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## THE PRINCIPLE OF LIMITING ABSORPTION FOR THE NON-SELFADJOINT SCHRÖDINGER OPERATOR IN $\mathbf{R}^2$

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### Introduction

The present paper is a continuation of [3] and is devoted to extending the results obtained in [3] to the non-selfadjoint Schrödinger operator in  $\mathbf{R}^2$ .

In the paper [3] we considered the non-selfadjoint Schrödinger operator

$$(0.1) \quad L = - \sum_{j=1}^N \left( \frac{\partial}{\partial x_j} + i b_j(x) \right)^2 + Q(x)$$

in  $\mathbf{R}^N$ , where  $N$  is a positive integer such that  $N \neq 2$ , and the complex-valued function  $Q(x)$  and the real-valued functions  $b_j(x)$  ( $j=1, 2, \dots, N$ ) are assumed to satisfy some asymptotic conditions at infinity. Among others we have shown the following: Let us define a Hilbert space  $L_{2,\beta} = L_{2,\beta}(\mathbf{R}^N)$  ( $\beta \in \mathbf{R}$ ) by

$$(0.2) \quad L_{2,\beta} = \{f(x) / (1 + |x|)^\beta f(x) \in L_2(\mathbf{R}^N)\}$$

with its inner product

$$(0.3) \quad (f, g)_\beta = \int_{\mathbf{R}^N} (1 + |x|)^{2\beta} f(x) \overline{g(x)} dx$$

and norm

$$(0.4) \quad \|f\|_\beta = [(f, f)_\beta]^{1/2}.$$

If  $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C} / \kappa \neq 0 \text{ and } \text{Im } \kappa \geq 0\}$  does not belong to an exceptional set which is called the set of the singular points of  $L$ , then the operator  $(L - \kappa^2)^{-1}$  is well-defined as a bounded linear operator from  $L_{2, (1+\varepsilon)/2}$  into  $L_{2, -(1+\varepsilon)/2}$  ( $\varepsilon > 0$ ) with the estimate

$$(0.5) \quad \|(L - \kappa^2)^{-1}\| = O(|\kappa|^{-1}) \quad (|\kappa| \rightarrow \infty).$$

Here  $u = (L - \kappa^2)^{-1} f \in L_{2, -(1+\varepsilon)/2}$  ( $f \in L_{2, (1+\varepsilon)/2}$ ) is a unique solution of the equation

$$(0.6) \quad (L - \kappa^2)u = f$$

with a sort of "radiation condition", and  $\|(L - \kappa^2)^{-1}\|$  means the operator norm

of  $(L - \kappa^2)^{-1}$  from  $L_{2, (1+\varepsilon)/2}$  into  $L_{2, -(1+\varepsilon)/2}^{1)}$ .

In this paper, modifying the method of [3], we shall show that the estimate (0.5) holds good for  $L$  defined in  $\mathbf{R}^2$  with  $b_j(x) = 0, j = 1, 2$ . In our case  $L$  takes the form

$$(0.7) \quad L = -\Delta + Q(x).$$

At the same time it will be shown that the other results obtained in [3] also hold for  $L$  in  $\mathbf{R}^2$ . Throughout this paper we shall use the same notations as in [3]<sup>2)</sup>. For example  $\partial_j u = \frac{\partial u}{\partial x_j}$ ,  $\mathcal{D}_j u = \mathcal{D}_j^{(\kappa)} u = \partial_j u + (\tilde{x}_j / (2r))u - i\kappa \tilde{x}_j u$ ,  $r = |x|$ ,  $\tilde{x}_j = x_j / r$ ,  $\mathcal{D}_r u = (\mathcal{D}_1 u) \tilde{x}_1 + (\mathcal{D}_2 u) \tilde{x}_2$  etc.

### 1. A priori estimates

Let us define a differential operator  $L$  in  $\mathbf{R}^2$  by (0.7), where  $Q(x)$  is a complex-valued function on  $\mathbf{R}^2$  and  $L$  is regarded as an operator from  $H_{2, loc}$  into  $L_{2, loc}$ . We decompose  $Q(x)$  as  $Q(x) = V_0(x) + V(x)$ . Throughout this paper the following is assumed<sup>3)</sup>:  $V_0(x)$  is a real-valued, measurable function such that the radial derivative exists and

$$(1.1) \quad |V_0(x)| \leq C(1 + |x|)^{-\delta}, \quad \frac{\partial V_0}{\partial |x|} \leq C(1 + |x|)^{-1-\delta} \quad (x \in \mathbf{R}^2).$$

$V(x)$  is a complex-valued, measurable function which satisfies

$$(1.2) \quad |V(x)| \leq C(1 + |x|)^{-1-\delta} \quad (x \in \mathbf{R}^2).$$

Here  $C$  and  $\delta$  are positive constants.

Now let us note that with no loss of generality  $V_0(x)$  can be assumed to satisfy

$$(1.3) \quad V_0(x) = 0 \quad (|x| \leq R)$$

by replacing  $V_0$  and  $V$  with  $\alpha V_0$  and  $(1 - \alpha)V_0 + V$ , respectively,  $\alpha(x)$  being a real-valued,  $C^\infty$ -function such that

$$(1.4) \quad \alpha(x) = \begin{cases} 0 & (|x| \leq R), \\ 1 & (|x| \geq R + 1). \end{cases}$$

Henceforth we assume (1.3) with  $R = 7$  as well as (1.1) and (1.2).

1) In this regard we note that Ikebe-Saitō [1] has shown the boundedness of  $\|(L - \kappa^2)^{-1}\|$  for  $\kappa$  moving in any compact set contained in  $\mathbf{C}_+$ , where  $L$  is a self-adjoint Schrödinger operator in  $\mathbf{R}^N$  and  $N$  is an arbitrary positive integer.

2) The list of the notation is given in the end of Introduction of [3].

3) This aptissumon is the same as the one imposed on  $Q(x)$  in [3].

Let  $\varepsilon$  be a positive number such that  $0 < \varepsilon \leq 1$  and  $0 < \varepsilon \leq \delta/2$ . As in Definition 1.2 of [3] we define by  $\Sigma = \Sigma(L) = \Sigma(L, \varepsilon)$  the set of the singular points of  $L$ . i.e.,  $\kappa \in \Sigma$  if and only if  $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C} / \kappa \neq 0, \text{Im } \kappa \geq 0\}$  and there exists a non-trivial solution  $u$  of the equation

$$(1.5) \quad \begin{cases} (L - \kappa^2)u = 0, & u \in H_{2, \text{loc}} \cap L_{2, -(1+\varepsilon)/2}, \\ \|\mathcal{D}u\|_{\mathbf{C}_{-(1+\varepsilon)/2}, E_1} < \infty^4, \end{cases}$$

where  $E_1 = \{x \in \mathbf{R}^2 / |x| \geq 1\}$ .

For  $\kappa \in \mathbf{C}_+$  with  $\text{Im } \kappa > 0$  and the above  $\varepsilon$  we put

$$(1.6) \quad D_{\kappa, \varepsilon} = D_\kappa = \{u \in H_{2, \text{loc}} \cap L_{2, -(1+\varepsilon)/2} / (L - \kappa^2)u \in L_{2, (1+\varepsilon)/2}\}.$$

As is easily seen, Lemma 2.1 and Proposition 2.3, (i), (ii) of [3] are true in  $\mathbf{R}^2$ , too, and hence we have

**Proposition 1.1.** *Let  $u \in D_\kappa$  with  $\kappa \in \mathbf{C}_+$  and  $\text{Im } \kappa > 0$ . Then  $u, \partial_1 u, \partial_2 u \in L_{2, (1+\varepsilon)/2}$  and the estimate*

$$(1.7) \quad \|u\|_{\mathbf{C}_{(1+\varepsilon)/2}} \leq C(\|u\|_{\mathbf{C}_{-(1+\varepsilon)/2}} + \|(L - \kappa^2)u\|_{\mathbf{C}_{(1+\varepsilon)/2}})$$

holds with a constant  $C = C(\kappa, L, \varepsilon)^5$ . As a function of  $\kappa$ ,  $C$  is bounded when  $\kappa$  moves in a compact set contained in  $\{\kappa \in \mathbf{C} / \text{Im } \kappa > 0\}$ .

The purpose of this section is to prove the following estimates for  $u \in D_\kappa$ .

**Theorem 1.2.** *Let  $M$  be an open set such that  $M \subset M_a = \{\kappa \in \mathbf{C} / |\kappa| > a, \text{Im } \kappa > 0\}$  with some  $a > 0$ . and  $\bar{M} \cap \Sigma = \emptyset$ ,  $\bar{M}$  being the closure of  $M$  in  $\mathbf{C}$ . Let  $\kappa \in M$  and let  $u \in D_\kappa$ . Then there exists a constant  $C = C(M, L, \varepsilon)$  such that we have the estimates*

$$(1.8) \quad \|\mathcal{D}u\|_{\mathbf{C}_{-(1+\varepsilon)/2}, E_1} \leq C\|f\|_{\mathbf{C}_{(1+\varepsilon)/2}},$$

$$(1.9) \quad \|u\|_{\mathbf{C}_{-(1+\varepsilon)/2}, E_\rho} \leq \frac{C}{|\kappa|} (1 + \rho)^{-\varepsilon/2} \|f\|_{\mathbf{C}_{(1+\varepsilon)/2}} \quad (\rho \geq 0),$$

where  $f = (L - \kappa^2)u$  and  $E_\rho = \{x \in \mathbf{R}^2 / |x| \geq \rho\}$ .

REMARK 1.3. Cf. Theorem 2.7 of [3]. In  $\mathbf{R}^2$  the relation  $\mathcal{D}_j u \in L_{2, \mathbf{C}_{-(1+\varepsilon)/2}}$  for  $u \in D_\kappa$  is not necessarily true, because  $u/|x|$  ( $u \in H_{2, \text{loc}}$ ) is not always square

4) As in [3] we put

$$\|\mathcal{D}u\|_{\mathbf{C}_{-(1+\varepsilon)/2}}^2 = \sum_{j=1}^2 \int_{|x| \geq 1} (1 + |x|)^{-1+\varepsilon} |\mathcal{D}_j u|^2 dx.$$

5) Here and in the sequel we mean by  $C = C(A, B, \dots)$  that  $C$  is a positive constant depending only on  $A, B, \dots$ .

integrable on a neighborhood of the origin  $x=0$ . But, of course, for  $u \in D_\kappa$   $\mathcal{D}_j u \in L_{2, (1+\varepsilon)/2}(E_r)$  ( $r > 0$ ).

In order to prove Theorem 1.2 we prepare several propositions. Let us first show that the above estimates (1.8) and (1.9) can be easily obtained for  $u \in D_\kappa$  if  $\text{Im } \kappa$  is sufficiently large. Set

$$(1.10) \quad \beta_0 = \max \left[ \left\{ 2 \left( \sup_x |V_0(x) + V_1(x)| + 1 \right) \right\}^{1/2}, \sup_x |V_2(x)| \right]$$

$$(V_1(x) = \text{Re } V(x), V_2(x) = \text{Im } V(x)).$$

Then we have

**Proposition 1.4.** *Let  $u \in D_\kappa$  with  $\text{Im } \kappa \geq \beta_0$ . Then the estimates*

$$(1.11) \quad \|u\|_{-\mu, E_\rho} \leq \frac{C_0}{|\kappa|} (1+\rho)^{-\mu} \|f\| \quad (\rho \geq 0, \mu \geq 0)$$

and

$$(1.12) \quad \|\mathcal{D}u\|_{E_1} \leq C_0 \|f\|$$

hold with a constant  $C_0 = C_0(\beta_0)$ , where  $f = (L - \kappa^2)u$  and  $\| \cdot \|$  means the usual  $L_2$ -norm.

*Proof.* Take the real and imaginary part of  $((L - \kappa^2)u, u) = (f, u)$  to obtain

$$(1.13) \quad \sum_{j=1}^2 (\partial_j u, \partial_j u) + ((\kappa_2^2 - \kappa_1^2 + V_0 + V_1)u, u) = \text{Re}(f, u),$$

$$(1.14) \quad ((V_2 - 2\kappa_1 \kappa_2)u, u) = \text{Im}(f, u),$$

where  $\kappa_1 = \text{Re } \kappa$ ,  $\kappa_2 = \text{Im } \kappa$ , and  $(, )$  is the  $L_2$ -inner product. It follows from (1.10) that

$$(1.15) \quad \kappa_2^2 - \kappa_1^2 + V_0(x) + V_1(x) \geq \kappa_2^2/2 \geq \frac{\beta_0}{2} \kappa_2 \quad (|\kappa_1| < 1, \kappa_2 \geq \beta_0)$$

and

$$(1.16) \quad |V_2(x) - 2\kappa_1 \kappa_2| \geq |\kappa_1| \kappa_2 \quad (|\kappa_1| \geq 1, \kappa_2 \geq \beta_0)$$

for all  $x \in \mathbf{R}^2$ . By the use of the relations (1.13)~(1.16) we can show

$$(1.17) \quad \|u\| \leq \frac{C_1}{|\kappa|} \|f\| \quad (C_1 = C_1(\beta_0)).$$

In fact, if  $|\kappa_1| < 1$  and  $\kappa_2 \geq \beta_0$ , we have from (1.13) and (1.15)

$$(1.18) \quad \|u\| \leq (2/(\beta_0 \kappa_2)) \|f\| \leq (4/(\beta_0 |\kappa|)) \|f\|,$$

where we should note that  $|\kappa| \leq |\kappa_1| + \kappa_2 \leq 2\kappa_2$ . If  $|\kappa_1| \geq 1$  and  $\kappa_2 \geq \beta_0$ , we can

see from (1.14) and (1.16) that

$$(1.19) \quad |\kappa_1| \kappa_2 \|u\| \leq \|f\|,$$

whence we obtain

$$(1.20) \quad \|u\| \leq (|\kappa_1| \kappa_2)^{-1} \|f\| \leq (|\kappa| - 1)^{-1} \|f\| \leq \frac{2}{|\kappa|} \|f\|.$$

(1.11) is a direct consequence of (1.17).

Next let us prove (1.12). Since

$$(1.21) \quad \|\mathcal{D}u\|_{E_1} \leq \left[ \sum_{j=1}^2 \|\partial_j u\|^2 \right]^{1/2} + \left\| \frac{1}{2|x|} u \right\|_{E_1} + \|i\kappa u\|_{E_1}$$

and (1.17) has been established, we have only to show

$$(1.22) \quad \left[ \sum_{j=1}^2 \|\partial_j u\|^2 \right]^{1/2} \leq C_2 \|f\| \quad (C_2 = C_2(\beta_0)).$$

This follows from (1.1), (1.2), (1.13) and (1.17).

Q.E.D.

In the rest of this section it is enough to consider  $u \in D_\kappa$  with  $0 < \text{Im } \kappa < \beta_0$ .

**Proposition 1.5.** *Let  $a$  be a positive number and let  $u \in D_\kappa$  with  $|\kappa| > a$  and  $0 < \text{Im } \kappa < \beta_0$ , where  $\beta_0$  is as above. Then the estimate*

$$(1.23) \quad \|\mathcal{D}u\|_{C_{-1+\varepsilon/2}, E_1} \leq C \{ \|u\|_{C_{-1+\varepsilon/2}} + \|f\|_{C_{1+\varepsilon/2}} \} \quad (f = (L - \kappa^2)u)$$

holds with a positive constant  $C = C(a, \beta_0, L, \varepsilon)$ .

**Proof.** It follows from the formula (2.21) given in Lemma 2.5 of [3], which is true in the case  $N=2$ , too, that

$$\begin{aligned} (1.24) \quad & \int_{B_{t1}} \left( \frac{\partial \phi}{\partial r} - \frac{\phi}{r} \right) |\mathcal{D}_r u|^2 dx + \int_{B_{t1}} \left( \kappa_2 \phi + \frac{\phi}{r} - \frac{1}{2} \cdot \frac{\partial \phi}{\partial r} \right) |\mathcal{D}u|^2 dx \\ & + \int_{B_{tT}} \left( \kappa_2 \phi + \frac{1}{2} \cdot \frac{\partial \phi}{\partial r} \right) |\mathcal{D}u|^2 dx + \int_{B_{tT}} \left( \frac{\phi}{r} - \frac{\partial \phi}{\partial r} \right) (|\mathcal{D}u|^2 - |\mathcal{D}_r u|^2) dx \\ & = \int_{B_{tT}} \frac{1}{4} \left\{ \frac{\phi}{r^2} \kappa_2 - \frac{1}{2} \cdot \frac{\partial}{\partial r} \left( \frac{\phi}{r^2} \right) \right\} |u|^2 dx \\ & + \int_{B_{tT}} \left\{ \frac{1}{2} \left( \frac{\partial \phi}{\partial r} V_0 + \phi \frac{\partial V_0}{\partial r} \right) - \kappa_2 \phi V_0 \right\} |u|^2 dx \\ & + \text{Re} \int_{B_{tT}} \phi V u (\overline{\mathcal{D}_r u}) dx + \text{Re} \int_{B_{tT}} \phi f (\mathcal{D}_r u) dx \\ & - \frac{1}{2} \left[ \int_{S_T} - \int_{S_t} \right] \phi \left\{ |\mathcal{D}u|^2 - 2|\mathcal{D}_r u|^2 + \left( V_0 - \frac{1}{4r^2} \right) |u|^2 \right\} dS, \end{aligned}$$

where  $0 < t < 1 < T < \infty$ ,  $B_{ps} = \{x \in \mathbf{R}^2 / p \leq |x| \leq s\}$ ,  $r = |x|$ ,  $\phi = \phi(r)$  is a real-valued, piecewise continuously differentiable function on  $[0, \infty]$ , and we put in

(2.21) of [3]  $c_N = c_2 = -1/4$  and  $B_{jk}(x) = 0$ . Set

$$(1.25) \quad \phi(r) = \begin{cases} r^2 & (0 \leq t \leq 1), \\ \frac{1}{2^\varepsilon} (1+r)^\varepsilon & (r > 1) \end{cases}$$

in (1.24). Then we estimate the both sides of (1.24) as follows:

(1.26) the left-hand side of (1.24)

$$\geq \int_{B_{1T}} r |\mathcal{D}_r u|^2 dx + \int_{B_{1T}} \frac{\varepsilon}{2^{1+\varepsilon}} (1+r)^{-1+\varepsilon} |\mathcal{D}u|^2 dx$$

and

(1.27) the right-hand side of (1.24)

$$\begin{aligned} & \leq \int_{B_{1T}} \frac{1}{4} \left\{ \beta_0 \frac{\phi}{r^2} - \frac{1}{2} \cdot \frac{\partial}{\partial r} \left( \frac{\phi}{r^2} \right) \right\} |u|^2 dx \\ & + \int_{B_{1T}} \frac{1}{2} \left( \frac{\partial \phi}{\partial r} |V_0| + \phi \frac{\partial V_0}{\partial r} \right) |u|^2 dx + \kappa_2 \int_{B_{1T}} \phi |V_0| |u|^2 dx \\ & + \int_{B_{1T}} \phi |V| |u| |\mathcal{D}_r u| dx + \int_{B_{1T}} \phi |f| |\mathcal{D}_r u| dx \\ & + \int_{S_t} \phi (|\mathcal{D}u|^2 + |V_0| |u|^2) dS \\ & + \int_{S_T} \phi \left\{ 2 |\mathcal{D}_r u|^2 + \left( |V_0| + \frac{1}{4r^2} \right) |u|^2 \right\} dS \\ & = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \end{aligned}$$

Let us estimate each  $J_k$ . First we obtain

$$(1.28) \quad J_k \leq C_k \|u\|_{-C_1+\varepsilon/2}^2 \quad (k = 1, 2),$$

where  $C_1 = C_1(\beta_0, \varepsilon)$ ,  $C_2 = C_2(L, \varepsilon)$  and we used (1.1). It follows from (1.1) and (1.3) with  $R=7$  that

$$(1.29) \quad \begin{aligned} J_3 & \leq C_3' \kappa_2 \|u\|_{-E/2, E_2}^2 \\ & \leq C_3' (\kappa_2 \|u\|_{C_1-\varepsilon/2, E_2}) \|u\|_{-C_1+\varepsilon/2} \quad (C_3' = C_3'(L, \varepsilon)). \end{aligned}$$

On the other hand in quite a similar way to the one used to prove Proposition 2.3, (iii) of [3] we can show

$$(1.30) \quad \begin{aligned} \kappa_2 \|u\|_{C_1-\varepsilon/2, E_2} & \leq C_3'' \{ \|u\|_{-C_1+\varepsilon/2} \\ & + \|\mathcal{D}u\|_{C_1+\varepsilon/2, E_1} + \|f\|_{C_1+\varepsilon/2} \} \quad (C_3'' = C_3''(a, L, \varepsilon)), \end{aligned}$$

which, together with (1.29), yields

$$(1.31) \quad J_3 \leq C_3 \{ \|u\|_{-(1+\varepsilon)/2}^2 + \|u\|_{-(1+\varepsilon)/2} \| \mathcal{D}u \|_{(1+\varepsilon)/2, E_1} + \|u\|_{-(1-\varepsilon)/2} \|f\|_{(1+\varepsilon)/2} \} \quad (C_3 = C_3(a, L, \varepsilon)).$$

As to  $J_4$  and  $J_5$  we have, using (1.2),

$$(1.32) \quad \begin{cases} J_4 \leq C_4 \|u\|_{-(1+\varepsilon)/2} (\|r^{1/2}(\mathcal{D}_r u)\|_{B_1} + \| \mathcal{D}u \|_{(1+\varepsilon)/2, E_1}), \\ J_5 \leq C_5 \|f\|_{(1+\varepsilon)/2} (\|r^{1/2}(\mathcal{D}_r u)\|_{B_1} + \| \mathcal{D}u \|_{(1+\varepsilon)/2, E_1}), \end{cases}$$

where  $C_k = C_k(L, \varepsilon)$ ,  $k=4, 5$ . Here we should note that  $r^{1/2}(\mathcal{D}_r u) \in L_2(\mathbf{R}^2)_{loc}$  because  $u \in H_2(\mathbf{R}^2)_{loc}$  is a continuous function on  $\mathbf{R}^2$  by the Sobolev lemma.  $\lim_{t \rightarrow 0} J_6 = 0$  and  $\lim_{T \rightarrow \infty} J_7 = 0$  follow from the fact that  $r(|\mathcal{D}u|^2 + |V_0| |u|^2)$  and  $r^\varepsilon \{ 2|\mathcal{D}_r u|^2 + (|V_0| + 1/(4r^2))|u|^2 \}$  are integrable on  $B_1$  and  $E_1$ , respectively. Summing up these estimates and letting  $t \rightarrow 0$  and  $T \rightarrow \infty$ , we arrive at

$$(1.33) \quad \begin{aligned} & \|r^{1/2}(\mathcal{D}_r u)\|_{B_1}^2 + (\varepsilon/2^{1+\varepsilon}) \| \mathcal{D}u \|_{(1+\varepsilon)/2, E_1}^2 \\ & \leq C' \{ \|u\|_{-(1+\varepsilon)/2}^2 + \|u\|_{-(1+\varepsilon)/2} \|f\|_{(1+\varepsilon)/2} \\ & \quad + \|r^{1/2}(\mathcal{D}_r u)\|_{B_1} (\|u\|_{-(1+\varepsilon)/2} + \|f\|_{(1+\varepsilon)/2}) \\ & \quad + \| \mathcal{D}u \|_{(1+\varepsilon)/2, E_1} (\|u\|_{-(1+\varepsilon)/2} + \|f\|_{(1+\varepsilon)/2}) \} \quad (C' = C'(a, \beta_0, L, \varepsilon)). \end{aligned}$$

(1.23) is a direct consequence of (1.33).

Q.E.D.

REMARK 1.6. In the present paper we assume that the magnetic potentials  $b_j(x) = 0$  ( $j=1, 2$ ). This assumption is used only to prove the above Proposition 1.5. Technically, it is possible to adopt a weaker assumption. For example it is enough to assume that  $B_{12}(x) = \partial_1 b_2(x) - \partial_2 b_1(x) = 0$  in a neighborhood of the origin  $x=0$ .

**Proposition 1.7.** *Let  $u \in D_\kappa$  with  $|\kappa| > a$  ( $a > 0$ ) and  $0 < \text{Im } \kappa < \beta_0$ . Then there exists a constant*

$C = C(a, \beta_0, L, \varepsilon)$  *such that we have*

$$(1.34) \quad \|u\|_{-(1+\varepsilon)/2, E_\rho}^2 \leq C(1+\rho)^{-\varepsilon} \left\{ \frac{1}{|\kappa|} \|u\|_{-(1+\varepsilon)/2}^2 + \frac{1}{|\kappa|} \|u\|_{-(1+\varepsilon)/2} \|f\|_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2} \|f\|_{(1+\varepsilon)/2}^2 \right\} \quad (\rho \geq 0),$$

where  $f = (L - \kappa^2)u$ .

Proof. The proof will be divided into two steps. In Step I the estimate (1.34) with  $\rho \geq 1$  will be proved, where Proposition 1.5 will be useful. In Step



II we shall show (1.34) with  $\rho=0$ . From these results we can easily obtain (1.34) for all  $\rho \geq 0$ .

Step I. We can find positive numbers  $b$  and  $c$  such that  $\{\kappa = \kappa_1 + i\kappa_2 / |\kappa| > a, 0 < \kappa_2 < \beta_0\} \subset K_1 \cup K_2$ , where

$$(1.35) \quad \begin{cases} K_1 = \left\{ \kappa = \kappa_1 + i\kappa_2 / |\kappa| > a, 0 < \kappa_2 < \beta_0, \kappa_1^2 - \frac{1}{2} \kappa_2^2 \geq b^2 \right\}, \\ K_2 = \left\{ \kappa = \kappa_1 + i\kappa_2 / |\kappa| > a, 0 < \kappa_2 < \beta_0, \kappa_2^2 - \kappa_1^2 \geq c^2 \right\}, \end{cases}$$

e.g., we may put  $b=a/4$  and  $c=a/2$ . Consider the case  $\kappa \in K_1$ . Then, proceeding as in the first half of the proof of Proposition 2.6 of [3], we arrive at (1.34) with  $\rho \geq 1$ . Next consider the case  $\kappa \in K_2$ . Then we can proceed as in the second half of the proof of Proposition 2.6 of [3] to obtain (1.34) with  $\rho \geq 1$ .

Step II. Set  $x_0 = (2, 0) \in \mathbf{R}^2$  and set for  $u \in D_\kappa$

$$(1.36) \quad \tilde{u}(x) = u(x - x_0).$$

It follows from the relation  $(L - \kappa^2)u = f$  that

$$(1.37) \quad (\tilde{L} - \kappa^2)\tilde{u} = \tilde{f},$$

where

$$(1.38) \quad \begin{cases} \tilde{L} = -\Delta + \tilde{V}_0(x) + \tilde{V}(x), \\ \tilde{V}_0(x) = V_0(x - x_0), \tilde{V}(x) = V(x - x_0), \\ \tilde{f}(x) = f(x - x_0). \end{cases}$$

Obviously  $\tilde{V}(x)$  satisfies (1.2) with the same  $\delta$  and some positive  $\tilde{C}$ . It can be also shown that  $\tilde{V}_0(x)$  satisfies (1.1). In fact we have

$$(1.39) \quad \begin{aligned} \frac{\partial \tilde{V}_0(x)}{\partial |x|} &= \frac{V_0(x - x_0)}{\partial |x - x_0|} \cdot \frac{\partial |x - x_0|}{\partial |x|} \\ &= \frac{\partial V_0}{\partial |x|}(x - x_0) \cdot \frac{|x| - |x_0| \cos \theta}{|x - x_0|}, \end{aligned}$$

$\theta$  being the angle between  $x$  and  $x_0$ . By (1.3) with  $R=7 \frac{\partial V_0}{\partial |x|}(x - x_0) = 0$  for  $|x - x_0| \leq 5$ , and for  $|x - x_0| > 5$  it follows that

$$(1.40) \quad 0 < \frac{|x| - |x_0| \cos \theta}{|x - x_0|} \leq \frac{1 - (|x_0|/|x|) \cos \theta}{1 - (|x_0|/|x|)} < \frac{1 + 2/3}{1 - 2/3} = 5,$$

where we have used the fact that  $|x_0|/|x| < 2/3$  if  $|x - x_0| > 5$ . Thus we obtain, together with (1.1),

$$(1.41) \quad \frac{\partial \tilde{V}_0(x)}{\partial |x|} \leq 5C(1+|x-x_0|)^{-1-\delta} \\ \leq \tilde{C}(1+|x|)^{-1-\delta} \quad \left( \tilde{C} = 5C \left[ \sup_x \left( \frac{1+|x|}{1+|x-x_0|} \right) \right]^{1+\delta} \right)$$

Hence the result obtained in Step I can be applied to  $\tilde{L}$  to show

$$(1.42) \quad \|\tilde{u}\|_{-(1+\varepsilon)/2, E_1}^2 \leq \tilde{C} \left\{ \frac{1}{|\kappa|} \|\tilde{u}\|_{-(1+\varepsilon)/2}^2 \right. \\ \left. + \frac{1}{|\kappa|} \|\tilde{u}\|_{-(1+\varepsilon)/2} \|\tilde{f}\|_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2} \|\tilde{f}\|_{(1+\varepsilon)/2}^2 \right\} \quad (\tilde{C} = \tilde{C}(a, \beta_0, \tilde{L}, \varepsilon)).$$

Since the unit disc  $B_1$  of  $\mathbf{R}^2$  is contained in the set  $\{x \in \mathbf{R}^2 / |x+x_0| \geq 1\}$ , we have

$$(1.43) \quad \|\tilde{u}\|_{-(1+\varepsilon)/2, E_1}^2 = \int_{|x| \geq 1} (1+|x|)^{-1-\varepsilon} |u(x-x_0)|^2 dx \\ = \int_{|x+x_0| \geq 1} (1+|x-x_0|)^{-1-\varepsilon} |u(x)|^2 dx \\ \geq \int_{B_1} (1+|x-x_0|)^{-1-\varepsilon} |u(x)|^2 dx.$$

Therefore it follows from (1.42) and the boundedness of  $(1+|x|)/(1+|x-x_0|)$  and  $(1+|x-x_0|)/(1+|x|)$  on the whole space  $\mathbf{R}^2$  that

$$(1.44) \quad \|u\|_{-(1+\varepsilon)/2, B_1}^2 \leq C' \left\{ \frac{1}{|\kappa|} \|u\|_{-(1+\varepsilon)/2}^2 \right. \\ \left. + \frac{1}{|\kappa|} \|u\|_{-(1+\varepsilon)/2} \|f\|_{(1+\varepsilon)/2} + \frac{1}{|\kappa|^2} \|f\|_{(1+\varepsilon)/2}^2 \right\} \quad (C' = C'(a, \beta_0, L, \varepsilon)).$$

(1.34) with  $\rho=0$  can be easily obtained from (1.44) and (1.34) with  $\rho=1$ .

Q.E.D.

Proof of Theorem 1.2. Set

$$(1.45) \quad \begin{cases} M_1 = \{\kappa = \kappa_1 + i\kappa_2 / \kappa \in M, \kappa_2 \geq \beta_0\} \\ M_2 = \{\kappa = \kappa_1 + i\kappa_2 / \kappa \in M, 0 < \kappa_2 \leq \beta_0\}. \end{cases}$$

We have  $M = M_1 \cup M_2$ . For  $u \in D_\kappa$  with  $\kappa \in M_1$  we have (1.8) and (1.9) from (1.12) and (1.11), respectively. Next suppose that  $u \in D_\kappa$  with  $\kappa \in M_2$ . Then, since the estimates (1.23) and (1.34) have been shown, we can proceed as in the proof of Theorem 2.7 of [3] to obtain

$$(1.46) \quad \|u\|_{-(1+\varepsilon)/2} \leq \frac{C}{|\kappa|} \|f\|_{(1+\varepsilon)/2} \quad (C = C(a, \beta_0, L, \varepsilon)).$$

(1.8) and (1.9) for  $u \in D_\kappa$  with  $\kappa \in M_2$  follow from (1.23), (1.34) and (1.46), which completes the proof.

Q.E.D.

**2. The limiting absorption principle for  $L$**

Now that a priori estimates for  $L$  have been established (Theorem 1.2), it can be shown by arguments quite similar to those used in §3 and §4 of [3] that the main results of [3] hold in our case, too. We sum up these in the following three theorems whose proof will be omitted.

**Theorem 2.1** (the properties of the set  $\Sigma$  of the singular points of  $L$ ). Let (1.1) and (1.2) be satisfied and let  $0 < \varepsilon \leq \max(1, \delta/2)$ .

(i) Then the set  $\Sigma = \Sigma(L, \varepsilon)$  of the singular points of  $L$  is a bounded set of  $C_+ = \{\kappa \in C / \kappa \neq 0, \text{Im } \kappa \geq 0\}$ .  $\Sigma_R = \Sigma \cap R$  is a bounded set with the Lebesgue measure 0.

(ii) For any  $a > 0$   $\Sigma \cap \{\kappa \in C_+ / |\kappa| \geq a\}$  is a compact set of  $C_+$ , Further,  $\Sigma - \Sigma_R$  is an isolated, bounded set having no limit point in  $\{\kappa \in C_+ / \text{Im } \kappa > 0\}$ .

(iii) Let  $\kappa \in C_+$  and  $\text{Im } \kappa > 0$ . Then  $\nu \in \Sigma$  if and only if  $\kappa^2$  belongs to the point spectrum of  $H$ , where  $H$  is a densely defined, closed linear operator in  $L_2$  given by

$$(2.1) \quad \begin{cases} D(H) = H_2^{(6)}, \\ Hu = Lu. \end{cases}$$

For  $\kappa \in C_+ - \Sigma$  with  $\text{Im } \kappa > 0$  belongs to the resolvent set of  $H$ .

**Theorem 2.2.** (the limiting absorption principle for  $L$ ). Let (1.1) and (1.2) be satisfied and let  $\varepsilon$  and  $\Sigma$  be as above. Assume that  $M$  is an open set of  $C$  such that  $\bar{M} \cap \Sigma = \emptyset$  and  $M \subset M_a$  with some  $a > 0$ ,  $\bar{M}$  being the closure of  $M$  and  $M_a$  being given as in Theorem 1.2.

(i) Then for any pair  $(\kappa, f) \in \bar{M} \times L_{2, (1+\varepsilon)/2}$  there exists a unique solution  $u = u(\kappa, f)$  of the equation

$$(2.2) \quad \begin{cases} (L - \kappa^2)u = f, & u \in H_{2, loc} \cap L_{2, -(1+\varepsilon)/2}, \\ \|\mathcal{D}u\|_{L_{2, -(1+\varepsilon)/2}, E_1} < \infty. \end{cases}$$

(ii) The solution  $u = u(\kappa, f)$ ,  $(\kappa, f) \in \bar{M} \times L_{2, (1+\varepsilon)/2}$ , satisfies the estimates

$$(2.3) \quad \begin{cases} \|u\|_{L_{2, -(1+\varepsilon)/2}} \leq \frac{C}{|\kappa|} \|f\|_{L_{2, (1+\varepsilon)/2}}, \\ \|\mathcal{D}u\|_{L_{2, -(1+\varepsilon)/2}, E_1} \leq D \|f\|_{L_{2, (1+\varepsilon)/2}}, \\ \|u\|_{L_{2, -(1+\varepsilon)/2}, E_\rho} \leq \frac{C}{|\kappa|} (1+\rho)^{-\varepsilon/2} \|f\|_{L_{2, (1+\varepsilon)/2}} \quad (\rho \geq 1) \end{cases}$$

with a positive constant  $C = C(M, L, \varepsilon)$ .

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6)  $D(T)$  is the domain of  $T$ .

(iii) If we define an operator  $(L - \kappa^2)^{-1}$  by

$$(2.4) \quad (L - \kappa^2)^{-1}f = u(\kappa, f) \quad (f \in L_{2, (1+\varepsilon)/2})$$

for  $\kappa \in \bar{M}$ , then  $(L - \kappa^2)^{-1}$  is a  $\mathbf{B}(L_{2, (1+\varepsilon)/2}, L_{2, -(1+\varepsilon)/2})$ -valued, continuous function on  $\bar{M}^\tau$ , and we have

$$(2.5) \quad \|(L - \kappa^2)^{-1}\| \leq \frac{C}{|\kappa|} \quad (\kappa \in \bar{M}, C = C(M, L, \varepsilon)),$$

where  $\|(L - \kappa^2)^{-1}\|$  means the operator norm of  $\mathbf{B}(L_{2, (1+\varepsilon)/2}, L_{2, -(1+\varepsilon)/2})$ .

(iv)  $(L - \kappa^2)^{-1} \in \mathbf{C}(L_{2, (1+\varepsilon)/2}, L_{2, -(1+\varepsilon)/2})^{\text{op}}$ . Moreover we have the following: let  $\{f_n\}$  be any bounded sequence of  $L_{2, (1+\varepsilon)/2}$  and let  $\{\kappa_n\}$  be any sequence contained in  $\bar{M}$ . Then the sequence  $\{(L - \kappa_n^2)^{-1}f_n\}$  is relatively compact in  $L_{2, -(1+\varepsilon)/2}$ .

(v)  $(L - \kappa^2)^{-1}$  is a  $\mathbf{B}(L_{2, (1+\varepsilon)/2}, L_{2, (1+\varepsilon)/2})$ -valued, analytic function on  $M$ .

Finally let us show some properties of the spectrum  $\sigma(H)$  of  $H$  defined by (2.1). Its point spectrum, continuous spectrum and residual spectrum are denoted by  $\sigma_p(H)$ ,  $\sigma_c(H)$  and  $\sigma_r(H)$ , respectively. We define the essential spectrum  $\sigma_e(H)$  of  $H$  as in [3]<sup>9)</sup>.

**Theorem 2.3** (the properties of  $\sigma(H)$ ). Let (1.1) and (1.2) be satisfied and let  $H$  be as defined in (2.1). Then we have the following (i)~(iv):

(i)  $\sigma_e(H) = [0, \infty)$ .

(ii)  $\sigma_r(H) = \phi^{10)}$ .

(iii)  $\sigma(H) \cap (\mathbf{C} - [0, \infty)) \subset \sigma_p(H)$  and  $\sigma_p(H) \cap (0, \infty) = \phi$ , and hence  $\sigma_c(H) \supset (0, \infty)$ .

(iv) The eigenvalues in  $\mathbf{C} - [0, \infty)$ , if they exist, are of finite multiplicity and they form an isolated, bounded set having no limit point in  $\mathbf{C} - [0, \infty)$ .

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7)  $B(X, Y)$  denotes the set of all bounded linear operators on  $X$  into  $Y$ ,  $X$  and  $Y$  being Banach spaces.

8)  $\mathbf{C}(X, Y)$  is all compact operators on  $X$  into  $Y$ , where  $X$  and  $Y$  are Banach spaces.

9) See (1.20) of [3].

10) (iv) can be obtained from the fact that in our case the relation  $Hu=0$  ( $u \in D(H)$ ) is equivalent to  $H^* \bar{u}=0$ , where  $H^*$  is the adjoint of  $H$  and  $\bar{u}(x)$  is the conjugate of  $u(x)$ . See Mochizuki [2], Remark 1.1 (p. 425). See also Remark 1.8 of [3].

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