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Marubayashi, H.
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PURE INJECTIVE MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS WITH ENOUGH INVERTIBLE IDEALS

HIDETOSHI MARUBAYASHI

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The notion of purity in abelian groups is one of the most important types in abelian groups. Many authors defined the concept of purity in modules over non-commutative rings and generalized some of well known results in pure injective abelian groups to the case of modules (cf. [4], [9], [10], [18], [19] and [23]).

In [10], the author gave a complete structure of pure injective modules over bounded Dedekind prime rings by using essentially abelian techniques and recently Talwar generalized these results to the case of modules over bounded hereditary noetherian prime rings. In case not bounded, it seems to me that we can not adopt the techniques used in abelian groups to study pure injective modules over hereditary noetherian prime rings. In this paper, we show how some results on pure injecrtive abelian groups can be carried over the case of modules over hereditary noetherian prime rings with enough invertible ideals, using the adjoint theorem which was used in [18] and from the point of view of localizations. More precisely; let F be a non trivial right additive topology on R and let F_l be the left additive topology corresponding to F (cf. [11]). Let F_l^n be the set of all left ideals containing a finite interection of elements in F_l, each of which has at most n as the length of composition series of its factor module. A submodule N of a right R-module M is said to be *F^n-pure* if $MJ \cap N=NJ$ for every $J \in F_l^n$. If N is F^n-pure for every natural number n, then it is called *F^ω-pure.*

If $R=\boldsymbol{Z}$ is the ring of integers, p is a prime number and F_p is the topology of all powers of p, then F_p^n-purity and F_p^ω-purity are equivalent to p^n-purity and p^ω-purity in the sense of [15], respectively. If F_0 is the topology of all ideals of $\boldsymbol{Z}$, then F_0^ω-purity is equivalent to the ordinary purity.

In Section 1, we shall summarize some elementary properties on these purities F^α ($\alpha \leq \omega$), F^∞-purities and EF^n-purities which were defined in [12] and [13] .

In Section 2, we shall give characterizations of F^α-pure projective modules and F^{α}-pure injective modules which are essentially a special case of the moie general results (cf. [18] and [23])

In Section 3, we shall give more detailed properties concerning with F^{α}-pure injective modules by using the results in Sections 1, 2 and some results in [1]. Firstly any F^{α}-pure injective module is a direct sum of an injective module and an F^{α}-reduced, F^{α}-pure injective module. F^{α}-reduced and F^{α}-pure injective modules are characterized as a direct summand of a compact and F^{α}-reduced modules. We shall also discuss relationships between $F^{\omega}(F^{n})$-pure injective modules and $F^{\infty}(EF^{n})$-pure injective modules (Theorems 3.5 and 3.6). An appendix contains some results on F^{∞} and EF^{n}-pure injective modules which need in Section 3.

1. We say that a ring R has *enough invertible ideals* if every non zero ideal of R contains an invertible ideal of R. Throughout this paper, R will be a hereditary noetherian prime ring with enough invertible ideals, Q will be the two-sided quotient ring of R and $K=Q/R\neq 0$. By a module we shall understand a unitary right R-module. In place of $\otimes_{R}$, Hom_{R}, Ext_{R} and Tor_{R}, we shall just write $\otimes$, Hom, Ext and Tor, respectively. Since R is hereditary $\mathrm{Tor}_{n}=0=\mathrm{Ext}^{n}$ for all $n>1$ and so we shall use Ext for Ext^{1} and Tor for Tor_{1}. A right additive topology F on R is called *trivial* if all modules are F-torsion or F-torsion-free. By the same way as in [11, p. 548], F is non trivial if and only if it consists of essential right ideals of R.

From now on, F denotes a fixed non trivial right additive topology on R. Let I be an essential one-sided ideal of R. Then R/I is an artinian module by Theorem 1.3 of [2]. So the length of the composition series of the module R/I is finite. We call it *the length of* I. For any positive integer n, let F^{n} be the set of all right ideals containing a finite interection of elements in F, each of which has at most n as the length. Then F^{n} induces a preradical on R in the sense of [20]. For a convenience, we let $F^{\omega}=\cup_{n}F^{n}$. Clearly $F=F^{\omega}$.

From now on, *α denotes a fixed natural number n or ω unless otherwise stated.*

Let M be a module. An element m of M is said to be *F^{α}-torsion* if $O(m)=\{r\in R\mid mr=0\}\in F^{\alpha}$ and we denote the submodule of all F^{α}-torsion elements in M by $M_{F^{\alpha}}$. So, under this notation, F^{ω}-torsion means F-torsion and $M_{F}=M_{F^{\omega}}$ is the F-torsion submodule of M. In a similar way we can define the concepts of *L^{α}-torsion elements* and *L^{α}-torsion submodule* for any left module and any non-trivial left additive topology L on R.

Let I be an essential right ideal of R. Define $I^{*}=\{q\in Q\mid qI\subseteq R\}$. Similarly $^{*}J=\{q\in Q\mid Jq\subseteq R\}$ for any essential left ideal J of R. We put $Q_{F}=\cup I^{*}(I\in F)$, the ring of quotients of R with respect to F. The family F_{l} of left ideals of R such that $Q_{F}J=Q_{F}$ is a left additive topology on R. We call it the *left additive topology corresponding to F*. We let $Q_{F^{\alpha}}=\cup I^{*}(I\in F^{\alpha})$

and $Q_{F_l^\alpha}=\cup^* J$ $(J\in F_l^\alpha)$. Then $Q_{F^\alpha}=Q_{F_l^\alpha}$ and $K_{F^\alpha}=Q_{F^\alpha}/R=K_{F_l^\alpha}$ by the same way as in Lemma 1.1 of [13]. The exact sequence $0\to R\xrightarrow{i_\alpha} Q_{F^\alpha}\to K_{F^\alpha}\to 0$ yields the exact sequences:

$$0\to \mathrm{Tor}(M, K_{F^\alpha})\to M\xrightarrow{i_{\alpha*}} M\otimes Q_{F^\alpha}\to M\otimes K_{F^\alpha}\to 0\,,$$

$$\mathrm{Hom}(K_{F^\alpha}, M)\to \mathrm{Hom}(Q_{F^\alpha}, M)\xrightarrow{i_\alpha^*} M\to \mathrm{Ext}(K_{F^\alpha}, M)\,,$$

where $i_{\alpha*}(m)=m\otimes 1$ and $i_\alpha^*(f)=f(1)$ ($m\in M$ and $f\in\mathrm{Hom}(Q_{F^\alpha}, M)$).

Lemma 1.1. (1) $\mathrm{Tor}(M, K_{F^\alpha}))\cong M_{F^\alpha}$.
(2) $\mathrm{Im}\, i_\alpha^*\subseteq \cap MJ$ $(J\in F_l^\alpha)$

Proof. As in Lemma 1.2 of [13].

We write Im i_α^* by MF^α in case $\alpha=n$ and $\cap_n \mathrm{Im}\, i_n^*$ by MF^ω. If $\alpha=\omega$, then we denote it by MF^∞; it is the maximal F_l-divisible submodule of M and is the maximal F-injective submodule of M. These are proved by the same way as in Lemma 2.5 of [11] and Lemma 1.1 of [12]. If $MF^\alpha=0$, then M is said to be *F^α-reduced.* Similarly, if $MF^\infty=0$, then we say that M is *F-reduced.* An exact sequence

$$\text{(E)}\qquad 0\to L\to M\to N\to 0$$

of modules is said to be *F^α-pure* if $MJ\cap L=LJ$ for every $J\in F_l^\alpha$. A module G is *F^α-pure injective* if it has the injective property relative to the class of F^α-pure exact sequences. Similarly we can define the concept of F^α-pure projective modules. Following [12], (E) is called *F^∞-pure* (cf. also [6]) if the induced sequence $0\to L_F\to M_F\to N_F\to 0$ is splitting exact. If (E) is F^∞-pure, then the induced sequence $0\to\mathrm{Ext}(K_F, L)\to\mathrm{Ext}(K_F, M)\to\mathrm{Ext}(K_F, N)\to 0$ is splitting exact by the same way as in Lemma 1.3 of [12]. (E) is said to be *EF^n-pure* if the induced sequence $0\to\mathrm{Exp}(K_{F^n}, L)\to\mathrm{Ext}(K_{F^n}, M)\to\mathrm{Ext}(K_{F^n}, N)\to 0$ is splitting exact. In an obvious way, we can define the concepts of *EF^n-pure* and *F^∞-pure injective modules.*

In the remainder of this section we shall summalize some elementary properties on these putities and injectivities which need in this paper (these results are implicitly known in [11], [12] and [13]).

Lemma 1.2. *The following conditions of a short exact sequence (E): $0\to L\to M\to N\to 0$ are equivalent:*

(1) *(E) is F^α-pure.*

(2) *For any finitely generated F^α-torsion module T, the natural homomorphism $Hom(T, M)\to Hom(T, N)\to 0$ is exact.*

(3) *For any F_l^α-torsion left module T, the natural homomorphism $0\to L\otimes T\to M\otimes T$ is exact.*

Proof. As in Lemma 2.2 of [13].

Lemma 1.3. *Let $0\to L\to M\to N\to 0$ be F^{α}-pure. Then the induced sequence $0\to L_{F^{\alpha}}\to M_{F^{\alpha}}\to N_{F^{\alpha}}\to 0$ is exact.*

Proof. This follows from Lemmas 1.1 and 1.2.

Lemma 1.4. *The folloming implications hold*:

$$\begin{tikzcd}[column sep=small]
 & (EF^{n}\text{-purity}) \arrow[dr, Rightarrow] & \\
(F^{\infty}\text{-purity}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (F^{n}\text{-purity}) \\
 & (F^{\omega}\text{-purity}) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & (EF^{n}\text{-pure injectives}) \arrow[dr, Rightarrow] & \\
(F^{n}\text{-pure injectives}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (F^{\infty}\text{-pure injectives}) \\
 & (F^{\omega}\text{-pure injectives}) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Proof. The lemma follows from the definitions and from the similar method as the proof in (1) of Lemma 2.1 of [13].

Lemma 1.5. *Let M be any module. Then*

(1) *$MF^{n}\supseteq MF^{m}\supseteq MF^{\infty}$ for any $n<m\leq\omega$.*

(1*) *(F^{n}-reduced)$\Rightarrow$(F^{m}-reduced)$\Rightarrow$(F-reduced).*

(2) *For any non-trivial right additive topologies F_1, F_2 on R such that $F_1\subset F_2$, we have $MF_1^{\infty}\supseteq MF_2^{\infty}$ and $MF_1^{\alpha}\supseteq MF_2^{\alpha}$.*

(2*) *(F_1-reduced)$\Rightarrow$(F_2-reduced) and (F_1^{α}-reduced)$\Rightarrow$(F_2^{α}-reduced).*

Proof. (1) The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & R & \to & Q_{F^n} & \to & K_{F^n} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & R & \to & Q_{F^m} & \to & K_{F^m} & \to & 0
\end{array}$$

yields the commutative diagram with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}(Q_{F^n}, M) & \to & M & \to & \mathrm{Ext}(K_{F^n}, M) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}(Q_{F^m}, M) & \to & M & \to & \mathrm{Ext}(K_{F^m}, M).
\end{array}$$

From this diagram, we get $MF^{n}\supseteq MF^{m}$. Similarly we have $MF^{m}\supseteq MF^{\infty}$, and the proof of (2) is also similar to one of (1).

Lemma 1.6. *Let G be a module. Then*

(1) *The following are equivalent:*

(a) *G is F^n-reduced and EF^n-pure injective.*

(b) *G is F^n-reduced and F^∞-pure injective.*

(c) *$G \cong Ext(K_{F^n}, G)$, where the isomorphism is the connecting homomorphism induced by the exact sequence $0 \to R \to Q_{F^n} \to K_{F^n} \to 0$.*

(2) *G is F-reduced and F^∞-pure injective if and only if $G \cong Ext(K_F, G)$.*

Proof. (1) As in Lemma 2.4 of [13].

(2) Let G be an F-reduced module. Then G is F^∞-pure injective if and only if it is F-cotorsion in the sense of [19] (cf. Proposition 1.4 of [12]). Thus the assertion follows from Proposition 5.2 of [19].

Let F_0 be the right additive topology consists of all essential right ideals of R.

Lemma 1.7. *A module is F_0^∞-pure injective if and only if it is pure injective in the sense of* [23].

Proof. Any finitely presented module over hereditary noetherian prime rings is a direct sum of a projective module and of forms R/I ($I \in F_0$) by Theorem 2.1 of [2] and Theorem 3.1 of [3]. So the lemma follows from Proposition 3 of [23] and Lemma 1.2.

2. Let **Mod**-R (R-**Mod**) be the category of right (left) modules and let **E** be the class of all F^α-pure exact sequences. Then by Lemma 1.2, it is a proper class and projectively closed in the sense of [18]. Let $\boldsymbol{O}$ be the set of finitely generated projective modules and of finitely generated F^α-torsion modules, and let $\pi^{-1}(\boldsymbol{O})$ denote the class of all those short exact sequences of **Mod**-R for which the objects in $\boldsymbol{O}$ are relative projectives. Then we have $\mathbf{E}=\pi^{-1}(\boldsymbol{O})$. Further any finitely generated torsion module is a direct sum of cyclic modules by Theorem 3.1 of [3]. Thus we have the following proposition which is a special case of Proposition 2.3 of [18] (cf. also Proposition 1 of [23]).

Proposition 2.1. (1) *For any module M there is an F^α-pure exact sequence $0 \to L \to P \to M \to 0$ such that P is an F^α-pure projective module.*

(2) *A module is F^α-pure projective if and only if it is a direct summand of a direct sum of a projective module and of modules R/I ($I \in F^\alpha$).*

Next we shall study F^α-pure injective module. To this purpose, we shall use a result of [18], which is essentially Theorem 7.2 of [14]. We define the functor S: R-**Mod** $\to$ **Mod**-R to be $\mathrm{Hom}_{\boldsymbol{Z}}(\ ,\ \boldsymbol{R}/\boldsymbol{Z})$, and define the functor T: **Mod**-$R \to R$-**Mod** to be $\mathrm{Hom}_{\boldsymbol{Z}}(\ ,\ \boldsymbol{R}/\boldsymbol{Z})$, where $\boldsymbol{Z}$ is the ring of integers and

$\boldsymbol{R}$ is the field of reals.[1] Since $\boldsymbol{R}/\boldsymbol{Z}$ is an injective cogenerator as $\boldsymbol{Z}$-modules, S and T are both exact and faithful. Further we have the duality formula:

$$(*) \qquad \operatorname{Hom}(M, S(P)) \cong \operatorname{Hom}_{\boldsymbol{Z}}(M \otimes P, \boldsymbol{R}/\boldsymbol{Z}) \cong \operatorname{Hom}(P, T(M)),$$

where $M \in$ **Mod**-R and $P \in R$-**Mod**. Let $\mathbf{E}_l$ be the class of all F_l^α-pure exact sequences. Then it is a projectively closed proper class of R-**Mod** with enough $\mathbf{E}_l$-projectives by Proposition 2.1. Let $T^{-1}(\mathbf{E}_l)$ be the class of those short exact sequences of **Mod**-R which are carried into $\mathbf{E}_l$ by T. By Lemma 1.2 and $(*)$, we have $\mathbf{E} = T^{-1}(\mathbf{E}_l)$. Thus, by the abjoint theorem (cf. [18, p. 161]), we have

Proposition 2.2. (1) *For any module M there is an F^α-pure exact sequence $0 \to M \to G \to N \to 0$ such that G is an F^α-pure injective module.*

(2) *A module is F^α-pure injective if and only if it is a direct summand of $\operatorname{Hom}_{\boldsymbol{Z}}(P, \boldsymbol{R}/\boldsymbol{Z})$ for some F_l^α-pure projective module P.*

3. In this section we shall give some detailed properties on F^α-pure injective modules by using the results in §§1 and 2. For any module M we define its *F_l^α-topology* by taking the subgroups of M of the form MJ ($J \in F_l^\alpha$) as a subbase of open neighborhoods of *0*. Under this topology R becomes a topological ring such that F_l^α is the set of all open left ideals, and M becomes a topological module (cf. [21, p 143–144] for the definition of topological modules). M is Hausdorff if and only if $0 = \cap MJ (J \in F_l^\alpha)$. We define $\hat{M}_{F_l^\alpha} = \varinjlim M/MJ$ ($J \in F_l^\alpha$). Then $\hat{R}_{F_l^\alpha}$ becomes a topological ring and $\hat{M}_{F_l^\alpha}$ is a topological $\hat{R}_{F_l^\alpha}$-module by the similar way as in §4 of [19]. M is said to be *F_l^α-complete* if it is Hausdorff and complete in its F_l^α-topology. Let $j_M \colon M \to \hat{M}_{F_l^\alpha}$ be the canonical map. Then $\operatorname{Ker} j_M = \cap MJ$ ($J \in F_l^\alpha$), and M is F_l^α-complete if and only if j_M is an isomorphism (cf. Proposition 4.2 of [19]).

Lemma 3.1. (1) *Let P be a F_l^α-torsion module. Then $\operatorname{Hom}_{\boldsymbol{Z}}(P, \boldsymbol{R}/\boldsymbol{Z})$ is F_l^α-complete and so it is F^α-reduced.*

(2) *Let P be a projective left module. Then $\operatorname{Hom}_{\boldsymbol{Z}}(P, \boldsymbol{R}/\boldsymbol{Z})$ is an injective module.*

Proof. (1) Let J be any element of F_l^α. Then $0 \to R \xrightarrow{\beta} {}^*J \to {}^*J/R \to 0$ be a projective resolution of ${}^*J/R$. We have the exact sequence $0 \to \operatorname{Tor}({}^*J/R, P) \to P \xrightarrow{\beta \otimes 1} {}^*J \otimes P \to {}^*J/R \otimes P \to 0$. By the same way as in Proposition 2.2 of [17], we get $\operatorname{Ker}(\beta \otimes 1) \subseteq P[J] = \{p \in P \mid Jp = 0\}$. Since ${}^*JJ \in 1$, we have $\operatorname{Ker}(\beta \otimes 1) \supseteq P[J]$ and therefore $\operatorname{Tor}({}^*J/R, P) = \operatorname{Ker}(\beta \otimes 1) = P[J]$. Furthermore, we have the following commutative diagram by Proposition 5.2′ of [1, Chap. II]:

1) To study compact modules, the field $\boldsymbol{R}$ of reals is more useful than the field $\boldsymbol{Q}$ of rationals. So we used here $\boldsymbol{R}$ instead of $\boldsymbol{Q}$ which was used in [18].

$$\begin{array}{ccc}
\mathrm{Hom}({}^*J, G) & \overset{s_1}{\cong} & \mathrm{Hom}_{\boldsymbol{Z}}({}^*J\otimes P, \boldsymbol{R}/\boldsymbol{Z}) \\
\downarrow \beta^* & & \downarrow (\beta\otimes 1)^* \\
\mathrm{Hom}(R, G) & \overset{s_2}{\cong} & \mathrm{Hom}_{\boldsymbol{Z}}(R\otimes P, \boldsymbol{R}/\boldsymbol{Z}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

, where $G=\mathrm{Hom}_{\boldsymbol{Z}}(P, \boldsymbol{R}/\boldsymbol{Z})$. From this diagram we get the isomorphism $\bar{s}_2$: $\mathrm{Hom}(R, G)/\mathrm{Im}\,\beta^*$ $(=\mathrm{Ext}(J^*/R, G))\cong\mathrm{Hom}_{\boldsymbol{Z}}(R\otimes P, \boldsymbol{R}/\boldsymbol{Z})/\mathrm{Im}(\beta\otimes 1)^*$. Furthermore a mapping γ: $\mathrm{Hom}_{\boldsymbol{Z}}(R\otimes P, \boldsymbol{R}/\boldsymbol{Z})/\mathrm{Im}(\beta\otimes 1)^*\to\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Ker}(\beta\otimes 1), \boldsymbol{R}/\boldsymbol{Z})$ given by $\gamma([f+\mathrm{Im}(\beta\otimes 1)^*])=f\,|\,\mathrm{Ker}(\beta\otimes 1)$, the restricted map of f to $\mathrm{Ker}(\beta\otimes 1)$, $(f\in\mathrm{Hom}_{\boldsymbol{Z}}(R\otimes P, \boldsymbol{R}/\boldsymbol{Z}))$, is an isomorphism, because $\boldsymbol{R}/\boldsymbol{Z}$ is $\boldsymbol{Z}$-injective. Thus we have an isomorphism $\mathrm{Hom}(R, G)/\mathrm{Im}\,\beta^*\cong\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Ker}(\beta\otimes 1), \boldsymbol{R}/\boldsymbol{Z})$ $(=\mathrm{Hom}_{\boldsymbol{Z}}(P[J], \boldsymbol{R}/\boldsymbol{Z}))$. By Proposition 3.2 of [19], G/GJ is naturally isomorphic to $\mathrm{Hom}(R, G)/\mathrm{Im}\,\beta^*$. Thus the mapping δ: $G/GJ\to\mathrm{Hom}_{\boldsymbol{Z}}(P[J], \boldsymbol{R}/\boldsymbol{Z})$, given by $\delta([g+GJ])=g\,|\,P[J]$ is an isomorphism. Hence we have $G=\mathrm{Hom}_{\boldsymbol{Z}}(\varinjlim P[J], \boldsymbol{R}/\boldsymbol{Z})\cong\varprojlim\mathrm{Hom}_{\boldsymbol{Z}}(P[J], \boldsymbol{R}/\boldsymbol{Z})\cong\varprojlim G/GJ=\hat{G}_{F_l^\alpha}$ $(J\in F_l^\alpha)$ and the isomorphism $G\cong\hat{G}_{F_l^\alpha}$ is the natural map. Hence G is F_l^α-complete.

(2) This is well known (cf. Theorem 3.25 of [16]).

In abelian groups, a reduced abelian group is pure injective if and only if it is complete in the $\boldsymbol{Z}$-adic topology (cf. Theorems 38.1 and 39.1 of [5]). For modules over a noetherian ring S, L. Fuchs proved that a pure injective module is complete in its S-adic topology (cf. Theorem 7.1 of [4]). In our case we have the following

Theorem 3.2. *Let G be an F^α-pure injective module. Then $G=E\oplus H$, where E is injective and H is F^α-reduced, F^α-pure injective and F_l^α-complete.*

Proof. By Propositions 2.1, 2.2 and Lemma 3.1, we have $G\oplus G'=D\oplus M$, where D is injective, and M is F^α-reduced, F^α-pure injective and F_l^α-complete. In particular, $\cap MJ=0$ $(J\in F_l^\alpha)$. Since D is a fully invariant submodule of $D\oplus M$, we have $D=(G\cap D)\oplus(G'\cap D)$ (cf. Lemma 9.3 of [5]). So $G\cap D$ is also injective. Thus we get $G\cong(G\cap D)\oplus\bar{G}$, where $\bar{G}=G/G\cap D$. Let $x=(d, m)$ be any element of G, where $d\in D$ and $m\in M$. Then the mapping f: $\bar{G}\to M$ given by $f(\bar{x})=m$ $(x=[\bar{x}+G\cap D])$ is a monomorphism. So $\bar{G}$ is an F^α-reduced, because $\cap MJ=0$ (cf. Lemma 1.1). Thus we have $G=E\oplus H$, where $E=G\cap D$ is injective and $H\cong\bar{G}$ is F^α-reduced, F^α-pure injective. Similarly $G'=E'\oplus H'$, where E' is injective and H' is F^α-reduced, F^α-pure injective. Thus we have $(E\oplus E')\oplus(H\oplus H')=D\oplus M$. $E+E'$ and D are both the unique maximal injective submodules of the module. So it follows that $E\oplus E'=D$ and hence $H\oplus H'\cong M$. Therefore H is F_l^α-complete.

REMARK. If G is F-reduced and F^{α}-pure injective, then $0=\cap GJ(J\in F_l^{\alpha})$.

Let n and α be any pair such that $n<\alpha\leq\omega$. Then, by the definition, any F^{n}-pure injective module is F^{α}-pure injective. Next we shall give a necessary and sufficient condition for an F^{α}-pure injective module G to be F^{n}-pure injective. By Theorem 3.2 we may assume that G is F^{α}-reduced.

Proposition 3.3. *Let n and α be any pair such that $n<\alpha$ $(\leq\omega)$ and let G be an F^{α}-reduced and F^{α}-pure injective module. Then G is F^{n}-pure injective if and only if it is F^{n}-reduced.*

Proof. The necessity is evident from Theorem 3.2. To prove the sufficiency we suppose that G is F^{α}-pure injective and F^{n}-reduced. Then, by Proposition 2.2, we get

$$(**)\qquad G\oplus G' = \mathrm{Hom}_{Z}(P_1, \boldsymbol{R}/\boldsymbol{Z})\oplus\mathrm{Hom}_{Z}(P_2, \boldsymbol{R}/\boldsymbol{Z}),$$

where G' is a module, P_1 is a projective left module and P_2 is a direct sum of the forms R/J $(J\in F_l^{\alpha})$. By Lemmas 1.4, 1.5 and 1.6, we have the commutative diagram with exact row and column:

$$\begin{array}{ccccc}
 & & G & \cong & \mathrm{Ext}(K_{F^{\alpha}}, G) \\
 & & \| & & \downarrow \\
0 & \rightarrow & G & \rightarrow & \mathrm{Ext}(K_{F^{n}}, G) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}$$

Thus $G\cong\mathrm{Ext}(K_{F^{n}}, G)$. Applying $\mathrm{Ext}(K_{F^{n}},\ )$ to the equality (**), we have $G\oplus\mathrm{Ext}(K_{F^{n}}, G')\cong\mathrm{Ext}(K_{F^{n}}, \mathrm{Hom}_{Z}(P_2, \boldsymbol{R}/\boldsymbol{Z}))\cong\mathrm{Hom}_{Z}(\mathrm{Tor}(K_{F^{n}}, P_2), \boldsymbol{R}/\boldsymbol{Z})$ by Proposition 5.1 of [1, Chp. VI]. $\mathrm{Tor}(K_{F^{n}}, P_2)$ is F_l^{n}-pure projective by Lemma 1.1 and Proposition 2.1. Hence G is F^{n}-pure injective by Proposition 2.2,

For any left module M we regard it as a discrete ablian group, then it is well known that $G=\mathrm{Hom}_{Z}(M, \boldsymbol{R}/\boldsymbol{Z})$ is a compact abelian group (cf. (19.1) of [8, p. 64)]. If M is an F_l^{α}-torsion module, then we can easily check that G is a topological module, where R is equipped with its F_l^{α}-topology. Thus, in this case, G is a compact module. In [23], Warfield proved that a module over a ring is pure injective if and only if it is a direct summand of a compact module by using the Bohr compactification (cf. Theorem 2 of [23]). For local theory, we have the following

Theorem 3.4. *Let G be an F-reduced module. Then G is F^{α}-pure injective if and only if it is a direct summand of a compact and F^{α}-reduced module, where R is equipped with its F_l^{α}-topology.*

Proof. Assume that G is F^{α}-pure injective and F-reduced. Then, by Proposition 2.2, we have $\mathrm{Hom}_{Z}(P, \boldsymbol{R}/\boldsymbol{Z})=G\oplus G'$, where P is an F_l^{α}-pure projective module and G' is a module. Since $G \cong \mathrm{Ext}(K_{F^\alpha}, G)$ by Lemmas 1.4, 1.6 and Theorem 3.2, applying $\mathrm{Ext}(K_{F^\alpha},\ )$ to the equality we get:

$$G \oplus \mathrm{Ext}(K_{F^\alpha}, G') \cong \mathrm{Ext}(K_{F^\alpha}, \mathrm{Hom}_{\boldsymbol{Z}}(P, \boldsymbol{R}/\boldsymbol{Z})) \cong \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Tor}(K_{F^\alpha}, P), \boldsymbol{R}/\boldsymbol{Z}).$$

Thus G is a direct summand of a compact and F^α-reduced module by Lemma 3.1 and the above remark. To prove the sufficiency we let X be a compact module with $XF^\alpha=0$. Then it suffices to prove that X is F^α-pure injective. Since any topological module is a topological module in the sense of [23], X is F^ω_0-pure injective by Theorem 2 of [23] and Lemma 1.7. So X is a direct summand of $\mathrm{Hom}_{\boldsymbol{Z}}(P', \boldsymbol{R}/\boldsymbol{Z})$, where P' is an F^ω_{0l}-pure projective module. It follows that X is a direct summand of $\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Tor}(K_{F^\alpha}, P'), \boldsymbol{R}/\boldsymbol{Z})$ by the same way as in the proof of Proposition 3.3. Hence X is F^α-pure injective by Proposition 2.2.

In abelian groups, the following two properties hold:

(i) Any torsion-free, pure injective abelian group is cotorsion.

(ii) An abelian group is cotorsion if and only if it is an epimorphic image of a pure injective abelian group.

In the remainder of this paper, we show how the results above can be carried over the case of modules over any (pre) topology F^α. Let $C(F^n)$ be the category of F^n-reduced and EF^n-pure injective modules, together with their homomorphisms. Then $C(F^n)$ is an abelian category (cf. Proposition 2.8 of [13]). Similarly let $C(F^\infty)$ be the abelian category of F-reduced and F^∞-pure injective modules with their homomorphisms. An object in $C(F^n)$ is said to be *$C(F^n)$-projective* if it is projective in the category $C(F^n)$. We can define $C(F^\infty)$-projective objects in the same way. Some properties about $C(F^n)$ ($C(F^\infty)$)-projectives are given in Porposition A.5 and its remark in the appendix.

Theorem 3.5. (1) *Any $C(F^\infty)$-projective object in $C(F^\infty)$ is F^ω-pure injective.*
(2) *Any $C(F^n)$-projective object in $C(F^n)$ is F^n-pure injective.*

Proof. (1) Let G be a $C(F^\infty)$-projective object in $C(F^\infty)$ and we cansider a diagram of a form

$$\begin{array}{ccccccccc} 0 & \to & L & \to & M & \to & N & \to & 0 \\ & & \downarrow{\scriptstyle f} & & & & & & \\ & & G\,, & & & & & & \end{array}$$

where the upper row is an F^ω-pure exact sequence. By Lemma 1.3, the induced sequence $0 \to L_F \to M_F \to N_F \to 0$ is exact. So the sequence $0 \to L/L_F \to M/M_F \to N/N_F \to 0$ is also exact. Further it is F^∞-pure, since N/N_F is F-torsion-free. By the remark to Proposition A.5 in the appendix, G is F-torsion-free. Hence we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & L \arrow[r] \arrow[d] \arrow[dll, "f"'] & M \arrow[d] \\
G \arrow[d, equal] & 0 \arrow[r] & L/L_F \arrow[r] \arrow[d] \arrow[dll, "\bar{f}"'] & M/M_F \arrow[d] \\
G \arrow[d, phantom, sloped, "\cong"] & 0 \arrow[r] & \mathrm{Ext}(K_F, L/L_F) \arrow[r] \arrow[dll, "\bar{f}_*"'] & \mathrm{Ext}(K_F, M/M_F) \\
\mathrm{Ext}(K_F, G) & & &
\end{tikzcd}$$

where $\bar{f}$ is the map induced from f and the bottom row splits by the same way as in Lemma 1.3 of [12]. By Lemma 1.6, $G\cong\mathrm{Ext}(K_F, G)$. So G is F^{ω}-pure injective.

(2) Let G be $C(F^n)$-projective. To prove that it is F^n-pure injective it suffices to prove that $\mathrm{Ext}(K_{F^n}, \sum\oplus R)$ is F^n-pure injective by Proposition A.5. By (1) and the remark to Proposition A.5, $\mathrm{Ext}(K_F, \sum\oplus R)$ is F^{ω}-pure injective. we have $\mathrm{Ext}(K_F, \sum\oplus R)\oplus X\cong\mathrm{Hom}_Z(P, \boldsymbol{R}/\boldsymbol{Z})$, where P is an F^{ω}_l-pure projective module and X is a module. By Proposition 3.5a of [1, Chap. VI] and Lemma 1.1, we have $\mathrm{Ext}(K_{F^n}, \mathrm{Ext}(K_F, \sum\oplus R))\cong\mathrm{Ext}(K_{F^n}, \sum\oplus R)$. Thus, applying $\mathrm{Ext}(K_{F^n},\ )$ to the isomorphism above we get $\mathrm{Ext}(K_{F^n}, \sum\oplus R)\oplus\mathrm{Ext}(K_{F^n}, X)\cong$ $\mathrm{Ext}(K_{F^n}, \mathrm{Hom}_Z(P, \boldsymbol{R}/\boldsymbol{Z}))\cong\mathrm{Hom}_Z(\mathrm{Tor}(K_{F^n}, P), \boldsymbol{R}/\boldsymbol{Z})$. Hence $\mathrm{Ext}(K_{F^n}, \sum\oplus R)$ is F^n-pure injective by Proposition 2.2.

Theorem 3.6. (1) *A module is F^{∞}-pure injective if and only if it is an epimorphic image of an F^{ω}-pure injective module.*

(2) *A module is EF^n-pure injective if and only if it is an epimorphic image of an F^n-pure injective module.*

Proof. (1) Assume that G is F^{∞}-pure injective, then $G=E(GF^{\infty})\oplus$ $\mathrm{Ext}(K_F, G)$ by Proposition A.4 ($E(GF^{\infty})$ denotes the injective hull of GF^{∞}). From an exact sequence $\sum\oplus R\to G\to 0$ we derive the exact sequence $\mathrm{Ext}(K_F, \sum\oplus R)\to\mathrm{Ext}(K_F, G)\to 0$. We let $H=E(GF^{\infty})\oplus\mathrm{Ext}(K_F, \sum\oplus R)$, which is F^{ω}-pure injective by the remark to Proposition A.5 and Theorem 3.5. Then we have an exact sequence $H\to G\to 0$. Sufficiency follows from Theorem 1, Lemma 3 of [6] and Lemma 1.4.

(2) The proof of the necessity proceeds just like that of (1). To prove the sufficiency, assume that there is an exact sequence $H\xrightarrow{f}G\to 0$, where H is F^n-pure injective. Applying $\mathrm{Ext}(K_F,\ )$ to this we get the exact sequence $0\to$ $\mathrm{Ker}\,f_*\to\mathrm{Ext}(K_F, H)\xrightarrow{f_*}\mathrm{Ext}(K_F, G)\to 0$. $\mathrm{Ext}(K_F, H)$ and $\mathrm{Ext}(K_F, G)$ are elements in $C(F^{\infty})$ by Lemmas 1.1, 1.6 and Proposition 3.5a of [1, Chap. VI].

So it follows that $\operatorname{Ker} f_* \in C(F^\infty)$, because $C(F^\infty)$ is an abelian category. Since H is F^n-pure injective, it is F^∞-pure injective and thus G is also F^∞-pure injective by Lemma 3 of [6]. We can write $H=E(HF^\infty)\oplus \operatorname{Ext}(K_F, H)$ and $G=E(GF^\infty)\oplus \operatorname{Ext}(K_F, G)$ by Proposition A.4. Since $E(HF^\infty)$ is the unique maximal injective submodule of H, we have $0=\cap \operatorname{Ext}(K_F, H)J$ $(J\in F_l^n)$ by Theorem 3.2 and so $0=\cap(\operatorname{Ker} f_*)J$. Thus $\operatorname{Ker} f_*$ and $\operatorname{Ext}(K_F, H)$ are both F^n-erduced by Lemma 1.1. Hence these are both elements in $C(F^n)$ by Lemma 1.6. Therefore $\operatorname{Ext}(K_F, G)$ is also an element in $C(F^n)$ and hence G is EF^n-pure injective.

Appendix

We shall present, in this appendix, some results on EF^n (F^∞)-pure injective modules which are obtained by modifying the methods used in the correspondence ones in modules over Dedekind prime rings or are implicitly known in [12] and [13].

For any module M, we let f_n: $M\to E(MF^n)$ be an extension of the inclusion map $MF^n\to E(MF^n)$ and δ_n: $M\to \operatorname{Ext}(K_{F^n}, M)$ be the connecting homomorphism induced by $0\to R\to Q_{F^n}\to K_{F^n}\to 0$. We define a map g_n: $M\to E(MF^n)\oplus \operatorname{Ext}(K_{F^n}, M)$ by $g_n(m)=(f_n(m), \delta_n(m))$ for every $m\in M$. Similarly we get a map g_∞: $M\to E(MF^\infty)\oplus \operatorname{Ext}(K_F, M)$.

Proposition A.1. *Let M be any module. Then*

(1) $0\to M\xrightarrow{g_n} E(MF^n)\oplus Ext(K_{F^n}, M)\to Coker\, g_n\to 0$ *is exact and EF^n-pure. $E(MF^n)\oplus Ext(K_{F^n}, M)$ is EF^n-pure injective and $Coker\, g_n$ is injective* (cf. [13, Lemma 2.5]).

(2) $0\to M\xrightarrow{g_\infty} E(MF^\infty)\oplus Ext(K_F, M)\to Coker\, g_\infty\to 0$ *is exact and F^∞-pure. $E(MF^\infty)\oplus Ext(K_F, M)$ is F^∞-pure injective and $Coker\, g_\infty$ is injective* ([12, Theorem 1.5]).

Proposition A.2. *Let M be a module. Then*
(1) *M/MF^n is F^n-reduced* (cf. [13, Corollary 2.7]).
(2) *M/MF^∞ is F-reduced* (cf. [12, Lemma 1.1]).

Proposition A.3. *Let M be a module. Then*
(1) *$Ext(K_{F^n}, M)\cong Ext(K_{F^n}, M/MF^n)$* (cf. [13, Lemma 2.6]).
(2) *$Ext(K_F, M)\cong Ext(K_F, M/MF^\infty)$.*

The proof of (2): From the exact sequence $0\to MF^\infty\to M\to M/MF^\infty\to 0$, we get the exact sequence $\operatorname{Ext}(K_F, MF^\infty)\to \operatorname{Ext}(K_F, M)\to \operatorname{Ext}(K_F, M/MF^\infty)\to 0$. But $\operatorname{Ext}(K_F, MF^\infty)=0$, since K_F is F-torsion and MF^∞ is an F-injective module. Thus $\operatorname{Ext}(K_F, M)\cong \operatorname{Ext}(K_F, M/MF^\infty)$.

Proposition A.4. *Let G be a module. Then*

(1) *G is an EF^n-pure injective module if and only if $G \cong E(GF^n) \oplus Ext(K_{F^n}, G)$.*

(2) *G is an F^∞-pure injective module if and only if $G \cong E(GF^\infty) \oplus Ext(K_F, G)$.*

Proof. (1) Sufficiency is clear from Lemma 1.6 and Lemma 1.5 of [13]. To prove the necessity let G be EF^n-pure injective. Then from Proposition A.1, we have the following splitting exact sequence $0 \to G \to E(GF^n) \oplus \mathrm{Ext}(K_{F^n}, G)$. Write $G \oplus X = E(GF^n) \oplus \mathrm{Ext}(K_{F^n}, G)$. Since $\mathrm{Ext}(K_{F^n}, G)$ is reduced, $E(GF^n)$ is the maximal injective submodule of $G \oplus X$ and so it is a fully invariant submodule of $G \oplus X$. This means that $E(GF^n) = (G \cap E(GF^n)) \oplus (X \cap E(GF^n))$. But $GF^n \cap X = 0$ and thus $E(GF^n) \cap X = 0$. It follows that $E(GF^n) \subseteq G$ and so $E(GF^n) = GF^n$. Hence G/GF^n is also EF^n-pure injective, and it is F^n-reduced by Proposition A.2. By Lemma 1.6 and Proposition A.3, we have $G/GF^n \cong \mathrm{Ext}(K_{F^n}, G)$. Therefore $G \cong E(GF^n) \oplus \mathrm{Ext}(K_{F^n}, G)$. The proof of (2) is similar to one of (1).

Proposition A.5. *Let G be a module. Then the following conditions are equivalent*:

(1) *G is $C(F^n)$-projective.*

(2) *G is isomorphic to a direct summand of $Ext(K_{F^n}, \sum \oplus R)$.*

(3) *G is isomorphic to a direct summand of $\prod \hat{R}_{F_i^n}$.*

(4) *G is isomorphic to $Ext(K_{F^n}, M)$, where M is an F-torsion-free module.*

(5) *G is isomorphic to a direct summand of $Ext(K_{F^n}, \sum \oplus \hat{R}_{F_i^n})$* (cf. Theorem 2.9 of [13]).

REMARK. The corresponding results to Proposition A.5 also hold for the category $C(F^\infty)$. Further a module G is $C(F^\infty)$-projective if and only if $G \in C(F^\infty)$ and G is F-torsion-free (cf. Remark to Theorem 2.9 of [13]).

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Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan