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Title	Generalized Alexander duality and applications
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Citation	Osaka Journal of Mathematics. 38(2) P.469-P.485
Issue Date	2001-06
Text Version	publisher
URL	https://doi.org/10.18910/4757
DOI	10.18910/4757
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Osaka J. Math.
38 (2001), 469–485

GENERALIZED ALEXANDER DUALITY AND APPLICATIONS

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(Received April 22, 1999)

Introduction

Let Δ be a simplicial complex on the vertex set $[n]$, K a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring and $K[\Delta]$ the Stanley-Reisner ring over S. In a series of papers ([4], [5], [7]) relations between Betti numbers of $K[\Delta]$ and those of the Stanley-Reisner ring $K[\Delta^*]$ of the Alexander dual Δ^* have been studied.

In this paper we extend these results to squarefree S-modules, which were introduced by Yanagawa in [6]. This will be accomplished by defining the dual of a squarefree S-module. The definition is a natural extension of the Alexander dual.

To define the generalized Alexander dual we will see that there is an equivalence of the categories of squarefree S-modules and squarefree E-modules, where E denotes the exterior algebra. In the category of squarefree E-modules we may consider the E-dual $M^* = \mathrm{Hom}_E(M, E)$. If M is the squarefree E-module corresponding to a squarefree S-module N, then we call the squarefree S-module corresponding to M^* the generalized Alexander dual of N. The construction which assigns to a squarefree S-module a squarefree E-module is described in [1].

For the applications it is important to consider the so called distinguished pairs (l, j) introduced by Aramova and Herzog in [3]. Distinguished pairs are homological invariants of modules over the exterior algebra. The definition is based on the Cartan homology, an analogue to the Koszul homology in the polynomial ring. We generalize this definition to homological distinguished pairs (l, j) and cohomological distinguished pairs (l, j).

We prove that a homological distinguished pair (l, j) of M corresponds to the cohomological distinguished pair $(l, n-j)$ of M^*, which in turn corresponds to the homological distinguished pair $(l, n-j-l+1)$ of M^*. These homological considerations lead to the following results about graded Betti numbers:

Let $\beta_{i,i+j}$ be the graded Betti number of a finitely generated graded S-module. Bayer, Charalambous and S. Popescu introduced in [4] a refinement of the Mumford-Castelnuovo regularity, the extremal Betti numbers. They call a Betti number $\beta_{i,i+j} \neq 0$ extremal if $\beta_{l,l+r} = 0$ for all $r \geq j$ and all $l \geq i$ with $(l, r) \neq (i, j)$. One of their results states the following: if $\beta_{i,i+j}(K[\Delta])$ is an extremal Betti number of $K[\Delta]$, then $\beta_{j+1,i+j}(K[\Delta^*])$ is an extremal number of $K[\Delta^*]$ and $\beta_{i,i+j}(K[\Delta]) = \beta_{j+1,i+j}(K[\Delta^*])$.

In this paper we will prove a similar result for any squarefree S-module N: if $\beta_{i,i+j}(N)$ is an extremal Betti number of N, then $\beta_{j,i+j}(N^*)$ is an extremal Betti number of N^* and $\beta_{i,i+j}(N) = \beta_{j,i+j}(N^*)$. In particular for any pair of squarefree ideals $I_\Delta \subseteq I_\Gamma$ one has: if $\beta_{i,i+j}(I_\Gamma/I_\Delta)$ is an extremal number of I_Γ/I_Δ, then $\beta_{j,i+j}(I_{\Delta^*}/I_{\Gamma^*})$ is an extremal number of $I_{\Delta^*}/I_{\Gamma^*}$ and $\beta_{i,i+j}(I_\Gamma/I_\Delta) = \beta_{j,i+j}(I_{\Delta^*}/I_{\Gamma^*})$.

The author is grateful to Prof. Herzog for the inspiring discussions on the subject of the paper.

1. Squarefree Modules and generalized Alexander Duality

We fix some notation and recall some definitions. For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, we say a is *squarefree* if $0 \le a_i \le 1$ for $i = 1, \dots, n$. We set $|a| = a_1 + \dots + a_n$ and $\operatorname{supp}(a) = \{i : a_i \neq 0\} \subseteq [n] := \{1, \dots, n\}$. Sometimes a squarefree vector a and $F = \operatorname{supp}(a)$ are identified. Let $\varepsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$ be the vector, where the one is at the ith position. For an element u of an $\mathbb{N}^n$-graded vector space $M = \bigoplus_{a\in\mathbb{N}^n} M_a$ the notation $\deg(u) = a$ is equivalent to $u \in M_a$; we set $\operatorname{supp}(\deg(u)) = \operatorname{supp}(u)$ and $|\deg(u)| = |u|$.

Let $S = K[x_1, \dots, x_n]$ be the symmetric algebra over a field K and $\mathfrak{m}$ the graded maximal ideal $(x_1, \dots, x_n)$ of S. Consider the natural $\mathbb{N}^n$-grading on S. For a monomial $x_1^{a_1} \dots x_n^{a_n}$ with $a = (a_1 \dots, a_n)$ we set x^a.

Let $E = K\langle e_1, \dots, e_n\rangle$ be the exterior algebra over an n-dimensional vector space V with basis $e_1, \dots, e_n$. We denote by $\mathcal{M}$ the category of finitely generated graded left and right E-modules M, satisfying $ax = (-1)^{|\deg(a)||\deg(x)|}xa$ for all homogeneous $a \in E$ and $x \in M$. For example every graded ideal $J \subseteq E$ belongs to $\mathcal{M}$. For an element $M \in \mathcal{M}$ we define $M^* = \operatorname{Hom}_E(M, E)$. Observe that $(\ )^*$ is an exact contravariant functor [2, 5.1 (a)] and $M^* \in \mathcal{M}$. For a K-vector space W we define $W^\vee := \operatorname{Hom}_K(W, K)$. The following was proved in [2, 5.1 (d)]: $(M^*)_i \cong (M_{n-i})^\vee$.

If $a \in \mathbb{N}^n$ is squarefree we set $e_a = e_{a_{j_1}} \wedge \dots \wedge e_{a_{j_k}}$, where $\operatorname{supp}(a) = \{j_1 < \dots < j_k\}$ and we say e_a is a monomial in E. For any $a \in \mathbb{N}^n$ we set $e_a = e_{\operatorname{supp}(a)}$. For monomials $u, v \in E$ with $\operatorname{supp}(v) \subseteq \operatorname{supp}(u)$ there exists an unique monomial $w \in E$ such that $vw = u$; then we set $w = v^{-1}u$. Notice that for monomials $u, v, w, z \in E$ the equalities below hold whenever the expressions are defined:

$$(v^{-1}u)w = v^{-1}(uw) \qquad \text{and} \qquad (z^{-1}v)(v^{-1}u) = z^{-1}u.$$

A simplicial complex Δ is a collection of subsets of $[n]$ such that $\{i\} \in \Delta$ for $i = 1, \dots, n$, and that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. Further we denote by $\Delta^* := \{F : \overline{F} \notin \Delta\}$ the Alexander dual of Δ. Then $K[\Delta] = S/I_\Delta$ is the Stanley-Reisner ring, where $I_\Delta = (x_{i_1} \dots x_{i_s} : \{i_1, \dots, i_s\} \notin \Delta)$, and $K\{\Delta\} := E/J_\Delta$ is the exterior face ring, where $J_\Delta := (e_{i_1} \wedge \dots \wedge e_{i_s} : \{i_1, \dots, i_s\} \notin \Delta)$.

The starting point of this section is a definition introduced by Yanagawa in [6].

DEFINITION 1.1. A finitely generated $\mathbb{N}^n$-graded S-module $N = \bigoplus_{a\in\mathbb{N}^n} N_a$ is *squarefree* if the multiplication maps $N_a \ni y \mapsto x_i y \in N_{a+\varepsilon_i}$ is bijective for all $a \in \mathbb{N}^n$ and all $i \in \operatorname{supp}(a)$.

For example the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex Δ is a squarefree module. It is easy to see that for $a \in \mathbb{N}^n$ and a squarefree module N we have $\dim_K N_a = \dim_K N_{\operatorname{supp}(a)}$ and N is generated by its squarefree part $\{N_a : a \subseteq [n]\}$. Yanagawa proved in [6, 2.3, 2.4] that if $\varphi : N \to N'$ is a $\mathbb{N}^n$-homogeneous homomorphism, where N, N' are squarefree modules, $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are again squarefree. It follows that every syzygy module $\operatorname{Syz}_i(N)$ in a multigraded minimal free S-resolution $F_\bullet$ of N is squarefree. Indeed the free S-module F_i is generated by elements f with $\deg(f)$ is squarefree and this is called a squarefree resolution. It follows that an S-module N is squarefree if and only if N has a squarefree resolution.

The following construction which is of crucial importance for this paper is due to Aramova, Avramov and Herzog [1]:
Let $(F_\bullet, \theta)$ be an acyclic complex of free $\mathbb{N}^n$-graded S-modules. Furthermore we assume that each F_i has a homogeneous basis B_i with $\deg(f)$ is squarefree for all $f \in B_i$.

For $a \in \mathbb{N}^n$ and $f \in B_i$ we let $y^{(a)} f$ be a symbol to which we assign $\deg(y^{(a)} f) = a + \deg(f)$. Now define the free $\mathbb{N}^n$-graded E-module $G_l \in \mathcal{M}$ with basis $y^{(a)} f$, where $a \in \mathbb{N}^n$, $f \in B_i$, $\operatorname{supp}(a) \subseteq \operatorname{supp}(f)$ and $l = |a| + i$. For $f \in B_i$ and

$$\theta(f) = \sum_{j: f_j \in B_{i-1}} \lambda_j x^{b-b_j} f_j \quad \text{with } \lambda_j \in K, \quad b = \deg(f), \quad b_j = \deg(f_j),$$

we define homomorphisms $G_l \to G_{l-1}$ of $\mathbb{N}^n$-graded E-modules by

$$\begin{aligned} \gamma(y^{(a)} f) &= (-1)^{|b|} \sum_{k\in\operatorname{supp}(a)} y^{(a-\varepsilon_k)} f e_k, \\ \vartheta(y^{(a)} f) &= (-1)^{|a|} \sum_{j: f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b. \end{aligned}$$

Set $\delta = \gamma + \vartheta : G_l \to G_{l-1}$, then $(G_\bullet, \delta)$ is a complex of free $\mathbb{N}^n$-graded E-modules in $\mathcal{M}$. Indeed if $(G'_\bullet, \delta)$ is the complex obtained by a different homogeneous basis B' of $F_\bullet$, then $G_\bullet$ and $G'_\bullet$ are isomorph as complexes of $\mathbb{N}^n$-graded modules. Now there is the important theorem [1, 1.3]

Theorem 1.2. *If $(F_\bullet, \theta)$ is the minimal free $\mathbb{N}^n$-graded S-resolution of a squarefree S-module N, then $(G_\bullet, \delta)$ is the minimal free $\mathbb{N}^n$-graded E-resolution of $N_E :=$* $\operatorname{Coker}(G_1 \to G_0)$.

Proof. See the proof of [1, 1.3]. There the theorem was proved for S/I, where I is a squarefree monomial ideal, but the proof works also in this more generalized situation. □

N and N_E can also be seen as $\mathbb{N}$-graded modules over S and E by defining $N_i = \bigoplus_{a:|a|=i} N_a$ and the same for N_E. By [1, 2.1] we get the following

Corollary 1.3. *Let N be a squarefree S-module, then*

$$\beta^E_{i,i+j}(N_E) = \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N),$$

where β^E denotes graded Betti-numbers over the exterior algebra and β^S graded Betti-numbers over the polynomial ring.

It is a natural question if this construction has an inverse. This means, given an E-module with its minimal free E-resolution we want to construct an S-module with a free S-resolution.

DEFINITION 1.4. A finitely generated $\mathbb{N}^n$-graded E-module $M = \bigoplus_{a\in\mathbb{N}^n} M_a$ is *square free* if it has only squarefree components.

For example the exterior face ring $K\{\Delta\}$ obtained from a simplicial complex Δ is a squarefree E-module. Also for any squarefree S-module N, the E-module N_E is squarefree as can be easily seen from its definition.

Now we consider the following inverse construction:
Let M be a squarefree E-module with the minimal free $\mathbb{N}^n$-graded E-resolution $(G_\bullet, \delta)$. Let B_i a homogeneous basis of G_i. Then we set $\tilde{B}_i := \{f \in B_i \mid \deg(f)$ is squarefree$\}$. We define a complex $(F_\bullet, \theta)$ of S-modules, where F_i is a free S-module with basis $\tilde{B}_i$. If $f \in \tilde{B}_i$ and

$$\delta(f) = \sum_{j:f_j\in\tilde{B}_{i-1}} f_j \lambda_j e_{b_j}^{-1} e_b \quad \text{with } b = \deg(f),\ b_j = \deg(f_j) \text{ and } \lambda_j \in K,$$

then we set

$$\theta(f) = \sum_{j:f_j\in\tilde{B}_{i-1}} f_j \lambda_j x^{b-b_j}.$$

It is easy to see that $(F_\bullet, \theta)$ is indeed a complex. But a little more is true:

Theorem 1.5. *If $(G_\bullet, \delta)$ is the minimal free $\mathbb{N}^n$-graded E-resolution of a squarefree E-module M, then $(F_\bullet, \theta)$ is the minimal free $\mathbb{N}^n$-graded S-resolution of $M_S :=$ $\mathrm{Coker}(F_1 \to F_0)$ and M_S is a squarefree S-module.*

Proof. Let $(\tilde{F}_\bullet, \tilde{\theta})$ be the minimal free $\mathbb{N}^n$-graded S-resolution of the S-module M_S. By the first construction (see 1.2) we get a minimal free $\mathbb{N}^n$-graded E-resolution $(\tilde{G}_\bullet, \tilde{\delta})$ of the E-module $(M_S)_E$. The definitions imply that $(M_S)_E = M$. Therefore $\tilde{G}_\bullet \cong G_\bullet$ as complexes, since both complexes are minimal free E-resolutions of M. If we apply the second construction for $(\tilde{G}_\bullet, \tilde{\delta})$ we get $(\tilde{F}_\bullet, \tilde{\theta})$. All in all it follows that $\tilde{F}_\bullet \cong F_\bullet$ as complexes and hence $(F_\bullet, \theta)$ is the minimal free $\mathbb{N}^n$-graded S-resolution of the S-module M_S. Since F_0, F_1 are squarefree S-modules, we see that M_S is a squarefree S-module. □

We get immediately

Corollary 1.6. *Let N be a squarefree S-module and M be a squarefree E-module. We denote by N_E the squarefree E-module defined in* 1.2 *and by M_S the squarefree S-module defined in* 1.5. *Then*

$$(M_S)_E \cong M \quad \textit{and} \quad (N_S)_E \cong N.$$

Now consider two squarefree S-modules N, N' and a $\mathbb{N}^n$-homogeneous homomorphism $\varphi : N \to N'$. Take the minimal free $\mathbb{N}^n$-graded S-resolution $(F_\bullet, \theta)$ of N and the minimal free $\mathbb{N}^n$-graded S-resolution $(F'_\bullet, \theta')$ of N' with homogeneous bases $B_\bullet$ and $B'_\bullet$. It is well known that φ induce a complex homomorphism $\varphi_\bullet : F_\bullet \to F'_\bullet$. By construction 1.2 we get minimal free $\mathbb{N}^n$-graded E-resolutions $(G_\bullet, \delta)$ and $(G'_\bullet, \delta')$ of N_E and N'_E, respectively. Let $f \in B_i$ and $\varphi_i(f) = \sum_{j:f'_j \in B'_i} \lambda_j x^{b-b'_j} f'_j$, where $b = \deg(f)$ and $b'_j = \deg(f')$. Then we define a complex homomorphism

$$\psi_\bullet : G_\bullet \longrightarrow G'_\bullet \quad y^{(a)} f \longmapsto \sum_{j:f'_j \in B'_i} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b,$$

for all $a \in \mathbb{N}^n$ and all $f \in B_i$. Now $\psi_\bullet$ induces a $\mathbb{N}^n$-homogeneous homomorphism $\psi : M_E \to M'_E$.

Similary two squarefree E-modules M, M' and a $\mathbb{N}^n$-homogeneous homomorphism $\psi : M \to M'$ induce a $\mathbb{N}^n$-homogeneous homomorphism $\varphi : M_S \to M'_S$.

It turns out that these assignments define functors. Denote by $\mathcal{SQ}(S)$ the abelian category of the squarefree S-modules, where the morphisms are the $\mathbb{N}^n$-homogeneous homomorphisms. Let $\mathcal{SQ}(E)$ be the abelian category of the squarefree E-modules, where the morphisms are again the $\mathbb{N}^n$-homogeneous homomorphisms. Then

$$F : \mathcal{SQ}(S) \longrightarrow \mathcal{SQ}(E), \quad N \longmapsto N_E$$

and

$$G : \mathcal{SQ}(E) \longrightarrow \mathcal{SQ}(S), \quad M \longmapsto M_S$$

are additive covariant exact functors of abelian categories. Hence we see by 1.6 that the categories $\mathcal{SQ}(S)$ and $\mathcal{SQ}(E)$ are equivalent.
We conclude the section with an example. Let $\Gamma \subseteq \Delta$ be simplicial complexes. Then I_Γ/I_Δ is an element of $\mathcal{SQ}(S)$ and J_Γ/J_Δ is an element of $\mathcal{SQ}(E)$.

Corollary 1.7. *With the notation introduced we have*

$$(I_\Gamma/I_\Delta)_E = J_\Gamma/J_\Delta \quad \textit{and} \quad (J_\Gamma/J_\Delta)_S = I_\Gamma/I_\Delta.$$

We further have

Lemma 1.8. $(J_\Gamma/J_\Delta)^* \cong J_{\Delta^*}/J_{\Gamma^*}$.

Proof. We see that $(E/J_\Delta)^* = \mathrm{Hom}_E(E/J_\Delta, E) \cong 0 :_E J_\Delta = J_{\Delta^*}$. Consider the exact sequence $0 \to J_\Gamma/J_\Delta \to E/J_\Delta \to E/J_\Gamma \to 0$. Since the functor $(\ )^*$ is exact we get the exact sequence $0 \to (E/J_\Gamma)^* \to (E/J_\Delta)^* \to (J_\Gamma/J_\Delta)^* \to 0$, and the assertion follows. $\square$

This lemma gives us the hint how to define the generalized Alexander dual for elements in $\mathcal{SQ}(S)$.

Definition 1.9. Let $N \in \mathcal{SQ}(S)$. Then we call

$$N^* = ((N_E)^*)_S$$

the *generalized Alexander dual* of N.

We note that

$$(\ )^* : \mathcal{SQ}(S) \longrightarrow \mathcal{SQ}(S), \quad N \longmapsto N^*,$$

is a contravariant exact functor on the category $\mathcal{SQ}(S)$.

2. The Cartan complex

In this section we recall the *Cartan complex* which provides a minimal free graded E-resolution of the residue class field K of the exterior algebra E.

For a sequence $\mathbf{v} = v_1, \dots, v_m \subseteq E_1$ the Cartan complex $C_\bullet(\mathbf{v}; E)$ is defined to be the free divided power algebra $E\langle x_1, \dots, x_m\rangle$ together with a differential δ. The free divided power algebra $E\langle x_1, \dots, x_m\rangle$ is generated over E by the divided powers $x_i^{(j)}$ for $i = 1, \dots, m$ and $j \geq 0$, satisfying the relations $x_i^{(j)} x_i^{(k)} = ((j+k)!/(j!k!))x_i^{(j+k)}$. We set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \dots, m$. Therefore $C_\bullet(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} \dots x_m^{(a_m)}$, $a \in \mathbb{N}^m$. We set $\deg x^{(a)} = i$ if $|a| = a_1 + \dots + a_m = i$ and

$C_i(\mathbf{v}; E) = \bigoplus_{|a|=i} Ex^{(a)}$. The E-linear differential δ on $C_\bullet(\mathbf{v}; E)$ is defined as follows: for $x^{(a)} = x_1^{(a_1)} \ldots x_m^{(a_m)}$ we set

$$\delta(x^{(a)}) = \sum_{a_i > 0} v_i x_1^{(a_1)} \ldots x_i^{(a_i - 1)} \ldots x_m^{(a_m)}.$$

Now one sees easily that $\delta \circ \delta = 0$ and $C_\bullet(\mathbf{v}; E)$ is indeed a complex.

DEFINITION 2.1. Let $M \in \mathcal{M}$, $\mathbf{v} = v_1, \ldots, v_m \subseteq E_1$. The complexes

$$C_\bullet(\mathbf{v}; M) = C_\bullet(\mathbf{v}; E) \otimes_E M, \quad C^\bullet(\mathbf{v}; M) = \operatorname{Hom}_E(C_\bullet(\mathbf{v}; E), M)$$

are called *Cartan complex* and *Cartan cocomplex* of $\mathbf{v}$ with values in M. We denote the homology of these complexes by

$$H_\bullet(\mathbf{v}; M), \quad H^\bullet(\mathbf{v}; M)$$

and call it the *Cartan homology* and *Cartan cohomology*.

One can see that the elements of $C^i(\mathbf{v}; M)$ can be identified with homogeneous polynomials $\sum m_a y^a$ in the variables $y_1, \ldots, y_m$ and coefficients $m_a \in M$, where $y^a = y^{a_1} \ldots y^{a_m}$ for $a \in \mathbb{N}^m$, $a = (a_1, \ldots, a_m)$. An element $m_a y^a \in C^\bullet(\mathbf{v}; M)$ is characterized by the following property

$$m_a y^a(x^{(b)}) = \begin{cases} m_a & b = a, \\ 0 & b \neq a. \end{cases}$$

Set $y_\mathbf{v} = \sum_{i=1}^n v_i y_i$, then

$$\delta^i : C^i(\mathbf{v}; M) \longrightarrow C^{i+1}(\mathbf{v}; M), \quad f \longmapsto y_\mathbf{v} f.$$

Now there is a naturally grading of the complexes and their homology. We set

$$\deg x_i = 1, \quad C_j(\mathbf{v}; M)_i := \operatorname{span}_K(m_a x^{(b)} : |a| + |b| = i, |b| = j),$$

and

$$\deg y_i = -1, \quad C^j(\mathbf{v}; M)_i := \operatorname{span}_K(m_a y^b : |a| - |b| = i, |b| = j).$$

In [3, 4.2] the following is shown

Proposition 2.2. *Let $M \in \mathcal{M}$. Then for all i there is an isomorphism of graded E-modules*

$$H_i(\mathbf{v}; M)^* \cong H^i(\mathbf{v}; M^*).$$

Cartan homology can be computed recursively. Let $\mathbf{v} = v_1, \ldots, v_m$ be a sequence in E_1. For $j = 1, \ldots, m-1$ the following sequence is exact

$$0 \to C_\bullet(v_1, \ldots, v_j; M) \xrightarrow{\iota} C_\bullet(v_1, \ldots, v_{j+1}; M) \xrightarrow{\tau} C_{\bullet-1}(v_1, \ldots, v_{j+1}; M)(-1) \to 0.$$

Here ι is a natural inclusion map, where τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \ldots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \ldots + g_k x_{j+1}^{(k-1)}$$

with $g_i \in C_{i-k}(v_1, \ldots, v_j; M)$. This implies (see [3, 4.1, 4.3])

Proposition 2.3. *Let $M \in \mathcal{M}$. Then for all $j = 1, \ldots, m-1$ there exist exact sequences of graded E-modules*

$$\begin{gathered} \ldots \longrightarrow H_i(v_1, \ldots, v_j; M) \xrightarrow{\alpha_i} H_i(v_1, \ldots, v_{j+1}; M) \xrightarrow{\beta_i} H_{i-1}(v_1, \ldots, v_{j+1}; M)(-1) \\ \xrightarrow{\delta_{i-1}} H_{i-1}(v_1, \ldots, v_j; M) \xrightarrow{\alpha_{i-1}} H_{i-1}(v_1, \ldots, v_{j+1}; M) \xrightarrow{\beta_{i-1}} \ldots, \end{gathered}$$

and

$$\begin{gathered} \ldots \longrightarrow H^{i-1}(v_1, \ldots, v_{j+1}; M) \longrightarrow H^{i-1}(v_1, \ldots, v_j; M) \longrightarrow H^{i-1}(v_1, \ldots, v_{j+1}; M)(+1) \\ \xrightarrow{y_{j+1}} H^i(v_1, \ldots, v_{j+1}; M) \longrightarrow H^i(v_1, \ldots, v_j; M) \longrightarrow \ldots. \end{gathered}$$

Here α_i is induced by ι, β_i by τ. For a cycle $z = g_0 + g_1 x_{j+1} + \ldots + g_{i-1} x_{j+1}^{(i-1)}$ in $C_{i-1}(v_1, \ldots, v_{j+1}; M)$ one has $\delta_{i-1}([z]) = [g_0 v_{j+1}]$.

Corollary 2.4. *Let $\mathbf{v} = v_1, \ldots, v_n$ be a basis for E_1. The Cartan complex $C_\bullet(\mathbf{v}; E)$ is a free resolution of the residue class field K of E. In particular for all $M \in \mathcal{M}$,*

$$\operatorname{Tor}_i^E(K, M) \cong H_i(\mathbf{v}; M), \quad \operatorname{Ext}_E^i(K, M) \cong H^i(\mathbf{v}; M)$$

as graded modules.

Proof. Use the exact sequences of 2.3 to show that $C_\bullet(\mathbf{v}; E)$ is a free resolution of K. The other statements follow then immediately. □

3. Distinguished Pairs

This section describes the behavior of the so called *distinguished pairs* introduced by Aramova and Herzog [3]. Let $M \in \mathcal{M}$ and $\mathbf{v} = v_1, \ldots, v_n$ be a basis for E_1. Consider the long exact homology sequence 2.3

$$\ldots \longrightarrow H_i(v_1,\ldots,v_{j-1};M) \longrightarrow H_i(v_1,\ldots,v_j;M) \longrightarrow H_{i-1}(v_1,\ldots,v_j;M)(-1)$$
$$\longrightarrow H_{i-1}(v_1,\ldots,v_{j-1};M) \longrightarrow H_{i-1}(v_1,\ldots,v_j;M) \longrightarrow \ldots$$

To simplify the notation we set: $H_i(k) = H_i(v_1,\ldots,v_k;M)$ for $i > 0$ and $H_0(k) = (0 :_{M/(v_1,\ldots,v_{k-1})M} v_k)/v_k(M/(v_1,\ldots,v_{k-1})M)$, where $(0 :_W e) = \{a \in W : ea = 0\}$ for an E-module W and $e \in E$. Let $H_i(0) = 0$ for $i > 0$. Notice that $H_0(k)$ is not the 0th Cartan homology of M with respect to $v_1,\ldots,v_k$. We obtain the exact sequence

$$\ldots \longrightarrow H_1(j-1) \longrightarrow H_1(j) \longrightarrow H_0(j)(-1) \longrightarrow 0$$

The following lemma leads to the concept of distinguished pairs [3, 9.5]

Lemma 3.1. *Let $1 \le l \le n$, $j \in \mathbb{N}$. The following statements are equivalent*:
(a) (1) $H_0(k)_j = 0$ *for* $k < l$ *and* $H_0(l)_j \neq 0$,
(2) $H_0(k)_{j'} = 0$ *for all* $j' > j$ *and all* $k \le l + j - j'$.
(b) *For all* $i \ge 0$
(1) $H_i(k)_{i+j} = 0$ *for* $k < l$ *and* $H_i(l)_{i+j} \neq 0$,
(2) $H_i(k)_{i+j'} = 0$ *for all* $j' > j$ *and all* $k \le l + j - j'$.
(c) *Condition* (b) *is satisfied for some* i.
Moreover, if the equivalent conditions hold, then $H_i(l)_{i+j} \cong H_0(l)_j$ *for all* $i \ge 0$.

DEFINITION 3.2. A pair of numbers (l, j) satisfying the equivalent conditions of 3.1 will be called a *homological distinguished pair* (for M).

Next we give a similar definition of distinguished pairs for the Cartan cohomology. Consider the long exact cohomology sequence

$$\ldots \longrightarrow H^{i-1}(v_1,\ldots,v_j;M) \longrightarrow H^{i-1}(v_1,\ldots,v_{j-1};M) \longrightarrow H^{i-1}(v_1,\ldots,v_j;M)(+1)$$
$$\longrightarrow H^i(v_1,\ldots,v_j;M) \longrightarrow H^i(v_1,\ldots,v_{j-1};M) \longrightarrow \ldots.$$

We define

$$H^0(k) = (0 :_{(0:_M(v_1,\ldots,v_{k-1}))} v_k)/v_k(0 :_M (v_1,\ldots,v_{k-1})) \quad \text{and}$$
$$H^i(k) = H^i(v_1,\ldots,v_k;M) \text{ for } i > 0.$$

Furthermore we set $H^i(0) = 0$ for $i > 0$. Notice that $H^0(k)$ is not the 0th Cartan cohomology of M with respect to $v_1,\ldots,v_k$. We obtain the exact sequence

$$0 \longrightarrow H^0(j)(+1) \longrightarrow H^1(j) \longrightarrow H^1(j-1) \longrightarrow \ldots.$$

There is a result similar to 3.1

Lemma 3.3. *Let $1 \leq l \leq n$, $j \in \mathbb{N}$. The following statements are equivalent*:
(a) (1) $H^0(k)_j = 0$ *for* $k < l$ *and* $H^0(l)_j \neq 0$,
(2) $H^0(k)_{j'} = 0$ *for all* $j' < j$ *and all* $k \leq l + j' - j$.
(b) *For all* $i \geq 0$
(1) $H^i(k)_{-i+j} = 0$ *for* $k < l$ *and* $H^i(l)_{-i+j} \neq 0$,
(2) $H^i(k)_{-i+j'} = 0$ *for all* $j' < j$ *and all* $k \leq l + j' - j$.
(c) *Condition* (b) *is satisfied for some* i.
Moreover, if the equivalent conditions hold, then $H^i(l)_{-i+j} \cong H^0(l)_j$ *for all* $i \geq 0$.

Proof. The proof is analog to the proof of [3, 9.5]. □

DEFINITION 3.4. A pair of numbers (l, j) satisfying the equivalent conditions of 3.3 will be called a *cohomological distinguished pair* (for M).

As a first corollary we get.

Corollary 3.5. *Let $M \in \mathcal{M}$. The following statements are equivalent*:
(a) (l, j) *is a homological distinguished pair for* M,
(b) $(l, n - j)$ *is a cohomological distinguished pair for* M^*.
Moreover, if the equivalent conditions hold, then $H_i(l; M)_{i+j} \cong H^i(l; M^*)_{-i+n-j}$ *for all* i.

Proof. Let $\mathbf{v} \subseteq E_1$ be a sequence of elements. Then by by 2.2 $H_i(\mathbf{v}; M)_{i+j} \cong (H_i(\mathbf{v}; M)_{i+j})^\vee \cong (H_i(\mathbf{v}; M)^*)_{n-i-j} \cong H^i(\mathbf{v}; M^*)_{-i+n-j}$. The claim follows directly from the definitions 3.2 and 3.4. □

We set $M_{\langle k \rangle} = (0 :_M (v_1, \ldots, v_k))$ and $M^{\langle k \rangle} = M/(v_1, \ldots, v_k)M$. We shall need the following two technical lemmata.

Lemma 3.6. (a) *For all t there exists a natural graded E-module homomorphism*

$$\alpha(t) : (0 :_{M^{\langle t-1 \rangle}} v_t) \longrightarrow (0 :_M (v_1, \ldots, v_t))(t - 1)$$

(b) *Suppose that for some t and j one has*

$$H_0(k)_j = 0 \text{ for } k < t, \quad H_0(k)_{j'} = 0 \text{ for all } j' > j \text{ and all } k \leq t + j - j'.$$

Then

$$\alpha(t)_j : (0 :_{M^{\langle t-1 \rangle}} v_t)_j \longrightarrow (0 :_M (v_1, \ldots, v_t))_{j+t-1}$$

is bijective and

$$\alpha(t)_{j+1} : (0 :_{M^{\langle t-1\rangle}} v_t)_{j+1} \longrightarrow (0 :_M (v_1, \dots, v_t))_{j+t}$$

is injective.

Proof. (a) We prove the existence of $\alpha(t) : (0 :_{M^{\langle t-1\rangle}} v_t) \to (0 :_M (v_1, \dots, v_t))(t-1)$ by induction on t. For $t = 1$ there is nothing to show. Now let $t > 1$ and consider the composition β of graded E-module homomorphisms

$$M^{\langle t-1\rangle} \xrightarrow{\beta_1} \frac{M^{\langle t-2\rangle}}{v_{t-1}M^{\langle t-2\rangle}} \xrightarrow{\beta_2} \frac{M^{\langle t-2\rangle}}{(0 :_{M^{\langle t-2\rangle}} v_{t-1})} \xrightarrow{\beta_3} (v_{t-1}M^{\langle t-2\rangle})(+1) \xrightarrow{\beta_4} (0 :_{M^{\langle t-2\rangle}} v_{t-1})(+1)$$

Here the β_i are defined as follows: by the definition of $M^{\langle t-1\rangle}$ we see that there is a natural graded E-module isomorphism β_1. Now consider $v_{t-1}M^{\langle t-2\rangle} \subseteq (0 :_{M^{\langle t-2\rangle}} v_{t-1})$. It follows that there is a natural graded surjective E-module homomorphism β_2 and a natural graded injective E-module homomorphism β_4. Furthermore β_3 is the natural graded E-module isomorphism.
The following diagram is commutative since β is an E-module homomorphism

$$\begin{array}{ccc} M^{\langle t-1\rangle} & \xrightarrow{v_t} & M^{\langle t-1\rangle} \\ \beta\downarrow & & \beta\downarrow \\ (0 :_{M^{\langle t-2\rangle}} v_{t-1})(+1) & \xrightarrow{v_t} & (0 :_{M^{\langle t-2\rangle}} v_{t-1})(+1) \end{array}$$

and therefore induces a natural graded E-module homomorphism between the two kernel of the multiplication homomorphisms with v_t:

(1) $$\alpha_1 : (0 :_{M^{\langle t-1\rangle}} v_t) \longrightarrow (0 :_{M^{\langle t-2\rangle}} (v_{t-1}, v_t))(+1).$$

By our induction hypothesis for $t-1$, we get a natural graded E-module homomorphism $\alpha(t-1) : (0 :_{M^{\langle t-2\rangle}} v_{t-1}) \to (0 :_M (v_1, \dots, v_{t-1}))(t-2)$, and the following diagram is commutative

$$\begin{array}{ccc} (0 :_{M^{\langle t-2\rangle}} v_{t-1}) & \xrightarrow{v_t} & (0 :_{M^{\langle t-2\rangle}} v_{t-1}) \\ \alpha(t-1)\downarrow & & \alpha(t-1)\downarrow \\ (0 :_M (v_1, \dots, v_{t-1}))(t-2) & \xrightarrow{v_t} & (0 :_M (v_1, \dots, v_{t-1}))(t-2) \end{array} .$$

Thus we get a graded E-module homomorphism between the two kernel of the multiplication homomorphisms:

(2) $$\alpha_2 : (0 :_{M^{\langle t-2\rangle}} (v_{t-1}, v_t)) \longrightarrow (0 :_M (v_1, \dots, v_t))(t-2).$$

We define the natural graded E-module homomorphism $\alpha(t) : (0 :_{M^{\langle t-1\rangle}} v_t) \to (0 :_M (v_1, \dots, v_t))(t-1)$ as the composition of the maps (1) and (2).

(b) This is again proved by induction on t. For $t = 1$ there is nothing to show. So let $t > 1$ and j, where

$$H_0(k)_j = 0 \text{ for } k < t, \ \ H_0(k)_{j'} = 0 \text{ for all } j' > j \text{ and all } k \leq t + j - j'.$$

Then for $t - 1$ and $j + 1$ we see that

$$\begin{aligned}&H_0(k)_{j+1} = 0 \text{ for } k < t - 1,\\ &H_0(k)_{j'} = 0 \text{ for all } j' > j + 1 \text{ and all } k \leq t - 1 + j + 1 - j'.\end{aligned}$$

By the induction hypothesis $\alpha(t - 1)$ is an isomorphism in degree $j + 1$:

$$\alpha(t-1)_{j+1} : (0 :_{M^{\langle t-2\rangle}} v_{t-1})_{j+1} \xrightarrow{\sim} (0 :_M (v_1, \dots, v_{t-1}))_{j+t-1}, \tag{3}$$

and $\alpha(t - 1)$ is injective in degree $j + 2$

$$\alpha(t-1)_{j+2} : (0 :_{M^{\langle t-2\rangle}} v_{t-1})_{j+2} \longrightarrow (0 :_M (v_1, \dots, v_{t-1}))_{j+t}. \tag{4}$$

Now consider for t the condition: $H_0(t - 1)_j = H_0(t - 1)_{j+1} = 0$. This is equivalent to

$$(0 :_{M^{\langle t-2\rangle}} v_{t-1})_j = v_{t-1}(M^{\langle t-2\rangle})_{j-1} \tag{5}$$

and

$$(0 :_{M^{\langle t-2\rangle}} v_{t-1})_{j+1} = v_{t-1}(M^{\langle t-2\rangle})_j. \tag{6}$$

With the decomposition of the E-module homomorphism β in (a) it follows for the pair (t, j):
From (5) and (6) we see that β_2 is bijective in degree j and $j + 1$. From (6) we get that the map β_4 is bijective in degree j. Thus it follows that in this case (1) $(0 :_{M^{\langle t-1\rangle}} v_t)_j \to (0 :_{M^{\langle t-2\rangle}} (v_{t-1}, v_t))_{j+1}$ is bijective. From (3) we see that (2) $(0 :_{M^{\langle t-2\rangle}} (v_{t-1}, v_t))_{j+1} \to (0 :_M (v_1, \dots, v_t))_{j+t-1}$ is bijective. Therefore the composition $\alpha(t)_j : (0 :_{M^{\langle t-1\rangle}} v_t)_j \to (0 :_M (v_1, \dots, v_t))_{j+t-1}$ is bijective.
Similar we get for the pair $(t, j + 1)$ that

$$\alpha(t)_{j+1} : (0 :_{M^{\langle t-1\rangle}} v_t)_{j+1} \longrightarrow (0 :_M (v_1, \dots, v_t))_{j+t}$$

is injective. □

Lemma 3.7. *Suppose that for some t and j one has*

$$H_0(k)_j = 0 \textit{ for } k < t, \ \ H_0(k)_{j'} = 0 \textit{ for all } j' > j \textit{ and all } k \leq t + j - j'.$$

Then it follows that $H_0(t)_j \cong H^0(t)_{j+t-1}$.

Proof. We see the following: in the proof of 3.6 we defined a natural graded E-module homomorphism $\beta : M^{\langle t-1\rangle} \to (0 :_{M^{\langle t-2\rangle}} v_{t-1})(+1)$. With the condition $0 = H_0(t-1)_j$ for the pair (t, j) we get $v_{t-1}(M^{\langle t-2\rangle})_{j-1} \cong (0 :_{M^{\langle t-2\rangle}} v_{t-1})_j$. We see that β is surjective in degree $j-1$, because β_4 is an isomorphism in degree $j-1$. The pair $(t-1, j)$ satisfies the assumption of 3.6 (b) and it follows that $(0 :_{M^{\langle t-2\rangle}} v_{t-1})_j \cong (0 :_M (v_1, \ldots, v_{t-1}))_{j+t-2}$. Therefore the composition

$$\psi : (M^{\langle t-1\rangle})_{j-1} \longrightarrow (0 :_M (v_1, \ldots, v_{t-1}))_{j+t-2}$$

is surjective.

Now there exists the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & (0 :_{M^{\langle t-1\rangle}} v_t)_{j-1} & \longrightarrow & (M^{\langle t-1\rangle})_{j-1} \\
 & & \varphi\big\downarrow & & \psi\big\downarrow \\
0 & \longrightarrow & (0 :_M (v_1, \ldots, v_t))_{j+t-2} & \longrightarrow & (0 :_M (v_1, \ldots, v_{t-1}))_{j+t-2}
\end{array}$$

The K-linear map ψ is surjective (see above) and the K-linear map φ is the E-module homomorphism $\alpha(t)$ in degree $j-1$ (see 3.6 (a)). Furthermore

$$(0 :_{M^{\langle t-1\rangle}} v_t) = \operatorname{Ker}(M^{\langle t-1\rangle} \xrightarrow{v_t} v_t(M^{\langle t-1\rangle}))$$

and

$$(0 :_M (v_1, \ldots, v_t)) = \operatorname{Ker}((0 :_M (v_1, \ldots, v_{t-1})) \xrightarrow{v_t} v_t(0 :_M (v_1, \ldots, v_{t-1}))).$$

It follows that the induced K-linear map between the cokernel

$$\gamma : v_t(M^{\langle t-1\rangle})_{j-1} \longrightarrow v_t(0 :_M (v_1, \ldots, v_{t-1}))_{j+t-2} = v_t(M_{\langle t-1\rangle})_{j+t-2}$$

is surjective. Now consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & v_t(M^{\langle t-1\rangle})_{j-1} & \longrightarrow & (0 :_{M^{\langle t-1\rangle}} v_t)_j & \longrightarrow & H_0(t)_j & \longrightarrow 0 \\
 & \gamma\big\downarrow & & \alpha(t)\big\downarrow & & & \\
0 \longrightarrow & v_t(M_{\langle t-1\rangle})_{j+t-2} & \longrightarrow & (0 :_M (v_1, \ldots, v_t))_{j+t-1} & \longrightarrow & H^0(t)_{j+t-1} & \longrightarrow 0
\end{array},$$

where γ is the induced map. Since the pair (t, j) satisfies the assumption of 3.6 (b) we get that $\alpha(t)$ is bijective in degree j. Now we see that a bijective K-linear map

$$H_0(t)_j \cong H^0(t)_{j+t-1}$$

is induced and therefore the claim of this lemma follows. □

We are able to prove the main theorem

Theorem 3.8. *Let $M \in \mathcal{M}$. The following statements are equivalent*:
(a) (l, j) *is a homological distinguished pair for* M,
(b) $(l, j + l - 1)$ *is a cohomological distinguished pair for* M.
Moreover, if the equivalent conditions hold, then $H_i(l)_{i+j} \cong H^i(l)_{-i+j+l-1}$ *for all* i.

Proof. (a)$\Longrightarrow$(b): We show condition 3.3 (a). Let (l, j) be a homological distinguished pair for M. We get

$$H_0(k)_j = 0 \text{ for } k < l \quad \text{and} \quad H_0(l)_j \neq 0, \tag{7}$$

and

$$H_0(k)_{j'} = 0 \text{ for all } j' > j \text{ and all } k \leq l + j - j'. \tag{8}$$

For $k \leq l$ and j we see that

$$H_0(s)_j = 0 \text{ for } s < k,\ H_0(s)_{j'} = 0 \text{ for all } j' > j \text{ and all } s \leq k + j - j'.$$

For $k \leq l + j - j'$ and $j' > j$ we have

$$H_0(s)_{j'} = 0 \text{ for } s < k,\ H_0(s)_{j''} = 0 \text{ for all } j'' > j' \text{ and all } s \leq k + j' - j''.$$

Using 3.7 it follows that

$$H_0(k)_j \cong H^0(k)_{j+k-1}, \tag{9}$$

and

$$H_0(k)_{j'} \cong H^0(k)_{j'+k-1}, \tag{10}$$

for the pairs (k, j) and (k, j').
To prove that $(l, j + l - 1)$ is a cohomological distinguished pair for M we must show that

$$H^0(k)_{j+l-1} = 0 \text{ for } k < l \quad \text{and} \quad H^0(l)_{j+l-1} \neq 0, \tag{11}$$

and

$$H^0(k)_{j'} = 0 \text{ for all } j' < j + l - 1 \text{ and all } k \leq l + j' - (j + l - 1) \tag{12}$$

First we prove (11). For $k = l$ and $j + l - 1$ it follows from (7) and (9) that

$$H^0(l)_{j+l-1} \cong H_0(l)_j \neq 0.$$

For $k < l$ and $j + l - 1$ it follows from (8) and (10) that

$$H^0(k)_{j+l-1} \cong H_0(k)_{j+l-k} = 0,$$

because $j' = j + l - k > j$ and $k \leq l + j - (j + l - k) = l + j - j'$.
Finally we prove (12). Let $j' < j + l - 1$ and $k \leq l + j' - (j + l - 1)$. We can write $j' = j + l - 1 - t$, where $t = 1, \dots, j + l - 1$ and therefore $k \leq l - t$.
Now let $k = l - t$. It follows from (7) and (9) that

$$H^0(k)_{j+l-1-t} \cong H_0(k)_{j+l-k-t} = 0.$$

because $j'' = j + l - k - t = j$ and $k = l - t < l$. For $k < l - t$ it follows from (8) and (10) that

$$H^0(k)_{j+l-1-t} \cong H_0(k)_{j+l-k-t} = 0.$$

because $j'' = j + l - k - t > j$ and $k < k + t = l + j - (j + l - k - t) = l + j - j''$.
Altogether we get that $(l, j + l - 1)$ is a cohomological distinguished pair for M.
(b)$\Longrightarrow$(a): Let $(l, j + l - 1)$ be a cohomological distinguished pair for M. The results of 3.5 and 3.8 imply that (l, j) is a homological distinguished pair for M.
The claim $H_i(l)_{i+j} \cong H^i(l)_{-i+j+l-1}$ follows also from 3.5 and 3.8. □

To avoid confusion we now set $H_i(l; W)$ for $H_i(l)$ to indicate that the homology is taken with values in W.

Corollary 3.9. *Let $M \in \mathcal{M}$. The following statements are equivalent*:
(a) *(l, j) is a homological distinguished pair for M,*
(b) *$(l, n - j - l + 1)$ is a homological distinguished pair for M^*.*
Moreover, if the equivalent conditions hold, then $H_i(l; M)_{i+j} \cong H_i(l; M^*)_{i+n-j-l+1}$ for all i.

Proof. This follows from 3.5 and 3.8. □

4. Applications

In this section we extend a theorem of Bayer, Charalambous and S. Popescu on extremal Betti numbers to squarefree S-modules.

We quote the following result of [3]

Proposition 4.1. *Let $M \in \mathcal{M}$ and $j \in \mathbb{Z}$. The formal power series $P_j(t) = \sum_{i\geq 0} \beta^E_{i,i+j}(M)t^i$ is the Hilbert series of a graded $K[y_1, \dots, y_n]$-module. In particular there exists a polynomial $Q_j(t) \in \mathbb{Z}[t]$ and an integer $d_j \in \mathbb{N}$ with $d_j \leq n$ such that*

$$P_j(t) = \frac{Q_j(t)}{(1-t)^{d_j}} \quad \textit{and} \quad e_j := Q_j(1) \neq 0.$$

Let $N \in \mathcal{SQ}(S)$, $k(j) = \max\{k : \beta^S_{k,k+j}(N) \neq 0\}$ and $P_j(t) = \sum_{i\geq 0} \beta^E_{i,i+j}(N_E)t^i$. Then 1.3 yields

$$P_j(t) = \frac{\sum_{k=0}^{k(j)} \beta^S_{k,k+j}(N)t^k(1-t)^{k(j)-k}}{(1-t)^{k(j)+j}}. \tag{13}$$

As in [3, 9.2, 9.3] we conclude

Proposition 4.2. *For $j \in \mathbb{N}$ and $N \in \mathcal{SQ}(S)$ we have $d_j(N_E) = k(j) + j$ and $e_j(N_E) = \beta^S_{k(j),k(j)+j}(N)$.*

Let $M \in \mathcal{M}$; as shown in [3] there exists a basis $\mathbf{v}$ of E_1 and an integer $i \gg 0$ such that $d_j(M) = n + 1 - \min\{k : H_i(k, M)_{i+j} \neq 0\}$ and $e_j(M) = \dim_K H_i(n - d_j + 1, M)_{i+j}$, where $H_i(k, M) = H_i(v_1 \dots, v_k; M)$ for all k. Thus if (l, j) is a homological distinguished pair for M, we have

$$d_j(M) = n + 1 - l \quad \text{and} \quad e_j(M) = \dim_K H_i(l, M)_{i+j}.$$

Therefore 3.9 implies

Corollary 4.3. *Let $M \in \mathcal{M}$. If (l, j) is a homological distinguished pair for M then $d_j(M) = d_{n-j-l+1}(M^*)$ and $e_j(N) = e_{n-j-l+1}(M^*)$.*

The definition of an extremal Betti number together with (13) imply

Corollary 4.4. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent*:
(a) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N,*
(b) *$i = k(j)$ and $d_{j'}(N_E) - d_j(N_E) < j' - j$ for all $j' > j$.*

Now we get

Corollary 4.5. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent*:
(a) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N,*
(b) *$(n + 1 - i - j, j)$ is a homological distinguished pair for N_E.*
Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(N) = \dim_K H_0(n + 1 - i - j)_j$.

Combining 4.5, 3.9 and 1.8 we get

Theorem 4.6. *Let $N \in \mathcal{SQ}(S)$. The following statements are equivalent*:
(a) *$\beta^S_{i,i+j}(N)$ is an extremal Betti number of N,*

(b) $\beta^S_{j,j+i}(N^*)$ *is an extremal Betti number of* N^*.
Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(N) = \beta^S_{j,j+i}(N^*)$.

In particular we have

Corollary 4.7. *Let* $\Gamma \subseteq \Delta$ *be simplicial complexes on the vertex set* $[n]$, K *a field. Let* $I_\Delta \subseteq I_\Gamma \subseteq S = K[x_1, \dots, x_n]$ *be the corresponding ideals in the polynomial ring. The following statements are equivalent*:
(a) $\beta^S_{i,i+j}(I_\Gamma/I_\Delta)$ *is an extremal Betti number of* I_Γ/I_Δ,
(b) $\beta^S_{j,j+i}(I_{\Delta^*}/I_{\Gamma^*})$ *is an extremal Betti number of* $I_{\Delta^*}/I_{\Gamma^*}$.
Moreover, if the equivalent conditions hold, then $\beta^S_{i,i+j}(I_\Gamma/I_\Delta) = \beta^S_{j,j+i}(I_{\Delta^*}/I_{\Gamma^*})$.

In the case $\Gamma = \emptyset$ this is a result in [4].

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