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ON A PROBLEM OF NAGATA RELATED TO ZARISKI'S PROBLEM*

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1. Introduction

Related to the problem proposed by Zariski[6] if the intersection $A \bigcap L$ of a normal affine ring A over a field k and a function field L over k is again an affine ring over k (we always understand that L is a subfield of a field containing A), Nagata obtained a characterization[3, Proposition 1], aiming at the affirmative answer, that the intersection $A \bigcap L$ of a normal affine ring A over a Dedekind domain k'(merely stated ground ring) and a function field L over k' is exactly an ideal transform of a normal affine ring over k'.

We recall that A is an affine ring over B if A is an integral domain containing B as a subring and is finitely generated over B and that L is a function field over B if L is the field of quotients of an affine ring over B.

Making use of this result, Rees constructed a counter example to Zariski's problem with an algebro-geometric consideration [5].

Recently, Nagata showed the following result[4, Theorem 2.1, 2.2], in view of the fact that the answer to Zariski's problem was negative and for generalizing the original results, where the derived normal ring of an integral domain A means the integral closure of A in its field of quotients.

Theorem 1.1 (Nagata). *Let B be a noetherian domain with the property $*$). Then the following on a ring R over B are equivalent.*
1) *The ring R has a form $\widetilde{A} \bigcap L$ with the derived normal ring $\widetilde{A}$ of an affine ring A over B and a function field L over B.*
2) *The ring R is the I-transform of the derived normal ring $\widetilde{C}$ of an affine ring C over B with an ideal I of $\widetilde{C}$.*

The property $*$) on B is the following,

$*$) *For every divisorial valuation ring D over B, the intersection $D \bigcap K$ of D*

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and the field of quotients K of B is again a divisorial valuation ring over B unless D contains K.

Here we say that D is a divisorial valuation ring over B if D is a localization $D = \widetilde{C}_{\mathfrak{p}}$ of the derived normal ring $\widetilde{C}$ of an affine ring C over B by a height one prime ideal $\mathfrak{p}$ of $\widetilde{C}$.

In the proof of the theorem, the assumption $*)$ is necessary only to show 2) under the condition 1) and Nagata left the following problem[4, Question 1].

Problem 1.2. What is the class of noetherian integral domains for which the condition $*)$ holds?

The purpose of this note is to show that every noetherian domain has this property.

All rings are assumed to be commutative with identity. Notation and terminology in [1] and [4] are used freely.

In particular, a ring with a unique maximal ideal is called qusi-local and we say A is a unibranched local domain if A is a noetherian domain with its derived normal ring being quasi-local.

2. Main result

Lemma 2.1. *Let $(A, \mathfrak{m})$ be a unibranched local domain with* $\dim A \geq 2$. *Then for any minimal prime P of the completion $\hat{A}$ of A, we have* $\dim \hat{A}/P \geq 2$.

Proof. The derived normal ring $\widetilde{A}$ of A is quasi-local with depth$\widetilde{A} \geq 2$ in the sence that A has a regular sequence of length two on $\widetilde{A}$. Really, if not and assuming by induction hypothyesis depth $\widetilde{A}_Q \geq 2$ for any non-maximal prime ideal Q of $\widetilde{A}$ such that ht$Q \geq 2$, we see easily that there exist elements a, b in $\widetilde{A}$ such that the radical of $a\widetilde{A} : b\widetilde{A}$ is the maximal ideal of $\widetilde{A}$. Then we see $a/b \notin \widetilde{A}$ and that a/b is integral over A, a contradiction.

On the other hand, $C = \widetilde{A} \bigotimes_A \hat{A}$ is qusi-local with depth $C \geq 2$ because C is expressed as an inductive limit of local rings.

Now for a minimal prime P of $\hat{A}$, since we have $P \bigcap A = 0$, P corresponds to a prime ideal P' of $K \bigotimes_A \hat{A}$ for the field of quotients K of A. So take a decomposition $0 = I' \bigcap J'$ in the noetherian ring $K \bigotimes_A \hat{A}$ where I' is the primary component belonging to P' and J' is the intersection of the ones belonging to primes other than P'.

Put $I = I' \bigcap C$ and $J = J' \bigcap C$. Then we have a decomposition $0 = I \bigcap J$ in C and an exact sequence of $\hat{A}$-modules

$$0 \to C \longrightarrow C/I \bigoplus C/J \longrightarrow C/(I+J) \to 0.$$

If $\dim \hat{A}/P = 1$, then since P is a minimal prime and $P \not\supseteq J$ we have $\dim C/(I+J) = 0$ and $\mathrm{Ext}^1_{\hat{A}}(A/\mathfrak{m}, C) \neq 0$, which means depth $C = 1$, a contradiction. □

Proposition 2.2. *Let $(A,\mathfrak{m})$ be a unibranched local domain with $\dim A \geq 2$ and let C be an affine ring over A. Then for any height one prime ideal P of C lying over $\mathfrak{m}$, we have*

$$\mathrm{tr.deg}_{\kappa(\mathfrak{m})}\kappa(P) > \mathrm{tr.deg}_K L$$

where $\kappa(P)$ and $\kappa(\mathfrak{m})$ are the residue fields at P and $\mathfrak{m}$, L and K are the fields of quotients of C and A respectively.

Proof. For the completion $\hat{A}$ of A, we see that $P' = P(\hat{A} \otimes_A C)$ is a height one prime ideal of $\hat{A} \otimes_A C$ by [1, Theorem 15.1] because $(\hat{A} \otimes_A C)/P' = \hat{A}/\mathfrak{m}\hat{A} \otimes_A C/P = C/P$ is an integral domain and $C \to \hat{A} \otimes_A C$ is a flat morhism. So take a minimal prime Q' of $\hat{A} \otimes_A C$ contained in P' such that ht $P'/Q' = 1$.

Put $\mathfrak{q} = Q' \bigcap \hat{A}$, then $\mathfrak{q}$ is a minimal prime of $\hat{A}$. Really, since we have $Q' \bigcap C = 0$, Q' and $\mathfrak{q}$ correspond to prime ideals of $\hat{A} \otimes_A L$ and $\hat{A} \otimes_A K$ respectively. Applying the going down theorem [1, Theorem 9.5] to the flat morphism $\hat{A} \otimes_A K \to \hat{A} \otimes_A L$, the assumption that $\mathfrak{q}$ is not minimal leads us to a contradiction that Q' is non-minimal.

Thus we have $\dim \hat{A}/\mathfrak{q} \geq 2$ by Lemma 2.1.

Now the complete local domain $\hat{A}/\mathfrak{q}$ is universally catenary by [1, Theorem 31.6] and we can apply dimension formula[1, Theorem 15.6] for $\hat{A}/\mathfrak{q} \to (\hat{A} \otimes C)/Q'$, we have

$$\mathrm{ht}\ P'/Q' = \dim \hat{A}/\mathfrak{q} + \mathrm{tr.deg}_{\kappa(\mathfrak{q})}\kappa(Q') - \mathrm{tr.deg}_{\kappa(\mathfrak{m})}\kappa(P).$$

Now since Q' corresponds to a minimal prime of $(\hat{A}/\mathfrak{q}) \otimes_A L$, we have

$$\mathrm{tr.deg}_{\kappa(\mathfrak{q})}\kappa(Q') = \mathrm{tr.deg}_K L.$$

Thus we have

$$\mathrm{tr.deg}_{\kappa(\mathfrak{m})}\kappa(P) = \mathrm{tr.deg}_K L + \dim \hat{A}/\mathfrak{q} - 1 > \mathrm{tr.deg}_K L.$$ □

Theorem 2.3. *For any divisorial valuation ring D over a noetherian domain B, the intersection $D \bigcap K$ with the field of quotients K of B is again a divisorial valuation ring over B unless D contains K.*

Proof. We may assume that D does not contain K. Let $\mathfrak{n}$ be the maximal ideal of D. Adding some elements of $D \bigcap K$, we have an affine ring A over B such that

$(D\bigcap K)/(\mathfrak{n}\bigcap K)$ is algebraic over $A/\mathfrak{m}$ with $\mathfrak{m}=A\bigcap\mathfrak{n}$. Adding more elements if necessary, we may assume the localization $A_{\mathfrak{m}}$ is a unibranched local domain by [2, Theorem(33.10)].

If we can prove that $\dim A_{\mathfrak{m}}=1$, then we see that $D\bigcap K$ is the derived normal ring of $A_{\mathfrak{m}}$ and we finish the proof of Theorem 2.3.

So suppose, on the contrary, that $\dim A_{\mathfrak{m}}\geq 2$. Since D is divisorial over B, we have an affine ring C over A such that $D=\widetilde{C}_{\widetilde{P}}$ where $\widetilde{C}$ is the derived normal ring of C and $\widetilde{P}$ is a height one prime ideal of $\widetilde{C}$. Adding some elements of $\widetilde{C}$ if necessary, we may assume C_P is a unibranched local domain with $P=\mathfrak{n}\bigcap C$ by [2, Theorem(33.10)]. Then we have $\mathrm{ht}P=1$ and $P\bigcap A=\mathfrak{n}\bigcap A=\mathfrak{m}$.

On the other hand, D is divisorial over $D\bigcap K$ because so is D over B, and since the dimension formula holds between discrete valuation rings $D\bigcap K$ and D by [1, Theorem 15.6], we have

$$\mathrm{tr.deg}_K L=\mathrm{tr.deg}_{(D\bigcap K)/(\mathfrak{n}\bigcap K)}D/\mathfrak{n}$$

with the field of quotients L of D.

Apply Proposition 2.2 and we have

$$\mathrm{tr.deg}_{\kappa(\mathfrak{m})}\kappa(P)>\mathrm{tr.deg}_K L=\mathrm{tr.deg}_{(D\bigcap K)/(\mathfrak{n}\bigcap K)}D/\mathfrak{n}\geq\mathrm{tr.deg}_{\kappa(\mathfrak{m})}\kappa(P)$$

where the last inequality holds because $(D\bigcap K)/(\mathfrak{n}\bigcap K)$ is algebraic over $\kappa(\mathfrak{m})$, a contradiction. □

Now Theorem 1.1 can be restated.

Corollary 2.4 (Nagata). *A ring R over a noetherian domain B has the form $\widetilde{A}\bigcap L$ with the derived normal ring $\widetilde{A}$ of an affine ring A over B and with a function field L over B if and only if R is the I-transform of the derived normal ring $\widetilde{C}$ of an affine ring C over B for an ideal I of $\widetilde{C}$.*

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