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ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

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1. Introduction

Let $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ be a probability space with an increasing right continuous family of $(\mathcal{F}_\infty, \boldsymbol{P})$-complete σ-algebras $(\mathcal{F}_t)$, and let $\mathcal{P}$ be the predictable σ-algebra induced on $\Omega \times \boldsymbol{R}_+$ by the family $(\mathcal{F}_t)$.

For $H \in \mathcal{P}$, we write H_s for the random variable $\omega \to 1_H(s, \omega)$. If $Z = N + B$ is a semi-martingale such that N is a square integrable martingale and B an adapted process with square integrable variation, the mapping

$$H \to \int_0^\infty H_s \, dZ_s \tag{1}$$

defines a σ-additive vector measure on $(\Omega \times \boldsymbol{R}_+, \mathcal{P})$ with values in $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$. It has been shown by several authors that conversely if μ is a σ-additive measure from $\mathcal{P}$ to $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$ given on the elementary predictable sets H of the form

$$H = h \times]s, t] \qquad 0 < s < t, \quad h \in \mathcal{F}_s$$

by

$$\mu(H) = 1_h(Z_t - Z_s) \tag{2}$$

for a mean square right-continuous adapted process Z, then there is a modification of Z which is a semi-martingale [2].

Nevertheless, if we consider an other probability space $(W, \mathcal{W}, \boldsymbol{Q})$, an adapted process $(\omega, t) \to Z_t(\omega, w)$ depending on $w \in W$, and a measure μ which satisfies (2) for elementary predictable sets, and if we replace σ-additivity in $L^2(\boldsymbol{P})$ for each $w \in W$ by σ-additivity in $L^2(\boldsymbol{P} \times \boldsymbol{Q})$, it becomes possible that Z_t fails to be a semi-martingale for fixed w.

In the example that we give, Z_t is, for fixed w, the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.

2. Random mixing of semi-martingales

Let $(U_\alpha(w))_{\alpha\in\boldsymbol{R}}$ be a second order process on $(W, \mathcal{W}, Q)$ which is right continuous in L^2, with orthogonal increments and $\mathcal{B}(\boldsymbol{R})\times\mathcal{W}$ measurable and let m be the positive Radon measure on $\boldsymbol{R}$ associated to U_α by

$$m(]\alpha, \beta]) = \boldsymbol{E}_Q(U_\beta - U_\alpha)^2\,, \quad \alpha<\beta\,.$$

Let $(M_t^\alpha(\omega))_{\alpha\in\boldsymbol{R}}$ be a family of right continuous and left limited martingales, and $(A_t^\alpha(\omega))_{\alpha\in\boldsymbol{R}}$ a family of continuous increasing adapted processes on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ such that the maps $(\alpha, \omega, s)\to M_s^\alpha(\omega)$ and $(\alpha, \omega, s)\to A_s^\alpha(\omega)$ are $\mathcal{B}(\boldsymbol{R})\times\mathcal{F}_t\times\mathcal{B}(\boldsymbol{R}_+)$ measurable on $\boldsymbol{R}\times\Omega\times[0, t]$ and such that

$$\int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_P[(M_\infty^\alpha)^2+(A_\infty^\alpha)^2]\, dm(\alpha)<+\infty\,. \tag{3}$$

Then we set $Z_t^\alpha(\omega) = M_t^\alpha(\omega)+A_t^\alpha(\omega)$ and

$$Z_t(\omega, w) = \int_{\alpha\in\boldsymbol{R}} Z_t^\alpha(\omega)\, dU_\alpha(w) \tag{4}$$

where the stochastic integral is of Wiener's type and exists for $\boldsymbol{P}$ almost all ω since by (3) $Z_t^\alpha(\omega)$ belongs to $L^2(\boldsymbol{R}, \mathcal{B}(\boldsymbol{R}), dm(x))$ for $\boldsymbol{P}$-almost all ω.

For $\boldsymbol{P}$-almost ω the process $Z_t(\omega, w)$ is right continuous and left limited in $L^2(W, \mathcal{W}, \boldsymbol{Q})$.

If G is an elementary predictable process on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ given by:

$$G_s(\omega) = G_0(\omega)\, 1_{]0,t_1]}(s)+\cdots+G_n(\omega)\, 1_{]t_n,t_{n+1}]}(s)$$

for $0<t_1<\cdots<t_{n+1}$, where G_i is a $\mathcal{F}_{t_i}$-measurable bounded random variable, it follows immediately

$$G_0(Z_{t_1}-Z_0)+\cdots+G_n(Z_{t_{n+1}}-Z_{t_n}) = \int_{\alpha\in\boldsymbol{R}} \left(\int_0^\infty G_s\, dZ_s^\alpha\right) dU_\alpha\,.$$

And we have:

Proposition 1. *The map $H\in\mathcal{P}\to\int_0^t H_s\, dZ_s$ defined by*

$$\int_0^t H_s\, dZ_s = \int_{\alpha\in\boldsymbol{R}} \left(\int_0^t H_s\, dZ_s^\alpha\right) dU_\alpha$$

is a σ-additive $L^2(\boldsymbol{P}\times\boldsymbol{Q})$ valued measure on $(\Omega\times\boldsymbol{R}_+, \mathcal{P})$.

Proof. Let $H^{(n)}$ be a sequence of disjoint predictable subsets of $\Omega\times\boldsymbol{R}_+$, we have

$$\boldsymbol{E}_P\,\boldsymbol{E}_Q\left[\int_{\alpha\in\boldsymbol{R}} \left(\int_0^t \sum_{n=N}^\infty H_s^{(n)}\, dZ_s^\alpha\right) dU_\alpha\right]^2$$

$$= \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E}_P\left(\int_0^t \sum_{n=N}^\infty H_s^{(n)}\, dZ_s^\alpha\right)^2 dm(\alpha)$$

which can be made arbitrarily small for N large enough because

$$\boldsymbol{E_P}\left(\int_0^t \sum_{n=N}^{\infty} H_s^{(n)}\, dZ_s^{\alpha}\right)^2$$

tends to zero and remains bounded by

$$2\,\boldsymbol{E_P}\,[(M_\infty^{\alpha})^2+(A_\infty^{\alpha})^2]<+\infty. \square$$

Set

$$Z_t^{(1)} = \int_{\alpha\in \boldsymbol{R}} M_t^{\alpha}\, dU_\alpha \quad \text{and} \quad Z_t^{(2)} = \int_{\alpha\in \boldsymbol{R}} A_t^{\alpha}\, dU_\alpha\,.$$

Lemma 2. *There is a $\boldsymbol{P}\times\boldsymbol{Q}$-modification $\tilde{Z}_t^{(1)}$ of $Z_t^{(1)}$ which is a $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ right continuous and left limited martingale for $\boldsymbol{Q}$-almost all w.*

Proof. Let $G\in\mathcal{F}_s$, the following equalities hold in $L^2(W, \mathcal{W}, \boldsymbol{Q})$ for $s<t$:

$$\boldsymbol{E_P}[1_G\, Z_t^{(1)}] = \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E_P}[1_G\, M_t^{\alpha}]\, dU_\alpha = \int_{\alpha\in\boldsymbol{R}} \boldsymbol{E_P}[1_G\, M_s^{\alpha}]\, dU_\alpha$$
$$= \boldsymbol{E_P}[1_G\, Z_s^{(1)}]\,,$$

therefore, if we choose a $\mathcal{F}_t\times\mathcal{W}$-measurable element $z_t^{(1)}(\omega, w)$ in the $L^2(\boldsymbol{P}\times\boldsymbol{Q})$ equivalence class of $Z_t^{(1)}$, for w outside a $\boldsymbol{Q}$-negligible set $\mathcal{N}$, $z_s^{(1)}$ is a $(\mathcal{F}_s, \boldsymbol{P})$-martingale for rational s.
Then, if we put $\tilde{Z}_t^{(1)}= \lim\limits_{\substack{s \text{ rational}\\ s\downarrow t}} z_s^{(1)}$, for $w\notin\mathcal{N}$, $\tilde{Z}_t^{(1)}$ is $\boldsymbol{P}$-almost surely a right continuous and left limited $(\mathcal{F}_t)$-martingale and

$$\tilde{Z}_t^{(1)} = Z_t^{(1)} \qquad \boldsymbol{P}\times\boldsymbol{Q}\text{-a.e.}$$

because $Z_t^{(1)}$ is right continuous in $L^2(\boldsymbol{P}\times\boldsymbol{Q})$. $\square$

As concerns $Z_t^{(2)}$, it is a zero energy process:

Lemma 3. *Let τ_n be a sequence of partitions of* $[0, t]$ *with diameter tending to zero, then*

$$\boldsymbol{E}_{\mathrm{Q}}\,\boldsymbol{E_P}\Big[\sum_{t_i\in\tau_n} (Z_{t_{i+1}}^{(2)}-Z_{t_i}^{(2)})^2\Big] \underset{n\uparrow\infty}{\rightarrow} 0\,.$$

Proof. The expression is equal to

$$\boldsymbol{E_P}\int_{\alpha\in\boldsymbol{R}} \sum_{\tau_n} (A_{t_{i+1}}^{\alpha}-A_{t_i}^{\alpha})^2\, dm(\alpha)\,,$$

and $\sum_{\tau_n} (A_{t_{i+1}}^{\alpha}-A_{t_i}^{\alpha})^2$ tends to zero, because A_t^{α} is continuous, and remains majorized by $(A_\infty^{\alpha})^2$, which gives the result by (3).

Nevertheless, in general $Z_t^{(2)}$ has no modification with finite variation, as shown by the following example:

Let X be a continuous martingale on $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ such that

$$\boldsymbol{E_P}\, X_\infty^2 < +\infty .$$

Let

$$M_t^\alpha = \int_0^t 1_{\{X_s > \alpha\}}\, dX_s$$

and $A_t^\alpha = \frac{1}{2} L_t^\alpha$

where L_t^α is the local time of X at α. Condition (3) is satisfied as soon as the measure m is finite. If we put

$$Z_t = \int_{\alpha\in\boldsymbol{R}} M_t^\alpha\, dU_\alpha + \int_{\alpha\in\boldsymbol{R}} A_t^\alpha\, dU_\alpha$$

we have, from Meyer-Tanaka's formula:

$$Z_t = \int_{\alpha\in\boldsymbol{R}} [(X_t-\alpha)^+ - (X_0-\alpha)^+]\, dU_\alpha = \int_{X_0}^{X_t} U_\lambda\, d\lambda \quad \boldsymbol{P}\times\boldsymbol{Q} \text{ a.e.}$$

If Z_t had a $\boldsymbol{P}\times\boldsymbol{Q}$-modification such that, for fixed $w\in W$, $\tilde{Z}_t$ were a $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ semi-martingale, then, since $\tilde{Z}_t$ and $\int_{X_0}^{X_t} U_\lambda d\lambda$ are both right continuous, $\int_{X_0}^{X_t} U_\lambda d\lambda$ would be a semi-martingale. So, from ([1], theorem 5, 6), if we took for X a real stopped brownian motion starting at 0, the map

$$x \overset{\psi}{\to} \int_0^x U_\lambda(w)\, d\lambda$$

would be the difference of two convex functions. But, if for example, U itself is a stopped brownian motion, that can be true only on a $\boldsymbol{Q}$-negligible set because almost all brownian sample paths have not finite variation. So, in this case, the $\boldsymbol{P}\times\boldsymbol{Q}$-modifications of Z_t are $\boldsymbol{Q}$ a.e. not semi-martingales.

References

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