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ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

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1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing right continuous family of \((\mathcal{F}_n, \mathbb{P})\)-complete \(\sigma\)-algebras \((\mathcal{F}_i)\), and let \(\mathcal{P}\) be the predictable \(\sigma\)-algebra induced on \(\Omega \times \mathbb{R}_+\) by the family \((\mathcal{F}_i)\).

For \(H \in \mathcal{P}\), we write \(H_s\) for the random variable \(\omega \mapsto 1_{H}(s, \omega)\). If \(Z = N + B\) is a semi-martingale such that \(N\) is a square integrable martingale and \(B\) an adapted process with square integrable variation, the mapping

\[
H \rightarrow \int_0^t H_s \, dZ_s
\]

defines a \(\sigma\)-additive vector measure on \((\Omega \times \mathbb{R}_+, \mathcal{P})\) with values in \(L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})\). It has been shown by several authors that conversely if \(\mu\) is a \(\sigma\)-additive measure from \(\mathcal{P}\) to \(L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})\) given on the elementary predictable sets \(H\) of the form

\[
H = h \times [s, t] \quad 0 < s < t, \quad h \in \mathcal{F}_s
\]

by

\[
\mu(H) = 1_{h}(Z_t - Z_s)
\]

for a mean square right-continuous adapted process \(Z\), then there is a modification of \(Z\) which is a semi-martingale [2].

Nevertheless, if we consider an other probability space \((W, \mathcal{W}, \mathbb{Q})\), an adapted process \((\omega, t) \mapsto Z_t(\omega, w)\) depending on \(w \in W\), and a measure \(\mu\) which satisfies (2) for elementary predictable sets, and if we replace \(\sigma\)-additivity in \(L^2(\mathbb{P})\) for each \(w \in W\) by \(\sigma\)-additivity in \(L^2(\mathbb{P} \times \mathbb{Q})\), it becomes possible that \(Z_t\) fails to be a semi-martingale for fixed \(w\).

In the example that we give, \(Z_t\) is, for fixed \(w\), the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.
2. Random mixing of semi-martingales

Let \((U_\alpha(\omega))_{\alpha \in \Lambda}\) be a second order process on \((W, \mathcal{W}, Q)\) which is right continuous in \(L^2\), with orthogonal increments and \(\mathcal{B}(R) \times \mathcal{W}\) measurable and let \(m\) be the positive Radon measure on \(R\) associated to \(U_\alpha\) by

\[
m(\{\alpha, \beta\}) = E_Q(U_\beta - U_\alpha)^2, \quad \alpha < \beta.
\]

Let \((M^\alpha_t(\omega))_{\alpha \in \Lambda}\) be a family of right continuous and left limited martingales, and \((A^\alpha_t(\omega))_{\alpha \in \Lambda}\) a family of continuous increasing adapted processes on \((\Omega, \mathcal{F}, \mathcal{P})\) such that the maps \((\alpha, \omega, s) \to M^\alpha_t(\omega)\) and \((\alpha, \omega, s) \to A^\alpha_t(\omega)\) are \(\mathcal{B}(R) \times \mathcal{F}_s \times \mathcal{B}(R_+)^{\alpha}\) measurable on \(R \times \Omega \times [0, t]\) and such that

\[
\int_{\alpha \in \Lambda} E_P[(M^\alpha_t)^2+(A^\alpha_t)^2] \, dm(\alpha) < +\infty.
\]

Then we set \(Z^\alpha_t(\omega) = M^\alpha_t(\omega) + A^\alpha_t(\omega)\) and

\[
Z_t(\omega, \omega) = \int_{\alpha \in \Lambda} Z^\alpha_t(\omega) \, dU_\alpha(\omega)
\]

where the stochastic integral is of Wiener's type and exists for \(\mathcal{P}\) almost all \(\omega\) since by (3) \(Z^\alpha_t(\omega)\) belongs to \(L^2(R, \mathcal{B}(R), dm(\alpha))\) for \(\mathcal{P}\)-almost all \(\omega\).

For \(\mathcal{P}\)-almost \(\omega\) the process \(Z^\alpha_t(\omega)\) is right continuous and left limited in \(L^2(W, \mathcal{W}, Q)\).

If \(G\) is an elementary predictable process on \((\Omega, \mathcal{F}, \mathcal{P})\) given by:

\[
G_t(\omega) = G_0(\omega) \, 1_{0, t_1} + \cdots + G_s(\omega) \, 1_{t_s, t_{s+1}}(s)
\]

for \(0 < t_1 < \cdots < t_{s+1}\), where \(G_t\) is a \(\mathcal{F}_t\)-measurable bounded random variable, it follows immediately

\[
G_0(Z_{t_1} - Z_0) + \cdots + G_s(Z_{t_{s+1}} - Z_{t_s}) = \int_{\alpha \in \Lambda} \left( \int_0^\infty G_\alpha \, dZ^\alpha_t \right) \, dU_\alpha.
\]

And we have:

**Proposition 1.** The map \(H \in \mathcal{P} \to \int_0^t H_s \, dZ_s\) defined by

\[
\int_0^t H_s \, dZ_s = \int_{\alpha \in \Lambda} \left( \int_0^t H_s \, dZ^\alpha_t \right) \, dU_\alpha
\]

is a \(\sigma\)-additive \(L^2(\mathcal{P} \times Q)\) valued measure on \((\Omega \times R_+, \mathcal{P})\).

**Proof.** Let \(H^{(s)}\) be a sequence of disjoint predictable subsets of \(\Omega \times R_+\), we have

\[
E_P \, E_Q \left[ \int_{\alpha \in \Lambda} \left( \int_0^t \sum_{s = -N}^s H^{(s)}_s \, dZ^\alpha_t \right) \, dU_\alpha \right]^2
\]

\[
= \int_{\alpha \in \Lambda} E_P \left( \int_0^t \sum_{s = -N}^s H^{(s)}_s \, dZ^\alpha_t \right)^2 \, dm(\alpha)
\]
which can be made arbitrarily small for $N$ large enough because

$$E_P \left( \int_0^t \sum_{n=1}^N H_i^{(n)} dZ_i^n \right)^2$$

tends to zero and remains bounded by

$$2 E_P \left[ (M_t^*)^2 + (A_t^*)^2 \right] < +\infty. \quad \square$$

Set

$$Z_i^{(1)} = \int_{s \in \mathcal{R}} M_s^* dU_s \quad \text{and} \quad Z_i^{(2)} = \int_{s \in \mathcal{R}} A_s^* dU_s.$$

**Lemma 2.** There is a $P \times Q$-modification $\hat{Z}_i^{(1)}$ of $Z_i^{(1)}$ which is a $(\Omega, \mathcal{F}_t, P)$ right continuous and left limited martingale for $Q$-almost all $w$.

**Proof.** Let $G \in \mathcal{F}_s$, the following equalities hold in $L^2(W, \mathcal{W}, Q)$ for $s < t$:

$$E_P \left[ \int_G Z_i^{(1)} \right] = \int_{s \in \mathcal{R}} E_P \left[ \int_G M_s^* dU_s \right] = \int_{s \in \mathcal{R}} E_P \left[ \int_G M_s^* dU_s \right],$$

therefore, if we choose a $\mathcal{F}_t \times \mathcal{W}$-measurable element $z_i^{(1)}(\omega, w)$ in the $L^2(P \times Q)$ equivalence class of $Z_i^{(1)}$, for $w$ outside a $Q$-negligible set $\mathcal{N}$, $z_i^{(1)}$ is a $(\mathcal{F}_t, P)$-martingale for rational $s$.

Then, if we put $\hat{Z}_i^{(1)} = \lim_{r \to \infty} z_i^{(1)}$, for $w \in \mathcal{N}$, $Z_i^{(1)}$ is $P$-almost surely a right continuous and left limited $(\mathcal{F}_t)$-martingale and

$$\hat{Z}_i^{(1)} = Z_i^{(1)} \quad P \times Q\text{-a.e.}$$

because $Z_i^{(1)}$ is right continuous in $L^2(P \times Q)$. \quad \square

As concerns $Z_i^{(2)}$, it is a zero energy process:

**Lemma 3.** Let $\tau_n$ be a sequence of partitions of $[0, t]$ with diameter tending to zero, then

$$E_Q \left[ \left( \sum_{i \in S} (Z_{i+1}^{(2)} - Z_i^{(2)})^2 \right)_{s \in S} \right] \to 0.$$

**Proof.** The expression is equal to

$$E_P \left[ \sum_{s \in \mathcal{R}} \sum_{\tau_n} (A_{s+1}^* - A_s^*)^2 \right] dm(\alpha),$$

and $\sum_{\tau_n} (A_{s+1}^* - A_s^*)^2$ tends to zero, because $A_s^*$ is continuous, and remains majorized by $(A_s^*)^2$, which gives the result by (3).

Nevertheless, in general $Z_i^{(2)}$ has no modification with finite variation, as shown by the following example:
Let $X$ be a continuous martingale on $(\Omega, \mathcal{F}, P)$ such that

$$E_P X_\infty^2 < +\infty.$$ 

Let

$$M_t^\alpha = \int_0^t 1_{\{X_s > \alpha\}} dX_s$$

and

$$A_t^\alpha = \frac{1}{2} L_t^\alpha$$

where $L_t^\alpha$ is the local time of $X$ at $\alpha$. Condition (3) is satisfied as soon as the measure $m$ is finite. If we put

$$Z_t = \int_{\alpha \in R} M_t^\alpha dU_\alpha + \int_{\alpha \in R} A_t^\alpha dU_\alpha$$

we have, from Meyer-Tanaka's formula:

$$Z_t = \int_{\alpha \in R} [(X_t - \alpha)^+ - (X_0 - \alpha)^+] dU_\alpha = \int_{X_0}^{X_t} U_\lambda d\lambda \quad P\times Q \text{ a.e.}$$

If $Z_t$ had a $P\times Q$-modification such that, for fixed $w \in W$, $\tilde{Z}_t$ were a $(\Omega, \mathcal{F}_t, P)$ semi-martingale, then, since $\tilde{Z}_t$ and $\int_{X_0}^{X_t} U_\lambda d\lambda$ are both right continuous, $\int_{X_0}^{X_t} U_\lambda d\lambda$ would be a semi-martingale. So, from ([1], theorem 5, 6), if we took for $X$ a real stopped brownian motion starting at 0, the map

$$x \rightarrow \int_0^x U_\lambda(w) d\lambda$$

would be the difference of two convex functions. But, if for example, $U$ itself is a stopped brownian motion, that can be true only on a $Q$-negligible set because almost all brownian sample paths have not finite variation. So, in this case, the $P\times Q$-modifications of $Z_t$ are $Q$ a.e. not semi-martingales.

References


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