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DECOMPOSITION OF RADICAL ELEMENTS OF A COMPACTLY GENERATED *cm*-LATTICE

Dedicated to Prof. Atuo Komatu for his 60th birthday

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The aim of the present paper is to define a completely prime element and a new type of radicals of elements in some compactly generated *cm*-lattice¹⁾, and to obtain meet-decompositions of radical elements of the *cm*-lattice.

1. Preliminaries

Let L be a complete (upper and lower) lattice. A non-void subset Σ of L is called a compact set of L if, whenever $x \leq \sup N$ for an element x of Σ and a subset N of Σ , there exists a finite number of elements x_1, \dots, x_n of N satisfying $x_1 \cup \dots \cup x_n \geq x$. Every element of Σ is said to be compact. If every element of L is a join of a finite or infinite number of elements of the compact set Σ of L , then L is said to be compactly generated, and Σ is called a compact generator of L ²⁾.

We can prove easily that a complete lattice has a compact generator if and only if it is compactly generated in the sense of Dilworth-Crawley³⁾. In this case, the set of the compact elements⁴⁾ of the lattice is the unique maximal compact generator under the set-inclusion.

Now we shall consider, throughout this paper, a *cm*-lattice L which is compactly generated as a lattice. In what follows, we suppose that L has a compact generator Σ which satisfies the following condition.

(*) *If $\inf S \leq a$ for a subset S of Σ and an element a of L , then $a = \inf (S \cup a)$, where $S \cup a$ means the set of the elements $s \cup a$ ($s \in S$).*

It is easily verified that the condition (*) holds for any infinitely meet-

1) Cf. [1; pp. 200-201].

2) Cf. [5; p. 105] and [6; p. 54].

3), 4) Cf. [2; p. 2] and [3; p. 11].

distributive *cm*-lattice⁵⁾ with a compact generator. But, there exists a *cm*-lattice which is not infinitely meet-distributive, and satisfies the condition (*). Let \mathfrak{L} be the *cm*-lattice of all ideals of any Dedekind domain \mathfrak{D} , and Σ the compact generator consisting of the principal ideals of \mathfrak{D} . Then, of course, \mathfrak{L} is not necessarily infinitely meet-distributive. But, it can be proved easily that the condition (*) holds for \mathfrak{L} and Σ .

Let Σ be any fixed compact generator of L and let $a \in L$. By the symbol Σ_a we shall mean the set $\{x \in \Sigma \mid x \leq a\}$. Then we can prove that $\sup A = \sup (\bigvee_{a \in A} \Sigma_a)$ for every non-void subset A of L , where \bigvee denotes the set-theoretical union.

2. Completely prime elements

For a non-void subset A of a *cm*-lattice L , \bar{A} will denote the multiplicative system (monoid) which is generated by A under the multiplication of L .

DEFINITION 1. An element p of a *cm*-lattice is said to be *completely prime* if whenever $\inf \bar{A}$ is contained in p , then at least one of the element of A is contained in p .

We can prove easily that completely primes are primes. But the converse is not true. In fact, we can find an example of *cm*-lattices with an element which is prime but not completely prime.

DEFINITION 2. A non-void subset Γ of the compact set Σ of L is called a *c-system*, if $\inf \bar{\Gamma}$ is a member of Γ . The void set is a *c-system*.

Lemma 1. *The following conditions are equivalent to one another.*

- (1) p is completely prime.
- (2) If $\inf \bar{\Delta} \leq p$ ($\phi^{(6)} \neq \Delta \subseteq \Sigma$), then there exists an element $x \in \Delta$ such that $x \leq p$.
- (3) $\Sigma \setminus \Sigma_p^{(7)}$ is a *c-system*.

Proof. (1) \Rightarrow (2) is evident. (2) \Rightarrow (1): Suppose that p is not completely prime. Then, there exists a subset A of L such that $\inf \bar{A} \leq p$ and $a \not\leq p$ for every $a \in A$. Therefore we can take an element $x_a \in \Sigma_a$ such as $x_a \not\leq p$. Put $\Delta = \{x_a \mid a \in A\}$. Then it is easy to see that $\inf \bar{\Delta} \leq p$. (1) \Rightarrow (3): Suppose that p is completely prime. It is then easily verified that p does not contain $\inf \overline{\Sigma \setminus \Sigma_p}$. Hence we can take an element u of $\Sigma \setminus \Sigma_p$ such that $u \leq \inf \overline{\Sigma \setminus \Sigma_p}$. (3) \Rightarrow (2): Let $\Sigma \setminus \Sigma_p$ be a *c-system*, and Γ a non-void subset of Σ . If we suppose that p con-

5) =the dual of a relatively pseudo-complemented *cm*-lattice.

6) ϕ will mean the void set.

7) \setminus will denote the set-difference.

tains no element of Γ , then $\Gamma \subseteq \Sigma \setminus \Sigma_p, \bar{\Gamma} \subseteq \overline{\Sigma \setminus \Sigma_p}$. Hence we have $\inf \overline{\Sigma \setminus \Sigma_p} \leq \inf \bar{\Gamma}$. On the other hand, $\inf \overline{\Sigma \setminus \Sigma_p}$ is not contained in p , since $\Sigma \setminus \Sigma_p$ is a c -system.

Lemma 2. *Let Γ be a c -system such that $\Gamma \wedge \Sigma_a = \phi$ ⁸⁾ for an element a of L . Then there exists a completely prime element p such that $p \geq a$ and $\Gamma \wedge \Sigma_p = \phi$.*

Proof. First we show that the set $S = \{c \in L \mid a \leq c, \Gamma \wedge \Sigma_c = \phi\}$ is inductive. Let $\{c_\lambda\}$ be any chain in S , and let $c^* = \sup \{c_\lambda\}$. If we suppose that $\Gamma \wedge \Sigma_{c^*}$ contains an element x , then $x \leq c^* = \sup (\bigvee_\lambda \Sigma_{c_\lambda})$. Hence there exists a finite number of elements x_1, \dots, x_n such that $x \leq x_1 \cup \dots \cup x_n$ and $x_i \in \bigvee_\lambda \Sigma_{c_\lambda}$. Since there exists c_m such that $x_i \leq c_m$ ($i=1, \dots, n$), we have that $\Gamma \wedge \Sigma_{c_m} \ni x$, which is a contradiction. Zorn's lemma assures therefore the existence of a maximal element p in S . We now prove that p is completely prime. Let Δ be a non-void subset of Σ such that $\inf \bar{\Delta} \leq p$. If we suppose that p contains no element of Δ , then since $p < p \cup u$ for every $u \in \Delta$, we can find an element $v(u)$ (depending on u) of Σ such that $v(u) \in \Gamma \wedge \Sigma_{p \cup u}$. Now we put $V = \{v(u) \mid u \in \Delta\}$. Then \bar{V} is contained in $\bar{\Gamma}$. We have therefore that $\inf \bar{V} \geq \inf \bar{\Gamma}$. Now let $u^* = \mathfrak{P}(u_1, \dots, u_n)$ be an arbitrary element of $\bar{\Delta}$, where $u_i \in \Delta$ ($i=1, \dots, n$), and \mathfrak{P} denotes a product-form (product-polynomial) of u_1, \dots, u_n . Then we have that

$$\mathfrak{P}(v(u_1), \dots, v(u_n)) \leq \mathfrak{P}(p \cup u_1, \dots, p \cup u_n) \leq p \cup \mathfrak{P}(u_1, \dots, u_n) = p \cup u^*.$$

Since $\mathfrak{P}(v(u_1), \dots, v(u_n)) \in \bar{V}$, we obtain that $\inf \bar{V} \leq \inf \bar{\Delta} \leq \inf \{p \cup u^* \mid u^* \in \bar{\Delta}\} = p \cup \inf \bar{\Delta} = p$. Hence we have that $\inf \bar{\Gamma} \leq p$. Therefore we obtain that $\inf \bar{\Gamma} \in \Gamma \wedge \Sigma_p$, which is a contradiction.

3. Radicals of elements

Let a be an arbitrary element of L , and let X_a be the set of the elements x of Σ such that every c -system containing x contains an element of Σ_a . We now put

DEFINITION 3. The suplemum of X_a is called a *radical* of a , and is denoted by $r(a)$. An element a of L is said to be *radical* if $r(a) = a$.

Theorem 1. *Let L be a compactly generated cm-lattice with the condition (*). Then the radical of any element a of L is decomposed into the meet of the completely prime elements containing a . In particular, so is any radical element of L .*

Proof. Let p be any completely prime element containing a . Then $r(a) \leq p$. For, if contrary, the c -system $\Sigma \setminus \Sigma_p$ contains an element $x \in \Sigma$ such that

8) \wedge will denote the intersection. We say, following McCoy [4], that Γ does not meet Σ_a , if $\Gamma \wedge \Sigma_a = \phi$.

$x \not\leq p$ and $x \leq r(a)$. Hence $(\Sigma \setminus \Sigma_p) \wedge \Sigma_a$ is not vacuous. Since Σ_a is contained in Σ_p , this is a contradiction. Therefore we obtain that $r(a) \leq p$, $r(a) \leq \bigcap_{a \leq p} p$. For the proof of the converse inclusion, it is sufficient to show that $x \not\leq r(a)$ ($x \in \Sigma$) implies $x \not\leq \bigcap_{a \leq p} p$. Since $x \not\leq r(a)$, there exists a c -system Γ such that $\Gamma \ni x$ and it does not meet Σ_a . Then, by Lemma 2, we can take a completely prime element p satisfying $a \leq p$ and Γ does not meet Σ_p . Hence x is not contained in $\bigcap_{a \leq p} p$. This completes the proof.

Corollary 1. *Let L be a compactly generated cm -lattice with the condition (*), and I the greatest element of L . If the ascending chain condition holds for the elements of the interval $I|a$, the radical of a has a unique irredundant meet decomposition into completely prime elements. In particular, a is so if it is radical.*

Lemma 3. *A completely prime element of any cm -lattice is completely irreducible.⁹⁾*

Proof. Let p be a completely prime element of any cm -lattice. If $p = \inf Q$, then $p \geq \inf \bar{Q}$. Hence there exists an element q in Q such that $q \leq p$. If $q < p$, then $p = \inf Q \leq q < p$. This is a contradiction.

Corollary 2. *Let D be an infinitely meet distributive lattice with a compact generator. Then every element of D , which is different from the greatest element, is decomposed into completely irreducible elements of D .*

Proof. Let a be any element of D , and let Σ be any compact generator of D . Then the set $\Gamma = \{x\}$ consisting of the single element x of $\Sigma_{r(a)}$, is a c -system satisfying $\inf \bar{\Gamma} = x$. Hence a is radical. By Theorem 1 and Lemma 3, we complete the proof.

4. Completely minimal primes

Let Γ_0 be any fixed c -system of L , and let a be an element of L such that Σ_a does not meet Γ_0 . Then it is easily verified that the family of c -systems Γ each of which contains Γ_0 and does not meet Σ_a is inductive. Zorn's lemma assures therefore the existence of maximal c -systems. Let $\mathfrak{M} = \{\Gamma_\lambda\}$ be the family of the maximal c -systems based on Γ_0 . Then we can see that $p_\lambda = \sup(\Sigma \setminus \Gamma_\lambda)$ is completely prime, and the completely prime element p satisfying $a \leq p$ and $\Gamma_\lambda \wedge \Sigma_p = \phi$ coincides with p_λ . Moreover

$$\begin{aligned} \Gamma_\lambda &\rightarrow p_\lambda = \sup(\Sigma \setminus \Gamma_\lambda), \\ p_\lambda &\rightarrow \Gamma_\lambda = \Sigma \setminus \Sigma_{p_\lambda} \end{aligned}$$

give a one-to-one correspondence between \mathfrak{M} and the set of the completely primes $\{p\}$ which satisfy $a \leq p$ and $\Gamma_\lambda \wedge \Sigma_p = \phi$.

9) Cf. [2; p. 3].

DEFINITION 4. An completely prime element p of a cm -lattice is said to be *completely minimal prime belonging to a* , if it satisfies (1) $p \geq a$ and (2) there exists no completely prime p' such that $a \leq p' < p$.

Lemma 4. *In order that an element p of L is a completely minimal prime belonging to a , it is necessary and sufficient that $\Sigma \setminus \Sigma_p$ is a maximal c -system which does not meet Σ_a .*

Proof. We suppose that p is completely minimal prime, and a is not completely prime. Then we can assume that $a < p$. Now, by Zorn's lemma, there exists a maximal c -system Γ such that it does not meet Σ_a . Then we have that $p_0 = \sup(\Sigma \setminus \Gamma) \leq \sup(\Sigma \setminus (\Sigma \setminus \Sigma_p)) = p$. Since p_0 is completely prime, we obtain $p = p_0$. This implies that $\Sigma \setminus \Gamma = \Sigma_p$. Therefore $\Gamma = \Sigma \setminus \Sigma_p$ is a maximal c -system which does not meet Σ_a . The converse is easy to see.

Theorem 2. *Let L be a compactly generated cm -lattice with the condition (*). Then the radical of any element a of L is decomposed into the meet of the completely minimal primes belonging to a . In particular, so is any radical element of L .*

Proof. Let $\{p_\lambda\}$ be the completely minimal primes belonging to a . In order to prove that $r(a) \geq \bigcap_\lambda p_\lambda$, it is sufficient to show that $x \not\leq r(a)$ implies $x \not\leq \bigcap_\lambda p_\lambda$, where $x \in \Sigma$. Now by the definition of $r(a)$, we can take a c -system Γ such that $x \in \Gamma$ and it does not meet Σ_a . Hence there exists a completely minimal prime p belonging to a such that Σ_p does not meet Γ . Evidently $x \not\leq p$. We have therefore $x \not\leq \bigcap_\lambda p_\lambda$. The converse inclusion is evident.

REMARK 1. Suppose that p is any completely prime containing a . Then there exists a completely minimal prime belonging to a which is contained in p . Because, $\Sigma \setminus \Sigma_p$ is a c -system which does not meet Σ_a ; and we can take a maximal c -system which contains $\Sigma \setminus \Sigma_p$ and does not meet Σ_a .

REMARK 2. If the ascending chain condition holds for elements in the interval I/a , then the meet-decomposition mentioned in Corollary 1 to Theorem 1 is the irredundant meet-decomposition into the completely minimal primes belonging to a .

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