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| Author(s)    | Wada, Kentaro                                                                                              |
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# BLOCKS OF CATEGORY $\mathcal{O}$ FOR RATIONAL CHEREDNIK ALGEBRAS AND OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r, p, n)$

KENTARO WADA

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## Abstract

We classify blocks of category  $\mathcal{O}$  for rational Cherednik algebras and of cyclotomic Hecke algebras of type  $G(r, p, n)$  by using the “residue equivalence” for multipartitions.

## 0. Introduction

Let  $V$  be a finite dimensional vector space over  $\mathbb{C}$ , and  $W \subset \mathrm{GL}(V)$  be a finite complex reflection group. The rational Cherednik algebra  $\mathcal{H} = \mathcal{H}(W)$  over  $\mathbb{C}$  associated to  $W$  was introduced by [7]. It is known that the category  $\mathcal{O}$  of  $\mathcal{H}$  is a highest weight category with standard modules  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ , where  $\Lambda^+$  is an index set of pairwise non-isomorphic simple  $W$ -modules over  $\mathbb{C}$  ([11], [9]). Let  $\mathcal{H} = \mathcal{H}(W)$  be the cyclotomic Hecke algebra associated to  $W$  with appropriate parameters. Let  $\mathrm{KZ}: \mathcal{O} \rightarrow \mathcal{H}\text{-mod}$  be the Knizhnik–Zamolodchikov functor defined in [9]. It is known that  $\mathcal{O}$  is a quasi-hereditary cover (highest weight cover) of  $\mathcal{H}$  in the sense of [21]. Put  $S(\lambda) = \mathrm{KZ}(\Delta(\lambda))$ . We see that there exists a one-to-one correspondence between the blocks of  $\mathcal{O}$  and of  $\mathcal{H}$  thanks to the double centralizer property. Moreover, we see that the classification of blocks of  $\mathcal{O}$  and of  $\mathcal{H}$  is given by the linkage classes on  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$  or on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$  (see §1 for details). Hence, in order to classify the blocks of  $\mathcal{O}$  and of  $\mathcal{H}$ , it is enough to determine the linkage classes on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ .

In the case where  $W$  is a complex reflection group of type  $G(r, 1, n)$ ,  $\mathcal{H}$  is also called the Ariki–Koike algebra. In this case,  $\Lambda^+$  is the set of  $r$ -partitions of size  $n$ , which we denote by  $\mathcal{P}_{n,r}$ . Then the linkage classes on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$  are given by the equivalence relation “ $\sim_R$ ”, the so called residue equivalence, on  $\mathcal{P}_{n,r}$  by [17]. (Note that the Specht module  $S^\lambda$  ( $\lambda \in \Lambda^+$ ) considered in [17] does not coincide with  $S(\lambda)$  in general. However, one sees that the linkage classes on  $\{S^\lambda \mid \lambda \in \Lambda^+\}$  coincide with the linkage classes on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ . See §3.)

Our purpose is to classify the blocks of  $\mathcal{O}$  and of  $\mathcal{H}$  in the case where  $W$  is a complex reflection group of type  $G(r, p, n)$ . As seen in the above, we should determine the linkage classes on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ . Let  $W^\dagger$  be the complex reflection group of type  $G(r, 1, n)$ , and we denote by adding the superscript  $\dagger$  for objects of type  $G(r, 1, n)$ . It is known that  $W$  is a normal subgroup of  $W^\dagger$  with the index  $p$ , and that  $\mathcal{H}$  is a subalgebra of  $\mathcal{H}^\dagger$ . An index set  $\Lambda^+$  of pairwise non-isomorphic simple  $W$ -modules over  $\mathbb{C}$  (thus,  $\Lambda^+$  is also an index set of standard modules of  $\mathcal{O}$ ) is given as the equivalence classes of  $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  under a certain equivalence relation “ $\sim_*$ ” on  $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  (see 4.3 for details). We denote by  $\lambda(i) \in \Lambda^+$  the equivalence class containing  $(\lambda, \bar{i}) \in \mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ .

Some relations between representations of  $\mathcal{H}$  and of  $\mathcal{H}^\dagger$  have been studied in [2], [8], [12], [13], [14], [15] and [16] by using the Clifford theory. Combining these results with some fundamental properties of quasi-hereditary covers, and with the classification of blocks of  $\mathcal{H}^\dagger$  by using the residue equivalence “ $\sim_R$ ”, we give the classification of the blocks of  $\mathcal{O}$  and of  $\mathcal{H}$  by using a certain equivalence relation “ $\approx$ ” on  $\mathcal{P}_{n,r}$  as follows.

Let “ $\approx$ ” be the equivalence relation on  $\mathcal{P}_{n,r}$  defined by  $\lambda \approx \mu$  if  $\lambda \sim_R \mu[j]$  for some  $j \in \mathbb{Z}$ , where  $\mu[j] \in \mathcal{P}_{n,r}$  is obtained from  $\mu \in \mathcal{P}_{n,r}$  by a certain permutation of components of  $\mu$  (see 4.3 for the precise definition of  $\mu[j]$ ). Put  $\Gamma = \{\lambda \in \mathcal{P}_{n,r} \mid \lambda \not\sim_R \mu \text{ for any } \mu \in \mathcal{P}_{n,r} \text{ such that } \mu \neq \lambda\}$ . Then our main theorem is the following.

**Theorem 4.11** (i) *If  $\lambda \in \Gamma$ , then  $\Delta(\lambda(i))$  (resp.  $S(\lambda(i))$ ) is a simple object of  $\mathcal{O}$  (resp. simple  $\mathcal{H}$ -module) for any  $i \in \mathbb{Z}$ . Moreover,  $\Delta(\lambda(i))$  (resp.  $S(\lambda(i))$ ) is a block of  $\mathcal{O}$  (resp. of  $\mathcal{H}$ ) itself.*

(ii) *For  $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$  and  $i, j \in \mathbb{Z}$ ,*

$$\begin{aligned} & \text{both of } \Delta(\lambda(i)) \text{ and } \Delta(\mu(j)) \text{ belong to the same block of } \mathcal{O} \\ & \Leftrightarrow \text{both of } S(\lambda(i)) \text{ and } S(\mu(j)) \text{ belong to the same block of } \mathcal{H} \\ & \Leftrightarrow \lambda \approx \mu. \end{aligned}$$

NOTATIONS. For an algebra  $\mathcal{A}$ , we denote by  $\mathcal{A}\text{-mod}$  the category of finitely generated  $\mathcal{A}$ -modules, and denote by  $\mathcal{A}\text{-proj}$  the full subcategory of  $\mathcal{A}\text{-mod}$  consisting of projective objects. Let  $K_0(\mathcal{A}\text{-mod})$  be the Grothendieck group of  $\mathcal{A}\text{-mod}$ . We denote by  $[M]$  the image of  $M$  in the  $K_0(\mathcal{A}\text{-mod})$  for  $M \in \mathcal{A}\text{-mod}$ . For  $M \in \mathcal{A}\text{-mod}$  and simple object  $L$  of  $\mathcal{A}\text{-mod}$ , we denote by  $[M : L]_{\mathcal{A}}$  the multiplicity of  $L$  in the composition series of  $M$ . We also denote by  $\mathcal{A}^{\text{opp}}$  the opposite algebra of  $\mathcal{A}$ .

### 1. Some properties of quasi-hereditary covers

In this section, we recall some notions of quasi-hereditary covers from [21], and review some fundamental properties.

**1.1.** Let  $\mathcal{A}$  be a quasi-hereditary algebra over a field. Take a projective object  $P$  in  $\mathcal{A}\text{-mod}$ , and put  $\mathcal{B} = \text{End}_{\mathcal{A}}(P)^{\text{opp}}$ . Then we have an exact functor  $F = \text{Hom}_{\mathcal{A}}(P, -): \mathcal{A}\text{-mod} \rightarrow \mathcal{B}\text{-mod}$ . Let  $X$  be a progenerator of  $\mathcal{A}\text{-mod}$  such that  $X = P \oplus P'$  for some projective object  $P'$  in  $\mathcal{A}\text{-mod}$ . Then  $\text{End}_{\mathcal{A}}(X)^{\text{opp}}$  is Morita equivalent to  $\mathcal{A}$ . We may suppose that  $\text{End}_{\mathcal{A}}(X)^{\text{opp}} \cong \mathcal{A}$  without loss of generality.

Throughout this section, we assume the following condition.

(A1): The functor  $F$  is fully faithful when we restrict to  $\mathcal{A}\text{-proj}$ .

Hence,  $\mathcal{A}$  is a quasi-hereditary cover of  $\mathcal{B}$  in the sense of [21]. Since  $X \in \mathcal{A}\text{-proj}$ , we have

$$(1.1.1) \quad \mathcal{A} \cong \text{End}_{\mathcal{A}}(X)^{\text{opp}} \cong \text{End}_{\mathcal{B}}(F(X))^{\text{opp}}.$$

Note that  $X = P \oplus P'$ . Let  $\varphi_P^o \in \text{End}_{\mathcal{A}}(X)$  be such that  $\varphi_P^o$  is the identity map on  $P$ , and 0-map on  $P'$ . We denote by  $\varphi_P$  the element of  $\mathcal{A} \cong \text{End}_{\mathcal{A}}(X)^{\text{opp}}$  corresponding to  $\varphi_P^o$ . It is clear that  $\varphi_P$  is an idempotent. Since

$$\begin{aligned} F(X) &\cong \text{Hom}_{\mathcal{A}}(P, P) \oplus \text{Hom}_{\mathcal{A}}(P, P') \\ &\cong \text{End}_{\mathcal{A}}(X)\varphi_P^o \\ &\cong \varphi_P\mathcal{A} \end{aligned}$$

as right  $\mathcal{A}$ -modules, we have the following isomorphisms of algebras:

$$\begin{aligned} \text{End}_{\mathcal{A}^{\text{opp}}}(F(X)) &\cong \text{End}_{\mathcal{A}^{\text{opp}}}(\varphi_P\mathcal{A}) \\ &\cong \varphi_P\mathcal{A}\varphi_P \\ &\cong (\varphi_P^o\text{End}_{\mathcal{A}}(X)\varphi_P^o)^{\text{opp}} \\ &\cong \text{End}_{\mathcal{A}}(P)^{\text{opp}} \\ &= \mathcal{B}. \end{aligned}$$

Thus, we have the double centralizer property:

$$(1.1.2) \quad \mathcal{A} \cong \text{End}_{\mathcal{B}}(F(X))^{\text{opp}}, \quad \mathcal{B} \cong \text{End}_{\mathcal{A}^{\text{opp}}}(F(X)).$$

This double centralizer property implies the isomorphism  $Z(\mathcal{A}) \rightarrow Z(\mathcal{B})$ , where  $Z(\mathcal{A})$  (resp.  $Z(\mathcal{B})$ ) is the center of  $\mathcal{A}$  (resp.  $\mathcal{B}$ ). Thus, there exists a bijection between blocks of  $\mathcal{A}$  and of  $\mathcal{B}$ .

**1.2.** Recall that  $\mathcal{A}$  is a quasi-hereditary algebra. Let  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$  be the set of standard modules, and  $\{\nabla(\lambda) \mid \lambda \in \Lambda^+\}$  be the set of costandard modules of  $\mathcal{A}$ . For  $\lambda \in \Lambda^+$ , let  $L(\lambda)$  be the unique simple top of  $\Delta(\lambda)$ , and  $P(\lambda)$  be the projective cover of  $L(\lambda)$ . Then  $\{L(\lambda) \mid \lambda \in \Lambda^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{A}$ -modules.

For  $\lambda \in \Lambda^+$ , put  $S(\lambda) = F(\Delta(\lambda))$ ,  $D(\lambda) = F(L(\lambda))$  and  $\Lambda_0^+ = \{\lambda \in \Lambda^+ \mid D(\lambda) \neq 0\}$ . Since  $\mathcal{B} \cong \varphi_P \mathcal{A} \varphi_P$  and  $F = \text{Hom}_{\mathcal{A}}(P, -) = \text{Hom}_{\mathcal{A}}(\mathcal{A} \varphi_P, -)$ , the following lemma is standard (see e.g. [6, Appendix]).

- Lemma 1.3.** (i) For  $\lambda \in \Lambda_0^+$ , we have  $D(\lambda) \cong \text{Top } F(P(\lambda)) \cong \text{Top } S(\lambda)$ .  
 (ii)  $\{F(P(\lambda)) \mid \lambda \in \Lambda_0^+\}$  gives a complete set of non-isomorphic indecomposable projective  $\mathcal{B}$ -modules.  
 (iii)  $\{D(\lambda) \mid \lambda \in \Lambda_0^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{B}$ -modules.

**1.4.** For  $\lambda, \mu \in \Lambda^+$ , we denote by  $P(\lambda) \sim P(\mu)$  if there exists a sequence  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_{k+1} = \mu$  ( $\lambda_i \in \Lambda^+$ ) such that  $P(\lambda_i)$  and  $P(\lambda_{i+1})$  have a common composition factor for any  $i = 1, \dots, k$ . Then “ $\sim$ ” gives an equivalence relation on  $\{P(\lambda) \mid \lambda \in \Lambda^+\}$ . It is well-known that  $P(\lambda) \sim P(\mu)$  if and only if  $P(\lambda)$  and  $P(\mu)$  belong to the same block of  $\mathcal{A}$ . Similarly, we define an equivalence relation “ $\sim$ ” on  $\{F(P(\lambda)) \mid \lambda \in \Lambda_0^+\}$ , and we have  $F(P(\lambda)) \sim F(P(\mu))$  if and only if  $F(P(\lambda))$  and  $F(P(\mu))$  belong to the same block of  $\mathcal{B}$ . Then the double centralizer property (1.1.2) implies the following lemma.

**Lemma 1.5.** For  $\lambda, \mu \in \Lambda_0^+$ , we have

$$P(\lambda) \sim P(\mu) \text{ if and only if } F(P(\lambda)) \sim F(P(\mu)).$$

Note that all the composition factors of  $\Delta(\lambda)$  belong to the same block of  $\mathcal{A}$  since  $\Delta(\lambda)$  is indecomposable. Then, the exact functor  $F$  combined with Lemma 1.5 implies the following corollary.

**Corollary 1.6.** For each  $\lambda \in \Lambda^+$ , all the composition factors of  $S(\lambda)$  belong to the same block of  $\mathcal{B}$ .

**1.7.** From now on, we assume the following additional condition:

(A2):  $[\Delta(\lambda)] = [\nabla(\lambda)]$  in  $K_0(\mathcal{A}\text{-mod})$  for any  $\lambda \in \Lambda^+$ .

By the general theory of quasi-hereditary algebras, for  $\lambda \in \Lambda^+$ ,  $P(\lambda)$  has a  $\Delta$ -filtration such that  $(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)]_{\mathcal{A}}$ , where  $(P(\lambda) : \Delta(\mu))$  is the multiplicity of  $\Delta(\mu)$  in a  $\Delta$ -filtration of  $P(\lambda)$ . Combining with the assumption (A2), we have

$$(1.7.1) \quad (P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]_{\mathcal{A}}.$$

This implies the following lemma.

**Lemma 1.8.** For  $\lambda, \mu \in \Lambda_0^+$ , we have

$$[F(P(\lambda)) : D(\mu)]_{\mathcal{B}} = \sum_{\nu \in \Lambda^+} [S(\nu) : D(\lambda)]_{\mathcal{B}} [S(\nu) : D(\mu)]_{\mathcal{B}}.$$

**1.9.** For  $\lambda, \mu \in \Lambda^+$ , we denote by  $S(\lambda) \sim S(\mu)$  if there exists a sequence  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_{k+1} = \mu$  ( $\lambda_i \in \Lambda^+$ ) such that  $S(\lambda_i)$  and  $S(\lambda_{i+1})$  have a common composition factor for any  $i = 1, \dots, k$ . Then “ $\sim$ ” gives an equivalence relation on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ . Similarly, we define an equivalence relation “ $\sim$ ” on  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ .

Corollary 1.6 and Lemma 1.8 imply the following proposition.

**Proposition 1.10.** *For  $\lambda, \mu \in \Lambda^+$  we have the following.*

- (i)  $S(\lambda) \sim S(\mu)$  if and only if  $S(\lambda)$  and  $S(\mu)$  belong to the same block of  $\mathcal{B}$ .
- (ii)  $\Delta(\lambda) \sim \Delta(\mu)$  if and only if  $\Delta(\lambda)$  and  $\Delta(\mu)$  belong to the same block of  $\mathcal{A}$ .

**1.11.** Finally, we assume the following additional condition:

(A3):  $S(\lambda) = F(\Delta(\lambda)) \neq 0$  for any  $\lambda \in \Lambda^+$ .

Thanks to Proposition 1.10, we can classify blocks of  $\mathcal{B}$  (resp. blocks of  $\mathcal{A}$ ) by equivalence classes of  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$  (resp.  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ ) with respect to the relation “ $\sim$ ”. Then Lemma 1.5 and Proposition 1.10 (under the assumption (A3)) imply the following proposition which gives a relation between blocks of  $\mathcal{A}$  and of  $\mathcal{B}$ .

**Proposition 1.12.** *For  $\lambda, \mu \in \Lambda^+$ , we have*

$$\Delta(\lambda) \sim \Delta(\mu) \text{ if and only if } S(\lambda) \sim S(\mu).$$

## 2. Rational Cherednik algebras

**2.1.** Let  $V$  be a finite dimensional vector space over  $\mathbb{C}$ , and  $W \subset \text{GL}(V)$  be a finite complex reflection group. Let  $\mathcal{A}$  be the set of reflecting hyperplanes of  $W$ , and  $\mathcal{A}/W$  be the set of  $W$ -orbits of  $\mathcal{A}$ . For  $H \in \mathcal{A}$ , let  $W_H$  be the subgroup of  $W$  fixing  $H$  pointwise, and put  $e_H = |W_H|$ . Take a set

$$\Omega = \{k_{H,i} \in \mathbb{C} \mid H \in \mathcal{A}/W, 0 \leq i \leq e_H \text{ such that } k_{H,0} = k_{H,e_H} = 0\}.$$

Let  $\mathcal{H}$  be the rational Cherednik algebra associated to  $W$  with parameters  $\Omega$  (see [9, 3.1] for definitions). By [7], it is known that  $\mathcal{H}$  has the triangular decomposition

$$\mathcal{H} \cong S(V^*) \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} S(V) \text{ as vector spaces,}$$

where  $S(V)$  (resp.  $S(V^*)$ ) is the symmetric algebra of  $V$  (resp. the dual space  $V^*$ ), and  $\mathbb{C}W$  is the group ring of  $W$  over  $\mathbb{C}$ .

Let  $\mathcal{O}$  be the category of finitely generated  $\mathcal{H}$ -modules which are locally nilpotent for the action of  $S(V) \setminus \mathbb{C}$ . Let  $\text{Irr}W = \{E^\lambda \mid \lambda \in \Lambda^+\}$  be a complete set of non-isomorphic simple  $\mathbb{C}W$ -modules. For  $\lambda \in \Lambda^+$ , put

$$\Delta(\lambda) = \mathcal{H} \otimes_{S(V) \rtimes W} E^\lambda,$$

where  $S(V) \rtimes W \cong S(V) \otimes_{\mathbb{C}} \mathbb{C}W$  is a subalgebra of  $\mathcal{H}$ , and we regard  $E^\lambda$  as a  $S(V) \rtimes W$ -module through the natural surjection  $S(V) \rtimes W \rightarrow \mathbb{C}W$ . It is known that  $\mathcal{O}$  turns out to be a highest weight category with standard modules  $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$  ([9], [11]). Let  $L(\lambda)$  be the unique simple top of  $\Delta(\lambda)$ , then  $\{L(\lambda) \mid \lambda \in \Lambda^+\}$  is a complete set of non-isomorphic simple objects in  $\mathcal{O}$ . For  $\lambda \in \Lambda^+$ , we denote by  $P(\lambda)$  the projective cover of  $L(\lambda)$ .

**2.2.** Let  $\mathcal{H}$  be the cyclotomic Hecke algebra of  $W$  corresponding  $\mathcal{H}$  (see [9, 5.2.5] for the choice of parameters). Then the Knizhnik–Zamolodchikov functor (simply, KZ functor)  $\text{KZ}: \mathcal{O} \rightarrow \mathcal{H}\text{-mod}$  is defined in [9, 5.3]. KZ functor is a exact functor, and represented by a projective object

$$P_{\text{KZ}} = \bigoplus_{\lambda \in \Lambda^+} P(\lambda)^{\oplus \dim \text{KZ}(L(\lambda))} \in \mathcal{O},$$

namely, we have  $\text{KZ} = \text{Hom}_{\mathcal{O}}(P_{\text{KZ}}, -)$ . Moreover, by [9, Theorem 5.15], we have

$$\mathcal{H} \cong (\text{End}_{\mathcal{O}}(P_{\text{KZ}}))^{\text{opp}}.$$

By [9, Theorem 5.16], KZ functor is fully faithful when we restrict to projective objects in  $\mathcal{O}$ . Thus,  $\mathcal{O}$  is a quasi-hereditary cover of  $\mathcal{H}$ .

Put  $\mathcal{A} = \text{End}_{\mathcal{O}}(X)$ ,  $\mathcal{B} = \mathcal{H}$  and  $F = \text{KZ}$ , where  $X$  is a progenerator of  $\mathcal{O}$  such that  $X = P_{\text{KZ}} \oplus P'$  for some projective object  $P'$  in  $\mathcal{O}$ . Then, these satisfy assumptions (A1), (A2), (A3) by [9]. Thus, all results in §1 hold for this setting. In particular, we put  $S(\lambda) = \text{KZ}(\Delta(\lambda))$  and  $D(\lambda) = \text{KZ}(L(\lambda))$  for  $\lambda \in \Lambda^+$ . Let  $\Lambda_0^+ = \{\lambda \in \Lambda^+ \mid D(\lambda) \neq 0\}$ , then  $\{D(\lambda) \mid \lambda \in \Lambda_0^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{H}$ -modules.

**2.3.** In the rest of this section, we recall a modular system and a decomposition map described in [9]. Let  $\mathbb{C}[\hat{\Omega}]$  be the polynomial ring over  $\mathbb{C}$  with indeterminates  $\hat{\Omega} = \{\mathbf{k}_{H,i} \mid H \in \mathcal{A}/W, 1 \leq i \leq e_H - 1\}$ . We have a homomorphism  $\varphi: \mathbb{C}[\hat{\Omega}] \rightarrow \mathbb{C}$  of  $\mathbb{C}$ -algebras such that  $\mathbf{k}_{H,i} \mapsto k_{H,i}$ . Put  $\mathfrak{m} = \text{Ker}\varphi$ . Let  $R$  be the completion of  $\mathbb{C}[\hat{\Omega}]$  at the maximal ideal  $\mathfrak{m}$ . Then  $R$  is a regular local ring with the unique maximal ideal  $\hat{\mathfrak{m}} = ((\mathbf{k}_{H,i} - k_{H,i})_{H \in \mathcal{A}/W, 1 \leq i \leq e_H - 1})$ . We have the canonical homomorphism  $R \rightarrow \mathbb{C}$  such that  $\mathbf{k}_{H,i} \mapsto k_{H,i}$ . Let  $K$  be the quotient field of  $R$ .

Let  $\mathcal{H}_R$  be the rational Cherednik algebra of  $W$  over  $R$  with parameters  $\hat{\Omega}$  (put  $\mathbf{k}_{H,0} = \mathbf{k}_{H,e_H} = 0$ ), and  $\mathcal{H}_R$  be the cyclotomic Hecke algebra over  $R$  associated to  $\mathcal{H}_R$ . Then we have  $\mathcal{H} = \mathbb{C} \otimes_R \mathcal{H}_R$  and  $\mathcal{H} = \mathbb{C} \otimes_R \mathcal{H}_R$ . Put  $\mathcal{H}_K = K \otimes_R \mathcal{H}_K$  and  $\mathcal{H}_K = K \otimes_R \mathcal{H}_R$ . We denote objects over  $X = R$  or  $K$  by adding subscript  $X$ , e.g.  $\mathcal{O}_X, \Delta(\lambda)_X, \text{KZ}_X, S(\lambda)_X, \dots$

Under the modular system  $(K, R, \mathbb{C})$ , we can define the decomposition map

$$d_{K,\mathbb{C}}: K_0(\mathcal{H}_K\text{-mod}) \rightarrow K_0(\mathcal{H}\text{-mod})$$

by  $[M] \mapsto [\mathbb{C} \otimes_R N]$ , where  $N$  is an  $\mathcal{H}_R$ -lattice of  $M$ . Thanks to [9, Theorem 5.13], we have the following lemma.

**Lemma 2.4.** *For  $\lambda \in \Lambda^+$ , we have*

$$d_{K,\mathbb{C}}([S_K(\lambda)]) = [S(\lambda)].$$

### 3. Case of type $G(r, \mathbf{1}, n)$

In this section, we consider the complex reflection group  $W$  of type  $G(r, \mathbf{1}, n)$ , i.e.  $W = \mathfrak{S}_n \times (\mathbb{Z}/r\mathbb{Z})^n$ . In this case,  $\mathcal{H}$  is often called the Ariki–Koike algebra, and many results for representations of  $\mathcal{H}$  are known by several authors.

**3.1.** In this section, we use the modular system  $(K, R, \mathbb{C})$  given in the previous section, and we take parameters as follows.

Let  $V$  be an  $n$  dimensional vector space over  $\mathbb{C}$  with a basis  $\{e_1, \dots, e_n\}$ . Then we have  $W \subset \mathrm{GL}(V)$ . Let  $s_1, t_1 \in W$  be reflections such that

$$(3.1.1) \quad s_1(e_k) = \begin{cases} e_2 & \text{if } k = 1, \\ e_1 & \text{if } k = 2, \\ e_k & \text{otherwise,} \end{cases} \quad t_1(e_k) = \begin{cases} \zeta e_1 & \text{if } k = 1, \\ e_k & \text{otherwise,} \end{cases} \quad (\zeta = \exp(2\pi\sqrt{-1}/r)),$$

and  $H_{s_1}$  (resp.  $H_{t_1}$ ) be the reflecting hyperplane corresponding to  $s_1$  (resp.  $t_1$ ). Then  $\{H_{s_1}, H_{t_1}\}$  gives a complete set of representatives of  $W$ -orbits of  $\mathcal{A}$ , and we have  $e_{H_{s_1}} = 2$  and  $e_{H_{t_1}} = r$ . Hence, we can take parameters  $\{h, k_1, \dots, k_{r-1}\}$  (resp.  $\{\mathbf{h}, \mathbf{k}_1, \dots, \mathbf{k}_{r-1}\}$ ) of  $\mathcal{H}$  (resp.  $\mathcal{H}_X$  ( $X = R$  or  $K$ )) such that  $h = k_{H_{s_1}, 1}$  (resp.  $\mathbf{h} = \mathbf{k}_{H_{s_1}, 1}$ ) and  $k_j = k_{H_{t_1}, j}$  (resp.  $\mathbf{k}_j = \mathbf{k}_{H_{t_1}, j}$ ) for  $1 \leq j \leq r-1$ . Then  $\mathcal{H}$  (resp.  $\mathcal{H}_R, \mathcal{H}_K$ ) is the associative algebra over  $\mathbb{C}$  (resp.  $R, K$ ) defined by generators  $T_0, T_1, \dots, T_{n-1}$  with defining relations:

$$(3.1.2) \quad \begin{aligned} (T_0 - 1)(T_0 - Q_1) \cdots (T_0 - Q_{r-1}) &= 0, \\ (T_0 - 1)(T_0 + q) &= 0, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-1), \\ T_i T_j &= T_j T_i \quad (|i - j| > 1), \end{aligned}$$

where  $Q_i = \exp(2\pi\sqrt{-1}(k_i + i/r))$ ,  $q = \exp(2\pi\sqrt{-1}h)$  (resp.  $Q_i = \exp(2\pi\sqrt{-1}(\mathbf{k}_i + i/r))$ ,  $q = \exp(2\pi\sqrt{-1}\mathbf{h})$ ).



**3.2.** Let

$$\mathcal{P}_{n,r} = \left\{ \lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \left| \begin{array}{l} \lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \dots) \in \mathbb{Z}_{\geq 0}^n \text{ with } \lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \dots \\ \sum_{k=1}^r \sum_{i \geq 1} \lambda_i^{(k)} = n \end{array} \right. \right\}$$

be the set of  $r$ -partitions of size  $n$ . It is well-known that the isomorphism classes of simple  $\mathbb{C}W$ -modules are indexed by  $\mathcal{P}_{n,r}$ , thus we have  $\Lambda^+ = \mathcal{P}_{n,r}$ .

**3.3.** By [5], it is known that  $\mathcal{H}_X$  ( $X = K, R$  or  $\mathbb{C}$ , we may omit the subscript  $X$  when  $X = \mathbb{C}$ ) is a cellular algebra with respect to a poset  $(\Lambda^+, \triangleright)$ , where “ $\triangleright$ ” is the dominance order on  $\Lambda^+$ . We denote by  $S_X^\lambda$  the Specht (cell) module for  $\lambda \in \Lambda^+$  constructed by using the cellular basis in [5].

It is known that  $\mathcal{H}_K$  is semi-simple, and  $\{S_K^\lambda \mid \lambda \in \Lambda^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{H}_K$ -modules.

By the general theory of cellular algebras (see [10] or [19]), we can define the canonical bilinear form  $\langle \cdot, \cdot \rangle: S^\lambda \times S^\lambda \rightarrow \mathbb{C}$  by using the cellular basis. Put  $\text{Rad } S^\lambda = \{x \in S^\lambda \mid \langle x, y \rangle = 0 \text{ for any } y \in S^\lambda\}$  and  $D^\lambda = S^\lambda / \text{Rad } S^\lambda$ . Let  $\mathcal{K}_{n,r}$  be the set of Kleshchev multi-partitions containing in  $\Lambda^+$  (see e.g. [3] and [18] for the definition). Then it is known that  $\{D^\lambda \mid \lambda \in \mathcal{K}_{n,r}\}$  gives a complete set of non-isomorphic simple  $\mathcal{H}$ -modules by [3].

It is known that all composition factor of  $S^\lambda$  belong to the same block of  $\mathcal{H}$ . Let “ $\sim$ ” be an equivalence relation on  $\{S^\lambda \mid \lambda \in \Lambda^+\}$  defined in a similar way as the equivalence relation “ $\sim$ ” on  $\{S(\lambda) \mid \lambda \in \Lambda^+\}$  in the previous section. Then it is known that

$$(3.3.1) \quad S^\lambda \sim S^\mu \quad \text{if and only if} \quad S^\lambda \text{ and } S^\mu \text{ belong to the same block of } \mathcal{H}.$$

By (3.3.1), we can classify the blocks of  $\mathcal{H}$  by the equivalence classes of  $\{S^\lambda \mid \lambda \in \Lambda^+\}$  with respect to “ $\sim$ ”, and such equivalence classes are described by using some combinatorics in [17] as follows. For  $\lambda \in \Lambda^+$ , put

$$[\lambda] = \{(i, j, k) \in \mathbb{Z}_{\geq 0}^3 \mid 1 \leq j \leq \lambda_i^{(k)}, 1 \leq k \leq r\}.$$

For  $x = (i, j, k) \in [\lambda]$ , we define

$$\text{res}(x) = \begin{cases} q^{j-i} Q_{k-1} & \text{if } q \neq 1 \text{ and } Q_{k-1} \neq 0, \\ (j-i, Q_{k-1}) & \text{if } q = 1 \text{ and } Q_{l-1} \neq Q_{k-1} \text{ for } k \neq l, \\ Q_{k-1} & \text{otherwise,} \end{cases}$$

where we put  $Q_0 = 1$ . Put  $\text{Res}(\Lambda^+) = \{\text{res}(x) \mid x \in [\lambda] \text{ for some } \lambda \in \Lambda^+\}$ . Then, we define an equivalence relation (called residue equivalence) “ $\sim_R$ ” on  $\Lambda^+$  by

$$\lambda \sim_R \mu \quad \text{if} \quad \#\{x \in [\lambda] \mid \text{res}(x) = a\} = \#\{y \in [\mu] \mid \text{res}(y) = a\} \quad \text{for all } a \in \text{Res}(\Lambda^+).$$

**Theorem 3.4** ([17, Theorem 2.11]). *For  $\lambda, \mu \in \Lambda^+$ , we have*

$$S^\lambda \sim S^\mu \text{ if and only if } \lambda \sim_R \mu.$$

**3.5.** We take  $\text{Irr}W = \{E^\lambda \mid \lambda \in \Lambda^+\}$  such that  $K \otimes_{\mathbb{C}} E^\lambda \cong S_K^\lambda$  via the isomorphism  $\mathcal{H}_K \cong K \otimes_{\mathbb{C}} \mathbb{C}W$ . Since  $S_K^\lambda = K \otimes_R S_R^\lambda$  and  $S^\lambda = \mathbb{C} \otimes_R S_R^\lambda$ , we have

$$(3.5.1) \quad d_{K,\mathbb{C}}([S_K^\lambda]) = [S^\lambda].$$

It is also well-known that  $K \otimes_{\mathbb{C}} E^\lambda \cong S_K(\lambda)$  via the isomorphism  $\mathcal{H}_K \cong K \otimes_{\mathbb{C}} \mathbb{C}W$  (see before [9, Theorem 5.13]). Thus, we have  $S_K^\lambda \cong S_K(\lambda)$  as  $\mathcal{H}_K$ -modules. Then Lemma 2.4 together with (3.5.1) implies that

$$(3.5.2) \quad [S(\lambda)] = [S^\lambda] \text{ in } K_0(\mathcal{H}\text{-mod}) \text{ for } \lambda \in \Lambda^+.$$

Note that  $S(\lambda) \not\cong S^\lambda$  as  $\mathcal{H}$ -modules in general. Hence,  $\text{Top } S(\lambda) \not\cong \text{Top } S^\lambda$  in general. Moreover,  $\Lambda_0^+ \neq \mathcal{K}_{n,r}$  in general. Let

$$\theta: \Lambda_0^+ \rightarrow \mathcal{K}_{n,r}$$

be the bijection such that  $D(\lambda) \cong D^{\theta(\lambda)}$  as  $\mathcal{H}$ -modules. Then we have the following proposition.

**Proposition 3.6.** *For  $\lambda \in \Lambda^+$  and  $\mu \in \Lambda_0^+$ , we have*

$$[\Delta(\lambda) : L(\mu)]_{\mathcal{O}} = [S(\lambda) : D(\mu)]_{\mathcal{H}} = [S^\lambda : D^{\theta(\mu)}]_{\mathcal{H}}.$$

*Proof.* The first equality is clear since the KZ functor is exact. By (3.5.2), we have  $[S(\lambda)] = [S^\lambda]$  in  $K_0(\mathcal{H}\text{-mod})$ , and  $D(\mu) \cong D^{\theta(\mu)}$ . Thus, we have the second equality. □

The following theorem gives a relation between blocks of  $\mathcal{O}$  and blocks of  $\mathcal{H}$ . In particular, we obtain the classification of blocks of  $\mathcal{O}$  by using the residue equivalence.

**Theorem 3.7.** *For  $\lambda, \mu \in \Lambda^+$ , we have*

$$\Delta(\lambda) \sim \Delta(\mu) \Leftrightarrow S(\lambda) \sim S(\mu) \Leftrightarrow S^\lambda \sim S^\mu \Leftrightarrow \lambda \sim_R \mu.$$

*Proof.* The first equivalence is Proposition 1.12. The second equivalence follows from (3.5.2). The third equivalence is Theorem 3.4. □

**REMARK 3.8.** By [21], under a certain condition for parameters, it is known that  $\mathcal{O}$  is equivalent to  $\mathcal{S}_{n,r}\text{-mod}$  as highest weight categories, where  $\mathcal{S}_{n,r}$  is the cyclotomic

$q$ -Schur algebra associated to  $\mathcal{H}$  defined in [5]. In this case, we have  $S(\lambda) \cong S^\lambda$ , and  $\theta$  is the identity map (in particular,  $\Lambda_0^+ = \mathcal{K}_{n,r}$ ). So, the above theorem is known by [17]. However, the above theorem claim that a classification of blocks of  $\mathcal{O}$  is given by the equivalence relation “ $\sim_R$ ” on  $\Lambda^+$  (residue equivalence) even if the case where  $\mathcal{O}$  is not equivalent to  $\mathcal{S}_{n,r}$ -mod.

**4. Case of type  $G(r, p, n)$**

In this section, we consider the case where  $W$  is the complex reflection group of type  $G(r, p, n)$ , where  $r = pd$  for some  $d \geq 1$ . It is well-known that the complex reflection group of type  $G(r, p, n)$  is a normal subgroup of the complex reflection group of type  $G(r, 1, n)$  with the index  $p$ , and we will study some relations between type  $G(r, 1, n)$  and type  $G(r, p, n)$ . Hence, we denote by  $W^\dagger$  the complex reflection group of type  $G(r, 1, n)$ , and we use the results in the previous section for  $W^\dagger$ . In this section, we use the notations in §2 for corresponding objects of type  $G(r, p, n)$ , and we denote by adding the superscript  $\dagger$  for corresponding objects of type  $G(r, 1, n)$ , e.g.  $\mathcal{H}^\dagger, \mathcal{H}^\dagger, \Delta^\dagger(\lambda), KZ^\dagger, S^\dagger(\lambda), \dots$ . Let  $\text{Irr}W^\dagger = \{E^{\dagger\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$  be a complete set of non-isomorphic simple  $\mathbb{C}W^\dagger$ -modules considered in the previous section.

**4.1.** Let  $V$  be an  $n$  dimensional vector space over  $\mathbb{C}$  with a basis  $\{e_1, \dots, e_n\}$ . Then we have  $W \subset \text{GL}(V)$ . Recall that  $s_1, t_1 \in \text{GL}(V)$  is a reflection defined in (3.1.1). Then  $s_1$  (resp.  $t_1^p$  in the case where  $p \neq r$ ) is a reflection contained in  $W$ , and let  $H_{s_1}$  (resp.  $H_{t_1^p}$ ) be the reflecting hyperplane corresponding to  $s_1$  (resp.  $t_1^p$ ). In the case where  $p \neq r$ ,  $\{H_{s_1}, H_{t_1^p}\}$  gives a complete set of representatives of  $W$ -orbits of  $\mathcal{A}$ , and we have  $e_{H_{s_1}} = 2$  and  $e_{H_{t_1^p}} = d$ . Hence, we can take parameters  $\{h, k_1, \dots, k_{d-1}\}$  (resp.  $\{\mathbf{h}, \mathbf{k}_1, \dots, \mathbf{k}_{d-1}\}$ ) of  $\mathcal{H}$  (resp.  $\mathcal{H}_X$  ( $X = R$  or  $K$ )) such that  $h = k_{H_{s_1,1}}$  (resp.  $\mathbf{h} = \mathbf{k}_{H_{s_1,1}}$ ) and  $k_j = k_{H_{t_1^p,j}}$  (resp.  $\mathbf{k}_j = \mathbf{k}_{H_{t_1^p,j}}$ ) for  $1 \leq j \leq d-1$ . On the other hand, in the case where  $r = p$ ,  $\mathcal{A}$  is the  $W$ -orbit of  $\mathcal{A}$  itself. Hence  $\mathcal{H}$  (resp.  $\mathcal{H}_X$  ( $X = R$  or  $K$ )) has a parameter  $\{h\}$  (resp.  $\{\mathbf{h}\}$ ).

Then  $\mathcal{H}$  (resp.  $\mathcal{H}_R, \mathcal{H}_K$ ) is the associative algebra over  $\mathbb{C}$  (resp.  $R, K$ ) defined by generators  $a_0, a'_1, a_1, a_2, \dots, a_{n-1}$  with defining relations:

$$\begin{aligned} &(a_0 - 1)(a_0 - x_1) \cdots (a_0 - x_{d-1}) = 0, \\ &(a'_1 - 1)(a'_1 + q) = 0, \quad (a_i - 1)(a_i + q) = 0 \quad (1 \leq i \leq n - 1), \\ &a_0 a'_1 a_1 = a'_1 a_1 a_0, \quad a'_1 a_2 a'_1 = a_2 a'_1 a_2, \quad (a_2 a'_1 a_1)^2 = (a'_1 a_1 a_2)^2, \\ &a_0 a_i = a_i a_0 \quad (2 \leq i \leq n - 1), \quad a'_1 a_j = a_j a'_1 \quad (3 \leq j \leq n - 1), \\ &\underbrace{a_1 a_0 a'_1 a_1 a'_1 a_1 a'_1 \cdots}_{p+1 \text{ factors}} = \underbrace{a_0 a'_1 a_0 a'_1 a_0 a'_1 a_0 \cdots}_{p+1 \text{ factors}}, \\ &a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad (1 \leq i \leq n - 2), \\ &a_i a_j = a_j a_i \quad (1 \leq i < j - 1 \leq n - 2), \end{aligned}$$

where  $x_i = \exp(2\pi\sqrt{-1}(k_i + i/d), q = \exp(2\pi\sqrt{-1}h)$  (resp.  $x_i = \exp(2\pi\sqrt{-1}(\mathbf{k}_i + i/d), q = \exp(2\pi\sqrt{-1}\mathbf{h})$ ) (see [4] or [2] for braid relations).

**4.2.** Put

$$k_{c \cdot p + j}^\dagger = k_c + \frac{c}{d} + \frac{j}{p} - \frac{c \cdot p + j}{r} \quad (0 \leq c \leq d - 1, 0 \leq j \leq p - 1),$$

$$\mathbf{k}_{c \cdot p + j}^\dagger = \mathbf{k}_c + \frac{c}{d} + \frac{j}{p} - \frac{c \cdot p + j}{r} \quad (0 \leq c \leq d - 1, 0 \leq j \leq p - 1),$$

where we set  $k_0 = \mathbf{k}_0 = 0$ .

Throughout this section, let  $\mathcal{H}^\dagger$  (resp.  $\mathcal{H}_X^\dagger$  ( $X = R$  or  $K$ )) be the rational Cherednik algebra associated to  $W^\dagger$  with parameters  $\{h, k_1^\dagger, \dots, k_{r-1}^\dagger\}$  (resp.  $\{\mathbf{h}, \mathbf{k}_1^\dagger, \dots, \mathbf{k}_{r-1}^\dagger\}$ ) such that  $h = k_{H_{s_1, 1}}^\dagger$  (resp.  $\mathbf{h} = \mathbf{k}_{H_{s_1, 1}}^\dagger$ ) and  $k_j^\dagger = k_{H_{t_1, j}}^\dagger$  (resp.  $\mathbf{k}_j^\dagger = \mathbf{k}_{H_{t_1, j}}^\dagger$ ) for  $1 \leq j \leq r - 1$ . Since

$$\begin{aligned} \exp\left(2\pi\sqrt{-1}\left(k_{c \cdot p + j}^\dagger + \frac{c \cdot p + j}{r}\right)\right) &= \exp\left(2\pi\sqrt{-1}\left(k_c + \frac{c}{d} + \frac{j}{p}\right)\right) \\ &= x_c \xi^j \quad (\xi = \exp(2\pi\sqrt{-1}/p)), \end{aligned}$$

where we put  $x_0 = 1$  (similar for  $\mathbf{k}_{c \cdot p + j}^\dagger$ ), the defining relation (3.1.2) of  $\mathcal{H}^\dagger$  (resp.  $\mathcal{H}_X^\dagger$ ) replaced by

$$(T_0^p - 1)(T_0^p - x_1^p) \cdots (T_0^p - x_{d-1}^p) = 0.$$

Since  $\mathcal{H}_K^\dagger$  is semi-simple (thus,  $\mathcal{O}_K^\dagger$  is also semi-simple) by [1], we can obtain any results for type  $G(r, 1, n)$  in the previous sections even if the case of these parameters.

By [2, Proposition 1.6], there is the injective algebra homomorphism  $\varphi: \mathcal{H}_X \rightarrow \mathcal{H}_X^\dagger$  ( $X = \mathbb{C}, R$  or  $K$ ) such that  $\varphi(a_0) = T_0^p, \varphi(a'_1) = T_0^{-1}T_1T_0, \varphi(a_i) = T_i$  ( $1 \leq i \leq n - 1$ ). Under this injective homomorphism  $\varphi$ , we regard  $\mathcal{H}_X$  as a subalgebra of  $\mathcal{H}_X^\dagger$ .

**4.3.** For  $M^\dagger \in \mathbb{C}W^\dagger\text{-mod}$ , we denote by  $M^\dagger \downarrow$  the restriction of the action to  $\mathbb{C}W$ . For  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$  and  $i \in \mathbb{Z}$ , we define  $\lambda[i] = (\lambda[i]^{(1)}, \dots, \lambda[i]^{(r)}) \in \mathcal{P}_{n,r}$  by

$$\lambda[i]^{(c \cdot p + j)} = \lambda^{(c \cdot p + k)} \quad (0 \leq c \leq d - 1, 1 \leq j \leq p),$$

where  $c \cdot p < c \cdot p + k \leq (c + 1) \cdot p$  such that  $k \equiv j + i \pmod p$ . For an example, if  $r = 6$  and  $p = 3$ , we have

$$\begin{aligned} \lambda[1] &= (\lambda^{(2)}, \lambda^{(3)}, \lambda^{(1)}, \dot{\vdots} \lambda^{(5)}, \lambda^{(6)}, \lambda^{(4)}), \\ \lambda[2] &= (\lambda^{(3)}, \lambda^{(1)}, \lambda^{(2)}, \dot{\vdots} \lambda^{(6)}, \lambda^{(4)}, \lambda^{(5)}), \\ \lambda[3] &= (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dot{\vdots} \lambda^{(4)}, \lambda^{(5)}, \lambda^{(6)}) = \lambda. \end{aligned}$$

Let  $\mathfrak{k}_\lambda$  be the minimum positive integer such that  $\lambda[\mathfrak{k}_\lambda] = \lambda$ . It is clear that  $\mathfrak{k}_\lambda \mid p$ . Put  $\mathfrak{d}_\lambda = p/\mathfrak{k}_\lambda$ . Then we have  $\lambda[i + \mathfrak{k}_\lambda] = \lambda[i]$ . Let  $\sim_*$  be the equivalence relation on  $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  defined by

$$(\lambda, \bar{j}) \sim_* (\lambda[i], \overline{c \cdot \mathfrak{d}_\lambda + j}) \quad (i, c \in \mathbb{Z}),$$

where we denote by  $\bar{m}$  the image of  $m \in \mathbb{Z}$  in  $\mathbb{Z}/p\mathbb{Z}$ . Let  $\Lambda^+$  be the set of equivalence classes of  $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  with respect to the relation  $\sim_*$ , and we denote by  $\lambda\langle j \rangle \in \Lambda^+$  the equivalence class containing  $(\lambda, \bar{j}) \in \mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ . Thus we have  $\lambda\langle j \rangle = \lambda[i]\langle j \rangle = \lambda\langle c \cdot \mathfrak{d}_\lambda + j \rangle = \lambda[i]\langle c \cdot \mathfrak{d}_\lambda + j \rangle$  for  $i, c \in \mathbb{Z}$ . Then it is known that,

$$(4.3.1) \quad E^{\dagger\lambda} \downarrow \cong E^{\dagger\lambda[i]} \downarrow \cong E^{\lambda(1)} \oplus \cdots \oplus E^{\lambda(\mathfrak{d}_\lambda)} \quad (i \in \mathbb{Z}) \text{ as } \mathbb{C}W\text{-modules,}$$

for some simple  $\mathbb{C}W$ -modules  $E^{\lambda(j)}$  ( $1 \leq j \leq \mathfrak{d}_\lambda$ ), and  $\{E^{\lambda(j)} \mid \lambda\langle j \rangle \in \Lambda^+\}$  gives a complete set of pairwise non-isomorphic simple  $\mathbb{C}W$ -modules. Hence, we have

$$\text{Irr}\mathbb{C}W = \{E^{\lambda(j)} \mid \lambda\langle j \rangle \in \Lambda^+\}.$$

Moreover, we have

$$(4.3.2) \quad E^{\lambda(j)} \uparrow \cong E^{\dagger\lambda[1]} \oplus \cdots \oplus E^{\dagger\lambda[\mathfrak{k}_\lambda]} \quad \text{as } W^\dagger\text{-modules } (1 \leq j \leq \mathfrak{d}_\lambda).$$

**4.4.** For  $M^\dagger \in \mathcal{H}_X^\dagger\text{-mod}$ , we denote by  $M^\dagger \downarrow$  the restriction of the action to  $\mathcal{H}_X$ . On the other hand, for  $N \in \mathcal{H}_X\text{-mod}$ , we denote by  $N \uparrow$  the induced module  $\mathcal{H}_X^\dagger \otimes_{\mathcal{H}_X} N$ . Then, by (4.3.1), we have

$$(4.4.1) \quad S_K^\dagger(\lambda) \downarrow \cong S_K(\lambda\langle 1 \rangle) \oplus \cdots \oplus S_K(\lambda\langle \mathfrak{d}_\lambda \rangle) \quad \text{for } \lambda \in \mathcal{P}_{n,r}$$

and, by (4.3.2), we have

$$(4.4.2) \quad S_K(\lambda\langle j \rangle) \uparrow \cong S_K^\dagger(\lambda[1]) \oplus \cdots \oplus S_K^\dagger(\lambda[\mathfrak{k}_\lambda]) \quad \text{for } \lambda\langle j \rangle \in \Lambda^+.$$

We define the group homomorphism  $\text{Res}_X: K_0(\mathcal{H}_X^\dagger\text{-mod}) \rightarrow K_0(\mathcal{H}_X\text{-mod})$  by  $[M^\dagger] \mapsto [M^\dagger \downarrow]$ . We also define the group homomorphism  $\text{Ind}_X: K_0(\mathcal{H}_X\text{-mod}) \rightarrow K_0(\mathcal{H}_X^\dagger\text{-mod})$  by  $[N] \mapsto [N \uparrow]$ . Since  $\mathcal{H}_X^\dagger$  is a free right  $\mathcal{H}_X$ -module, induced functor from  $\mathcal{H}_X\text{-mod}$  to  $\mathcal{H}_X^\dagger\text{-mod}$  is exact. Thus  $\text{Ind}_X$  is well-defined. Then we have the following lemma.

**Lemma 4.5.** (i) For  $\lambda \in \mathcal{P}_{n,r}$ , we have

$$[S^\dagger(\lambda) \downarrow] = [S(\lambda\langle 1 \rangle)] + \cdots + [S(\lambda\langle \mathfrak{d}_\lambda \rangle)] \quad \text{in } K_0(\mathcal{H}\text{-mod}).$$

(ii) For  $\lambda\langle j \rangle \in \Lambda^+$ , we have

$$[S(\lambda\langle j \rangle) \uparrow] = [S^\dagger(\lambda[1])] + \cdots + [S^\dagger(\lambda[\mathfrak{k}_\lambda])] \quad \text{in } K_0(\mathcal{H}^\dagger\text{-mod}).$$

Proof. (i) By Lemma 2.4 and (4.4.1), we have

$$\begin{aligned} d_{K,C}([S_K^\dagger(\lambda)\downarrow]) &= d_{K,C}([S_K(\lambda(1))] + \cdots + [S_K(\lambda(\mathfrak{d}_\lambda))]) \\ &= [S(\lambda(1))] + \cdots + [S(\lambda(\mathfrak{d}_\lambda))]. \end{aligned}$$

On the other hand, by the definition of decomposition maps, we have

$$d_{K,C}([S_K^\dagger(\lambda)\downarrow]) = [S^\dagger(\lambda)\downarrow].$$

Then (i) was proven. By using (4.4.2) together with Lemma 2.4, we have (ii) in a similar way as in (i). □

**4.6.** We recall some relations between simple  $\mathcal{H}$ -modules and simple  $\mathcal{H}^\dagger$ -modules which have been studied in [8] and [16]

Let  $\{S^{\dagger\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$  be the set of Specht modules of  $\mathcal{H}^\dagger$  constructed in [5] as seen in the previous section. Then  $\{D^{\dagger\lambda} \mid \lambda \in \mathcal{K}_{n,r}\}$  is a complete set of simple  $\mathcal{H}^\dagger$ -modules.

Let  $\sigma$  be the algebra automorphism of  $\mathcal{H}^\dagger$  defined by  $\sigma(T_0) = \xi T_0$  ( $\xi = \exp(2\pi\sqrt{-1}/p)$ ),  $\sigma(T_i) = T_i$  for  $i = 1, \dots, n-1$ . Then we see that the restriction  $\sigma|_{\mathcal{H}}$  of  $\sigma$  to  $\mathcal{H}$  is the identity map on  $\mathcal{H}$ . We also define the algebra automorphism  $\tau$  of  $\mathcal{H}^\dagger$  by  $\tau(x) = T_0^{-1}xT_0$  for  $x \in \mathcal{H}^\dagger$ . Then we have  $\tau(\mathcal{H}) = \mathcal{H}$ .

For  $M^\dagger \in \mathcal{H}^\dagger\text{-mod}$ , let  $(M^\dagger)^\sigma$  be the twisted  $\mathcal{H}^\dagger$ -module of  $M$  via  $\sigma$ . Since  $\sigma|_{\mathcal{H}}$  is identity map, we have  $(M^\dagger)^\sigma \downarrow \cong M^\dagger \downarrow$  as  $\mathcal{H}$ -modules. Similarly, for  $N \in \mathcal{H}\text{-mod}$ , let  $N^\tau$  be the twisted  $\mathcal{H}$ -module of  $N$  via  $\tau$ .

For  $\lambda \in \mathcal{K}_{n,r}$  and  $i \in \mathbb{Z}$ , we define  $\lambda[i]^b$  by  $(D^{\dagger\lambda})^{\sigma^i} \cong D^{\dagger\lambda[i]^b}$ . Let  $\mathfrak{k}_\lambda^b$  be the minimum positive integer such that  $\lambda[\mathfrak{k}_\lambda^b]^b = \lambda$  (thus  $(D^{\dagger\lambda})^{\sigma^{\mathfrak{k}_\lambda^b}} \cong D^{\dagger\lambda}$ ), and put  $\mathfrak{d}_\lambda^b = p/\mathfrak{k}_\lambda^b$ . Let  $D$  be a simple  $\mathcal{H}$ -submodule of  $D^{\dagger\lambda} \downarrow$ . Then by [8, Lemma 2.2],  $\mathfrak{d}_\lambda^b$  is the minimum positive integer such that  $D^{\tau^{\mathfrak{d}_\lambda^b}} \cong D$ . Moreover we have, for  $\lambda \in \mathcal{K}_{n,r}$  and  $i = 1, \dots, \mathfrak{k}_\lambda^b$ ,

$$(4.6.1) \quad D^{\dagger\lambda} \downarrow \cong D^{\dagger\lambda[i]^b} \downarrow \cong D \oplus D^\tau \oplus \cdots \oplus D^{\tau^{\mathfrak{d}_\lambda^b - 1}} \quad \text{as } \mathcal{H}\text{-modules.}$$

Let  $\sim_\star$  be the equivalence relation on  $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  defined by

$$(\lambda, \bar{j}) \sim_\star (\lambda[i]^b, \overline{c \cdot \mathfrak{d}_\lambda^b + j}) \quad (i, c \in \mathbb{Z}).$$

We denote by  $(\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_\star$  the set of equivalence classes of  $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$  with respect to the relation  $\sim_\star$ , and we denote by  $\lambda\langle j \rangle^b \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_\star$  the equivalence class containing  $(\lambda, \bar{j}) \in \mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ . Then, by [8, Lemma 2.2] (see also [16, Proposition 2.4]),

$$\{D^{\lambda\langle j \rangle^b} \mid \lambda\langle j \rangle^b \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_\star\}$$

gives a complete set of pairwise non-isomorphic simple  $\mathcal{H}$ -modules, where we put  $D^{\lambda\langle j \rangle^b} = D^{\tau^j}$  for some simple  $\mathcal{H}$ -submodule  $D$  of  $D^{\dagger\lambda} \downarrow$  (see (4.6.1)).

By [8, Lemma 2.2], we also have, for  $\lambda\langle j \rangle^b \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$ ,

$$(4.6.2) \quad D^{\lambda\langle j \rangle^b} \uparrow \cong D^{\dagger\lambda[1]^b} \oplus \dots \oplus D^{\dagger\lambda[\mathfrak{k}_\lambda^b]^b} \quad \text{as } \mathcal{H}^\dagger\text{-modules.}$$

REMARKS 4.7. (i) For  $\lambda \in \mathcal{K}_{n,r}$ ,  $\lambda[i]^b$  ( $1 \leq i \leq \mathfrak{k}_\lambda^b$ ) is described in [13] (the case of type D), [15] (the case of type  $G(r, r, n)$ ) and [8], [16] (general case).

(ii) Recall that  $\{D(\lambda\langle i \rangle') \mid \lambda\langle i \rangle' \in \Lambda_0^+\}$  gives a complete set of non-isomorphic simple  $\mathcal{H}$ -modules (Lemma 1.3). Hence, there exists the bijection  $\eta: \Lambda_0^+ \rightarrow (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_*$  such that  $D(\lambda\langle i \rangle') \cong D^{\eta(\lambda\langle i \rangle')}$ .

Now we have the following proposition.

**Proposition 4.8.** *For  $\lambda \in \mathcal{P}_{n,r}$  and  $\mu \in \mathcal{K}_{n,r}$ , we have the following.*

- (i)  $\sum_{s=1}^{\mathfrak{d}_\lambda} [S(\lambda\langle s \rangle) : D^{\mu(i)^b}]_{\mathcal{H}} = \sum_{t=1}^{\mathfrak{k}_\mu^b} [S^{\dagger\lambda[j]} : D^{\dagger\mu[t]^b}]_{\mathcal{H}^\dagger}$  ( $1 \leq i \leq \mathfrak{d}_\mu^b$ ,  $1 \leq j \leq \mathfrak{k}_\lambda$ ).
- (ii)  $\sum_{s=1}^{\mathfrak{d}_\mu^b} [S(\lambda\langle i \rangle) : D^{\mu(s)^b}]_{\mathcal{H}} = \sum_{t=1}^{\mathfrak{k}_\lambda} [S^{\dagger\lambda[t]} : D^{\dagger\mu[j]^b}]_{\mathcal{H}^\dagger}$  ( $1 \leq i \leq \mathfrak{d}_\lambda$ ,  $1 \leq j \leq \mathfrak{k}_\mu^b$ ).

Proof. Let

$$S^{\dagger\lambda[j]} = M_k \supset M_{k-1} \supset \dots \supset M_1 \supset M_0 = 0$$

be a composition series of  $S^{\dagger\lambda[j]}$  in  $\mathcal{H}^\dagger$ -mod such that  $M_l/M_{l-1} \cong D^{\dagger\mu_l}$ . Applying the restriction functor, we have the filtration

$$S^{\dagger\lambda[j]} \downarrow = M_k \downarrow \supset M_{k-1} \downarrow \supset \dots \supset M_1 \downarrow \supset M_0 \downarrow = 0$$

such that  $M_l \downarrow / M_{l-1} \downarrow \cong D^{\dagger\mu_l} \downarrow$  in  $\mathcal{H}$ -mod. Thus, by (4.6.1), we have

$$(4.8.1) \quad [S^{\dagger\lambda[j]} \downarrow : D^{\mu(i)^b}]_{\mathcal{H}} = \sum_{t=1}^{\mathfrak{k}_\mu^b} [S^{\dagger\lambda[j]} : D^{\dagger\mu[t]^b}]_{\mathcal{H}^\dagger}.$$

On the other hand, by (3.5.2) and Lemma 4.5 (i) together with  $S^{\dagger\lambda} \downarrow \cong S^{\dagger\lambda[j]} \downarrow$ ,

$$(4.8.2) \quad [S^{\dagger\lambda[j]} \downarrow : D^{\mu(i)^b}]_{\mathcal{H}} = \sum_{s=1}^{\mathfrak{d}_\lambda} [S(\lambda\langle s \rangle) : D^{\mu(i)^b}]_{\mathcal{H}}.$$

(4.8.1) and (4.8.2) imply (i). Next we prove (ii). Let

$$S(\lambda\langle i \rangle) = N_k \supset N_{k-1} \supset \dots \supset N_1 \supset N_0 = 0$$

be a composition series of  $S(\lambda\langle i \rangle)$  in  $\mathcal{H}$ -mod such that  $N_l/N_{l-1} \cong D^{\mu\langle ji \rangle^b}$ . Applying the induced functor, we have the filtration

$$S(\lambda\langle i \rangle)\uparrow = N_k\uparrow \supset N_{k-1}\uparrow \supset \cdots \supset N_1\uparrow \supset N_0\uparrow = 0$$

such that  $N_l\uparrow/N_{l-1}\uparrow \cong D^{\mu\langle ji \rangle^b}\uparrow$  in  $\mathcal{H}^\dagger$ -mod. Thus, by (4.6.2), we have

$$(4.8.3) \quad [S(\lambda\langle i \rangle)\uparrow : D^{\dagger\mu\langle ji \rangle^b}]_{\mathcal{H}^\dagger} = \sum_{s=1}^{\mathfrak{d}_\mu^\dagger} [S(\lambda\langle i \rangle) : D^{\mu\langle s \rangle^b}]_{\mathcal{H}}.$$

On the other hand, by (3.5.2) and Lemma 4.5 (ii), we have

$$(4.8.4) \quad [S(\lambda\langle i \rangle)\uparrow : D^{\dagger\mu\langle ji \rangle^b}]_{\mathcal{H}^\dagger} = \sum_{t=1}^{\mathfrak{k}_\lambda} [S^{\dagger\lambda\langle t \rangle} : D^{\dagger\mu\langle ji \rangle^b}]_{\mathcal{H}^\dagger}.$$

(4.8.3) and (4.8.4) imply (ii). □

**4.9.** Recall that “ $\sim_R$ ” is the residue equivalence on  $\mathcal{P}_{n,r}$  defined in the previous section. We define an equivalence relation “ $\approx$ ” on  $\mathcal{P}_{n,r}$  by  $\lambda \approx \mu$  if  $\lambda \sim_R \mu\langle j \rangle$  for some  $j \in \mathbb{Z}$ . Put

$$\Gamma = \{\lambda \in \mathcal{P}_{n,r} \mid \lambda \not\sim_R \mu \text{ for any } \mu \in \mathcal{P}_{n,r} \text{ such that } \mu \neq \lambda\}.$$

We see easily that  $\lambda \sim_R \mu$  if and only if  $\lambda\langle i \rangle \sim_R \mu\langle i \rangle$  for any  $i \in \mathbb{Z}$ . Thus, we have  $\lambda\langle i \rangle \in \Gamma$  if  $\lambda \in \Gamma$ . Then we have the following proposition.

**Proposition 4.10.** *For  $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$ , we have*

$$S(\lambda\langle 1 \rangle) \sim S(\lambda\langle 2 \rangle) \sim \cdots \sim S(\lambda\langle \mathfrak{d}_\lambda \rangle).$$

**Proof.** If  $\mathfrak{k}_\lambda = p$ , there is nothing to prove since  $\mathfrak{d}_\lambda = 1$ . Hence, we assume that  $\mathfrak{k}_\lambda \neq p$ . First, we show the following claim.

**Claim.** *For  $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$  such that  $\mathfrak{k}_\lambda \neq p$ , we can take  $\mu \in \mathcal{P}_{n,r}$  such that  $\lambda \sim_R \mu$ , and that  $\mathfrak{k}_\mu = p$  (thus  $\mathfrak{d}_\mu = 1$ ).*

Since  $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$ , we can take  $\mu \in \mathcal{P}_{n,r}$  such that  $\lambda \sim_R \mu$  and  $\mu \neq \lambda$ . By [17, Theorem 2.11], it is known that  $\lambda \sim_R \mu$  if and only if  $\lambda \sim_J \mu$ , where “ $\sim_J$ ” is the Jantzen equivalence on  $\mathcal{P}_{n,r}$  (see [17, Definition 2.8] for definitions). By the definition of the Jantzen equivalence, we may assume that  $\mu$  obtained by unwrapping a rim hook  $r_x^\lambda$  from  $\lambda$ , and wrapping another rim hook  $r_y^\mu$  from  $[\lambda] \setminus r_x^\lambda$ . Namely, we have



$[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$  (See [17] for notations here). Suppose that  $x \in \lambda^{(i)}$  and  $y \in \mu^{(j)}$ . Then  $[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$  implies that

$$(4.10.1) \quad \lambda^{(i)} \neq \mu^{(i)}, \lambda^{(j)} \neq \mu^{(j)} \quad \text{and} \quad \lambda^{(l)} = \mu^{(l)} \quad \text{for } l \neq i, j.$$

Note that  $\mu^{(i)} \neq \mu^{(j)}$  if  $\lambda^{(i)} = \lambda^{(j)}$  and  $i \neq j$ . Thus, we have  $\mu^{(i)} \neq \mu^{(l)}$  for any  $l \neq i$  such that  $l \equiv i \pmod{\mathfrak{k}_\lambda}$  and  $c \cdot p < l \leq (c + 1) \cdot p$  when  $c \cdot p < i \leq (c + 1) \cdot p$ . This implies that

$$(4.10.2) \quad \mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu \quad \text{unless} \quad \mathfrak{k}_\mu = p.$$

In the case where  $p$  is a prime number, (4.10.2) implies  $\mathfrak{k}_\mu = p$  since  $\mathfrak{k}_\lambda = 1$  by  $\mathfrak{k}_\lambda \mid p$  and  $\mathfrak{k}_\lambda \neq p$ . In the case where  $p = 4$ , one can easily check that  $\mathfrak{k}_\mu = p$  directly. Let  $p \geq 6$  be not a prime number. Assume that  $\mathfrak{k}_\mu \neq p$ . Then we have  $\mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu$  by (4.10.2). In a similar way as in the above arguments, we have  $\mathfrak{k}_\mu \nmid \mathfrak{k}_\lambda$  (note that  $\mathfrak{k}_\lambda \neq p$ ). By the conditions  $p \geq 6$ ,  $\mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu$  and  $\mathfrak{k}_\mu \nmid \mathfrak{k}_\lambda$ , one sees that there are at least three integers  $x_1, x_2, x_3$  such that  $\lambda^{(x_l)} \neq \mu^{(x_l)}$  ( $l = 1, 2, 3$ ). However, this contradicts to (4.10.1). Thus we have  $\mathfrak{k}_\mu = p$ , and the claim was proved.

Thanks to the claim, we can take  $\mu \in \mathcal{P}_{n,r}$  such that  $\lambda \sim_R \mu$ , and that  $\mathfrak{d}_\mu = 1$ . Then we can take a sequence  $\lambda = \lambda_0, \dots, \lambda_k = \mu$  satisfying the following two conditions:

- $S^{\dagger\lambda_{i-1}}$  and  $S^{\dagger\lambda_i}$  have a common composition factor  $D^{\dagger\nu_i}$ .
- There exists an integer  $l$  such that  $\mathfrak{d}_{\lambda_i} \neq 1$  for any  $i < l$ , and that  $\mathfrak{d}_{\lambda_l} = 1$ .

By Proposition 4.8 (i), one sees that  $S(\lambda_l \langle 1 \rangle)$  has a composition factor  $D^{\nu_l \langle i \rangle^b}$  for any  $i \in \{1, \dots, \mathfrak{d}_{\nu_l}^b\}$  (note that  $\mathfrak{d}_{\lambda_l} = 1$ ). On the other hand, by Proposition 4.8 (ii), one sees that  $S(\lambda_{l-1} \langle j \rangle)$  ( $1 \leq j \leq \mathfrak{d}_{\lambda_{l-1}}$ ) has a composition factor  $D^{\nu_l \langle i \rangle^b}$  for some  $i \in \{1, \dots, \mathfrak{d}_{\nu_l}^b\}$ . Thus, we have  $S(\lambda_l \langle 1 \rangle) \sim S(\lambda_{l-1} \langle j \rangle)$  for any  $j = 1, \dots, \mathfrak{d}_{\lambda_{l-1}}$ . This implies that  $S(\lambda_{l-1} \langle 1 \rangle) \sim S(\lambda_{l-1} \langle 2 \rangle) \sim \dots \sim S(\lambda_{l-1} \langle \mathfrak{d}_{\lambda_{l-1}} \rangle)$ . By using the (backward) inductive argument combined with Proposition 4.8, we have the proposition.  $\square$

**Theorem 4.11.** (i) For  $\lambda \in \Gamma$  and  $i = 1, \dots, \mathfrak{d}_\lambda$ , we have  $S(\lambda \langle i \rangle)$  (resp.  $\Delta(\lambda \langle i \rangle)$ ) is a simple  $\mathcal{H}$ -module (resp. a simple object of  $\mathcal{O}$ ). Moreover,  $S(\lambda \langle i \rangle)$  (resp.  $\Delta(\lambda \langle i \rangle)$ ) is a block of  $\mathcal{H}$  (resp. of  $\mathcal{O}$ ) itself.

(ii) For  $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$  and  $i, j \in \mathbb{Z}$ , we have

$$\Delta(\lambda \langle i \rangle) \sim \Delta(\mu \langle j \rangle) \Leftrightarrow S(\lambda \langle i \rangle) \sim S(\mu \langle j \rangle) \Leftrightarrow \lambda \approx \mu.$$

*Proof.* Suppose that  $S(\lambda \langle i \rangle)$  and  $S(\mu \langle j \rangle)$  have a common composition factor  $D^{\nu \langle k \rangle^b}$ . Then, by Proposition 4.8 (ii),  $S^{\dagger\lambda \langle i' \rangle}$  and  $S^{\dagger\mu \langle j' \rangle}$  have a common composition factor  $D^{\dagger\mu}$  for some  $i', j'$ . This implies that

$$(4.11.1) \quad S(\lambda \langle i \rangle) \sim S(\mu \langle j \rangle) \quad \text{only if} \quad \lambda \approx \mu.$$

(i) Suppose that  $\lambda \in \Gamma$ , then  $S^{\dagger\lambda}$  is a simple  $\mathcal{H}^{\dagger}$ -module from the definition of  $\Gamma$ . If  $S(\lambda\langle i \rangle) \sim S(\mu\langle j \rangle)$  for some  $\mu\langle j \rangle \in \Lambda^+$ , we have  $\lambda \approx \mu$  by (4.11.1). This implies that there exists an integer  $l$  such that  $\lambda = \mu[l]$  since  $\lambda \in \Gamma$ . Thus, we have  $\mu\langle j \rangle = \mu[l]\langle j \rangle = \lambda\langle j \rangle$  from the definition of  $\Lambda^+$ . Now we may assume that  $S(\lambda\langle i \rangle)$  and  $S(\lambda\langle j \rangle)$  have a common composition factor  $D^{\mu\langle k \rangle^b}$ . If  $\lambda\langle i \rangle \neq \lambda\langle j \rangle$  (i.e.  $i \not\equiv j \pmod{\mathfrak{d}_\lambda}$ ), we have  $\sum_{s=1}^{\mathfrak{d}_\lambda} [S(\lambda\langle s \rangle) : D^{\mu\langle k \rangle^b}]_{\mathcal{H}^{\dagger}} \geq 2$ . On the other hand, we have  $\sum_{i=1}^{\mathfrak{e}_\mu^b} [S^{\dagger\lambda} : D^{\dagger\mu\langle l \rangle^b}]_{\mathcal{H}^{\dagger}} \leq 1$  since  $S^{\dagger\lambda}$  is simple. These contradict to Proposition 4.8 (i). Thus we have  $\lambda\langle i \rangle = \lambda\langle j \rangle = \mu\langle j \rangle$ . This implies (i).

Next we prove (ii). For  $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$ , suppose that  $S^{\dagger\lambda}$  and  $S^{\dagger\mu}$  have a common composition factor  $D^{\dagger\nu}$ . Then, by Proposition 4.8 (i),  $S(\lambda\langle i \rangle)$  and  $S(\mu\langle j \rangle)$  have a common composition factor  $D^{\nu\langle l \rangle^b}$  for some  $i, j$  (and for any  $l$ ). Thus,  $S(\lambda\langle i \rangle) \sim S(\mu\langle j \rangle)$ . Combining Proposition 4.10 and (4.11.1), we obtain the theorem.  $\square$

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Graduate School of Mathematics  
Nagoya University  
Furocho, Chikusaku, Nagoya, 464-8602  
Japan  
e-mail: kentaro-wada@math.nagoya-u.ac.jp

Current address:  
Department of Mathematics  
Faculty of Science  
Shinshu University  
Asahi 3-1-1, Matsumoto 390-8621  
Japan  
e-mail: wada@math.shinshu-u.ac.jp