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Author(s)	Murata, Kentaro
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Murata, K.  
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## DECOMPOSITION OF RADICAL ELEMENTS OF A COMPACTLY GENERATED *cm*-LATTICE

Dedicated to Prof. Atuo Komatu for his 60th birthday

KENTARO MURATA

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The aim of the present paper is to define a completely prime element and a new type of radicals of elements in some compactly generated *cm*-lattice<sup>1)</sup>, and to obtain meet-decompositions of radical elements of the *cm*-lattice.

### 1. Preliminaries

Let  $L$  be a complete (upper and lower) lattice. A non-void subset  $\Sigma$  of  $L$  is called a compact set of  $L$  if, whenever  $x \leq \sup N$  for an element  $x$  of  $\Sigma$  and a subset  $N$  of  $\Sigma$ , there exists a finite number of elements  $x_1, \dots, x_n$  of  $N$  satisfying  $x_1 \cup \dots \cup x_n \geq x$ . Every element of  $\Sigma$  is said to be compact. If every element of  $L$  is a join of a finite or infinite number of elements of the compact set  $\Sigma$  of  $L$ , then  $L$  is said to be compactly generated, and  $\Sigma$  is called a compact generator of  $L$ <sup>2)</sup>.

We can prove easily that a complete lattice has a compact generator if and only if it is compactly generated in the sense of Dilworth-Crawley<sup>3)</sup>. In this case, the set of the compact elements<sup>4)</sup> of the lattice is the unique maximal compact generator under the set-inclusion.

Now we shall consider, throughout this paper, a *cm*-lattice  $L$  which is compactly generated as a lattice. In what follows, we suppose that  $L$  has a compact generator  $\Sigma$  which satisfies the following condition.

(\*) *If  $\inf S \leq a$  for a subset  $S$  of  $\Sigma$  and an element  $a$  of  $L$ , then  $a = \inf(S \cup a)$ , where  $S \cup a$  means the set of the elements  $s \cup a$  ( $s \in S$ ).*

It is easily verified that the condition (\*) holds for any infinitely meet-

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1) Cf. [1; pp. 200-201].

2) Cf. [5; p. 105] and [6; p. 54].

3), 4) Cf. [2; p. 2] and [3; p. 11].

distributive *cm*-lattice<sup>5)</sup> with a compact generator. But, there exists a *cm*-lattice which is not infinitely meet-distributive, and satisfies the condition (\*). Let  $\mathfrak{L}$  be the *cm*-lattice of all ideals of any Dedekind domain  $\mathfrak{O}$ , and  $\Sigma$  the compact generator consisting of the principal ideals of  $\mathfrak{O}$ . Then, of course,  $\mathfrak{L}$  is not necessarily infinitely meet-distributive. But, it can be proved easily that the condition (\*) holds for  $\mathfrak{L}$  and  $\Sigma$ .

Let  $\Sigma$  be any fixed compact generator of  $L$  and let  $a \in L$ . By the symbol  $\Sigma_a$  we shall mean the set  $\{x \in \Sigma \mid x \leq a\}$ . Then we can prove that  $\sup A = \sup (\bigvee_{a \in A} \Sigma_a)$  for every non-void subset  $A$  of  $L$ , where  $\bigvee$  denotes the set-theoretical union.

## 2. Completely prime elements

For a non-void subset  $A$  of a *cm*-lattice  $L$ ,  $\bar{A}$  will denote the multiplicative system (monoid) which is generated by  $A$  under the multiplication of  $L$ .

**DEFINITION 1.** An element  $p$  of a *cm*-lattice is said to be *completely prime* if whenever  $\inf \bar{A}$  is contained in  $p$ , then at least one of the element of  $A$  is contained in  $p$ .

We can prove easily that completely primes are primes. But the converse is not true. In fact, we can find an example of *cm*-lattices with an element which is prime but not completely prime.

**DEFINITION 2.** A non-void subset  $\Gamma$  of the compact set  $\Sigma$  of  $L$  is called a *c-system*, if  $\inf \bar{\Gamma}$  is a member of  $\Gamma$ . The void set is a *c*-system.

**Lemma 1.** *The following conditions are equivalent to one another.*

- (1)  $p$  is completely prime.
- (2) If  $\inf \bar{\Delta} \leq p$  ( $\phi^6) \neq \Delta \subseteq \Sigma$ ), then there exists an element  $x \in \Delta$  such that  $x \leq p$ .
- (3)  $\Sigma \setminus \Sigma_p^7)$  is a *c*-system.

**Proof.** (1) $\Rightarrow$ (2) is evident. (2) $\Rightarrow$ (1): Suppose that  $p$  is not completely prime. Then, there exists a subset  $A$  of  $L$  such that  $\inf \bar{A} \leq p$  and  $a \not\leq p$  for every  $a \in A$ . Therefore we can take an element  $x_a \in \Sigma_a$  such as  $x_a \not\leq p$ . Put  $\Delta = \{x_a \mid a \in A\}$ . Then it is easy to see that  $\inf \bar{\Delta} \leq p$ . (1) $\Rightarrow$ (3): Suppose that  $p$  is completely prime. It is then easily verified that  $p$  does not contain  $\inf \Sigma \setminus \Sigma_p$ . Hence we can take an element  $u$  of  $\Sigma \setminus \Sigma_p$  such that  $u \leq \inf \Sigma \setminus \Sigma_p$ . (3) $\Rightarrow$ (2): Let  $\Sigma \setminus \Sigma_p$  be a *c*-system, and  $\Gamma$  a non-void subset of  $\Sigma$ . If we suppose that  $p$  con-

5) =the dual of a relatively pseudo-complemented *cm*-lattice.

6)  $\phi$  will mean the void set.

7) \ will denote the set-difference.

tains no element of  $\Gamma$ , then  $\Gamma \subseteq \Sigma \setminus \Sigma_p$ ,  $\bar{\Gamma} \subseteq \overline{\Sigma \setminus \Sigma_p}$ . Hence we have  $\inf \overline{\Sigma \setminus \Sigma_p} \leq \inf \bar{\Gamma}$ . On the other hand,  $\inf \overline{\Sigma \setminus \Sigma_p}$  is not contained in  $p$ , since  $\Sigma \setminus \Sigma_p$  is a  $c$ -system.

**Lemma 2.** *Let  $\Gamma$  be a  $c$ -system such that  $\Gamma \wedge \Sigma_a = \phi$ <sup>8)</sup> for an element  $a$  of  $L$ . Then there exists a completely prime element  $p$  such that  $p \geq a$  and  $\Gamma \wedge \Sigma_p = \phi$ .*

Proof. First we show that the set  $S = \{c \in L \mid a \leq c, \Gamma \wedge \Sigma_c = \phi\}$  is inductive. Let  $\{c_\lambda\}$  be any chain in  $S$ , and let  $c^* = \sup \{c_\lambda\}$ . If we suppose that  $\Gamma \wedge \Sigma_{c^*}$  contains an element  $x$ , then  $x \leq c^* = \sup (\vee_\lambda \Sigma_{c_\lambda})$ . Hence there exists a finite number of elements  $x_1, \dots, x_n$  such that  $x \leq x_1 \cup \dots \cup x_n$  and  $x \in \vee_\lambda \Sigma_{c_\lambda}$ . Since there exists  $c_m$  such that  $x_v \leq c_m$  ( $v=1, \dots, n$ ), we have that  $\Gamma \wedge \Sigma_{c_m} \ni x$ , which is a contradiction. Zorn's lemma assures therefore the existence of a maximal element  $p$  in  $S$ . We now prove that  $p$  is completely prime. Let  $\Delta$  be a non-void subset of  $\Sigma$  such that  $\inf \bar{\Delta} \leq p$ . If we suppose that  $p$  contains no element of  $\Delta$ , then since  $p < p \cup u$  for every  $u \in \Delta$ , we can find an element  $v(u)$  (depending on  $u$ ) of  $\Sigma$  such that  $v(u) \in \Gamma \wedge \Sigma_{p \cup u}$ . Now we put  $V = \{v(u) \mid u \in \Delta\}$ . Then  $\bar{V}$  is contained in  $\bar{\Gamma}$ . We have therefore that  $\inf \bar{V} \geq \inf \bar{\Gamma}$ . Now let  $u^* = \mathfrak{P}(u_1, \dots, u_n)$  be an arbitrary element of  $\bar{\Delta}$ , where  $u_i \in \Delta$  ( $i=1, \dots, n$ ), and  $\mathfrak{P}$  denotes a product-form (product-polynomial) of  $u_1, \dots, u_n$ . Then we have that

$$\mathfrak{P}(v(u_1), \dots, v(u_n)) \leq \mathfrak{P}(p \cup u_1, \dots, p \cup u_n) \leq p \cup \mathfrak{P}(u_1, \dots, u_n) = p \cup u^*.$$

Since  $\mathfrak{P}(v(u_1), \dots, v(u_n)) \in \bar{V}$ , we obtain that  $\inf \bar{V} \leq \inf \bar{\Delta} \leq \inf \{p \cup u^* \mid u^* \in \bar{\Delta}\} = p \cup \inf \bar{\Delta} = p$ . Hence we have that  $\inf \bar{\Gamma} \leq p$ . Therefore we obtain that  $\inf \bar{\Gamma} \in \Gamma \wedge \Sigma_p$ , which is a contradiction.

### 3. Radicals of elements

Let  $a$  be an arbitrary element of  $L$ , and let  $X_a$  be the set of the elements  $x$  of  $\Sigma$  such that every  $c$ -system containing  $x$  contains an element of  $\Sigma_a$ . We now put

**DEFINITION 3.** The supremum of  $X_a$  is called a *radical* of  $a$ , and is denoted by  $r(a)$ . An element  $a$  of  $L$  is said to be *radical* if  $r(a) = a$ .

**Theorem 1.** *Let  $L$  be a compactly generated cm-lattice with the condition (\*). Then the radical of any element  $a$  of  $L$  is decomposed into the meet of the completely prime elements containing  $a$ . In particular, so is any radical element of  $L$ .*

Proof. Let  $p$  be any completely prime element containing  $a$ . Then  $r(a) \leq p$ . For, if contrary, the  $c$ -system  $\Sigma \setminus \Sigma_p$  contains an element  $x \in \Sigma$  such that

8)  $\wedge$  will denote the intersection. We say, following McCoy [4], that  $\Gamma$  does not meet  $\Sigma_a$ , if  $\Gamma \wedge \Sigma_a = \phi$ .

$x \not\leq p$  and  $x \leq r(a)$ . Hence  $(\Sigma \setminus \Sigma_p) \wedge \Sigma_a$  is not vacuous. Since  $\Sigma_a$  is contained in  $\Sigma_p$ , this is a contradiction. Therefore we obtain that  $r(a) \leq p$ ,  $r(a) \leq \bigcap_{a \leq p} p$ . For the proof of the converse inclusion, it is sufficient to show that  $x \not\leq r(a)$  ( $x \in \Sigma$ ) implies  $x \not\leq \bigcap_{a \leq p} p$ . Since  $x \not\leq r(a)$ , there exists a  $c$ -system  $\Gamma$  such that  $\Gamma \ni x$  and it does not meet  $\Sigma_a$ . Then, by Lemma 2, we can take a completely prime element  $p$  satisfying  $a \leq p$  and  $\Gamma$  does not meet  $\Sigma_p$ . Hence  $x$  is not contained in  $\bigcap_{a \leq p} p$ . This completes the proof.

**Corollary 1.** *Let  $L$  be a compactly generated cm-lattice with the condition  $(*)$ , and  $I$  the greatest element of  $L$ . If the ascending chain condition holds for the elements of the interval  $I/a$ , the radical of  $a$  has a unique irredundant meet decomposition into completely prime elements. In particular,  $a$  is so if it is radical.*

**Lemma 3.** *A completely prime element of any cm-lattice is completely irreducible.<sup>9)</sup>*

Proof. Let  $p$  be a completely prime element of any cm-lattice. If  $p = \inf Q$ , then  $p \geq \inf \bar{Q}$ . Hence there exists an element  $q$  in  $Q$  such that  $q \leq p$ . If  $q < p$ , then  $p = \inf Q \leq q < p$ . This is a contradiction.

**Corollary 2.** *Let  $D$  be an infinitely meet distributive lattice with a compact generator. Then every element of  $D$ , which is different from the greatest element, is decomposed into completely irreducible elements of  $D$ .*

Proof. Let  $a$  be any element of  $D$ , and let  $\Sigma$  be any compact generator of  $D$ . Then the set  $\Gamma = \{x\}$  consisting of the single element  $x$  of  $\Sigma_{r(a)}$ , is a  $c$ -system satisfying  $\inf \bar{\Gamma} = x$ . Hence  $a$  is radical. By Theorem 1 and Lemma 3, we complete the proof.

#### 4. Completely minimal primes

Let  $\Gamma_0$  be any fixed  $c$ -system of  $L$ , and let  $a$  be an element of  $L$  such that  $\Sigma_a$  does not meet  $\Gamma_0$ . Then it is easily verified that the family of  $c$ -systems  $\Gamma$  each of which contains  $\Gamma_0$  and does not meet  $\Sigma_a$  is inductive. Zorn's lemma assures therefore the existence of maximal  $c$ -systems. Let  $\mathfrak{M} = \{\Gamma_\lambda\}$  be the family of the maximal  $c$ -systems based on  $\Gamma_0$ . Then we can see that  $p_\lambda = \sup(\Sigma \setminus \Gamma_\lambda)$  is completely prime, and the completely prime element  $p$  satisfying  $a \leq p$  and  $\Gamma_\lambda \wedge \Sigma_p = \emptyset$  coincides with  $p_\lambda$ . Moreover

$$\begin{aligned}\Gamma_\lambda &\rightarrow p_\lambda = \sup(\Sigma \setminus \Gamma_\lambda), \\ p_\lambda &\rightarrow \Gamma_\lambda = \Sigma \setminus \Sigma_{p_\lambda}\end{aligned}$$

give a one-to-one correspondence between  $\mathfrak{M}$  and the set of the completely primes  $\{p\}$  which satisfy  $a \leq p$  and  $\Gamma_\lambda \wedge \Sigma_p = \emptyset$ .

9) Cf. [2; p. 3].

**DEFINITION 4.** An completely prime element  $p$  of a *cm*-lattice is said to be *completely minimal prime belonging to a*, if it satisfies (1)  $p \geq a$  and (2) there exists no completely prime  $p'$  such that  $a \leq p' < p$ .

**Lemma 4.** *In order that an element  $p$  of  $L$  is a completely minimal prime belonging to  $a$ , it is necessary and sufficient that  $\Sigma \setminus \Sigma_p$  is a maximal *c*-system which does not meet  $\Sigma_a$ .*

Proof. We suppose that  $p$  is completely minimal prime, and  $a$  is not completely prime. Then we can assume that  $a < p$ . Now, by Zorn's lemma, there exists a maximal *c*-system  $\Gamma$  such that it does not meet  $\Sigma_a$ . Then we have that  $p_0 = \sup(\Sigma \setminus \Gamma) \leq \sup(\Sigma \setminus (\Sigma \setminus \Sigma_p)) = p$ . Since  $p_0$  is completely prime, we obtain  $p = p_0$ . This implies that  $\Sigma \setminus \Gamma = \Sigma_p$ . Therefore  $\Gamma = \Sigma \setminus \Sigma_p$  is a maximal *c*-system which does not meet  $\Sigma_a$ . The converse is easy to see.

**Theorem 2.** *Let  $L$  be a compactly generated *cm*-lattice with the condition (\*). Then the radical of any element  $a$  of  $L$  is decomposed into the meet of the completely minimal primes belonging to  $a$ . In particular, so is any radical element of  $L$ .*

Proof. Let  $\{p_\lambda\}$  be the completely minimal primes belonging to  $a$ . In order to prove that  $r(a) \geq \bigcap_\lambda p_\lambda$ , it is sufficient to show that  $x \not\leq r(a)$  implies  $x \not\leq \bigcap_\lambda p_\lambda$ , where  $x \in \Sigma$ . Now by the definition of  $r(a)$ , we can take a *c*-system  $\Gamma$  such that  $x \in \Gamma$  and it does not meet  $\Sigma_a$ . Hence there exists a completely minimal prime  $p$  belonging to  $a$  such that  $\Sigma_p$  does not meet  $\Gamma$ . Evidently  $x \not\leq p$ . We have therefore  $x \not\leq \bigcap_\lambda p_\lambda$ . The converse inclusion is evident.

**REMARK 1.** Suppose that  $p$  is any completely prime containing  $a$ . Then there exists a completely minimal prime belonging to  $a$  which is contained in  $p$ . Because,  $\Sigma \setminus \Sigma_p$  is a *c*-system which does not meet  $\Sigma_a$ ; and we can take a maximal *c*-system which contains  $\Sigma \setminus \Sigma_p$  and does not meet  $\Sigma_a$ .

**REMARK 2.** If the ascending chain condition holds for elements in the interval  $I/a$ , then the meet-decomposition mentioned in Corollary 1 to Theorem 1 is the irredundant meet-decomposition into the completely minimal primes belonging to  $a$ .

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