

Title	Pseudo-Anosov maps and fixed points of boundary homeomorphisms compatible with a Fuchsian group
Author(s)	Zhang, Chaohui
Citation	Osaka Journal of Mathematics. 2009, 46(3), p. 783-798
Version Type	VoR
URL	<a href="https://doi.org/10.18910/4725">https://doi.org/10.18910/4725</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

# PSEUDO-ANOSOV MAPS AND FIXED POINTS OF BOUNDARY HOMEOMORPHISMS COMPATIBLE WITH A FUCHSIAN GROUP

CHAOHUI ZHANG

(Received January 24, 2008, revised May 27, 2008)

## Abstract

Let  $\tilde{S}$  be a Riemann surface of type  $(p, n)$  with  $3p - 3 + n > 0$ . Let  $F$  be a pseudo-Anosov map of  $\tilde{S}$  defined by two filling simple closed geodesics on  $\tilde{S}$ . Let  $a \in \tilde{S}$ , and  $S = \tilde{S} - \{a\}$ . For any map  $f: S \rightarrow S$  that is generated by two simple closed geodesics and is isotopic to  $F$  on  $\tilde{S}$ , there corresponds to a configuration  $\tau$  of invariant half planes in the universal covering space of  $\tilde{S}$ . We give a necessary and sufficient condition (with respect to the configuration) for those  $f$  to be pseudo-Anosov maps. As a consequence, we obtain infinitely many pseudo-Anosov maps  $f$  on  $S$  that are isotopic to  $F$  on  $\tilde{S}$  as  $a$  is filled in.

## 1. Statement of results

Let  $\tilde{S}$  be a Riemann surface of type  $(p, n)$ , where  $p$  is the genus of  $\tilde{S}$  and  $n$  is the number of punctures of  $\tilde{S}$ . Assume that  $3p - 3 + n > 0$ . Let  $a \in \tilde{S}$ , and  $S = \tilde{S} - \{a\}$ . Let  $F$  be a pseudo-Anosov map on  $\tilde{S}$  in the sense that there exists a pair  $(\mathcal{F}_+, \mathcal{F}_-)$  of transverse measured foliations of  $\tilde{S}$  with  $F(\mathcal{F}_+) = \lambda\mathcal{F}_+$  and  $F(\mathcal{F}_-) = (1/\lambda)\mathcal{F}_-$  for some  $\lambda > 1$ . (See also FLP [7] and Penner [15].) In [10], Kra investigated the problem of finding pseudo-Anosov maps  $f$  on  $S$  so that  $f$  is isotopic to  $F$  on  $\tilde{S}$  as  $a$  is filled in. He showed that if  $\tilde{S}$  is compact with genus  $p \geq 2$ , then for some integer  $k$ , there is a pseudo-Anosov map  $f$  on  $S$  so that  $f$  is isotopic to  $F^k$  on  $\tilde{S}$ . In this article, we show that there always exist infinitely many pseudo-Anosov maps  $f$  on  $S$  so that  $f$  is isotopic to a pseudo-Anosov map  $F$  on  $\tilde{S}$  that is obtained from Thurston's construction [17].

To illustrate, let  $\tilde{\alpha}_1, \tilde{\alpha}_2 \subset \tilde{S}$  be two filling simple closed geodesics, that is, each component of  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  is a disk or an once punctured disk. Let  $t_{\tilde{\alpha}_i}$  denote the positive Dehn twist along  $\tilde{\alpha}_i$ . It is well known [17] (see also [2, 12, 16] for some variations) that a finite product

$$(1.1) \quad \tilde{\zeta} = \prod_{i=1}^N t_{\tilde{\alpha}_2}^{r_i} \circ t_{\tilde{\alpha}_1}^{-s_i}, \quad N, r_i, s_i \in \mathbb{Z}^+$$

is isotopic to a pseudo-Anosov map  $F$  on  $\tilde{S}$ . Throughout the article we denote by  $H_t(\cdot): \tilde{S} \rightarrow \tilde{S}$ ,  $0 \leq t \leq 1$ , the isotopy between  $\tilde{\zeta}$  and  $F$ . Note that  $\tilde{\alpha}_1, \tilde{\alpha}_2$  can be viewed as curves on  $S$  (call them  $\alpha_1$  and  $\alpha_2$ , respectively), and thus the maps  $\tilde{\zeta}$  are also defined on  $S$ . Clearly, if  $\tilde{S}$  is compact,  $S - \{\alpha_1, \alpha_2\}$  consists of disks and only one once punctured disk. Hence  $\tilde{\zeta}$  intimately represents a pseudo-Anosov mapping class on  $S$  that has the required property. However, if  $\tilde{S}$  is non-compact and in particular, if  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  only consists of once punctured disks, then one component of  $S - \{\alpha_1, \alpha_2\}$  is a twice punctured disk (enclosing the puncture  $a$ ), which means that the map (1.1) on  $S$  does not represent a pseudo-Anosov mapping class.

A question arises as to whether or not we can take another geodesic  $\alpha'_2$  on  $S$  as a substitution of  $\alpha_2$  so that  $\{\alpha_1, \alpha'_2\}$  fills  $S$  and  $\alpha'_2$  is still homotopic to  $\alpha_2$  on  $\tilde{S}$  as  $a$  is filled in. In [19], we constructed such a geodesic  $\alpha'_2$  with the required properties by utilizing topological methods. As a consequence, we showed that there exist infinitely many distinct pseudo-Anosov maps on  $S$  isotopic to on  $\tilde{S}$ .

Let  $\mathcal{F}$  be the set of isotopy classes of maps of  $S$  that are isotopic to the identity on  $\tilde{S}$ . The main purpose of this article is to develop a tool to detect in general whether or not a pair  $\{\alpha_1, \alpha'_2\}$ , where  $\alpha'_2 = f(\alpha_2)$  for some  $f \in \mathcal{F}$ , fills  $S$ ; we will give a necessary and sufficient condition for the pair  $\{\alpha_1, \alpha_2\}$  of geodesics on  $S$  to fill  $S$ . To do this, we need to transform the view of Dehn twists on  $S$  to the view of some special fiber-preserving automorphisms on the Bers fiber space  $F(\tilde{S})$ . (See Bers [4] and Kra [10] for more details.)

Let  $\mathbb{H} = \{z \in \mathbb{C}: \text{Im } z > 0\}$  be a hyperbolic plane, and let  $\varrho: \mathbb{H} \rightarrow \tilde{S}$  be a universal covering with the covering group  $G$ . It is well known [4, 6] that  $G$  is isomorphic (via an isomorphism  $\varphi^*$ , see Bers [4]) to  $\mathcal{F}$ . Further,  $\varphi^*$  naturally extends to an isomorphism (call  $\varphi^*$  also) of the group of fiber preserving automorphisms of  $F(\tilde{S})$  onto the group of mapping classes on  $S$  fixing the puncture  $a$ .

Let  $\hat{\alpha}_i \subset \mathbb{H}$ ,  $i = 1, 2$ , be geodesics such that  $\varrho(\hat{\alpha}_i) = \tilde{\alpha}_i$ . Let  $\{D_i, D'_i\} = \mathbb{H} - \{\hat{\alpha}_i\}$ . As we will explain in Section 3, the Dehn twist  $t_{\tilde{\alpha}_i}: \tilde{S} \rightarrow \tilde{S}$  can be lifted to a quasi-conformal map  $\tau_i: \mathbb{H} \rightarrow \mathbb{H}$  with respect to  $D_i$ . The map  $\tau_i$  determines a disjoint union of invariant half planes  $D_i(j)$  with the property that the restriction of  $\tau_i$  to the complement

$$(1.2) \quad H_i = \mathbb{H} - \bigcup_j D_i(j)$$

is the identity. Furthermore, the map  $\tau_i$  induces a fiber-preserving automorphism  $[\tau_i]$  of  $F(\tilde{S})$  such that, if  $\hat{\alpha}_i \in \{\varrho^{-1}(\tilde{\alpha}_i)\}$  (and hence  $D_i$ ) is chosen properly,  $\varphi^*([\tau_i]) = t_{\tilde{\alpha}_i}$  (see Lemma 3.3). Our main result is the following:

**Theorem 1.1.** *Let  $\tilde{\alpha}_1, \tilde{\alpha}_2 \subset \tilde{S}$  be arbitrary two simple closed geodesics so that  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  only consists of once punctured disks. Let  $\alpha_i$ ,  $i = 1, 2$ , be two simple closed geodesics on  $S$  homotopic to  $\tilde{\alpha}_i$  on  $\tilde{S}$ , and let  $\tau_i$  be the corresponding lifts with*

$\varphi^*([\tau_i]) = t_{\alpha_i}$ . Then the map

$$(1.3) \quad \prod_{i=1}^N t_{\alpha_2}^{r_i} \circ t_{\alpha_1}^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+,$$

represents a pseudo-Anosov mapping class on  $S$  (and projects to  $\tilde{\zeta}$  on  $\tilde{S}$ ) if and only if the intersection  $H_1 \cap H_2$ , where  $H_i$  are defined as in (1.2), is empty.

Given any pair  $\{\tau_1, \tau_2\}$  of lifts of  $t_{\tilde{\alpha}_1}$  and  $t_{\tilde{\alpha}_2}$  with  $H_1 \cap H_2 \neq \emptyset$ , it is easy to replace  $\tau_2$  with a  $G$ -conjugation  $\tau'_2$  so that  $H_1 \cap H'_2 = \emptyset$ . Therefore, via the Bers isomorphism, we are able to construct infinitely many pairs  $\{\alpha_1, \alpha'_2\}$  that fill  $S$ , where  $\alpha'_2 = f(\alpha_2)$  for some  $f \in \mathcal{F}$ . There are several applications of Theorem 1.1.

We now assume that  $\tilde{S}$  is an  $F$ -minimal surface in the sense that  $\mathcal{F}_{\pm}$  are defined by a quadratic differential  $\phi$  on  $\tilde{S}$  (see Bers [5] for the definitions and terminology). If the genus  $p \geq 2$ , then by the Riemann-Roch theorem (see [9] for instance), there exists a finite number of zeros of  $\phi$  on the compactification of  $\tilde{S}$ . Note that some zeros could be punctures of  $\tilde{S}$ .

If  $\phi$  has non-puncture zeros  $z_0$ , we may choose  $\hat{z}_0 \in \mathbb{H}$  with  $\varrho(\hat{z}_0) = z_0$ , and thereby a pair  $\{\tau_1, \tau_2\}$  of configurations of invariant half planes under the lifts of the Dehn twists such that  $H_1 \cap H_2 \neq \emptyset$  and  $\hat{z}_0 \in H_1 \cap H_2$ . This implies that the map

$$(1.4) \quad \zeta = \prod_{i=1}^N t_2^{r_i} \tau_1^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+$$

fixes  $\hat{z}_0 \in \mathbb{H}$ . It is important to note that  $\zeta: \mathbb{H} \rightarrow \mathbb{H}$  is a quasiconformal map compatible with  $G$ . It naturally extends to a map of  $\overline{\mathbb{H}}$  onto itself, which is also denoted by  $\zeta$ .

If  $\phi$  has no non-puncture zeros, then some punctures (call  $z_0$  also) must be zeros of  $\phi$ . In this case, we can still choose a pair  $\{\tau_1, \tau_2\}$  of lifts of the Dehn twists such that  $H_1 \cap H_2 \neq \emptyset$  and  $\zeta$  fixes  $\hat{z}_0 \in \hat{\mathbb{R}}$ .

Under certain conditions  $\zeta$  can be replaced with a pseudo-Anosov map  $\hat{F}$  so that  $\varrho \circ \hat{F} = F \circ \varrho$ ,  $\hat{F}(\hat{z}_0) = \hat{z}_0$  and  $\hat{F}|_{\partial H} = \zeta|_{\partial H}$ . Lemma 5.4 of Marden-Strebel [13] then asserts that  $\zeta$  does not fix any other fixed points of  $G$  on  $\hat{\mathbb{R}}$  (except for  $\hat{z}_0$  in the second case). Consider the maps  $h\zeta$  for  $h \in G$ . Unfortunately, the existence of fixed points of  $h\zeta$  is not guaranteed, and a question arises as to whether  $h\zeta$  fixes some fixed points of  $G$  on  $\hat{\mathbb{R}}$ . It is easy to show that for certain elements  $h$  of  $G$ ,  $h\zeta$  fix some points on  $\hat{\mathbb{R}}$  that may not be fixed points of  $G$ . Our second result states:

**Theorem 1.2.** *Let  $\tilde{S}$  be an  $F$ -minimal surface of genus  $p \geq 2$  and  $n > 0$ . Let  $z_0$  be a zero of the corresponding quadratic differential  $\phi$  which may or may not be a puncture of  $\tilde{S}$ . Then associated to each  $\hat{z}_0 \in \overline{\mathbb{H}}$  with  $\varrho(\hat{z}_0) = z_0$ , there exists a pair  $\{\tau_1, \tau_2\}$  of lifts of the Dehn twists  $t_{\tilde{\alpha}_1}$  and  $t_{\tilde{\alpha}_2}$  with  $H_1 \cap H_2 \neq \emptyset$ , and hence a map  $\zeta$  such that  $h_n\zeta$  does not fix any fixed points of  $G$  on  $\hat{\mathbb{R}}$  for an infinite sequence  $\{h_n\} \subset G$ .*

We call  $\zeta_1$  and  $\zeta_2$  with forms (1.4) are conjugate if there is an element  $h \in G$  such that  $\zeta_1 = h\zeta_2h^{-1}$ , which is equivalent to saying that  $\varphi^*([\zeta_1])$  and  $\varphi^*([\zeta_2])$  with forms (1.3) are conjugate if there is a map  $f \in \mathcal{F}$  so that  $\varphi^*([\zeta_1])$  is isotopic to  $f \circ \varphi^*([\zeta_2]) \circ f^{-1}$ . As a consequence of Theorem 1.1 and Theorem 1.2, we have:

**Theorem 1.3.** *Let  $\tilde{S}$  be a Riemann surface of type  $(p, n)$  with  $p \geq 2$ , and  $n > 0$ . Let  $\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  be a pair of filling simple closed geodesics on  $\tilde{S}$ . Let  $\zeta$  be defined by (1.4) via an  $F$ -minimal surface and a pair  $\{r_1, r_2\}$  with  $H_1 \cap H_2 \neq \emptyset$ . Then there are infinitely many mapping classes  $\omega_j^*$  on  $S$  with these properties:*

- (1) *all  $\omega_j^*$  are pseudo-Anosov,*
  - (2) *every  $\omega_j^*$  fixes  $a$  and projects to the mapping class represented by (1.1) as  $a$  is filled in,*
  - (3) *every  $\omega_j^*$  is represented by two filling simple loops on  $S$  and is of form (1.3).*
- If in addition we assume that  $z_0$  is a non-puncture zero of  $\phi$  so that  $F(z_0) = z_0$  and the curve  $H_t(z_0)$ ,  $0 \leq t \leq 1$ , is a trivial loop, then:*
- (4)  *$\varphi^*(\zeta)$  is pseudo-Anosov if  $z_0$  is a non-puncture zero of  $\phi$ ,*
  - (5)  *$\varphi^*(\zeta)$  is not conjugate to any  $\omega_j^*$ , and*
  - (6) *all  $\omega_j^*$  lie in different conjugacy classes.*

This article is organized as follows. In Section 2, we establish a correspondence between the set of pseudo-Anosov maps of  $S$  (that are isotopic to  $\tilde{\zeta}$  on  $\tilde{S}$ ) and the set  $\mathcal{L}$  of lifts of  $\tilde{\zeta}$  that fix no fixed points of  $G$ . It follows from Lemma 5.4 of [13] (see Lemma 2.2 for a different approach) that elements in  $\mathcal{L}$  that do not fix any parabolic fixed points of  $G$  must be pseudo-Anosov mapping classes on  $S$ . Details appear in Sections 3. Sections 4, 5, and 6 are devoted to the proofs of the results.

## 2. Notation and background

To establish notation and terminology, we begin with an overview of relevant Teichmüller theory. For more information, we refer to [4, 10].

Let  $\tilde{S}_1$  be a Riemann surface with the same type  $(p, n)$ . A marking of  $\tilde{S}_1$  is a homeomorphism  $f_1: \tilde{S} \rightarrow \tilde{S}_1$ . By  $(f_1: \tilde{S} \rightarrow \tilde{S}_1)$  we denote a marked Riemann surface. The Teichmüller space  $T(\tilde{S})$  is defined as a set of marked Riemann surfaces  $(f_1: \tilde{S} \rightarrow \tilde{S}_1)$  quotient by an equivalent relation “ $\sim$ ”, where  $(f_1: \tilde{S} \rightarrow \tilde{S}_1) \sim (f_2: \tilde{S} \rightarrow \tilde{S}_2)$  if and only if there is a conformal map  $h: \tilde{S}_1 \rightarrow \tilde{S}_2$  such that  $h \circ f_1$  is isotopic to  $f_2$ .

We denote by  $[f_1: \tilde{S} \rightarrow \tilde{S}_1]$  the equivalence class of the marked surface  $(f_1: \tilde{S} \rightarrow \tilde{S}_1)$ . Every marked surface  $(f_1: \tilde{S} \rightarrow \tilde{S}_1)$  defines a new conformal structure  $\mu_1$  on  $\tilde{S}$  via pullbacks. Two conformal structures  $\mu_1$  and  $\mu_2$  are called equivalent if and only if  $(f_1: \tilde{S} \rightarrow \tilde{S}_1) \sim (f_2: \tilde{S} \rightarrow \tilde{S}_2)$ . Let  $[\mu]$  denote the equivalence class of a conformal structure  $\mu$  on  $\tilde{S}$ . By Ahlfors-Bers [1], every conformal structure  $\mu$  on  $\tilde{S}$  determines a quasiconformal mapping  $w^\mu$  of  $\mathbb{C}$  that fixes 0, 1 and is conformal on  $\mathbb{H}^* = \{z \in \mathbb{C}: \text{Im } z < 0\}$ . The region  $w^\mu(\mathbb{H})$  is a Jordan domain that only depends on  $[\mu]$ .

The Bers fiber space  $F(\tilde{S})$  is defined as a collection  $\{([\mu], z); [\mu] \in T(\tilde{S}), z \in w^\mu(\mathbb{H})\}$  of pairs endowed with a product structure. The natural projection  $\pi: F(\tilde{S}) \rightarrow T(\tilde{S})$  defined by sending each point  $([\mu], z)$  to  $[\mu]$  is holomorphic. From Theorem 9 of Bers [4], There is an isomorphism  $\varphi: F(\tilde{S}) \rightarrow T(S)$  such that

$$(2.1) \quad \pi = \iota \circ \varphi,$$

where  $\iota: T(S) \rightarrow T(\tilde{S})$  is the natural forgetful map.

The group of isotopy classes of self-maps  $f$  of  $\tilde{S}$  is the mapping class group  $\text{Mod}_{\tilde{S}}$ , which naturally acts on  $T(\tilde{S})$  as holomorphic automorphisms. Let  $\text{mod } \tilde{S}$  denote the full group of fiber preserving holomorphic automorphisms of  $F(\tilde{S})$  that projects to  $\text{Mod}_{\tilde{S}}$ . Elements of  $\text{mod } \tilde{S}$  are of forms  $[\hat{f}]$ , where  $\hat{f}: \mathbb{H} \rightarrow \mathbb{H}$  is a lift of a self-map  $f$  of  $\tilde{S}$ .  $[\hat{f}]$  only depends on the boundary values  $\hat{f}|_{\mathbb{R}}$ . The Bers isomorphism  $\varphi: F(\tilde{S}) \rightarrow T(S)$  induces an isomorphism  $\varphi^*$  of  $\text{mod } \tilde{S}$  onto a group  $\text{Mod}_S^a$  of mapping classes of  $S$  fixing the puncture  $a$ .

An element  $\theta \in \text{Mod}_S^a$  is called a reducible mapping class if there is a curve system  $\mathcal{C} = \{c_1, \dots, c_s\}$ ,  $s \geq 1$ , of independent and disjoint simple closed geodesics on  $S$  with  $f(\{c_1, \dots, c_s\}) = \{c_1, \dots, c_s\}$  for certain representative  $f$  of  $\theta$ . There is a smallest positive integer  $K$  such that  $f^K$  maps each loop in  $\mathcal{C}$  to itself and the restriction of  $f^K$  to each component of  $S - \{c_1, \dots, c_s\}$  is either the identity or a pseudo-Anosov map.  $\theta$  is called pure if  $K = 1$ .

We now assume that  $\theta$  is reducible and projects to a pseudo-Anosov mapping class  $\tilde{\theta}$  on  $\tilde{S}$  that is induced by a map  $F$ . By Lemma 5.1 and 5.2 of [18], the curve system  $\mathcal{C}$  consists of only one curve  $c_1$  that bounds a twice punctured disk enclosing  $a$  and another puncture of  $\tilde{S}$ , which is equivalent to that  $c_1$  is peripheral on  $\tilde{S}$ . If we write  $\varphi^{*-1}(\theta) = [\hat{f}]$ , then  $\hat{f}: \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$  fixes a parabolic fixed point of  $G$ . Conversely, each element  $[\hat{f}]$  fixing the fixed point of a parabolic element of  $G$  corresponds to a reducible mapping class in  $\text{Mod}_S^a$  which is reduced by a single simple closed geodesic that is trivial on  $\tilde{S}$ . For hyperbolic fixed points, we have

**Lemma 2.1** (Marden-Strebel [13]). *Assume that  $\tilde{S}$  is  $F$ -minimal. Let  $z_0$  be a zero of  $\phi$ , and let  $\hat{z}_0 \in \overline{\mathbb{H}}$  be such that  $\varrho(\hat{z}_0) = z_0$ . Suppose that  $\hat{f}: \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$  fixes  $\hat{z}_0$ . Then  $\hat{f}$  does not fix any hyperbolic fixed point of  $G$ .*

To proof our theorems, we need a slightly general version of the lemma that states:

**Lemma 2.2.** *Let  $\hat{f}: \overline{\mathbb{H}} \rightarrow \overline{\mathbb{H}}$  be any lift of a pseudo-Anosov map  $F: \tilde{S} \rightarrow \tilde{S}$ . Then  $\hat{f}$  does not fix any hyperbolic fixed point of  $G$ .*

**REMARK 2.1.** If  $\tilde{S}$  is  $F$ -minimal, the lemma is covered by the argument of Lemma 5.4 of [13]. Our approach however does not assume that  $\tilde{S}$  is  $F$ -minimal.

Proof of Lemma 2.1. Suppose that  $\hat{f}(x) = x$  for a fixed point of a primitive hyperbolic element  $g$  of  $G$ . Let  $\sigma : S \rightarrow S$  be a map that induces the mapping class  $\varphi^*([\hat{f}])$ . There are three cases to consider.

CASE 1.  $g \in G$  is simple hyperbolic (the axis of  $g$  projects to a simple closed geodesic on  $\tilde{S}$ ). We claim that  $\sigma$  is reduced by a pair  $\{\alpha, \beta\}$  of geodesics which bounds a punctured cylinder enclosing  $a$  (throughout  $\alpha$  and  $\beta$  are called parallel geodesics). Note that  $g' = \hat{f}g\hat{f}^{-1}$  is also an element of  $G$  that fixes  $x$ .  $g'$  cannot be parabolic. For otherwise,  $g'$  and  $g$  would share the same fixed point  $x$ , it would follow that  $\langle g, g' \rangle \subset G$  is not discrete. We see that  $g'$  is also hyperbolic. If  $g$  and  $g'$  share only one fixed point  $x$ , by Theorem 4.3.5 of Beardon [3], the commutator  $[g, g']$  is parabolic whose fixed point is  $x$ . From Theorem 5.1.2 of [3],  $\langle g, [g, g'] \rangle \subset G$  is not discrete. This is a contradiction. We conclude that  $g$  and  $g'$  share both fixed points. It follows that  $g' = g^k$ , where  $k = \pm 1$  since  $g' = \hat{f}g\hat{f}^{-1}$  and  $g$  is primitive in  $G$ .

Let  $h^* : S \rightarrow S$  denote a map that induces the mapping class  $\varphi^*(h)$  for an element  $h \in G$ . From Theorem 2 of [10] or Theorem 2 of [14], we can write  $g^* = t_\beta^{-1} \circ t_\alpha$ , where  $\alpha, \beta$  are parallel geodesics. Hence  $g'^* = g^{k*} = t_\beta^{-k} \circ t_\alpha^k$ . Recall that  $g' = \hat{f}g\hat{f}^{-1}$ , we thus obtain

$$t_\beta^{-k} \circ t_\alpha^k = \sigma \circ (t_\beta^{-1} \circ t_\alpha) \circ \sigma^{-1} = t_{\sigma(\beta)}^{-1} \circ t_{\sigma(\alpha)}.$$

This means that  $\sigma(\{\alpha, \beta\}) = \{\alpha, \beta\}$ , which says that  $\sigma$  is reduced by  $\{\alpha, \beta\}$ .

Observe that both  $\alpha$  and  $\beta$  project to a non-trivial geodesic  $\tilde{\alpha}$  on  $\tilde{S}$  as  $a$  is filled in.  $\theta$  projects to  $\tilde{\theta}$  that is reduced by  $\tilde{\alpha}$ . Hence  $\tilde{\theta}$  is reducible, contradicting the hypothesis.

REMARK 2.2. Conversely, if  $\sigma$  is reduced by a pair  $\{\alpha, \beta\}$  of parallel geodesics, then we claim that  $\hat{f}$  fixes a hyperbolic fixed point of  $G$ . In fact,  $\sigma$  commutes with  $t_\beta^{-1} \circ t_\alpha$ . From Theorem 2 of [10] or Theorem 2 of [14], there is a simple hyperbolic element  $g \in G$  so that  $g^* = t_\beta^{-1} \circ t_\alpha$ . We see that  $\hat{f}$  commutes with  $g$ . That is,

$$(2.2) \quad g = \hat{f}g\hat{f}^{-1}.$$

Denote  $\{x, y\}$  the attracting and repelling fixed points of  $g$ . It follows from (2.2) that  $\hat{f}(\{x, y\}) = \{x, y\}$ . If  $\hat{f}(x) = y$ , then by (2.2) again, for any integer  $k$ ,

$$(2.3) \quad g^k(\hat{f}(z)) = \hat{f}(g^k(z))$$

for a  $z \in \mathbb{H}$ . As  $k \rightarrow +\infty$ ,  $g^k(\hat{f}(z)) \rightarrow x$  and  $g^k(z) \rightarrow x$ . It follows that  $\hat{f}(g^k(z)) \rightarrow y$ . This contradicts to (2.3).

CASE 2.  $g$  is essential hyperbolic (the axis of  $g$  projects to a filling geodesic on  $\tilde{S}$ ). Then by Theorem 2 of [10],  $g^*$  is pseudo-Anosov. Using the same argument as in Case 1, we have  $\hat{f}g\hat{f}^{-1} = g^k$  for  $k = \pm 1$ .

If  $k = 1$ , then  $\hat{f}$  commutes with  $g$ . So  $\sigma$  commutes with  $g^*$ . Suppose that  $\sigma$  is pseudo-Anosov. Since  $g^*$  is pseudo-Anosov, by Theorem 7.5.A of [8], there are integers  $i, j$  such that  $\sigma^i = g^{*j}$ . This implies that  $\sigma^i$  projects to the trivial mapping class on  $\tilde{S}$ . But  $\sigma^i$  projects to the pseudo-Anosov mapping class represented by the map (1.1). This is impossible. Suppose that  $\sigma$  is reduced by a simple loop  $c$  on  $S$  which is peripheral on  $\tilde{S}$ . Recall that  $\hat{f} = g\hat{f}g^{-1}$ . We obtain  $\sigma = g^* \circ \sigma \circ g^{*-1}$ . This implies that  $\sigma$  is also reduced by a unique loop  $g^*(c)$ . It follows that  $g^*(c) = c$ , which says  $g^*$  is reducible. This is also a contradiction.

If  $k = -1$ , then we have  $g = \hat{f}^2 g \hat{f}^{-2}$  instead of (2.2). That is,  $\hat{f}^2$  commutes with  $g$ . The similar argument as above can be applied in this case.

CASE 3.  $g \in G$  is a non-simple and non-essential hyperbolic element. By Theorem 2 of [10],  $g^*$  is a pure mapping class with a single component  $R$  on which  $g^*$  is pseudo-Anosov. Write  $g^* = f_R$ . If  $g = \hat{f}g\hat{f}^{-1}$ , then  $f_R = \sigma \circ f_R \circ \sigma^{-1} = f_{\sigma(R)}$ . We conclude that  $\sigma$  keeps  $R$  invariant. Since  $\sigma$  is reduced by only one loop  $c$  which bounds a twice punctured disk  $\Delta$ ,  $c$  is the only boundary of  $R$ . That is,  $R = S - \Delta$ . Both  $f_R$  and  $\sigma$  restrict to commuting mapping classes on  $R$ . By Theorem 7.5.A of [8] again, there are integers  $i, j$  such that  $f_R^i = \sigma^j$ . That is,  $\sigma^j$  projects to the trivial mapping class on  $\tilde{S}$ . But  $\sigma$  projects to the pseudo-Anosov mapping class represented by (1.1). This is also impossible. The case that  $g^{-1} = \hat{f}g\hat{f}^{-1}$  can be handled in the same way. □

### 3. Special cases

In this section, we consider those elements in mod  $\tilde{S}$  that come from some special mapping classes on  $\tilde{S}$ . We assume that  $\tilde{S}$  contains some punctures.

For  $i = 1, 2$ , let  $\hat{\alpha}_i \subset \mathbb{H}$  be a geodesic with  $\varrho(\hat{\alpha}_i) = \tilde{\alpha}_i$ , where  $\tilde{\alpha}_i$  are filling simple closed geodesics on  $\tilde{S}$  as introduced in Section 1. Let  $D_i, D'_i$  be the components of  $\mathbb{H} - \hat{\alpha}_i$ . The Dehn twist  $t_{\tilde{\alpha}_i}$  can be lifted to a quasiconformal mapping  $\tau_i$  of  $\mathbb{H}$  with respect to  $D_i$ . The construction is as follows. Let  $g_i \in G$  be the primitive simple hyperbolic element keeping both  $D_i$  and  $D'_i$  invariant. Throughout we assume that  $g_i$  is oriented as shown in Fig. 1.

In the figure, the arrow on  $\hat{\alpha}_i$  indicates the orientation of  $g_i$  that points from the repelling fixed point to the attracting fixed point of  $g_i$ . We take an earthquake  $g_i$ -shift on  $D_i$  and leave  $D'_i$  fixed. Then we define  $\tau_i: \mathbb{H} \rightarrow \mathbb{H}$  via  $G$ -invariance, which gives rise to a collection  $\mathcal{U}_i$  of layered half planes in  $\mathbb{H}$  in a partial order. In Fig. 1, the arrow underneath  $\hat{\alpha}_i$  points to the direction of the motion of  $\tau_i$  on  $D_i$ .

There are infinitely many disjoint maximal elements  $D_i(j)$  of  $\mathcal{U}_i$  each of which is invariant under  $\tau_i$  ( $D_i$  is just one of them). Recall that  $H_i$  is defined as in (1.2). From the definition, the restriction  $\tau_i|_{H_i} = \text{id}$ . Since  $\tau_i$  defined in this way is quasiconformal, it extends continuously to act on  $\overline{\mathbb{H}}$ . In particular,  $\tau_i|_{\hat{\mathbb{R}}}$  is quasisymmetric if we normalize so that “ $\infty$ ” lies outside of all maximal elements of  $\mathcal{U}_i$ .



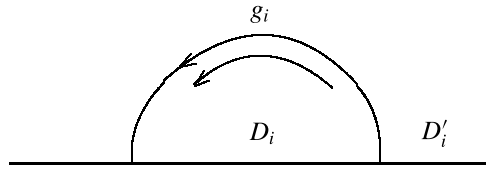


Fig. 1.

**Lemma 3.1.** *Let  $x \in \hat{\mathbb{R}}$  be fixed by a parabolic element of  $G$ . There are only finitely many elements of  $\mathcal{U}_i$  that cover  $x$ .*

Proof. Let  $D_i(0)$  be the maximal element of  $\mathcal{U}_i$  that covers  $x$ . Pick a point  $z$  in  $H_i$ , and draw a geodesic ray  $\hat{\Gamma}$  connecting  $z$  to  $x$ .  $\hat{\Gamma}$  projects to a geodesic ray  $\tilde{\Gamma}$  connecting  $\varrho(z)$  to the puncture  $b$  of  $\tilde{S}$  corresponding to  $x$ .

Let  $\tilde{U}$  be a punctured disk around  $b$ .  $\tilde{U}$  is chosen so small that is disjoint from  $\tilde{\alpha}_i$  and  $\tilde{U} \cap \tilde{\Gamma}$  is a single ray. Observe that  $\tilde{\Gamma} \cap (\tilde{S} - \tilde{U})$  has finite hyperbolic length. It intersects  $\tilde{\alpha}_i$  only finitely many times. So  $\tilde{\Gamma}$  intersects  $\tilde{\alpha}_i$  only finitely many times. This implies that  $\hat{\Gamma}$  meets finite number of elements  $D_i(0), \dots, D_i(m)$  of  $\mathcal{U}_i$  and the horodisk  $U$  at  $x$  that corresponds to  $\tilde{U}$  is included in all  $D_i(j)$ .  $\square$

For each parabolic fixed point  $z \in \hat{\mathbb{R}}$ , by Lemma 3.1, let  $D_i(0) \supset D_i(1) \supset \dots \supset D_i(m)$ ,  $D_i(j) \in \mathcal{U}_i$ , cover  $z$ . Let  $g_{ik}$ ,  $k = 0, 1, \dots, m$ , denote the primitive simple hyperbolic elements of  $G$  that keep  $D_i(j)$  invariant and take the same orientation as  $g_{i0}$  (here we refer to Fig. 1 for  $D_i = D_i(0)$  and  $g_i = g_{i0}$ ). Then  $\tau_i(z)$  is defined as

$$(3.1) \quad \tau_i(z) = g_{i0}g_{i1} \cdots g_{im}(z).$$

For each  $z \in \hat{\mathbb{R}}$  not covered by any element of  $\mathcal{U}_i$ ,  $\tau_i(z) = z$ . Let  $x \in \hat{\mathbb{R}}$  be arbitrary. Since the set of parabolic fixed points of  $G$  is dense on  $\hat{\mathbb{R}}$ , we choose a sequence  $\{x_j\}$  of parabolic fixed points so that  $x_j \rightarrow x$ . We see that

$$(3.2) \quad \tau_i(x) = \lim_{j \rightarrow \infty} \tau_i(x_j).$$

We summarize some additional properties of  $\tau_i$  which are derived from the definition:

- (1) If  $\tau_i$  is with respect to  $D_i$ , then  $\tau'_i = g_i^{-1}\tau_i = \tau_i g_i^{-1}$  is also a lift of  $t_{\tilde{\alpha}_i}$  and  $\tau'_i$  is with respect to  $D'_i$ .
- (2) For any point  $x$  covered by a maximal element  $D_i$  of  $\mathcal{U}_i$ ,  $\tau_i^m(x)$  and  $\tau_i^{-m}(x)$ ,  $m \rightarrow \infty$ , tend to the attracting and repelling fixed point of  $g_{i0}$ , respectively, and if  $g_{i0}$  is oriented as in Fig. 1, we have

$$\tau_i^{m+1}(x) < \tau_i^m(x), \quad \text{for } m \geq 1.$$

(3) For any  $x, y \in \hat{\mathbb{R}}, x \leq y$  implies  $\tau_i(x) \leq \tau_i(y)$ , and  $\tau_i(x) = x$  if and only if  $x$  does not lie in the interior of any maximal element of  $\mathcal{U}_i$ .

(4) For each hyperbolic element  $h \in G$  and each maximal element  $D_i$  of  $\mathcal{U}_i, h(D_i) \in \mathcal{U}_i$  if the repelling fixed point of  $h$  does not lie in  $D_i$ ; and  $h(\mathbb{H} - D_i) \in \mathcal{U}_i$  if  $D_i$  covers the repelling but not the attracting fixed point of  $h$ . Furthermore,  $h(D_i)$  is also a maximal element of  $\mathcal{U}_i$  if  $D_i$  does not contain any fixed points of  $h$ .

We observe that the map  $\tau_i$  determines a fiber-preserving automorphism  $[\tau_i]$  of the Bers fiber space  $F(\tilde{S})$ . Let  $\Delta \subset \mathbb{H}$  denote a fundamental region of  $G$  such that  $\Delta \cap \hat{\alpha}_i \neq \emptyset$ . Let  $\hat{a} = \varrho^{-1}(a) \cap \Delta$ . Since a Bers isomorphism  $\varphi: F(\tilde{S}) \rightarrow T(\tilde{S} - \{a\})$  is defined by picking up any point  $a \in \tilde{S}$ , we may choose a point  $a \in \tilde{S}$  so that  $\hat{a} \in D'_i$ . Under the isomorphism  $\varphi$  we then obtain a mapping class  $\varphi^*([\tau_i]) \in \text{Mod}_S^a$ .

**Lemma 3.2.** (1)  $\varphi^*([\tau_i])$  is represented by the Dehn twist  $t_{\alpha_i}$ , where  $\alpha_i$  is homotopic to  $\tilde{\alpha}_i$  on  $\tilde{S}$  as  $a$  is filled in.

(2) For any simple closed geodesic  $\alpha_i$  on  $S$ , let  $\tilde{\alpha}_i \subset \tilde{S}$  be the geodesic homotopic to  $\alpha_i$  on  $\tilde{S}$ . Then a geodesic  $\hat{\alpha}_i$  in  $\{\varrho^{-1}(\tilde{\alpha}_i)\}$ , and thus a component  $D_i$  of  $\mathbb{H} - \hat{\alpha}_i$  can be selected so that the map  $\tau_i$  with respect to  $D_i$  satisfies the condition that  $\varphi^*([\tau_i]) = t_{\alpha_i}$ .

Proof. For simplicity, we denote  $\tau = \tau_i$  and  $g = g_i$ . Since  $\varphi^*([\tau])$  is a mapping class, we denote by  $f: S \rightarrow S$  the map that represents  $\varphi^*([\tau])$ . By construction,  $\tau$  commutes with  $g$ . Thus  $\varphi^*([\tau])$  commutes with  $g^* = \varphi^*(g)$ . By Theorem 2 of [10] or Theorem 2 of [14],  $g^* = \varphi^*(g)$  is represented by  $t_\beta^{-1} \circ t_\alpha$ , where  $\{\alpha, \beta\}$  bounds a punctured cylinder  $P$  containing  $a$ . we obtain

$$f \circ (t_\beta^{-1} \circ t_\alpha) \circ f^{-1} = t_\beta^{-1} \circ t_\alpha.$$

That is,

$$(3.3) \quad t_{f(\beta)}^{-1} \circ t_{f(\alpha)} = t_\beta^{-1} \circ t_\alpha.$$

From (3.3) we conclude that  $f(P) = P$ , i.e.,  $f$  keeps  $\{\alpha, \beta\}$  invariant.

Let  $\tilde{f}: \tilde{S} \rightarrow \tilde{S}$  be the map isotopic to  $f$  as  $a$  is filled in. Since  $P$  is a cylinder containing  $a$ , it projects to a simple geodesic  $\tilde{\alpha}$ .  $\tilde{\alpha}$  is the projection of the axis of  $g$ . It follows that  $\tilde{f}$  keeps  $\tilde{\alpha}$  invariant. Thus it defines a map  $\tilde{f}_0$  on  $\tilde{S} - \{\tilde{\alpha}\}$ .

On the other hand, by (2.1), we know that  $f$  projects to the Dehn twist along  $\tilde{\alpha}$ . So  $\tilde{f} = t_{\tilde{\alpha}}$ . That is,  $\tilde{f}_0 = \text{id}$ , which says that  $f|_{S-P}$  is isotopic to the identity. In particular, this implies that  $f(\alpha) = \alpha$  and  $f(\beta) = \beta$ . Hence,  $f$  can be written as  $t_\beta^{-k+1} \circ t_\alpha^k$ , where we may assume that  $k \geq 1$ .

To show that  $k = 1$ , we consider  $\tau' = g^{-1}\tau$ . By Property (1),  $\tau'$  is with respect to  $D'$ , and is also a lift of  $t_{\tilde{\alpha}}$ . By the same argument as above,  $\varphi^*([\tau'])$  is represented by  $t_\beta^m \circ t_\alpha^{-m+1}$  for  $m \geq 1$ . Thus  $\varphi^*([\tau'^{-1}])$  is represented by  $t_\beta^{-m} \circ t_\alpha^{m-1}$ . Since  $\tau'^{-1}\tau$  coincides with  $g$  on  $\partial\mathbb{R}$ ,  $\varphi^*([\tau'^{-1}\tau])$  is represented by  $t_\beta^{-m-k+1} \circ t_\alpha^{m+k-1}$ . Once again,

by Theorem 2 of [10] or Theorem 2 of [14],  $\varphi^*(g)$  is represented by  $t_\beta^{-1} \circ t_\alpha$ . We see that

$$t_\beta^{-m-k+1} \circ t_\alpha^{m+k-1} = t_\beta^{-1} \circ t_\alpha^1.$$

It follows that  $m + k - 1 = 1$ . Since  $m \geq 1$  and  $k \geq 1$ , we conclude that  $m = k = 1$ . This proves (1).

From (1), we see that either  $\varphi^*([\tau])$  or  $\varphi^*([g^{-1}\tau])$  is represented by a Dehn twist  $t_{\alpha_1}$  along a simple closed geodesic  $\alpha_1$  on  $S$  for which there is an element  $\zeta \in \mathcal{F}$  such that  $\zeta(\alpha_1) = \alpha$ . Since  $\varphi^*(G) = \mathcal{F}$ , there is an element  $h \in G$  such that  $\varphi^*(h) = \zeta$ . Now it is easy to see that  $h(\hat{\alpha}) \subset \mathbb{H}$  is the desired geodesic, and thus either  $h\tau h^{-1}$  or  $hg^{-1}\tau h^{-1}$  is the desired lift of  $t_{\hat{\alpha}}$ . This proves (2). □

**Lemma 3.3.** *Let  $\tau_1$  and  $\tau_2$  be any lifts of  $t_{\hat{\alpha}_1}$  and  $t_{\hat{\alpha}_2}$  with  $H_1 \cap H_2 = \emptyset$ . Then for sufficiently large integers  $r, s$ , the map  $\tau_2^r \tau_1^{-s}$  does not fix any parabolic fixed points of  $G$ .*

*Proof.* Suppose that  $x \in \partial\mathbb{H} = \hat{\mathbb{R}}$  is a parabolic fixed point that is fixed by  $\tau_2^r \tau_1^{-s}$ . There is a parabolic element  $T \in G$  so that  $T(x) = x$ .

Notice that  $H_i$  is closed. Thus  $H_1 \cap H_2$  is also closed. If  $x$  lies outside of any maximal elements of  $\mathcal{U}_1$  and  $\mathcal{U}_2$  (in the sense that  $x$  does not belong to any closed half plane in  $\mathcal{U}_1$  and  $\mathcal{U}_2$ ), then  $x$  lies in the closure  $(H_1 \cap H_2) \cap \hat{\mathbb{R}}$ . There is a fundamental region  $\Delta \subset \mathbb{H}$  that takes  $x$  as a cusp and has an overlap with  $H_1 \cap H_2$ . This in particular implies that  $H_1 \cap H_2$  is not empty. This is a contradiction.

Assume that  $x \in D_2$  for a maximal element  $D_2$  of  $\mathcal{U}_2$ . If  $x$  does not lie in any maximal elements of  $\mathcal{U}_1$ , then  $\tau_1^{-s}(x) = x$ . Thus  $\tau_2^r \tau_1^{-s}(x) = \tau_2^r(x) \neq x$ . If  $x \in D_1$  for a maximal element  $D_1$  of  $\mathcal{U}_1$ , but not lie in any maximal elements of  $\mathcal{U}_2$ , we use the same argument to prove that  $(\tau_2^r \tau_1^{-s})^{-1}(x) \neq x$ .

For any half plane  $D$  in  $\mathcal{U}_1$  or  $\mathcal{U}_2$ , let  $\partial D$  denote the boundary of  $D$  in  $\mathbb{H}$ . Let  $h \in G$  be a simple hyperbolic element so that  $h(D) = D$ . If  $x$  is a vertex of  $D$ , i.e.,  $x \in \partial D \cap \mathbb{H}$ , then  $T$  and  $h$  would share a common fixed point  $x$ , and this would contradict to that  $G$  is discrete.

By the above discussion, we are left with the possibility that  $x \in D_2 \cap D_1$  for a maximal element  $D_1$  of  $\mathcal{U}_1$  and a maximal element  $D_2$  of  $\mathcal{U}_2$ . If  $\partial D_2$  intersects  $\partial D_1$ , the intersection point is in  $\mathbb{H}$ . It follows that  $H_1 \cap H_2 \neq \emptyset$ , contradicting the hypothesis.

Now we assume that  $D_2 \subset D_1$ . Let  $g_i \in G$  be hyperbolic such that  $g_i(D_i) = D_i$ ,  $i = 1, 2$ . Since  $x \in D_2$ , from (3.2),

$$(3.4) \quad g_1^{-s}(x) \leq \tau_1^{-s}(x).$$

From Property (4), we know that  $g_1^{-s}(D_2) \in \mathcal{U}_2$  is also maximal, and Property (2) says that  $\tau_2$  keeps  $g_1^{-s}(D_2)$  invariant. Thus  $\tau_2^r g_1^{-s}(x) \in g_1^{-s}(D_2)$ . Since  $\partial D_2$  projects to a

simple closed geodesic  $\tilde{\alpha}_2$ ,  $D_2 \cap g_1^{-s}(D_2) = \emptyset$ . We assert that

$$(3.5) \quad x < \tau_2^r g_1^{-s}(x).$$

By Property (3),  $\tau_i$  is monotonic. It follows from (3.4) and (3.5) that

$$x < \tau_2^r g_1^{-s}(x) \leq \tau_2^r \tau_1^{-s}(x).$$

In particular,  $\tau_2^r \tau_1^{-s}(x) \neq x$ . If  $x \in D_1 \cap D_2$  and  $D_1 \subset D_2$ , by using the same argument above, we conclude that the inverse  $(\tau_2^r \tau_1^{-s})^{-1}$  does not fix any parabolic fixed point of  $G$ , which is equivalent to that  $\tau_2^r \tau_1^{-s}(x) \neq x$ . Finally, we assume that  $x \in D_1 \cap D_2$  where  $\partial D_1 \cap \partial D_2 = \emptyset$  and neither  $D_1 \subset D_2$  nor  $D_2 \subset D_1$ . In this case, we can use Lemma 3.1 to prove that for large integers  $r$  and  $s$ ,  $\tau_2^r \tau_1^{-s}(s) \in D_1 \cap D_2$  is covered by more elements of  $\mathcal{U}_1$  than  $x$  is, from which we derive  $\tau_2^r \tau_1^{-s}(x) \neq x$ . Details are omitted. See [20] for more information. The lemma is proved.  $\square$

#### 4. Proof of Theorem 1.1

For the sufficient condition, suppose that  $H_1 \cap H_2 \neq \emptyset$ . Choose a point  $\hat{z} \in H_1 \cap H_2$  and let  $z = \varrho(\hat{z})$ , where  $\varrho: \mathbb{H} \rightarrow \tilde{S}$  is the universal covering. Then  $z$  belongs to a component of  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ . By hypothesis,  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  consists of once punctured disks  $\{Q_1, \dots, Q_k\}$ .

Assume that  $z \in Q_1$ , say. Let  $x_0$  be the puncture of  $Q_1$ . In  $Q_1$ , we connect  $z$  and the puncture  $x_0$  by an arc  $\gamma$  that avoids  $\tilde{\alpha}_1$  and  $\tilde{\alpha}_2$ . Obviously,  $\gamma$  can be lifted to an arc  $\hat{\gamma} \subset \mathbb{H}$  connecting  $\hat{z}$  and a parabolic cusp  $\hat{x}_0$ . Since  $\gamma \subset Q_1$ ,  $\hat{\gamma} \subset H_1 \cap H_2$ . But  $\zeta|_{H_1 \cap H_2} = \text{id}$ . It follows that  $\zeta|_{\hat{\gamma}} = \text{id}$ . Since  $\zeta$  has a continuous extension to  $\overline{\mathbb{H}}$ , we see that  $\zeta(\hat{x}_0) = \hat{x}_0$ . Therefore, according to the discussion in Section 2,  $\varphi^*([\zeta])$  is reducible by a single reduced loop on  $S$  that is the boundary of a twice punctured disk enclosing  $a$ .

For the necessary condition, we assume that  $H_1 \cap H_2 = \emptyset$ . By Lemma 3.2, the mapping class  $\varphi^*([\tau_i])$  is induced by the Dehn twist  $t_{\alpha_i}$ , where  $\alpha_i$  is a geodesic on  $S$  homotopic to  $\tilde{\alpha}_i$  on  $\tilde{S}$ . It follows that  $\varphi^*([\zeta]) \in \text{Mod}_S^g$  is represented by (1.3).

From Lemma 3.3,  $\tau_2^r \tau_1^{-s}$  does not fix any parabolic fixed point of  $G$  for large  $r$  and  $s$ , which says that if  $\varphi^*([\tau_2^r \tau_1^{-s}])$  is reducible, it must be reduced by a loop  $c$  that is also non-trivial on  $\tilde{S}$ . It follows that  $\varphi^*([\tau_2^r \tau_1^{-s}])$  projects to a reducible map  $F_0$  that is reduced by  $\tilde{c}$ . But since  $\tau_i$  is a lift of  $t_{\tilde{\alpha}_i}$ ,  $F_0$  is isotopic to  $t_{\tilde{\alpha}_2}^r \circ t_{\tilde{\alpha}_1}^{-s}$ . By hypothesis,  $\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  fills  $\tilde{S}$ . Thus  $t_{\tilde{\alpha}_2}^r \circ t_{\tilde{\alpha}_1}^{-s}$  is isotopic to a pseudo-Anosov map. It follows that  $F_0$  can not be reducible. This is a contradiction.

We conclude that  $\varphi^*([\tau_2^r \tau_1^{-s}])$  is pseudo-Anosov. Hence  $\{\alpha_1, \alpha_2\}$  fills  $S$ . Now by the Theorem of [17, 2, 12], for any integers  $N, r_i, s_i \in \mathbb{Z}^+$ ,

$$\zeta^* = \prod_{i=1}^N t_{\alpha_2}^{r_i} \circ t_{\alpha_1}^{-s_i}$$

is pseudo-Anosov. This completes the proof of Theorem 1.1. □

**5. Proof of Theorem 1.2**

Let  $\{\tau_1, \tau_2\}$  be such that  $H_1 \cap H_2 \neq \emptyset$ . Let  $D_i \in \mathcal{U}_i$  be maximal half planes such that  $\partial D_1 \cap \partial D_2 \neq \emptyset$ . Let  $D'_1 \in \mathcal{U}_1$  be another maximal element that is disjoint from both  $D_1$  and  $D_2$ . From Lemma 5.3.8 of Beardon [3], we can choose a hyperbolic element  $h \in G$  whose repelling fixed point lies in  $D_2$  and whose attracting fixed point lies in  $D'_1$ . For  $j \geq 1$ ,  $h^j(D_2)$  is a maximal half plane for  $h^j \tau_2 h^{-j}$  and the complement of  $h^j(D_2)$  is contained in  $D'_1$ . Thus for large  $j \geq 1$ , all pairs  $(\tau_1, h^j \tau_2 h^{-j})$  satisfies the condition of Theorem 1.1. From the theorem we conclude that

$$(5.1) \quad \varphi^* \left( \left[ \prod_i h^j \tau_2^{r_i} h^{-j} \tau_1^{-s_i} \right] \right)$$

is a pseudo-Anosov mapping class projecting to the class represented by  $\tilde{\zeta}$ . This is equivalent to that

$$(5.2) \quad \prod_{i=1}^N h^j \tau_2^{r_i} h^{-j} \tau_1^{-s_i}$$

does not fix any parabolic fixed point of  $G$  on  $\hat{\mathbb{R}}$ .

REMARK 5.1. To understand the mapping class (5.1) in topological term, we notice that the map that represents (5.1) is generated by the two geodesics  $\alpha_1$  and  $f(\alpha_2)$  where  $f \in \mathcal{F}$  is determined by an element  $h$  of  $G$ . To see how the curve  $\alpha_2$  is altered to  $f(\alpha_2)$ , we refer to Theorem 2 of Kra [10]. For example, if  $h$  is a simple hyperbolic, then  $f = \varphi^*(h)^j$  is a multiple of a spin map, written as  $t_c^j \circ t_{c_0}^{-j}$ , where both  $c$  and  $c_0$  are homotopic to  $\tilde{c}$ , the projection of the axis of  $h$ . if  $h$  is parabolic, then  $f$  is an ordinary power of the Dehn twist along the boundary of a twice punctured disk on  $S$  enclosing  $a$ .

Note that  $\tau_i$  determines an isomorphism  $\chi_i: G \rightarrow G$  that is defined by

$$(5.3) \quad \tau_i h = \chi_i(h) \tau_i.$$

It follows from (5.3) that (5.2) can be written as  $g_j \zeta$  for  $g_j \in G$ .

We claim that for sufficiently large  $j$ ,

$$g_{j+1} \zeta \neq g_j \zeta.$$

Indeed, as discussed above, for large  $j$ , the complement of  $D_3 = h^j(D_2)$  is contained in  $D'_1$ . This implies that  $D_3 \supset D_1$ . Let  $D_4 = h(D_3)$ . We have  $D_4 \supset D_3 \supset D_1$  and  $D_4$  is

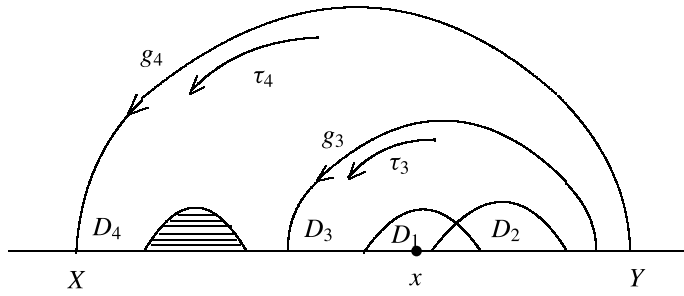


Fig. 2.

a maximal element for the map  $h^{j+1}\tau_2h^{-j-1}$ . For simplicity, we denote  $\tau_3 = h_j\tau_2h_j^{-1}$  and  $\tau_4 = h_{j+1}\tau_2h_{j+1}^{-1}$ . Let

$$\omega_j = \prod_{i=1}^N (\tau_3^{r_i} \tau_1^{-s_i})$$

and

$$\omega_{j+1} = \prod_{i=1}^N (\tau_4^{r_i} \tau_1^{-s_i}).$$

Pick any point  $x \in D_1 \cap \hat{\mathbb{R}}$ . Since  $D_1$  is maximal for  $\tau_1$ ,  $y = \tau_1^{-s_1}(x) \in D_1 \cap \hat{\mathbb{R}}$ . Note also that  $D_3$  is maximal for  $\tau_3$ . Since  $\tilde{\alpha}_1$  is simple, by Property (3),  $\tau_3$  is monotonic on  $D_3 \cap \hat{\mathbb{R}}$ . By (3.2), we conclude that for all positive integers  $r_1$ ,  $\tau_3^{r_1}(y) \in (D_3 - D_1) \cap \hat{\mathbb{R}}$ . By induction process, one can show that  $\omega_j(x) \in (D_3 - D_1) \cap \hat{\mathbb{R}}$ .

On the other hand, since  $D_4 \supset D_3$  is a maximal element for  $\tau_4$  and hence is invariant under the action of  $\tau_4$ . Denote  $\partial D_4 \cap \hat{\mathbb{R}} = \{X, Y\}$  as shown in Fig. 2. Let  $g_4 \in G$  be the element that keeps  $D_4$  invariant and takes the same orientation as in Fig. 1.

Now  $y = \tau_1^{-s_1}(x) \in D_1 \cap \hat{\mathbb{R}} \subset D_3 \cap \hat{\mathbb{R}}$ . Hence we get that  $g_4^{r_1}(y) \in g_4^{r_1}(D_3)$ . Observe that  $g_4^{r_1}(D_3)$  is disjoint from  $D_3$  (the shaded region in Fig. 2).

It is obvious that  $g_4^{r_1}(D_3)$  contains  $g_4^{r_1}(D_1)$ . By Property (4),  $g_4^{r_1}(D_1)$  is a maximal element of  $\mathcal{U}_1$ , which means that  $\tau_1^{-s_2}$  keeps  $g_4^{r_1}(D_1)$  invariant. It follows that

$$(5.4) \quad \tau_1^{-s_2} g_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}.$$

On the other hand, (3.2) and (3.2) along with Property (2) yield

$$X < \tau_4^{r_1}(y) < g_4^{r_1}(y).$$

Hence by Property (3) again, we obtain

$$X < \tau_1^{-s_2} \tau_4^{r_1}(y) < \tau_1^{-s_2} g_4^{r_1}(y).$$

It follows from (5.4) that  $\tau_1^{-s_2} \tau_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$  and that  $\tau_4^{r_2} \tau_1^{-s_2} \tau_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$ . That is,  $\tau_4^{r_2} \tau_1^{-s_2} \tau_4^{r_1} \tau_1^{-s_1}(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$ .

By induction process, we can show that

$$(\tau_4^{r_N} \tau_1^{-s_N}) \cdots (\tau_4^{r_2} \tau_1^{-s_2})(\tau_4^{r_1} \tau_1^{-s_1})(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}.$$

That is,  $\omega_{j+1}(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$ . In particular, we conclude that  $\omega_{j+1}(x) \neq \omega_j(x)$ . Similar argument yields that  $\omega_{j+k}(x) \neq \omega_{j+l}(x)$  for  $k \neq l$  and  $k, l \geq 0$ . This completes the proof of Theorem 1.2.  $\square$

### 6. Proof of Theorem 1.3

From Theorem 1.2, there are infinitely many elements  $h_j \in G$  so that  $\omega_j = h_j \zeta$  do not fix any fixed points of  $G$ . Hence all  $\omega_j^* = \varphi^*([\omega_j])$  are pseudo-Anosov mapping class of  $S$  projecting to (1.1). From the construction, each  $\omega_j^*$  is induced by a map with from (1.3). By the argument of Theorem 1.2, there are infinitely many distinct elements in the sequence  $\{\omega_j^*\}$ . This proves (1)–(3) of Theorem 1.3.

To prove (4)–(6) of Theorem 1.3, we choose a point in the Teichmüller space  $T(\tilde{S})$  represented  $F$ -minimal surface denoted by  $\tilde{S}$ . Let  $G$  be the Fuchsian group so that  $\mathbb{H}/G = \tilde{S}$ . Let  $z_0$  be a zero of  $\phi$  ( $\phi$  is defined by the pseudo-Anosov map  $F$ ), so that  $F(z_0) = z_0$  and  $H_t(z_0)$ ,  $0 \leq t \leq 1$ , is trivial if  $z_0$  is not a puncture. It may or may not be a puncture of  $\tilde{S}$ .

Associated to each  $\hat{z}_0 \in \overline{\mathbb{H}}$  with  $\varrho(\hat{z}_0) = z_0$ , there is a map  $\zeta$  defined by (1.3). To see that  $\varphi^*([\zeta])$  is pseudo-Anosov if  $z_0$  is a non-puncture zero, we refer to [18] and outline the proof as follows. Let  $l$  denote the (unique) Teichmüller geodesic in  $T(\tilde{S})$  determined by  $\tilde{\zeta}$  defined as (1.1). Let  $\hat{l} \subset F(\tilde{S})$  be a lift of  $l$  defined by

$$\hat{l} = \{([t\mu], w^{t\mu}(\hat{z}_0)), t \in (-1, 1)\} \subset F(\tilde{S}).$$

Clearly,  $\hat{l}$  is a line in  $F(\tilde{S})$  passing through  $\hat{z}_0$ . By [10],  $\varphi(\hat{l}) \subset T(S)$  is a Teichmüller geodesic in  $T(S)$ . By the argument of Proposition 3 and Corollary 2 of [10],  $\hat{l}$  is invariant under a lift  $\hat{F}$  of  $F$  with  $\hat{F}(\hat{z}_0) = \hat{z}_0$ . From the assumption,  $H_t(z_0)$  is a trivial loop. This implies that  $\zeta|_{\hat{\mathbb{R}}} = \hat{F}|_{\hat{\mathbb{R}}}$ . Therefore,  $\varphi(\hat{l})$  is invariant under the action of  $\varphi^*([\zeta])$ . So by Bers [5],  $\varphi^*([\zeta])$  is pseudo-Anosov.

We need to prove that all  $\omega_n^*$ , for large  $j$ , are not conjugate to  $\varphi^*([\zeta])$ .

Suppose that for some  $j \geq 1$ , there is  $h_0 \in G$  such that

$$(6.1) \quad \prod_{i=1}^N (h^j \tau_2 h^{-j})^{r_i} \tau_1^{-s_i} = h_0 \zeta h_0^{-1},$$

where  $\zeta$  is defined in (1.4). Note that  $\{\tau_1, \tau_2\}$  possesses the property that  $H_1 \cap H_2 \neq \emptyset$ . If  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  consists of once punctured disks only,  $\zeta$  fixes a parabolic fixed point of

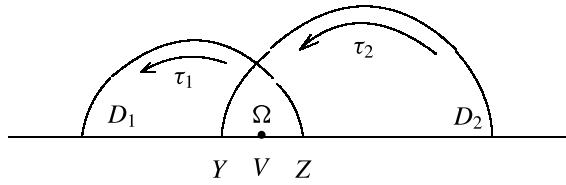


Fig. 3.

$G$  on  $\mathbb{R}$ . However, if  $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$  contains some disk components, for some  $\zeta$  defined as (1.4), the existence of fixed points of  $\zeta$  that are also fixed by elements of  $G$  is not clear. In this case, we use a different approach as follows.

For  $i = 1, 2$ , let  $D_i$  be maximal elements of  $\mathcal{U}_i$  so that  $\partial D_1$  intersects with  $\partial D_2$ . Let  $\Omega = D_1 \cap D_2$  and  $\Lambda = \Omega \cap \hat{\mathbb{R}}$ . See Fig. 3.

Now we consider the action of  $\zeta$  on  $\Lambda$ . The endpoint  $Y$  is moved to the right while the point  $Z$  is moved to the left. Since the action of  $\zeta$  on  $\Lambda$  is continuous, there is a point  $V \in \Lambda$  so that  $\zeta(V) = V$  (according to Lemma 2.2,  $V$  is not a hyperbolic fixed point of  $G$ , but it could be a parabolic fixed point of  $G$ ). Thus  $h_0 \zeta h_0^{-1}$  fixes  $h_0(V) \in \hat{\mathbb{R}}$ .

On the other hand,  $\{\tau_1, h^j \tau_2 h^{-j}\}$  has the property that  $H_1 \cap H_2 = \emptyset$ . By the same argument of Lemma 3.3,

$$\prod_{i=1}^N (h^j \tau_2 h^{-j})^{r_i} \tau_1^{-s_i}$$

does not fix any point on  $\hat{\mathbb{R}}$ . This contradicts to (6.1). The same methods can be used to prove that all  $\omega_j^*$  lie in different conjugacy classes. Details are omitted.  $\square$

ACKNOWLEDGMENT. I am grateful to the referee(s) for pointing out errors in the original version of this paper; and particularly for their helpful comments and thoughtful suggestions.

---

**References**

- [1] L.V. Ahlfors and L. Bers: *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.
- [2] M. Bauer: *Examples of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **330** (1992), 333–359.
- [3] A. Beardon: *The Geometry of Discrete Groups*, Springer, New York, 1983
- [4] L. Bers: *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
- [5] L. Bers: *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Math. **141** (1978), 73–98.



- [6] J.S. Birman: Braids, Links, and Mapping Class Groups, Ann. of Math. Stud. **82**, Princeton Univ. Press, Princeton, N.J., 1974.
- [7] A. Faithi, F. Laudenbach and V. Poenaru: Travaux de Thurston sur les Surfaces, Seminaire Orsay, Asterisque, **66–67**, Soc. Math. de France, 1979.
- [8] N.V. Ivanov: Mapping Class Groups, manuscript, 1998.
- [9] H.M. Farkas and I. Kra: Riemann Surfaces, Springer, New York, 1980.
- [10] I. Kra: *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.
- [11] O. Lehto: Univalent Functions and Teichmüller Spaces, Graduate Texts in Mathematics **109**, Springer, New York, 1987.
- [12] D.D. Long: *Constructing pseudo-Anosov maps*; in Knot Theory and Manifolds (Vancouver, B.C., 1983), Lecture Notes in Math. **1144**, Springer, Berlin, 1985, 108–114.
- [13] A. Marden and K. Strebel: *Pseudo-Anosov Teichmüller mappings*, J. Analyse Math. **46** (1986), 194–220.
- [14] S. Nag: *Non-geodesic discs embedded in Teichmüller spaces*, Amer. J. Math. **104** (1982), 399–408.
- [15] R.C. Penner: *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **310** (1988), 179–197.
- [16] R.C. Penner: *Probing mapping class groups using arcs*, manuscript, 2005.
- [17] W.P. Thurston: *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431.
- [18] C. Zhang: *Singularities of quadratic differentials and extremal Teichmüller mappings defined by Dehn twists*, to appear in J. Aust. Math. (2009).
- [19] C. Zhang: *Remarks on Thurston's construction of pseudo-Anosov maps of Riemann surfaces*, Chin. Ann. Math. Ser. B **29** (2008), 85–94.
- [20] C. Zhang: *Dehn twists combined with pseudo-Anosov maps*, preprint (2009).

Department of Mathematics  
Morehouse College  
Atlanta, GA 30314  
USA  
e-mail: czhang@morehouse.edu