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<th>Decomposition of radical elements of a compactly generated cm-lattice</th>
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The aim of the present paper is to define a completely prime element and a new type of radicals of elements in some compactly generated cm-lattice, and to obtain meet-decompositions of radical elements of the cm-lattice.

1. Preliminaries

Let \( L \) be a complete (upper and lower) lattice. A non-void subset \( \Sigma \) of \( L \) is called a compact set of \( L \) if, whenever \( x \leq \sup N \) for an element \( x \) of \( \Sigma \) and a subset \( N \) of \( \Sigma \), there exists a finite number of elements \( x_1, \ldots, x_n \) of \( N \) satisfying

\[
x_1 \cup \cdots \cup x_n \geq x.
\]

Every element of \( \Sigma \) is said to be compact. If every element of \( L \) is a join of a finite or infinite number of elements of the compact set \( \Sigma \) of \( L \), then \( L \) is said to be compactly generated, and \( \Sigma \) is called a compact generator of \( L \).

We can prove easily that a complete lattice has a compact generator if and only if it is compactly generated in the sense of Dilworth-Crawley. In this case, the set of the compact elements of the lattice is the unique maximal compact generator under the set-inclusion.

Now we shall consider, throughout this paper, a cm-lattice \( L \) which is compactly generated as a lattice. In what follows, we suppose that \( L \) has a compact generator \( \Sigma \) which satisfies the following condition.

\((*)\) If \( \inf S \leq a \) for a subset \( S \) of \( \Sigma \) and an element \( a \) of \( L \), then \( a = \inf (S \cup a) \), where \( S \cup a \) means the set of the elements \( s \cup a \) (\( s \in S \)).

It is easily verified that the condition \((*)\) holds for any infinitely meet-
distributive *cm*-lattice with a compact generator. But, there exists a *cm*-lattice which is not infinitely meet-distributive, and satisfies the condition (\*). Let \( \mathfrak{L} \) be the *cm*-lattice of all ideals of any Dedekind domain \( \mathfrak{O} \), and \( \Sigma \) the compact generator consisting of the principal ideals of \( \mathfrak{O} \). Then, of course, \( \mathfrak{L} \) is not necessarily infinitely meet-distributive. But, it can be proved easily that the condition (\*) holds for \( \mathfrak{L} \) and \( \Sigma \).

Let \( \Sigma \) be any fixed compact generator of \( L \) and let \( a \in L \). By the symbol \( \Sigma_a \) we shall mean the set \( \{ x \in \Sigma | x \leq a \} \). Then we can prove that \( \sup A = \sup (\bigvee_{a \in A} \Sigma_a) \) for every non-void subset \( A \) of \( L \), where \( \bigvee \) denotes the set-theoretical union.

### 2. Completely prime elements

For a non-void subset \( A \) of a *cm*-lattice \( L \), \( \bar{A} \) will denote the multiplicative system (monoid) which is generated by \( A \) under the multiplication of \( L \).

**Definition 1.** An element \( p \) of a *cm*-lattice is said to be **completely prime** if whenever \( \inf A \) is contained in \( p \), then at least one of the element of \( A \) is contained in \( p \).

We can prove easily that completely primes are primes. But the converse is not true. In fact, we can find an example of *cm*-lattices with an element which is prime but not completely prime.

**Definition 2.** A non-void subset \( \Gamma \) of the compact set \( \Sigma \) of \( L \) is called a **c-system**, if \( \inf \Gamma \) is a member of \( \Gamma \). The void set is a c-system.

**Lemma 1.** The following conditions are equivalent to one another.

1. \( p \) is completely prime.
2. If \( \inf \Delta \leq p \) \( (\phi^6 = \Delta \subseteq \Sigma) \), then there exists an element \( x \in \Delta \) such that \( x \leq p \).
3. \( \Sigma \setminus \Sigma_p \) is a c-system.

**Proof.** (1)\( \Rightarrow \) (2): is evident. (2)\( \Rightarrow \) (1): Suppose that \( p \) is not completely prime. Then, there exists a subset \( A \) of \( L \) such that \( \inf \bar{A} \leq p \) and \( a \leq p \) for every \( a \in A \). Therefore we can take an element \( x_a \in \Sigma_a \) such as \( x_a \leq p \). Put \( \Delta = \{ x_a | a \in A \} \). Then it is easy to see that \( \inf \Delta \leq p \). (1)\( \Rightarrow \) (3): Suppose that \( p \) is completely prime. It is then easily verified that \( p \) does not contain \( \inf \Sigma \setminus \Sigma_p \). Hence we can take an element \( u \) of \( \Sigma \setminus \Sigma_p \) such that \( u \leq \inf \Sigma \setminus \Sigma_p \). (3)\( \Rightarrow \) (2): Let \( \Sigma \setminus \Sigma_p \) be a c-system, and \( \Gamma \) a non-void subset of \( \Sigma \). If we suppose that \( p \) con-

5) = the dual of a relatively pseudo-complemented *cm*-lattice.
6) \( \phi \) will mean the void set.
7) \( \setminus \) will denote the set-difference.
tains no element of $\Gamma$, then $\Gamma \subseteq \Sigma \setminus \Sigma_p$, $\overline{\Gamma} \subseteq \overline{\Sigma \setminus \Sigma_p}$. Hence we have $\inf \overline{\Sigma \setminus \Sigma_p} \leq \inf \overline{\Gamma}$. On the other hand, $\inf \overline{\Sigma \setminus \Sigma_p}$ is not contained in $p$, since $\Sigma \setminus \Sigma_p$ is a $c$-system.

Lemma 2. Let $\Gamma$ be a $c$-system such that $\Gamma \wedge \Sigma_a = \phi$ for an element $a$ of $L$. Then there exists a completely prime element $p$ such that $p \geq a$ and $\Gamma \wedge \Sigma_p = \phi$.

Proof. First we show that the set $S = \{c \in L | a \leq c, \Gamma \wedge \Sigma_c = \phi\}$ is inductive. Let $\{c_\lambda\}$ be any chain in $S$, and let $c^* = \sup \{c_\lambda\}$. If we suppose that $\Gamma \wedge \Sigma_{c^*}$ contains an element $x$, then $x \leq c^* = \sup (\bigvee \Sigma_{c_\lambda})$. Hence there exists a finite number of elements $x_1, \ldots, x_n$ such that $x \leq x_1 \cup \cdots \cup x_n$ and $x_i \in \bigvee \Sigma_{c_\lambda}$. Since there exists $c_m$ such that $x_i \leq c_m (i = 1, \ldots, n)$, we have that $\Gamma \wedge \Sigma_{c_m} \ni x$, which is a contradiction. Zorn's lemma assures therefore the existence of a maximal element $p$ in $S$. We now prove that $p$ is completely prime. Let $\Delta$ be a non-void subset of $\Sigma$ such that $\inf \Delta \leq p$. If we suppose that $p$ contains no element of $\Delta$, then since $p < p \cup u$ for every $u \in \Delta$, we can find an element $v(u)$ (depending on $u$) of $\Sigma$ such that $v(u) \in \Gamma \wedge \Sigma_{p \cup u}$. Now we put $V = \{v(u) | u \in \Delta\}$. Then $V$ is contained in $\overline{\Gamma}$. We have therefore that $\inf V \geq \inf \overline{\Gamma}$. Now let $u^* = \Psi(u_1, \ldots, u_n)$ be an arbitrary element of $\overline{\Delta}$, where $u_i \in \Delta (i = 1, \ldots, n)$, and $\Psi$ denotes a product-form (product-polynomial) of $u_1, \ldots, u_n$. Then we have that $\Psi(v(u_1), \ldots, v(u_n)) \leq \Psi(p \cup u_1, \ldots, p \cup u_n) \leq p \cup \Psi(u_1, \ldots, u_n) = p \cup u^*$. Since $\Psi(v(u_1), \ldots, v(u_n)) \in V$, we obtain that $\inf V \leq \inf \overline{\Delta} \leq \inf \{p \cup u^* | u^* \in \Delta\} = p \cup \inf \overline{\Delta} = p$. Hence we have that $\inf \overline{\Gamma} \leq p$. Therefore we obtain that $\inf \overline{\Gamma} \leq p$. Consequently, we obtain that $\inf \overline{\Gamma} = p$. Hence we have that $\inf \Gamma \subseteq p$. Therefore we obtain that $\inf \Gamma \subseteq p$. This is a contradiction.

3. Radicals of elements

Let $a$ be an arbitrary element of $L$, and let $X_a$ be the set of the elements $x$ of $\Sigma$ such that every $c$-system containing $x$ contains an element of $\Sigma_a$. We now put

Definition 3. The suplemum of $X_a$ is called a radical of $a$, and is denoted by $r(a)$. An element $a$ of $L$ is said to be radical if $r(a) = a$.

Theorem 1. Let $L$ be a compactly generated cm-lattice with the condition $(\ast)$. Then the radical of any element $a$ of $L$ is decomposed into the meet of the completely prime elements containing $a$. In particular, so is any radical element of $L$.

Proof. Let $p$ be any completely prime element containing $a$. Then $r(a) \leq p$. For, if contrary, the $c$-system $\Sigma \setminus \Sigma_p$ contains an element $x \in \Sigma$ such that

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8) $\wedge$ will denote the intersection. We say, following McCoy [4], that $\Gamma$ does not meet $\Sigma_a$, if $\Gamma \wedge \Sigma_a = \phi$. 

\(x \leq r(a)\). Hence \((\Sigma \setminus \Sigma_p) \cap \Sigma_a\) is not vacuous. Since \(\Sigma_a\) is contained in \(\Sigma_p\), this is a contradiction. Therefore we obtain that \(r(a) \leq p\), \(r(a) \leq \cap_{a \leq p} p\). For the proof of the converse inclusion, it is sufficient to show that \(x \leq r(a)\) \((x \in \Sigma)\) implies \(x \leq \cap_{a \leq p} p\). Since \(x \leq r(a)\), there exists a \(c\)-system \(\Gamma\) such that \(\Gamma \ni x\) and it does not meet \(\Sigma_a\). Then, by Lemma 2, we can take a completely prime element \(p\) satisfying \(a \leq p\) and \(\Gamma\) does not meet \(\Sigma_p\). Hence \(x\) is not contained in \(\cap_{a \leq p} p\). This completes the proof.

**Corollary 1.** Let \(L\) be a compactly generated cm-lattice with the condition \((*)\), and \(I\) the greatest element of \(L\). If the ascending chain condition holds for the elements of the interval \(I[a]\), the radical of \(a\) has a unique irredundant meet decomposition into completely prime elements. In particular, \(a\) is so if it is radical.

** Lemma 3.** A completely prime element of any cm-lattice is completely irreducible. \(^9\)

Proof. Let \(p\) be a completely prime element of any cm-lattice. If \(p = \inf Q\), then \(p \geq \inf \overline{Q}\). Hence there exists an element \(q\) in \(Q\) such that \(q \leq p\). If \(q < p\), then \(p = \inf Q \leq q < p\). This is a contradiction.

** Corollary 2.** Let \(D\) be an infinitely meet distributive lattice with a compact generator. Then every element of \(D\), which is different from the greatest element, is decomposed into completely irreducible elements of \(D\).

Proof. Let \(a\) be any element of \(D\), and let \(\Sigma\) be any compact generator of \(D\). Then the set \(\Gamma = \{x\}\) consisting of the single element \(x\) of \(\Sigma_{r(a)}\), is a \(c\)-system satisfying \(\inf \overline{\Gamma} = x\). Hence \(a\) is radical. By Theorem 1 and Lemma 3, we complete the proof.

### 4. Completely minimal primes

Let \(\Gamma_0\) be any fixed \(c\)-system of \(L\), and let \(a\) be an element of \(L\) such that \(\Sigma_a\) does not meet \(\Gamma_0\). Then it is easily verified that the family of \(c\)-systems \(\Gamma\) each of which contains \(\Gamma_0\) and does not meet \(\Sigma_a\) is inductive. Zorn’s lemma assures therefore the existence of maximal \(c\)-systems. Let \(\mathcal{M} = \{\Gamma_\lambda\}\) be the family of the maximal \(c\)-systems based on \(\Gamma_0\). Then we can see that \(p_\lambda = \sup (\Sigma \setminus \Gamma_\lambda)\) is completely prime, and the completely prime element \(p\) satisfying \(a \leq p\) and \(\Gamma_\lambda \cap \Sigma_p = \emptyset\) coincides with \(p_\lambda\). Moreover

\[
\Gamma_\lambda \rightarrow p_\lambda = \sup (\Sigma \setminus \Gamma_\lambda),
\]

\[
p_\lambda \rightarrow \Gamma_\lambda = \Sigma \setminus \Sigma_{p_\lambda}
\]
give a one-to-one correspondence between \(\mathcal{M}\) and the set of the completely primes \(\{p\}\) which satisfy \(a \leq p\) and \(\Gamma_\lambda \cap \Sigma_p = \emptyset\).

\(^9\) Cf. [2; p. 3].
DEFINITION 4. An completely prime element $p$ of a $cm$-lattice is said to be completely minimal prime belonging to $a$, if it satisfies (1) $p \geq a$ and (2) there exists no completely prime $p'$ such that $a \leq p' < p$.

Lemma 4. In order that an element $p$ of $L$ is a completely minimal prime belonging to $a$, it is necessary and sufficient that $\Sigma \setminus \Sigma_p$ is a maximal $c$-system which does not meet $\Sigma_a$.

Proof. We suppose that $p$ is completely minimal prime, and $a$ is not completely prime. Then we can assume that $a < p$. Now, by Zorn's lemma, there exists a maximal $c$-system $\Gamma$ such that it does not meet $\Sigma_a$. Then we have that $p_0 = \sup (\Sigma \setminus \Gamma) \leq \sup (\Sigma \setminus (\Sigma \setminus \Sigma_p)) = p$. Since $p_0$ is completely prime, we obtain $p = p_0$. This implies that $\Sigma \setminus \Gamma = \Sigma_p$. Therefore $\Gamma = \Sigma \setminus \Sigma_p$ is a maximal $c$-system which does not meet $\Sigma_a$. The converse is easy to see.

Theorem 2. Let $L$ be a compactly generated $cm$-lattice with the condition $(\ast)$. Then the radical of any element $a$ of $L$ is decomposed into the meet of the completely minimal primes belonging to $a$. In particular, so is any radical element of $L$.

Proof. Let $\{p_\lambda\}$ be the completely minimal primes belonging to $a$. In order to prove that $r(a) > \cap \lambda p_\lambda$, it is sufficient to show that $x \leq r(a)$ implies $x \not\leq \cap \lambda p_\lambda$, where $x \in \Sigma$. Now by the definition of $r(a)$, we can take a $c$-system $\Gamma$ such that $x \in \Gamma$ and it does not meet $\Sigma_a$. Hence there exists a completely minimal prime $p$ belonging to $a$ such that $\Sigma_p$ does not meet $\Gamma$. Evidently $x \not\leq p$. We have therefore $x \not\leq \cap \lambda p_\lambda$. The converse inclusion is evident.

Remark 1. Suppose that $p$ is any completely prime containing $a$. Then there exists a completely minimal prime belonging to $a$ which is contained in $p$. Because, $\Sigma \setminus \Sigma_p$ is a $c$-system which does not meet $\Sigma_a$; and we can take a maximal $c$-system which contains $\Sigma \setminus \Sigma_p$ and does not meet $\Sigma_a$.

Remark 2. If the ascending chain condition holds for elements in the interval $I/a$, then the meet-decomposition mentioned in Corollary 1 to Theorem 1 is the irredundant meet-decomposition into the completely minimal primes belonging to $a$.

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References