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GENERALIZED EICHLER-SHIMURA ISOMORPHISMS FOR COMPACT LOCALLY SYMMETRIC SPACES

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1. Introduction

Let $\Gamma \subset SL(2,\mathbb{R})$ be a Fuchsian group of the first kind, and let V_k be the k-th symmetric power of the standard representation of $SL(2,\mathbb{R})$ on $\mathbb{C}^2$. If $S_{k+2}(\Gamma)$ denotes the space of cusp forms of weight $k+2$ for Γ and if $H^1_P(\Gamma, V_k)$ is the parabolic cohomology space for Γ with coefficients in V_k, then there is a canonical isomorphism

$$H^1_P(\Gamma, V_k) \cong S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)}$$

known as the Eichler-Shimura isomorphism (cf. [1], [11]). If $\mathcal{H}$ is the Poincaré upper half plane and if $\widetilde{V}_k$ is the locally constant sheaf on the Riemann surface $X = \Gamma\backslash\mathcal{H} \cup \{\text{cusps}\}$ associated to V_k, then $H^1_P(\Gamma, V_k)$ can be identified with $H^1(X, \widetilde{V}_k)$. Thus the Eichler-Shimura isomorphism describes the cohomology of the Riemann surface X with coefficients in $\widetilde{V}_k$ in terms of cusp forms for Γ. The purpose of this paper is to investigate similar descriptions for the cohomology of more general locally symmetric spaces.

Let G (resp. G') be a semisimple Lie group, $K \subset G$ (resp. $K' \subset G'$) a maximal compact subgroup, and $D = G/K$ (resp. $D' = G'/K'$) the associated symmetric space. We assume that the associated symmetric space has a G-invariant complex structure. Consider an equivariant pair (ρ, τ) consisting of a homomorphism $\rho : G \to G'$ and a holomorphic map $\tau : D \to D'$ satisfying $\tau(gz) = \rho(g)\tau(z)$ for all $g \in G$ and $z \in D$ (cf. [10]). Let $\Gamma \subset G$ and $\Gamma' \subset G'$ be torsion-free cocompact discrete subgroups with $\rho(\Gamma) \subset \Gamma'$. If V and V' are finite-dimensional complex vector spaces and if $J : \Gamma \times D \to GL(V)$ and $J' : \Gamma' \times D' \to GL(V')$ are automorphy factors, then we denote by $\mathcal{M}_{\rho,\tau}(\Gamma, J, J')$ the space of mixed automorphic forms for Γ of type (J, J', ρ, τ), that is, holomorphic functions $f : D \to V \otimes V'$ satisfying

$$f(\gamma z) = (J(\gamma, z) \otimes J'(\rho(\gamma), \tau(z)))f(z)$$

for all $\gamma \in \Gamma$ and $z \in D$ (cf. [6]; see also [5]).

Let J_0 be the automorphy factor given by the Jacobian determinant of D. In this paper we show that there is a canonical antilinear isomorphism between the space $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ and the quotient of cohomology spaces of $X = \Gamma\backslash D$ with coefficients in certain sheaves. We also describe some examples which, in particular, show that the above isomorphism generalizes the Eichler-Shimura isomorphism for parabolic cohomology of Fuchsian groups in the cocompact case.

2. Automorphic vector bundles

Let G be a semisimple Lie group, K a maximal compact subgroup of G, and $D = G/K$ the associated symmetric space as in Section 1. We assume that D has a G-invariant complex structure so that D becomes a Hermitian symmetric domain. Let Γ be a torsion-free cocompact discrete subgroup of G, and let V be a finite-dimensional complex vector space. Let $J : \Gamma \times D \to GL(V)$ be an automorphy factor of Γ, that is, a map such that the function $D \to GL(V)$, $x \mapsto J(\gamma, x)$ is holomorphic for each $\gamma \in \Gamma$ and

$$J(\gamma\gamma', x) = J(\gamma, \gamma' x) J(\gamma', x)$$

for all $x \in D$ and $\gamma, \gamma' \in \Gamma$. Let G' be another semisimple Lie group, $D' = G'/K'$ the associated symmetric domain. Let $\rho : G \to G'$ be a homomorphism, and let $\tau : D \to D'$ be a holomorphic map such that $\tau(gz) = \rho(g)\tau(z)$ for all $g \in G$ and $z \in D$. Various aspects of such equivariant pairs were discussed extensively in [10]. Let Γ' be a torsion-free cocompact discrete subgroup with $\rho(\Gamma) \subset \Gamma'$, V' a finite-dimensional complex vector space, and $J' : \Gamma' \times D' \to GL(V')$ an automorphy factor of Γ'.

DEFINITION 2.1. A *mixed automorphic form for* Γ *of type* (J, J', ρ, τ) is a holomorphic function $f : D \to V \otimes V'$ satisfying

$$f(\gamma z) = (J(\gamma, z) \otimes J'(\rho(\gamma), \tau(z))) f(z)$$

for all $\gamma \in \Gamma$ and $z \in D$.

We shall denote by $\mathcal{M}_{\rho,\tau}(\Gamma, J, J')$ the space of mixed automorphic forms for Γ of type (J, J', ρ, τ).

EXAMPLE 2.2. Let $\Gamma \subset SL(2, \mathbb{R})$ be a Fuchsian group of the first kind, $\chi : \Gamma \to SL(2, \mathbb{R})$ a homomorphism, and $\omega : \mathcal{H} \to \mathcal{H}$ a holomorphic map such that $\omega(\gamma z) = \chi(\gamma)\omega(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, where $\mathcal{H}$ is a Poincaré upper half plane. Let $\Gamma' = \chi(\Gamma)$, and for nonnegative integers k and l let $J : \Gamma \times \mathcal{H} \to \mathbb{C}$, $J' : \Gamma' \times \mathcal{H} \to \mathbb{C}$ be automorphy factors given by

$$J(\gamma, z) = (cz + d)^k, \qquad J'(\gamma', z') = (c'z' + d')^l$$

for $z, z' \in \mathcal{H}$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \qquad \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma'.$$

Then a mixed automorphic form for Γ of type (J, J', χ, ω) is a mixed automorphic form of type (k, l) associated to Γ, ω and χ in the sense of [3] if the condition of boundedness at the cusps is added. Similar examples can be considered in the Siegel modular case (cf. [4]) and the Hilbert modular case (cf. [8]).

For each $\gamma \in \Gamma$, let $z \mapsto J_0(\gamma, z)$ be the determinant of the Jacobian matrix of the holomorphic map $z \mapsto \gamma z$ of the Hermitian symmetric domain D, and set

$$\mathcal{L}_{J_0} = \Gamma \backslash D \times \mathbb{C},$$

where the quotient is taken with respect to the action of Γ on $D \times \mathbb{C}$ given by

$$\gamma \cdot (z, \lambda) = (\gamma z, J_0(\gamma, z)\lambda)$$

for $\gamma \in \Gamma$ and $(z, \lambda) \in D \times \mathbb{C}$. Similarly, to the automorphy factor $J' : \Gamma' \times D' \to GL(V')$ we associate an operation of Γ' on $D' \times V'$ by

$$\gamma' \cdot (z', v') = (\gamma' z', J'(\gamma', z')v')$$

for all $\gamma' \in \Gamma'$, $z' \in D'$ and $v' \in V'$. Then the quotient $\mathcal{V}'_{J'} = \Gamma' \backslash D' \times V'$ with respect to the above operation is a vector bundle over the locally symmetric space $X' = \Gamma' \backslash D'$ with its fiber isomorphic to V'.

Since ρ and τ are equivariant and $\rho(\Gamma) \subset \Gamma'$, the holomorphic map $\tau : D \to D'$ induces a map $\tau_X : X \to X'$ of compact complex manifolds. Let $\tau_X^* \mathcal{V}'_{J'}$ be the vector bundle over X obtained by pulling back $\mathcal{V}'_{J'}$ via τ_X. If $\mathcal{V}$ is a vector bundle we shall denote by $\widetilde{\mathcal{V}}$ the associated sheaf of sections.

If n is the complex dimension of X, then the Serre duality determines the map

$$H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}) \times H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1}) \to \mathbb{C}$$

that is given by

$$([\omega], \varphi) \mapsto \int_X \varphi\omega \wedge dz,$$

where $[\omega]$ is the cohomology class of a differential n-form ω with coefficients in $\tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}$, φ is a section of the sheaf $\tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1}$, and dz is a volume form on X.

Lemma 2.3. *The space $H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$ of sections of the vector bundle $\tau_X^* \mathcal{V}'_{J'} \otimes \mathcal{L}_{J_0}^{-1}$ is canonically isomorphic to the space $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ of mixed automorphic forms for Γ of type (J_0, J', ρ, τ).*

Proof. A section of $\mathcal{V}'_{J'}$ can be identified with a function $f' : D' \to V'$ such that $f'(\gamma' z') = J'(\gamma', z')f'(z')$ for all $\gamma' \in \Gamma'$ and $z' \in D'$, and a section of $\tau_X^* \mathcal{V}'_{J'}$ can be regarded as a function $f : D \to V'$ of the form $f = f' \circ \tau$ for such a function f'. Thus we have

$$\begin{aligned} f(\gamma z) = f'(\tau(\gamma z)) = f'(\rho(\gamma)\tau(z)) &= J'(\rho(\gamma), \tau(z))f'(\tau(z)) \\ &= J'(\rho(\gamma), \tau(z))f(z). \end{aligned}$$

Now the lemma follows from the fact that a section of $\mathcal{L}_{J_0}^{-1}$ can be identified with a function $g : D \to \mathbb{C}$ such that $g(\gamma z) = J_0(\gamma, z)g(z)$ for $\gamma \in \Gamma$ and $z \in D$. $\square$

3. The generalized Eichler-Shimura isomorphism

Let G, G', $X = \Gamma \backslash D$, $X' = \Gamma' \backslash D'$, $\rho : G \to G'$ and $\tau : D \to D'$ be as in Section 2. Let $r : G' \to GL(W)$ be a representation of G' on a finite-dimensional complex vector space W equipped with a nondegenerate inner product $\langle \, , \rangle$ that is invariant under the action of $\rho(G) \subset G'$ via r. Then the discrete subgroup Γ of G acts on $D \times W$ by

$$\gamma \cdot (z, w) = (\gamma z, r \circ \rho(\gamma)w)$$

for $\gamma \in \Gamma$ and $(z, w) \in D \times W$. Let $\mathcal{W} = \Gamma \backslash D \times W$ be the quotient of $D \times W$ with respect to this action of Γ, and let $\pi : \mathcal{W} \to X = \Gamma \backslash D$ be the map induced by the natural projection $D \times W \to D$. Then $\mathcal{W}$ is a vector bundle over X with fiber map π whose fiber is isomorphic to W. The inner product $\langle \, , \rangle$ on W induces a pairing $\langle \, , \rangle : \mathcal{W} \oplus \mathcal{W} \to X \times \mathbb{C}$ given by

$$\langle w_x, w'_x \rangle = (x, \langle w_x, w'_x \rangle)$$

for all $x \in X$ and $w_x, w'_x \in \pi^{-1}(x)$.

We fix a section $\xi_0 \in H^0(X, \widetilde{\mathcal{W}} \otimes \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee})$ of the vector bundle $\mathcal{W} \otimes \tau_X^* \mathcal{V}_{J'}'^{\vee}$ over $X = \Gamma \backslash D$. Then ξ_0 can be regarded as a map $\xi_0 : D \to V' \otimes W$ satisfying the relation

$$\xi_0(\gamma z) = (J'(\rho(\gamma), \tau(z))^{-1} \otimes r \circ \rho(\gamma))\xi_0(z)$$

for all $z \in D$ and $\gamma \in \Gamma$. We assume that for each nonempty open set U in D the subspace of $V' \otimes W$ spanned by the set $\{\xi_0(z) \mid z \in U\}$ is of the form $V'' \otimes W$ for some subspace V'' of V'. We now consider the vector bundle $\tau_X^* \mathcal{V}'_{J'} \otimes \mathcal{L}_{J_0}^{-1}$ over X and define an inner product $\langle\!\langle \, , \rangle\!\rangle$ on the space $H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$ by

$$\langle\!\langle f, g \rangle\!\rangle = \int_X f \overline{g} \langle \overline{\xi}_0, \xi_0 \rangle d\overline{z} \wedge dz$$

for all $f, g \in H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$. Let ω be a differential n-form that determines an element $[\omega] \in H^n(X, \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'})$. Then for each $\varphi \in H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$ we can find a unique section ψ_ω of $\tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1}$ such that the complex conjugate $\overline{\psi}_\omega$ of ψ_ω satisfies the relation

$$\langle\!\langle \varphi, \overline{\psi}_\omega \rangle\!\rangle = \int_X \varphi\omega \wedge dz.$$

Thus we obtain an antilinear isomorphism

$$H^n(X, \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'}) \cong H^0(X, \tau_X^* \widetilde{\mathcal{V}}'_{J'} \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$$

given by $[\omega] \mapsto \psi_\omega$. Since the fiber-wise pairing $\langle\,,\rangle : \mathcal{W} \oplus \mathcal{W} \to X \times \mathbb{C}$ described above induces a map

$$\langle\,,\rangle : \widetilde{\mathcal{W}} \oplus (\widetilde{\mathcal{W}} \otimes \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'}) \to \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'},$$

we obtain the map $\nu : \widetilde{\mathcal{W}} \to \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'}$ given by $\nu(s) = \langle \overline{s}, \xi_0 \rangle$.

Proposition 3.1. *The map $\nu : \widetilde{\mathcal{W}} \to \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'}$ described above is injective.*

Proof. Suppose $\langle \overline{s}, \xi_0 \rangle = 0$ with $s \in \Gamma(U, \widetilde{\mathcal{W}})$ for an open set $U \subset X$. Recall that the bundle $\mathcal{W}$ can be considered as the quotient of the trivial vector bundle $D \times W \to D$ by Γ with respect to the action

$$\gamma \cdot (z, x) = (\gamma z, r \circ \rho(\gamma) x)$$

for $\gamma \in \Gamma$ and $(z, x) \in D \times W$. Thus we have a commutative diagram of the form

$$\begin{array}{ccc} D \times W & \xrightarrow{p_W} & \mathcal{W} = \Gamma \backslash D \times W \\ \downarrow & & \downarrow \pi \\ D & \xrightarrow{p} & X = \Gamma \backslash D, \end{array}$$

where p_W and p are natural projection maps. Let $s' \in \Gamma(p^{-1}(U), D \times W)$ be a locally constant section of the bundle $D \times W$ on $p^{-1}(U)$. For each point $v \in p^{-1}(U)$ there is a neighborhood $U' \subset D$ of v such that there exists an element $w_0 \in W$ with $s'(z) = w_0$ for all $z \in U'$. Therefore we have

$$\langle \overline{w}_0, \xi_0(z) \rangle = \langle \overline{s}'(z), \xi_0(z) \rangle = \langle \overline{s}(p(z)), \xi_0(z) \rangle = 0$$

for all $z \in U'$. When ξ_0 is considered as a map from D to $V' \otimes W$, by our assumption the set $\{\xi_0(z) \mid z \in U'\}$ generates a subspace of $V' \otimes W$ of the form $V'' \otimes W$ with $V'' \subset V'$; hence we have $\overline{w}_0 = 0$ and $w_0 = 0$. Thus it follows that $s = 0$, and therefore ν is injective. □

From the injectivity of the map ν in Proposition 3.1 we obtain the short exact sequence

$$0 \to \widetilde{\mathcal{W}} \to \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} \to \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} / \widetilde{\mathcal{W}} \to 0$$

of sheaves, and consequently we can consider the associated long exact sequence

$$\begin{aligned} \cdots \to H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee}) &\to H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} / \widetilde{\mathcal{W}}) \\ &\to H^n(X, \widetilde{\mathcal{W}}) \to H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee}) \to \cdots \end{aligned}$$

of cohomology of the locally symmetric space X. Now we state our main theorem of this paper.

Theorem 3.2. *Let $\delta : H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} / \widetilde{\mathcal{W}}) \to H^n(X, \widetilde{\mathcal{W}})$ be the connecting homomorphism in the above exact sequence for $n = \dim_{\mathbb{C}} X$. Then there is a canonical antilinear isomorphism*

$$\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J') \cong H^n(X, \widetilde{\mathcal{W}}) / \delta(H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} / \widetilde{\mathcal{W}})),$$

where $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ is the space of mixed automorphic forms for Γ of type (J_0, J', ρ, τ).

Proof. First, the map $\nu : \widetilde{\mathcal{W}} \to \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee}$ determines the associated map

$$\nu^* : H^n(X, \widetilde{\mathcal{W}}) \to H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee})$$

of cohomology spaces. Using the isomorphism in Lemma 2.3 and the antilinear isomorphism

$$H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee}) \cong H^0(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}' \otimes \widetilde{\mathcal{L}}_{J_0}^{-1})$$

described above, we obtain the map

$$\Psi_* : H^n(X, \widetilde{\mathcal{W}}) \to \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J').$$

Let f be a mixed automorphic form in $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$, and regard f as a section of the sheaf $\tau_X^* \widetilde{\mathcal{V}}_{J'}' \otimes \widetilde{\mathcal{L}}_{J_0}^{-1}$. If ξ_0 is the fixed section of the sheaf $\widetilde{\mathcal{W}} \otimes \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee}$ as before, then $f\xi_0 dz$ is a differential n-form on X with values in the sheaf $\widetilde{\mathcal{W}}$, and hence $f\xi_0 dz$ determines a cocycle in $H^n(X, \widetilde{\mathcal{W}})$. We shall now show that the map $\Psi^* : \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J') \to H^n(X, \widetilde{\mathcal{W}})$ defined by $\Psi^*(f) = [f\xi_0 dz]$ for all $f \in \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ is injective. Indeed, for each nonnegative integer p let $\mathcal{A}^p$ denote the sheaf of differential p-forms on X, and define the map $\nu_p : \widetilde{\mathcal{W}} \otimes \mathcal{A}^p \to \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} \otimes \mathcal{A}^p$ by

$$\nu_p(\omega) = \langle \overline{\omega_{p,0}}, \xi_0 \rangle \in \Gamma(U, \tau_X^* \widetilde{\mathcal{V}}_{J'}'^{\vee} \otimes \mathcal{A}^p)$$

for each $\omega \in \Gamma(U, \widetilde{\mathcal{W}} \otimes \mathcal{A}^p)$, where U is an open subset of X and $\omega_{p,0}$ denotes the $(p,0)$-component of ω. Then ν_p is an extension of the map $\nu : \widetilde{\mathcal{W}} \to \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}$, since ν_p coincides with ν when $p = 0$. If f is an element of $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$, then the differential $f\xi_0 dz$ is a section of the sheaf $\widetilde{\mathcal{W}} \otimes \mathcal{A}^n$ and we have $\Psi^*(f) = [f\xi_0 dz]$. Since $f\xi_0 dz$ is a holomorphic form, we have $(f\xi_0 dz)_{(n,0)} = f\xi_0 dz$, and it follows that

$$(\nu_n)_* \Psi^*(f) = [\langle \overline{f\xi_0 dz}, \xi_0 \rangle] \in H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}).$$

Using the antilinear isomorphism $H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}) \cong \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$, we can choose an element $f_1 \in \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ such that $\langle\langle g, f_1 \rangle\rangle = \langle\langle g, f \rangle\rangle$ for each $g \in \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$. Thus we obtain $\Psi_*(\Psi^* f) = f_1 = f$, and therefore the composite $\Psi_* \circ \Psi^*$ is the identity map on the cohomology space $H^n(X, \widetilde{\mathcal{W}})$; hence it follows that Ψ^* is injective. Now the theorem follows by applying this and the antilinear isomorphism

$$H^n(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}) \cong \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$$

to the long exact sequence above. $\square$

4. Examples

In this section we describe the results in Section 3 for a few specific equivariant pairs (ρ, τ) and indicate that the isomorphism given in Theorem 3.2 may indeed be regarded as a generalization of the Eichler-Shimura isomorphism for elliptic modular forms.

EXAMPLE 4.1. Let $M_m(\mathbb{C})$ denote the set of $m \times m$ matrices with entries in $\mathbb{C}$, and set

$$\Psi_m = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} \;\middle|\; U, V \in M_m(\mathbb{C}), \quad {}^tUV = {}^tVU, \quad \operatorname{rank} \begin{pmatrix} U \\ V \end{pmatrix} = m \right\}.$$

Given an element $\left(\begin{smallmatrix} U_0 \\ V_0 \end{smallmatrix}\right) \in \Psi_m$ and a nonnegative integer k we define $\widehat{\eta}_k \left(\begin{smallmatrix} U_0 \\ V_0 \end{smallmatrix}\right)$ to be the map

$$\widehat{\eta}_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} : \Psi_m \to \mathbb{C}$$

on Ψ_m given by

$$\widehat{\eta}_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \det{}^k \begin{pmatrix} U_0 & U \\ V_0 & V \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} U \\ V \end{pmatrix} \in \Psi_m.$$

Let W_k be the vector space over $\mathbb{C}$ generated by the functions $\phi : \Psi_m \to \mathbb{C}$ of the form $\widehat{\eta}_k \left(\begin{smallmatrix} U_0 \\ V_0 \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} U_0 \\ V_0 \end{smallmatrix}\right) \in \Psi_m$. We set $G' = Sp(m, \mathbb{R})$ so that D' can be identified with the Siegel upper half space $\mathcal{H}_m$ of degree m. In this case the corresponding equivariant pair (ρ, τ) induces a family of abelian varieties parameterized by the locally symmetric space $\Gamma\backslash D$. Such families of abelian varieties are known as Kuga fiber varieties, and they play an important role in number theory (see e.g. [2], [10, Chapter 4], [4], [6]). We also set $W = W_k$ and define the representation $r : G' \to GL(W)$ of G' in W by

$$r(\sigma)\widehat{\eta}_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \widehat{\eta}_k\left(\sigma \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}\right)$$

for $\sigma \in G'$, $\widehat{\eta}_k \in W$ and $\left(\begin{smallmatrix} U_0 \\ V_0 \end{smallmatrix}\right) \in \Psi_m$. Let G, D, $\rho : G \to G' = Sp(m, \mathbb{R})$ and $\tau : D \to D' = \mathcal{H}_m$ as in Section 2, and let Γ be a torsion-free cocompact arithmetic subgroup of G such that ρ is contained in an arithmetic subgroup Γ' of $Sp(m, \mathbb{Q})$. Let $J' : \Gamma' \times \mathcal{H}^m \to \mathbb{C}$ be the automorphy factor defined by

$$J'(\gamma', Z) = \det(C'Z + D')^k \quad \text{for} \quad \gamma' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma' \subset Sp(m, \mathbb{Q})$$

and $Z \in \mathcal{H}^m$, and let $\mathcal{V}'_{J'}$ be the associated vector bundle described in Section 2. We set $\mathcal{W} = \Gamma\backslash D \times W$, where the quotient is taken with respect to the action

$$\gamma \cdot (z, w) = (\gamma z, \rho(\gamma) w)$$

for $\gamma \in \Gamma$ and $(z, w) \in D \times W$, and define a bilinear pairing $\langle\, ,\, \rangle : W_k \times W_k \to \mathbb{C}$ on W_k by extending linearly to the whole vector space W_k the map

$$\left\langle \widehat{\eta}_k \begin{pmatrix} U \\ V \end{pmatrix}, \widehat{\eta}_k \begin{pmatrix} U' \\ V' \end{pmatrix} \right\rangle = \det{}^k \begin{pmatrix} U & U' \\ V & V' \end{pmatrix}$$

for the generators $\widehat{\eta}_k \left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$ and $\widehat{\eta}_k \left(\begin{smallmatrix} U' \\ V' \end{smallmatrix}\right)$ of W_k with $\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} U' \\ V' \end{smallmatrix}\right) \in \Psi_m$. Then this induces a fiber-wise pairing $\mathcal{W} \oplus \mathcal{W} \to X \times \mathbb{C}$. In this case we have

$$H^j(X, \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'}) = 0$$

for $j < n$, and therefore it follows that

$$\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J') \cong H^n(X, \widetilde{\mathcal{W}}) / H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'} / \widetilde{\mathcal{W}})$$

(see [7] for details).

EXAMPLE 4.2. In Example 4.1, let $G = Sp(m, \mathbb{R})$, $D = \mathcal{H}_m$, and $\rho = \mathrm{id}$. Then $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ becomes the space of Siegel modular forms of weight $k+m+1$, and the associated short exact sequence

$$0 \to H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}'^{\vee}_{J'} / \widetilde{\mathcal{W}}) \to H^n(X, \widetilde{\mathcal{W}}) \to \mathcal{M}_{\rho,\tau}(\Gamma, J_0, J') \to 0$$

may be regarded as the Eichler-Shimura isomorphism for Siegel modular forms. This case was considered by Nenashev in [9]. He also considered the cases where Γ is non-cocompact by using a compactification of $\Gamma\backslash\mathcal{H}_m$.

EXAMPLE 4.3. If we let $m = 1$ in Example 4.2, then both of the spaces $\mathcal{M}_{\rho,\tau}(\Gamma, J_0, J')$ and $H^{n-1}(X, \tau_X^* \widetilde{\mathcal{V}}_{J'}^{\prime\vee}/\widetilde{\mathcal{W}})$ are isomorphic to the space of cusp forms of weight $k+2$ and the corresponding short exact sequence reduces to the usual Eichler-Shimura isomorphism for elliptic modular forms (see [9] for details).

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