad \alpha(x) = \begin{cases} 0 & (|x| \leq R), \\ 1 & (|x| \geq R+1). \end{cases}
\]

Henceforth we assume (1.3) with \(R = 7\) as well as (1.1) and (1.2).

1) In this regard we note that Ikebe-Saito [1] has shown the boundedness of \(||(L - \epsilon^2)^{-1}||\) for \(\epsilon\) moving in any compact set contained in \(C_+\), where \(L\) is a self-adjoint Schrödinger operator in \(\mathbb{R}^2\) and \(N\) is an arbitrary positive integer.

2) The list of the notation is given in the end of Introduction of [3].

3) This assumption is the same as the one imposed on \(Q(x)\) in [3].
Let $\varepsilon$ be a positive number such that $0 < \varepsilon \leq 1$ and $0 < \varepsilon \leq \delta/2$. As in Definition 1.2 of [3] we define by $\Sigma = \Sigma(L) = \Sigma(L, \varepsilon)$ the set of the singular points of $L$. i.e., $\kappa \in \Sigma$ if and only if $\kappa \in C_+ = \{\kappa \in C/\kappa \neq 0, \text{Im} \kappa \geq 0\}$ and there exists a non-trivial solution $u$ of the equation

\[ (L - \kappa^2)u = 0, \quad u \in H_{2, \text{loc}} \cap L_2, -(\varepsilon + \varepsilon^2)/2, \]

\[ \|\partial u\|_{(1+\varepsilon^2)/2, B_1} < \infty, \]

where $E_1 = \{x \in \mathbb{R}^2/|x| \geq 1\}$.

For $\kappa \in C_+$, with $\text{Im} \kappa > 0$ and the above $\varepsilon$ we put

\[ D_\kappa = D_\kappa = \{u \in H_{2, \text{loc}} \cap L_2, -(\varepsilon + \varepsilon^2)/2, (L - \kappa^2)u \in L_2, -(\varepsilon + \varepsilon^2)/2\}. \]

As is easily seen, Lemma 2.1 and Proposition 2.3, (i), (ii) of [3] are true in $\mathbb{R}^2$, too, and hence we have

**Proposition 1.1.** Let $u \in D_\kappa$ with $\kappa \in C_+$ and $\text{Im} \kappa > 0$. Then $u, \partial u, \partial_2 u \in L_2, -(\varepsilon + \varepsilon^2)/2$ and the estimate

\[ \|u\|_{(1+\varepsilon^2)/2} \leq C(\|u\|_{-\varepsilon^2/2} + \|L - \kappa^2\|u\|_{1+\varepsilon^2/2}) \]

holds with a constant $C = C(\kappa, L, \varepsilon)$. As a function of $\kappa$, $C$ is bounded when $\kappa$ moves in a compact set contained in $\{\kappa \in C/\text{Im} \kappa > 0\}$.

The purpose of this section is to prove the following estimates for $u \in D_\kappa$.

**Theorem 1.2.** Let $M$ be an open set such that $M \subset M_\alpha = \{\kappa \in C/|\kappa| > \alpha, \text{Im} \kappa > 0\}$ with some $\alpha > 0$ and $\bar{M} \cap \Sigma = \phi$, $\bar{M}$ being the closure of $M$ in $C$. Let $\kappa \in M$ and let $u \in D_\kappa$. Then there exists a constant $C = C(M, L, \varepsilon)$ such that we have the estimates

\[ \|\partial u\|_{-\varepsilon^2/2, E_1} \leq C\|f\|_{1+\varepsilon^2/2}, \]

\[ \|u\|_{-\varepsilon^2/2, E_\rho} \leq \frac{C}{|\kappa|} (1 + \rho)^{-\varepsilon/2}\|f\|_{1+\varepsilon^2/2} \quad (\rho \geq 0), \]

where $f = (L - \kappa^2)u$ and $E_\rho = \{x \in \mathbb{R}^2/|x| \geq \rho\}$.

**Remark 1.3.** Cf. Theorem 2.7 of [3]. In $\mathbb{R}^2$ the relation $\partial u \in L_{2, -(\varepsilon + \varepsilon^2)/2}$ for $u \in D_\kappa$ is not necessarily true, because $u/|x|$ ($u \in H_{2, \text{loc}}$) is not always square

4) As in [3] we put

\[ \|\partial u\|_{-(\varepsilon + \varepsilon^2)/2} = \sum_{|j| \geq 1} \left( \frac{1}{|x|} \right)^{1+\varepsilon} |\partial_j u|^2 dx. \]

5) Here and in the sequel we mean by $C = C(A, B, \ldots)$ that $C$ is a positive constant depending only on $A, B, \ldots$. 


integrable on a neighborhood of the origin \(x=0\). But, of course, for \(u \in D_\kappa\), \(\partial_j u \in L_{2,1+\epsilon}(E_\kappa)\) \((r > 0)\).

In order to prove Theorem 1.2 we prepare several propositions. Let us first show that the above estimates (1.8) and (1.9) can be easily obtained for \(u \in D_\kappa\) if \(\Im \kappa\) is sufficiently large. Set

\[
\beta_0 = \max \left\{ \frac{1}{2} (\sup_x |V_0(x) + V_1(x)| + 1)^{1/2}, \sup_x |V_2(x)| \right\}
\]

\((V_1(x) = \Re V(x), V_2(x) = \Im V(x))\).

Then we have

**Proposition 1.4.** Let \(u \in D_\kappa\) with \(\Im \kappa \geq \beta_0\). Then the estimates

\[
|u|_{L_2} \leq C_0 \left( 1 + \rho \right)^{-\mu} ||f|| \quad (\rho \geq 0, \mu \geq 0)
\]

and

\[
||\partial u||_{L_2} \leq C_0 ||f||
\]

hold with a constant \(C_0 = C_0(\beta_0)\), where \(f = (L - \kappa^2) u\) and \(||\cdot||\) means the usual \(L_2\)-norm.

Proof. Take the real and imaginary part of \(( (L - \kappa^2) u, w ) = (f, u) \) to obtain

\[
\sum_{j=1}^{2} (\partial_j u, \partial_j u) + (\kappa_2^2 - \kappa_1^2 + V_0 + V_1) u, u \rangle = \Re (f, u),
\]

\[
\Re (f, u),
\]

where \(\kappa_1 = \Re \kappa, \kappa_2 = \Im \kappa\), and \((, )\) is the \(L_2\)-inner product. It follows from (1.10) that

\[
\kappa_2^2 - \kappa_1^2 + V_0(x) + V_1(x) \geq \frac{\beta_0}{2} \kappa_2
\]

\((|\kappa_1| < 1, \kappa_2 \geq \beta_0)\)

and

\[
|V_2(x) - 2 \kappa_1 \kappa_2| \geq |\kappa_1| \kappa_2
\]

\((|\kappa_1| \geq 1, \kappa_2 \geq \beta_0)\)

for all \(x \in \mathcal{R}^2\). By the use of the relations (1.13)~(1.16) we can show

\[
||u|| \leq \frac{C_1}{|\kappa|} ||f|| \quad (C_1 = C_1(\beta_0)).
\]

In fact, if \(|\kappa_1| < 1\) and \(\kappa_2 \geq \beta_0\), we have from (1.13) and (1.15)

\[
||u|| \leq \frac{2}{(\beta_0 \kappa_2)} ||f|| \leq \frac{4}{(|\kappa| \kappa)} ||f||,
\]

where we should note that \(|\kappa| \leq |\kappa_1| + \kappa_2 \leq 2\kappa_2\). If \(|\kappa_1| \geq 1\) and \(\kappa_2 \geq \beta_0\), we can
see from (1.14) and (1.16) that
\[(1.19) \quad |\kappa_1|\kappa_2||u|| \leq ||f|| , \]
whence we obtain
\[(1.20) \quad ||u|| \leq (|\kappa_1|\kappa_2)^{-1}||f|| \leq (|\kappa| - 1)^{-1}||f|| \leq \frac{2}{|\kappa|} ||f|| . \]

(1.11) is a direct consequence of (1.17).

Next let us prove (1.12). Since
\[(1.21) \quad ||\mathcal{D}u||_{B_2} \leq \left[ \sum_{j=1}^{2} ||\partial_j u||^{2}_{r^2} + \frac{1}{2|x|} u \right]_{B_2} + ||icu||_{B_1} \]
and (1.17) has been established, we have only to show
\[(1.22) \quad \left[ \sum_{j=1}^{2} ||\partial_j u||^{2}_{r^2} \right]^{1/2} \leq C_2 ||f|| \quad (C_2 = C_4(\beta_0)) . \]

This follows from (1.1), (1.2), (1.13) and (1.17).

In the rest of this section it is enough to consider \( u \in D_\kappa \) with \( 0 < \text{Im} \kappa < \beta_0 \).

**Proposition 1.5.** Let \( a \) be a positive number and let \( u \in D_\kappa \) with \( |\kappa| > a \) and \( 0 < \text{Im} \kappa < \beta_0 \), where \( \beta_0 \) is as above. Then the estimate
\[(1.23) \quad ||\mathcal{D}u||_{L^{1+\sigma/2}, \kappa} \leq C \{ ||u||_{L^{1+\sigma/2}} + ||f||_{L^{1+\sigma/2}} \} \quad (f = (L - \kappa^2)u) \]
holds with a positive constant \( C = C(a, \beta_0, L, \sigma) \).

**Proof.** It follows from the formula (2.21) given in Lemma 2.5 of [3], which is true in the case \( N = 2 \), too, that
\[(1.24) \quad \int_{B_1} \left( \frac{\partial f}{\partial r} - \frac{\phi}{r} \right) |\mathcal{D}_\kappa u|^2 dx + \int_{B_1} \left( \kappa_1 \phi + \frac{\phi}{r} - \frac{1}{2} \frac{\partial \phi}{\partial r} \right) |\mathcal{D}u|^2 dx \]
\[+ \int_{B_1} \left( \frac{\phi}{r} + \frac{1}{2} \frac{\partial \phi}{\partial r} \right) |\mathcal{D}u|^2 dx + \int_{B_2} \left( \phi \frac{\partial \phi}{\partial r} \right) |\mathcal{D}_\kappa u|^2 - |\mathcal{D}_{\kappa} u|^2 dx \]
\[= \int_{B_2} \left( \frac{\phi}{r^2} \kappa - \frac{\partial \phi}{\partial r} \right) |u|^2 dx \]
\[+ \int_{B_1} \left( \frac{\partial \phi}{\partial r} V_0 + \phi \frac{\partial V_0}{\partial r} - \kappa \phi V_0 \right) |u|^2 dx \]
\[+ \text{Re} \int_{B_1} \phi V_0 (\mathcal{D}_\kappa u) dx + \text{Re} \int_{B_1} \phi f (\mathcal{D}_\kappa u) dx \]
\[- \frac{1}{2} \int_{s_T} \left( \int_{s_T} \phi \left( |\mathcal{D}u|^2 - 2 |\mathcal{D}_\kappa u|^2 + \left( V_0 - \frac{1}{4r^2} \right) \right)^2 ds , \]
where \( 0 < \tau < 1 < T < \infty \), \( B_\rho = \{ x \in \mathbb{R}^3 | \rho \leq |x| \leq \delta \} \), \( r = |x| \), \( \phi = \phi(r) \) is a real-valued, piecewise continuously differentiable function on \( [0, \infty] \), and we put in
of [3] $c_N = c_2 = -1/4$ and $B_{jk}(x) = 0$. Set

\begin{equation}
(1.25) \quad \phi(r) = \begin{cases} r^t & (0 \leq t \leq 1), \\
\frac{1}{2^t} (1+r)^t & (r > 1)
\end{cases}
\end{equation}

in (1.24). Then we estimate the both sides of (1.24) as follows:

(1.26) the left-hand side of (1.24)

\[
\geq \int_{B_{t_1}} r |u|^2 \, dx + \int_{B_{t_2}} \frac{\varepsilon}{2} (1+r)^{-1+s} |Du|^2 \, dx
\]

and

(1.27) the right-hand side of (1.24)

\[
\leq \int_{B_{t_1}} \frac{1}{4} \left\{ \beta_2 \frac{\phi}{r^2} - \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\phi}{r^2} \right) \right\} |u|^2 \, dx
\]

\[
+ \int_{B_{t_2}} \frac{1}{2} \left( \frac{\partial \phi}{\partial t} |V_0| + \phi \frac{\partial V_0}{\partial r} \right) |u|^2 \, dx + \kappa_2 \int_{B_{t_2}} \phi |V_0| |u|^2 \, dx
\]

\[
+ \int_{B_{t_1}} \phi \left( |\nabla u|^2 + |V_0| |u|^2 \right) \, dx + \int_{B_{t_2}} \phi |f| |\nabla u| \, dx
\]

\[
+ \int_{S_{t_1}} \phi \left( 2 |\nabla u|^2 + |V_0| + \frac{1}{4r^2} \right) |u|^2 \, dS
\]

\[
= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\]

Let us estimate each $J_k$. First we obtain

(1.28) \quad $J_k \leq C_k \|u\|_{L^2(1+\varepsilon)^/2}^2 \quad (k = 1, 2),$

where $C_1 = C_1(\beta_0, \varepsilon)$, $C_2 = C_2(L, \varepsilon)$ and we used (1.1). It follows from (1.1) and (1.3) with $R = 7$ that

(1.29) \quad $J_3 \leq C_3' \|u\|_{L^2, E_2}^2$

\[
\leq C_3' \left( \kappa_2 \|u\|_{L^2, E_2} \right) \|u\|_{-1+\varepsilon/2} (C_3' = C_3'(L, \varepsilon)).
\]

On the other hand in quite a similar way to the one used to prove Proposition 2.3, (iii) of [3] we can show

(1.30) \quad $\kappa_2 \|u\|_{L^2, E_2} \leq C_3'' \|u\|_{-1+\varepsilon/2}$

\[
+ \|\nabla u\|_{-1+\varepsilon/2} + \|f\|_{L^2(1+\varepsilon)^/2} \quad (C_3'' = C_3''(a, L, \varepsilon)),
\]

which, together with (1.29), yields
(1.31) \[ J_3 \leq C_3 \left\{ ||u||_{L^{(1+\varepsilon)/2}}^2 + ||u||_{-\varepsilon} ||Du||_{-\varepsilon} + ||u||_{-\varepsilon} ||f||_{(1+\varepsilon)/2} \right\} \quad (C_3 = C_3 (a, L, \varepsilon)). \]

As to \( J_4 \) and \( J_5 \) we have, using (1.2),

\[
\begin{align*}
J_4 & \leq C_4 ||u||_{-\varepsilon} ||r^{1/2}(D,u)||_{B_1} + ||Du||_{-\varepsilon} \quad (C_4 = C_4 (a, L, \varepsilon)), \\
J_5 & \leq C_5 ||f||_{(1+\varepsilon)/2} ||r^{1/2}(D,u)||_{B_1} + ||Du||_{-\varepsilon} \quad (C_5 = C_5 (a, L, \varepsilon)),
\end{align*}
\]

where \( C_k = C_k (L, \varepsilon) \), \( k = 4, 5 \). Here we should note that \( r^{1/2}(D,u) \in L^p (\mathbb{R}^2) \) because \( u \in H^1 (\mathbb{R}^2) \) is a continuous function on \( \mathbb{R}^2 \) by the Sobolev lemma.

\[ \lim_{t \to 0} J_6 = 0 \quad \text{and} \quad \lim_{t \to 0} J_7 = 0 \]

follow from the fact that \( r (|Du|^2 + |V_0| |u|^2) \) and \( r^2 (|D,u|^2 + (|V_0| + 1/(4 \pi^2)) |u|^2) \) are integrable on \( B_1 \) and \( E_\varepsilon \), respectively.

Summing up these estimates and letting \( t \to 0 \) and \( y \to \infty \), we arrive at

\[
(1.33) \quad \|r^{1/2}(D,u)\|_{B_1}^2 + (\varepsilon/2 + \varepsilon) \|Du\|_{-\varepsilon}^2 \leq C' \left\{ ||u||_{L^{(1+\varepsilon)/2}}^2 + ||u||_{-\varepsilon} ||f||_{(1+\varepsilon)/2} + ||Du||_{-\varepsilon} ||f||_{(1+\varepsilon)/2} \right\} \quad (C' = C'(a, \beta_0, L, \varepsilon)).
\]

(1.23) is a direct consequence of (1.33).

**Remark 1.6.** In the present paper we assume that the magnetic potentials \( b_j (x) = 0 \) \( (j = 1, 2) \). This assumption is used only to prove the above Proposition 1.5. Technically, it is possible to adopt a weaker assumption. For example it is enough to assume that \( B_i (x) = \partial b_i (x) - \partial b_j (x) = 0 \) in a neighborhood of the origin \( x = 0 \).

**Proposition 1.7.** Let \( u \in D_\kappa \) with \( |\kappa| > a (a > 0) \) and \( 0 < \text{Im} \kappa < \beta_0 \). Then there exists a constant \( C = C(a, \beta_0, L, \varepsilon) \) such that we have

\[
(1.34) \quad \|u\|_{L^{(1+\varepsilon)/2}}^2 \leq C (1 + \rho) \left\{ \frac{1}{|\kappa|} \|u\|_{L^{(1+\varepsilon)/2}}^2 + \frac{1}{|\kappa|} \|f\|_{L^{(1+\varepsilon)/2}}^2 \right\} \quad (\rho \geq 0),
\]

where \( f = (L - \kappa^2) u \).

**Proof.** The proof will be divided into two steps. In Step I the estimate (1.34) with \( \rho \geq 1 \) will be proved, where Proposition 1.5 will be useful. In Step
II we shall show (1.34) with \( \rho = 0 \). From these results we can easily obtain (1.34) for all \( \rho \geq 0 \).

Step I. We can find positive numbers \( b \) and \( c \) such that 
\[
\{ \kappa = \kappa_1 + i\kappa_2 \mid |\kappa| > a, \quad 0 < \kappa_2 < \beta_0 \} \subset K_1 \cup K_2,
\]
where
\[
\begin{align*}
K_1 &= \left\{ \kappa = \kappa_1 + i\kappa_2 \mid |\kappa| > a, \quad 0 < \kappa_2 < \beta_0, \quad \kappa_1^2 - \frac{1}{2} \kappa_2^2 \geq b^2 \right\}, \\
K_2 &= \left\{ \kappa = \kappa_1 + i\kappa_2 \mid |\kappa| > a, \quad 0 < \kappa_2 < \beta_0, \quad \kappa_1^2 - \kappa_2^2 \geq c^2 \right\},
\end{align*}
\]
e.g., we may put \( b = a/4 \) and \( c = a/2 \). Consider the case \( \kappa \in K_1 \). Then, proceeding as in the first half of the proof of Proposition 2.6 of [3], we arrive at (1.34) with \( \rho \geq 1 \). Next consider the case \( \kappa \in K_2 \). Then we can proceed as in the second half of the proof of Proposition 2.6 of [3] to obtain (1.34) with \( \rho \geq 1 \).

Step II. Set \( x_0 = (2, 0) \in \mathbb{R}^2 \) and set for \( u \in D_x \)
\[
\begin{align*}
(1.36) \quad \tilde{u}(x) &= u(x-x_0).
\end{align*}
\]
It follows from the relation \((L-\kappa^2)u = f \) that
\[
(1.37) \quad (\tilde{L}-\kappa^2)\tilde{u} = \tilde{f},
\]
where
\[
\begin{align*}
\tilde{L} &= -\Delta + \tilde{V}_0(x) + \tilde{V}(x), \\
\tilde{V}_0(x) &= V_d(x-x_0), \quad \tilde{V}(x) = V(x-x_0), \\
\tilde{f}(x) &= f(x-x_0).
\end{align*}
\]
Obviously \( \tilde{V}(x) \) satisfies (1.2) with the same \( \delta \) and some positive \( C \). It can be also shown that \( \tilde{V}_0(x) \) satisfies (1.1). In fact we have
\[
\begin{align*}
(1.39) \quad \frac{\partial \tilde{V}_0(x)}{\partial |x|} &= \frac{V_d(x-x_0)}{\partial |x-x_0|} \frac{\partial |x-x_0|}{\partial |x|} \\
&= \frac{\partial V_d(x-x_0)}{\partial |x|} \cdot \frac{|x| - |x_0| \cos \theta}{|x-x_0|},
\end{align*}
\]
\( \theta \) being the angle between \( x \) and \( x_0 \). By (1.3) with \( R = 7 \frac{\partial V_d}{\partial |x|}(x-x_0) = 0 \) for \( |x-x_0| \leq 5 \), and for \( |x-x_0| > 5 \) it follows that
\[
(1.40) \quad 0 < \frac{|x| - |x_0| \cos \theta}{|x-x_0|} \leq \frac{1 - (|x_0|/|x|)}{1 - (|x_0|/|x|)} \cos \theta < \frac{1 + 2/3}{1 - 2/3} = 5,
\]
where we have used the fact that \( |x_0|/|x| < 2/3 \) if \( |x-x_0| > 5 \). Thus we obtain, together with (1.1),
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\[ \frac{\partial \tilde{V}_0(x)}{\partial |x|} \leq 5C(1 + |x - x_0|^1)^{-1-\delta} \]

\[ \leq \tilde{C}(1 + |x|)^{-1-\delta} \quad \left( \tilde{C} = 5C\left[ \sup_{x} \left( \frac{1 + |x|}{1 + |x - x_0|} \right)^{1+\delta} \right) \right)\]

Hence the result obtained in Step I can be applied to \( L \) to show

\[ \|\tilde{u}\|_{-1+\delta/2, \mathcal{E}_L} \leq \tilde{C} \left\{ \frac{1}{|\kappa|} \|\tilde{u}\|_{-1+\delta/2} \right\} \]

\[ + \frac{1}{|\kappa|} \|\tilde{u}\|_{-1+\delta/2} \|f\|_{1+\delta/2} + \frac{1}{|\kappa|} \|\tilde{f}\|_{1+\delta/2} \} \right) \quad (\mathcal{C} = C(a, \beta_0, L, \varepsilon)). \]

Since the unit disc \( B_1 \) of \( \mathbb{R}^2 \) is contained in the set \( \{x \in \mathbb{R}^2/|x+x_0| \geq 1\} \), we have

\[ \|\tilde{u}\|_{-1+\delta/2, \mathcal{E}_L} = \int_{|x| \geq 1} (1 + |x|)^{-1-\delta} |u(x-x_0)|^2 \, dx \]

\[ = \int_{|x+x_0| \geq 1} (1 + |x-x_0|)^{-1-\delta} |u(x)|^2 \, dx \]

\[ \geq \int_{B_1} (1 + |x-x_0|)^{-1-\delta} |u(x)|^2 \, dx. \]

Therefore it follows from (1.42) and the boundedness of \((1 + |x|)/(1 + |x-x_0|)\) and \((1 + |x-x_0|)/(1 + |x|)\) on the whole space \( \mathbb{R}^2 \) that

\[ \|u\|_{-1+\delta/2, \mathcal{E}_L} \leq C' \left\{ \frac{1}{|\kappa|} \|u\|_{-1+\delta/2} \right\} \]

\[ + \frac{1}{|\kappa|} \|u\|_{-1+\delta/2} \|f\|_{1+\delta/2} + \frac{1}{|\kappa|} \|f\|_{1+\delta/2} \} \quad (C' = C'(a, \beta_0, L, \varepsilon)). \]

(1.34) with \( \rho=0 \) can be easily obtained from (1.44) and (1.34) with \( \rho=1 \).

Q.E.D.

Proof of Theorem 1.2. Set

\[ M_1 = \{ \kappa = \kappa_1 + i\kappa_2 | \kappa \in M, \kappa_2 \geq \beta_0 \} \]

\[ M_2 = \{ \kappa = \kappa_1 + i\kappa_2 | \kappa \in M, 0 < \kappa_2 \leq \beta_0 \}. \]

We have \( M = M_1 \cup M_2 \). For \( u \in D_\kappa \) with \( \kappa \in M_1 \) we have (1.8) and (1.9) from (1.12) and (1.11), respectively. Next suppose that \( u \in D_\kappa \) with \( \kappa \in M_2 \). Then, since the estimates (1.23) and (1.34) have been shown, we can proceed as in the proof of Theorem 2.7 of [3] to obtain

\[ \|u\|_{-1+\delta/2} \leq \frac{C}{|\kappa|} \|f\|_{1+\delta/2} \quad (C = C(a, \beta_0, L, \varepsilon)). \]

(1.8) and (1.9) for \( u \in D_\kappa \) with \( \kappa \in M_2 \) follow from (1.23), (1.34) and (1.46), which completes the proof.

Q.E.D.
2. The limiting absorption principle for \( L \)

Now that a priori estimates for \( L \) have been established (Theorem 1.2), it can be shown by arguments quite similar to those used in §3 and §4 of [3] that the main results of [3] hold in our case, too. We sum up these in the following three theorems whose proof will be omitted.

**Theorem 2.1** (the properties of the set \( \Sigma \) of the singular points of \( L \)). Let (1.1) and (1.2) be satisfied and let \( 0 < \varepsilon \leq \max (1, \delta/2) \).

(i) Then the set \( \Sigma = \Sigma (L, \varepsilon) \) of the singular points of \( L \) is a bounded set of \( C_+ = \{ \kappa \in C/\kappa \neq 0, \im \kappa \geq 0 \} \). \( \Sigma_R = \Sigma \cap R \) is a bounded set with the Lebesgue measure 0.

(ii) For any \( a > 0 \) \( \sum (\kappa \in C_+ | \kappa | \geq a) \) is a compact set of \( C_+ \), Further, \( \Sigma = \sum \kappa \) is an isolated, bounded set having no limit point in \( \{ \kappa \in C_+ | \im \kappa > 0 \} \).

(iii) Let \( \kappa \in C_+ \) and \( \im \kappa > 0 \). Then \( \nu \in \sum \) if and only if \( \kappa \) belongs to the point spectrum of \( H \), where \( H \) is a densely defined, closed linear operator in \( L_a \) given by

\[
\begin{align*}
\{ & D(H) = H_z^0, \\
& Hu = Lu.
\end{align*}
\]

For \( \kappa \in C_+ - \sum \) with \( \im \kappa > 0 \) belongs to the resolvent set of \( H \).

**Theorem 2.2.** (the limiting absorption principle for \( L \)). Let (1.1) and (1.2) be satisfied and let \( \varepsilon \) and \( \sum \) be as above. Assume that \( M \) is an open set of \( C \) such that \( M \cap \sum = \emptyset \) and \( M \subset M_a \) with some \( a > 0 \), \( \bar{M} \) being the closure of \( M \) and \( M_a \) being given as in Theorem 1.2.

(i) Then for any pair \( (\kappa, f) \in \bar{M} \times L_{(1+\varepsilon)/2} \) there exists a unique solution \( u = u(\kappa, f) \) of the equation

\[
(L - \kappa^2)u = f, \quad u \in H_{z, \loc} \cap L_{- (1+\varepsilon)/2},
\]

\[
||Dz|| \leq \frac{C}{|\kappa|} ||f||_{(1+\varepsilon)/2}, E_1 < \infty .
\]

(ii) The solution \( u = u(\kappa, f) \), \( (\kappa, f) \in \bar{M} \times L_{(1+\varepsilon)/2} \), satisfies the estimates

\[
\begin{align*}
& ||u||_{- (1+\varepsilon)/2, E_1} \leq \frac{C}{|\kappa|} ||f||_{(1+\varepsilon)/2}, \\
& ||Dz||_{- (1+\varepsilon)/2, E_1} \leq D ||f||_{(1+\varepsilon)/2}, \\
& ||u||_{- (1+\varepsilon)/2, E_\rho} \leq \frac{C}{|\kappa|} (1+\rho)^{-\gamma/2} ||f||_{(1+\varepsilon)/2} \quad (\rho \geq 1)
\end{align*}
\]

with a positive constant \( C = C(M, L, \varepsilon) \).

6) \( D(T) \) is the domain of \( T \).
(iii) If we define an operator \((L-\kappa^2)^{-1}\) by

\[
(L-\kappa^2)^{-1}f = u(\kappa, f) \quad (f \in L_2, (1+\varepsilon)/\beta)
\]

for \(\kappa \in \overline{M}\), then \((L-\kappa^2)^{-1}\) is a \(B(L_2, (1+\varepsilon)/2, L_2, -(1+\varepsilon)/2)\)-valued, continuous function on \(\overline{M}\), and we have

\[
||(L-\kappa^2)^{-1}|| \leq \frac{C}{|\kappa|} \quad (\kappa \in \overline{M}, C = C(M, L, \varepsilon)),
\]

where \(||(L-\kappa^2)^{-1}||\) means the operator norm of \(B(L_2, (1+\varepsilon)/2, L_2, -(1+\varepsilon)/2)\).

(iv) \((L-\kappa^2)^{-1} \in C(L_2, (1+\varepsilon)/2, L_2, -(1+\varepsilon)/2)\). Moreover we have the following: let \(\{f_n\}\) be any bounded sequence of \(L_2, (1+\varepsilon)/2\) and let \(\{\kappa_n\}\) be any sequence contained in \(\overline{M}\). Then the sequence \(\{(L-\kappa_n^2)^{-1}f_n\}\) is relatively compact in \(L_2, -(1+\varepsilon)/2\).

(v) \((L-\kappa^2)^{-1}\) is a \(B(L_2, (1+\varepsilon)/2, L_2, -(1+\varepsilon)/2)\)-valued, analytic function on \(M\).

Finally let us show some properties of the spectrum \(\sigma(H)\) of \(H\) defined by (2.1). Its point spectrum, continuous spectrum and residual spectrum are denoted by \(\sigma_p(H), \sigma_c(H)\) and \(\sigma_r(H)\), respectively. We define the essential spectrum \(\sigma_e(H)\) of \(H\) as in [3].

**Theorem 2.3** (the properties of \(\sigma(H)\)). Let (1.1) and (1.2) be satisfied and let \(H\) be as defined in (2.1). Then we have the following (i)~(iv):

(i) \(\sigma_p(H) = [0, \infty)\).

(ii) \(\sigma_c(H) = \emptyset\).

(iii) \(\sigma(H) \cap (C - [0, \infty)) \subset \sigma_p(H)\) and \(\sigma_p(H) \cap (0, \infty) = \emptyset\), and hence \(\sigma_c(H) \supset (0, \infty)\).

(iv) The eigenvalues in \(C - [0, \infty)\), if they exist, are of finite multiplicity and they form an isolated, bounded set having no limit point in \(C - [0, \infty)\).

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**References**


7) \(B(X, Y)\) denotes the set of all bounded linear operators on \(X\) into \(Y\), \(X\) and \(Y\) being Banach spaces.

8) \(C(X, Y)\) is all compact operators on \(X\) into \(Y\), where \(X\) and \(Y\) are Banach spaces.

9) See (1.20) of [3].

10) (iv) can be obtained from the fact that in our case the relation \(Hu = 0 (u \in D(H))\) is equivalent to \(H^*u = 0\), where \(H^*\) is the adjoint of \(H\) and \(\overline{u}(x)\) is the conjugate of \(u(x)\). See Mochizuki [2], Remark 1.1 (p. 425). See also Remark 1.8 of [3].