Kasahara, Y.
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REDUCIBILITY AND ORDERS OF PERIODIC AUTOMORPHISMS OF SURFACES

YASUSHI KASAHARA

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Introduction

Det Σ_g be a closed oriented surface of genus $g\geq 2$. By an automorphism of Σ_g, we mean an element of the mapping class group $\mathcal{M}_g$ of Σ_g, which is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. The Nielsen-Thurston theory classifies the automorphisms of Σ_g into the following three types ([11]); (i) periodic, (ii) reducible, and (iii) pseudo-Anosov (the necessary definitions will be recalled in §1).

It is easy to see that the types (i) and (ii) have some overlap, although the type (iii) does not have any intersection with (i) nor (ii). The geometric characterization of this overlap was first obtained by Gilman [2] (Proposition 2.1). Recently, the author obtained the same characterization by a different approach making use of hyperbolic geometry ([4]).

In this paper, we apply the geometric characterization to consider the relationship between reducibility and orders of periodic automorphisms of Σ_g. Intuitively speaking, periodic automrophisms would tend to be irreducible when their orders grow since the number of components of an essential 1-submanifold, which should be invariant under reducible autuomorphisms, is known to be at most $3g-3$.

Recalling some definitions and necessary results, we shall proceed to justify the naive argument above by getting both the minimum order of periodic and irreducible automorphisms and the maximum order of periodic and reducible ones. The main result is given in §4. While the former value is obtained as a direct consequence of the geometric characterization, the latter requires some complicated calculations, all of which are elementary, however.

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1. The geometric characterization

We first recall some definitions mainly from [1] and [9]. An automorphism of Σ_g is *periodic* if it has finite order in the group $\mathcal{M}_g$. Let $\mathfrak{f}\in\mathcal{M}_g$ be a periodic automorphism of order $N>1$. By the Nielsen realization theorem, there exists a diffeomorphism $f\colon \Sigma_g\to\Sigma_g$ representing $\mathfrak{f}$ such that the N-th iteration f^N is the identity ([5], [7]). We call f simply a *realization* of $\mathfrak{f}$. The diffeomorphism f induces an effective $\boldsymbol{Z}_N$-action on Σ_g, and the quotient space, denoted by O_f, has naturally a structure of hyperbolic orbifold. Such an orbifold is topologically characterized by the genus of the underlying surface $|O_f|$, which is closed and orientable, and the cone type singular points $x_1, x_2, \cdots, x_n$, with their indices $m_1, \cdots, m_n$, respectively. For convenience, we write $O_f=\Sigma_\gamma(m_1, \cdots, m_n)$ where γ is the genus of $|O_f|$. We also write $O_f=S^2(m_1, \cdots, m_n)$ when $\gamma=0$.

Let $\pi\colon\Sigma_g\to O_f$ denote the canonical projection. Then we have the so-called Riemann-Hurwitz formula:

$$\frac{2-2g}{N}=2-2\gamma+\sum_{i=1}^{n}\left(\frac{1}{m_i}-1\right). \tag{1.1}$$

For more details of the orbifold, see [9], [10, chapter 13].

The orbifold type of O_f and the projection $\pi\colon\Sigma_g\to O_f$ does not depend on the choice of the realization f. In fact, making use of the main theorem of Nielsen [8], for any other realization f', it can be seen that there exists an orientation preserving homemorophism $h\colon\Sigma_g\to\Sigma_g$ such that $f'=h\circ f\circ h^{-1}$([6]). Henceforth, we shall denote $O_{\mathfrak{f}}$ by O_f with some realization f specified, which would make no confusions.

An *essential* 1-*submanifold* of Σ_g is a disjoint union of simple closed curves in Σ_g each component of which does not bound a disk in Σ_g, and no two components of which are homotopic. An automorphism of Σ_g is *reducible* if it has a representative which leaves some essential 1-submanifold of Σ_g invariant. An *irreducible* automorphism is the one which is not reducible.

The next theorem, which characterizes the reducibility of a periodic automorphism $\mathfrak{f}\in\mathcal{M}_g$, was first given by Gilman [2].

Proposition 1.1. *A periodic automorphism $\mathfrak{f}\in\mathcal{M}_g$ is irreducible if and only if the underlying surface of its quotient orbifold $O_{\mathfrak{f}}$ is homeomorphic to the two-sphere S^2 and the singular locus of $O_{\mathfrak{f}}$ consists of three cone points.*

2. Cyclic orbifolds

In this section, we recall Harvey's result on cyclic orbiofolds. By a $\boldsymbol{Z}_N$-cyclic orbifold, we mean a quotient orbifold O_f where f is an orientation preserving periodic diffeomorphism of order N on some closed orientable surface Σ. The next theorem gives a necessary and sufficient condition for a two-orbifold to be $\boldsymbol{Z}_N$-cyclic one. Let M denote the least common multiple of $\{m_1, m_2, \cdots, m_n\}$, and M_i denote $lcm(m_1, \cdots, \hat{m}_i, \cdots, m_n)$ where $\hat{m}_i$ indicates that m_i is omitted.

Proposition 2.1. (**Harvey** [3]).

A hyperbolic two-orbifold $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ is $\boldsymbol{Z}_N$-cyclic if and only if the following conditions are satisfied:

(*i*) *the lcm-condition*: $M_i = M (i=1, 2, \cdots, n)$,

(*ii*) *M divides N, and if $\gamma=0$, $M=N$,*

(*iii*) *$n \neq 1$, and if $\gamma=0$, $n \geq 3$,*

(*iv*) *if $2|M$, then the number of m_i's divisible by the maximum power of 2 dividing M is even.*

Therefore, for any periodic automorphism $\mathfrak{f} \in \mathcal{M}_g$ of order N, the quotient orbifold $O_{\mathfrak{f}}$ must satisfy the conditions (i)-(iv).

Conversely, given any hyperbolic orbifold $O=\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ satisfying (i)-(i), we can construct a periodic automorphism of order N. In fact, Proposition 2.1 assures that there exists a certain closed orientable surface Σ and an effective $\boldsymbol{Z}_N$-action on Σ such that the quotient orbifold $\Sigma/\boldsymbol{Z}_N$ is isomorphic to O. The genus g of Σ is uniquely determined by the Riemann-Hurwitz formula (1.1):

$$g = 1+N(\gamma-1)+\frac{1}{2}N\sum_{i=1}^{n}\left(1-\frac{1}{m_i}\right). \tag{2.1}$$

Any generator of the $\boldsymbol{Z}_N$-action is the one we need.

Table 1

EXAMPLE	O	order	genus of Σ
2.2	$S^2(2, 2, 2g, 2g)$	$2g$	g
2.3	$S^2(2g+1, 2g+1, 2g+1)$	$2g+1$	g
2.4	$S^2(g+1, 2g+2, 2g+2)$	$2g+2$	g
2.5	$S^2(2, 2, g+1, g+1)$ (g: even)	$2g+2$	g

Table 2

Periodic automorphisms on Σ of order
$N=p_1^{r_1}\cdots p_k^{r_k}$(prime decomposition; $p_1<\cdots<p_k$)

EXAMPLE	O	genus of Σ
2.6	$S^2(p_1, p_1, N, N)$	$\frac{N}{p_1}(p_1-1)$
2.7	$S^2(p_1, p_2, p_3, N)$ $(N=p_1p_2p_3)$	$N-\frac{1}{2}\left(\frac{N}{p_1}+\frac{N}{p_2}+\frac{N}{p_3}+\frac{1}{N}\right)$
2.8	$S^2(p_1, p_1, \frac{N}{p_1}, \frac{N}{p_1})$ $(r_1=1, k\geq 2)$	$\left(\frac{N}{p_1}-1\right)(p_1-1)$

Now, we give some examples of periodic automorphism of surface Σ by this construction in Tables 1 and 2. The reducibility of each automorphism would be seen directly by Proposition 1.1. The examples will assure later that our estimation for order will be best possible.

REMARK 2.9. Note that the periodic automorphisms given in each example may not be unique even if up to power. In fact, an effective $\boldsymbol{Z}_N$-action on Σ_g coresponds to a pair of a possible $\boldsymbol{Z}_N$-cyclic orbifold O and an epimorphism of the orbifold fundamental group of O to $\boldsymbol{Z}_N$ preserving torsion order ([8], [3]). In general, such epimorphisms may not be unique.

REMARK 2.10. Example 2.5 was given in [12], which deals with periodic and reducible automorphisms with a connected essential 1-submanifold invariant.

3. Minimum genus of periodic and reducible automorphisms

Now, we begin to estimate orders of periodic automorphisms. In this section, we establish the following crucial step:

Theorem 3.1. *Let N be an integer ≥ 2 with prime decomposition $p_1^{r_1}\cdots p_k^{r_k}$ where each p_i is prime, each $r_i\geq 1$, and $p_1<p_2<\cdots<p_k$. Then, the minimum genus $g_{\min}(N)$ of surfaces which admit a periodic and reducible automorphism of order N is given by*

$$(i)\quad g_{\min}(N)=\max\left\{2,(p_1-1)\frac{N}{p_1}\right\},\quad \text{if } r_1>1 \text{ or } N \text{ is prime,}$$

$$(ii)\quad g_{\min}(N)=N-\frac{1}{2}\left(\frac{N}{p_1}+\frac{N}{p_2}+\frac{N}{p_3}-1\right),\quad \text{if } N=p_1p_2p_5$$
$$\text{and } p_3\leq\frac{p_1p_2-2p_1+1}{p_2-p_1},$$

$$(iii)\quad g_{\min}(N)=(p_1-1)\left(\frac{N}{p_1}-1\right),\quad \text{otherwise.}$$

To prove Theorem 3.1, except only for $N=2$, it is sufficient to estimate the value of g in (2.1) where the orbifold $\Sigma_\gamma(m_1,\cdots,m_n)$ varies all $\boldsymbol{Z}\mu$-cyclic orbifolds with $\gamma\neq 0$ or $n\neq 3$ by the argument after Proposition 2.1. The exceptional case $N=2$ is considered in the beginning of the proof. The estimation for general case is divided into five cases according to the number n of the cone singular points; $n=0$, 2, 3, 4, and ≥ 5. Most of the difficulty lies in the case $n=4$.

PROOF OF THEOREM 3.1. Let first $N=2$. Since we restrict our attention to the case $g\geq 2$, we should have $g_{\min}(2)\geq 2$. Furthermore, it is well known that any involution on Σ_g is reducible (see also Theorem 4.1 (I), the proof of which is independent of this section). Hence, we have $g_{\min}(2)=2$ as stated.

Now, we shall consider in turn the lower bound for g in (2.1). We may assume $N\geq 3$.

(I) $n=0$: Equation (2.1) is $g=1+N(\gamma-1)$. The lower bound is $N+1$ since we consider only the case $g\geq 2$.

(II) $n=2$: Equation (2.1) is $g=1+N\gamma-\frac{1}{2}N\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$. This implies $\gamma>0$ since $g\geq 2$. Therefore $g\geq 1+N-\frac{N}{p_1}=1+N\left(1-\frac{1}{p_1}\right)$ (because $m_i\geq p_1$).

(III) $n=3$: In this case, $g=1+N\left(\gamma+\frac{1}{2}\right)-\frac{N}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}\right)$. By Proposition 1.1, we obtain $\gamma>0$. Therefore $g\geq 1+\frac{3}{2}\left(N-\frac{N}{p_1}\right)$ (because $\gamma\geq 1$, $m_i\geq p_1$).

(IV) $n=4$: By equation (2.1), $g=1+N(\gamma+1)-\frac{1}{2}NA_4$ where $A_4=\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}$.

If $\gamma>0$, then g has the lower bound $1+2\left(N-\frac{1}{p_1}\right)$ when $\gamma=1$ and each $m_i=p_1$. This is larger than the minimum in the case of (II).

If $\gamma=0$, then $g=1+N-\frac{N}{2}A_4$. In this case, the bound for g corresponds to the upper bound for $A_4=\frac{1}{m_1}+\cdots+\frac{1}{m_4}$. The *lcm*-condition for $(m_1,\cdots,m_4)$ is $lcm(m_2,m_3,m_4)=lcm(m_3,m_4,m_1)=lcm(m_4,m_1,m_2)=lcm(m_1,m_2,m_3)=N$.

We define a function A_n: $\boldsymbol{N}^n\rightarrow\boldsymbol{Q}$ by

$$A_n(m_1,\cdots,m_n)=\sum_{i=1}^{n}\frac{1}{m_i}\quad\text{for}\quad(m_1,\cdots,m_n)\in\boldsymbol{N}^n\,.$$

For an arbitrary subset F_n of $\boldsymbol{N}^n$, we write $\Delta_n(F_n)=\max_{F_n} A_n$. We also write $E_n=\{(m_1,\cdots,m_n)\in\boldsymbol{N}^n;\ m_i\geq 2,\ M_i=N \text{ for each } i=1,2,\cdots,n\}$, and $\Delta_n=\Delta_n(E_n)$.

The upper bound for A_4, denoted by Δ_4, is given by the next theorem. Recall that the prime decomposition of N is given by $p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ where $p_1<p_2<\cdots<p_k$.

Theorem 3.2.

(*i*) *if* $r_1>1$ *or* N *is prime, then*

$$\Delta_4=\frac{1}{p_1}+\frac{1}{p_1}+\frac{1}{N}+\frac{1}{N}\,.$$

(*ii*) *if* $N=p_1p_2p_3$ *and* $p_3\leq\frac{p_1p_2-2p_1+1}{p_2-p_1}$, *then*

$$\Delta_4=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{N}\,.$$

(*iii*) *otherwise, we have*

$$\Delta_4=\frac{1}{p_1}+\frac{1}{p_1}+\frac{p_1}{N}+\frac{p_1}{N}.$$

For an integer $N>1$, we denote by $f(N)$ the excepted value for Δ_4.

REMARK 3.3. In case (ii), the condition that $p_3\leq\frac{p_1p_2-2p_1+1}{p_2-p_1}$ is equivalent to that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{N}\geq\frac{1}{p_1}+\frac{1}{p_1}+\frac{p_1}{N}+\frac{p_1}{N}$. Therefore, $f(N)\geq\frac{1}{p_1}+\frac{1}{p_1}+\frac{p_1}{N}+\frac{p_1}{N}$ if $r_1=1$ and N is not prime.

Proof. At first, we see that there exists an element of E_4 attaining $f(N)$ by putting $(m_1, \cdots, m_4)=(p_1, p_1, N, N)$, (p_1, p_2, p_3, N), $\left(p_1, p_1, \frac{N}{p_1}, \frac{N}{p_1}\right)$ corresponding to cases (i), (ii), (iii), respectively. So, it is sufficient to prove $\Delta_4\leq f(N)$.

If N is a power of a prime $p_1^{r_1}$, at least two of m_i's should be equal to N for any $(m_1, \cdots, m_4)\in E_4$. We may assume that m_1 and m_2 are equal to N without loss of generality. Since $(N, N, m_3, m_4)\in E_4$ for any m_3 and m_4 which divide N and are larger than 1, we have $\Delta_4(E_4)=\frac{1}{N}+\frac{1}{N}+\max\left(\frac{1}{m_3}+\frac{1}{m_4}\right)=\frac{1}{N}+\frac{1}{N}+\frac{1}{p_1}+\frac{1}{p_1}$, which is equal to $f(N)$.

Hereafter, we may assume $k\geq 2$. We denote by E_4^α the subset of E_4:

$$E_4^\alpha=\{(m_1, \cdots, m_4)\in E_4; m_1=\alpha\}.$$

Thus, $E_4=\bigcup_\alpha E_4^\alpha$, and $\Delta_4(E_4)=\max_\alpha \Delta_4(E_4^\alpha)$. For $(m_1, \cdots, m_4)\in E_4$, suppose that none of m_i's is prime. Then we have $m_i\geq p_1^2\geq 2p_1$. Therefore, $A_4=\sum_{i=1}^{4}\frac{1}{m_i}\leq\frac{2}{p_1}$, which is less than $f(N)$ by Remark 3.3. Hence, we have $\Delta_4=\max_{1\leq i\leq k}\Delta_4(E_4^{p_i})$. Therefore the calculation of Δ_4 is reduced to that of $\Delta_4(E_4^{p_i})$'s.

The next theorem is fundamenal.

Proposition 3.4. (Harvey [3]).

(a) *Let* $R_L=\{(x, y); x, y\in \boldsymbol{N}, lcm(x, y)=L\}$. *Then,*

$$\Delta_2(R_L)=\max_{R_L}\left(\frac{1}{x}+\frac{1}{y}\right)=1+\frac{1}{L}.$$

(b) *Let* $R_L'=\{(x, y)\in R_L; x>1, y>1\}$ *for* $L=p_1^{l_1}\cdots p_\lambda^{l_\lambda}>1$ *where* $p_1^{l_1}\cdots p_\lambda^{l_\lambda}$ *is the prime decomposition and* $p_1<\cdots<p_\lambda$. *Then* $\Delta_2(R_L')$ *is given by*

$$\Delta_2(R_L')=\begin{cases}\dfrac{1}{p_1}+\dfrac{p_1}{L}, & \text{if } l_1=1 \text{ and } L \text{ is not prime}\\[2ex] \dfrac{1}{p_1}+\dfrac{1}{L}, & \text{if } l_1>1 \text{ or } L \text{ is prime.}\end{cases}$$

(c)

$$\Delta_3(E_3)=\begin{cases}\dfrac{1}{N}+\dfrac{1}{p_1}+\dfrac{p_1}{N}, & \textit{if } r_1=1 \textit{ and } N \textit{ is not prime,}\\[2ex] \dfrac{1}{N}+\dfrac{1}{N}+\dfrac{1}{p_1}, & \textit{if } r_1>1 \textit{ or } N \textit{ is prime.}\end{cases}$$

The calculation of $\Delta_4(E_4^{p_i})$ is divided into two cases according to $r_i>1$ or $r_i=1$, and subcases indicated.

(α) Assume that $r_i>1$. Since $(p_i, m_2, m_3, m_4)\in E_4^{p_i}$ if and only if $(m_2, m_3, m_4)\in E_3$, we can apply Proposition 3.4 (c) to obtain $\Delta_4(E_4^{p_i})=\dfrac{1}{p_i}+\Delta_3(E_3)\leq f(N)$ (note that N is not a prime since $k\geq 2$).

(β) Suppose next that $r_i=1$. We denote F_1^i, F_2^i by subsets of $E_4^{p_i}$:

$$F_1^i=\{(m_1, \cdots, m_4)\in E_4^{p_i};\, lcm(m_2, m_3)=lcm(m_2, m_4)=lcm(m_3, m_4)=N\}\ ,$$

$$F_2^i=\{(m_1, \cdots, m_4)\in E_4^{p_i};\, lcm(m_2, m_3)=lcm(m_2, m_4)=N,\, lcm(m_3, m_4)=\frac{N}{p_i}\}\,.$$

Then, considering the *lcm*-condition for $E_4^{p_i}$, we can check that any $(m_1, \cdots, m_4)\in E_4^{p_i}$ can be transformed to an element of F_1^i or F_2^i by permuting the m_i's adequately. Therefore, we have $\Delta_4(E_4^{p_i})=\max(\Delta_4(F_1^i), \Delta_4(F_2^i))$. For F_1^i, as in case (α), we can apply Proposition 3.4 (c) to obtain $\Delta_4(F_1^i)=\dfrac{1}{p_i}+\Delta_3(E_3)\leq f(N)$.

Now, we have reduced the estimation of $\Delta_4(E_4^{p_i})$ to that of $\Delta_4(F_2^i)$. For any divisor m of N, we denote by $P(m)$ the minimum positive integer satisfying $lcm(m, P(m))=N$. If $m=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ such that $a_{i_1}<r_{i_1}, \cdots, a_{i_s}<r_{i_s}$, and $a_j=r_j$ for any $j\neq i_1, \cdots, i_s$, then $P(m)=\prod_{t=1}^{s} p_{i_t}^{r_{i_t}}$. Suppose that $(m_1, \cdots, m_4)\in F_2^i$. We write $P=P(m_2)$. Then it holds that $p_i\nmid P$ since $p_i\mid m_2$ and $r_i=1$. We can also see that $P=1$ if and only if $m_2=N$. Therefore, F_2^i separates into $G_1^i\cup G_2^i$ where

$$G_1^i=\{(p_i, m_2, m_3, m_4)\in F_2^i;\, P(m_2)>1\}\ ,$$
$$G_2^i=\{(p_i, m_2, m_3, m_4)\in F_2^i;\, m_2=N\}\ .$$

Hence, we have $\Delta_4(F_2^i)=\max(\Delta_4(G_1), \Delta_4(G_2))$.

For $(p_i, m_2, m_3, m_4)\in G_1$, the *lcm*-condition requires that $P\mid m_3$, and $P\mid m_4$. So we can write $m_3=Pm_3'$ and $m_4=Pm_4'$. Then, $lcm(m_3, m_4)=\dfrac{N}{p_i}$ if and only if $lcm(m_3', m_4')=\dfrac{N}{Pp_i}$. Therefore,

$$\Delta_4(G_1)=\frac{1}{p_i}+\max_{\substack{p_i\mid m_2,\\ m_2\mid N, m_2\neq N}}\left(\frac{1}{m_2}+\frac{1}{P(m_2)}\max_{lcm(m_2', m_3')=N/Pp_i}\left(\frac{1}{m_2'}+\frac{1}{m_3'}\right)\right).$$

By Proposition 3.4 (a) together with $P(m_2)>1$, we have

$$\Delta_4(G_1)=\frac{1}{p_i}+\frac{p_i}{N}+\max_{\substack{p_i\mid m_2,\\ m_2\mid N, m_2\neq N}}\left(\frac{1}{m_2}+\frac{1}{P(m_2)}\right).$$

If $p_j \mid m_2$ and $p_j^{r_j} \nmid m_2$, then $P(m_2)=P(m_2')$ for $m_2'=\frac{m_2}{p_j}$, and $\frac{1}{m_2'}+\frac{1}{P(m_2')}>\frac{1}{m_2}+\frac{1}{P(m_2)}$. Therefore, $\frac{1}{m_2}+\frac{1}{P(m_2)}$ has the maximum value when m_2 is a product of $p_j^{r_j}$'s and then P is equal to $\frac{N}{m_2}$. Therefore,

$$\max\left(\frac{1}{m_2}+\frac{1}{P(m_2)}\right) = \max_{\left(\substack{p_i \mid m_2, m_2 \mid N, m_2 \neq N, \\ m_2 \text{ is a product of } p_j^{r_j}\text{'s}}\right)} \left(\frac{1}{m_2}+\frac{m_2}{N}\right)$$

$$\leq \max_{\left(\substack{p_1 \leq m_2 \leq N/p_1 \\ m_2 \text{ is a product of } p_j^{r_j}\text{'s}}\right)} \left(\frac{1}{m_2}+\frac{m_2}{N}\right)$$

$$= \frac{1}{q_1}+\frac{q_1}{N},$$

where $q_1=\min_{1\leq j\leq k} p_j^{r_j}$. Hence, we obtain

$$\Delta_4(G_1) \leq \frac{1}{p_i}+\frac{p_i}{N}+\frac{1}{q_1}+\frac{q_1}{N}. \tag{3.1}$$

If $r_1=1$, then $q_1=p_1$, and the right-hand term is less than or equal to $f(N)$.

If $r_1>1$, then we have $f(N)-\left(\frac{1}{p_i}+\frac{p_i}{N}+\frac{1}{q_1}+\frac{q_1}{N}\right)=\left\{\left(\frac{1}{p_1}+\frac{1}{N}\right)-\left(\frac{1}{p_i}+\frac{p_i}{N}\right)\right\} +\left\{\left(\frac{1}{p_1}+\frac{1}{N}\right)-\left(\frac{q_1}{N}+\frac{1}{q_1}\right)\right\}\geq 0$. Therefore, we have $\Delta_4(G_1)\leq f(N)$ by (3.1).

Now, we estimate $\Delta_4(G_2)$. By definition,

$$\Delta_4(G_2) = \frac{1}{p_i}+\frac{1}{N}+\max_{\left(\substack{m_3>1 m_4>1 \\ l_{cm}(m_3,m_4)=\frac{N}{p_i}}\right)} \left(\frac{1}{m_3}+\frac{1}{m_4}\right).$$

Since $\frac{N}{p_i}>1$, we can apply Proposition 3.4 (b) to obtain

$$\Delta_4(G_2) = \begin{cases} \frac{1}{p_i}+\frac{1}{N}+\frac{1}{p_1}+\frac{p_1 p_i}{N}, & \text{if } i\geq 2,\ r_1=1, \text{ and } \frac{N}{p_i} \text{ is not prime} & (a) \\ \frac{1}{p_i}+\frac{1}{N}+\frac{1}{p_1}+\frac{p_i}{N}, & \text{if } i\geq 2 \text{ and } (r_1>1 \text{ or } \frac{N}{p_i} \text{ is prime}) & (b) \\ \frac{1}{p_1}+\frac{1}{N}+\frac{1}{p_2}+\frac{p_1 p_2}{N}, & \text{if } i=1,\ r_1=1,\ r_2=1, \text{ and } \frac{N}{p_i} \text{ is not prime} & (c) \\ \frac{1}{p_1}+\frac{1}{N}+\frac{1}{p_2}+\frac{p_1}{N}, & \text{if } i=1,\ r_1=1, \text{ and } (r_2>1 \text{ or } \frac{N}{p_1} \text{ is prime}) & (d) \end{cases}$$

(note that we have assumed $r_i=1$). Now, we estimate these values in turn.

(*a*) Since $\left(\frac{1}{p_2}+\frac{p_1p_2}{N}\right)-\left(\frac{1}{p_i}+\frac{p_1p_i}{N}\right)=\frac{1}{p_2p_iN}(p_i-p_2)(N-p_1p_2p_i)\geq 0$, we have $\Delta_4(G_2)\leq\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{N}+\frac{p_1p_2}{N}$. If $N=p_1p_2p_3$, then the right- hand term is less than or equal to $f(N)$ (Remark 3.3).

Now, suppose that $N\neq p_1p_2p_3$. Since $r_i=1$, we have $k\geq 3$, and then $N\geq p_1p_2^2p_i$. Hence, we have

$$
\begin{aligned}
f(N)-\Delta_4(G_2) &= \frac{1}{p_1p_iN}\{(p_i-p_1)N+p_1p_i(2p_1-p_1p_i-1)\}\\
(\because\ N\geq p_1p_2^2p_i)\quad &\geq\frac{1}{N}\{p_i(p_2^2-p_1)-p_1(p_2^2-1)+p_1-1\}\\
(\because\ p_i\geq p_2)\quad &\geq\frac{1}{N}\{p_2^3-p_1(p_2-1)(p_2+2)-1\}\\
(\because\ p_1\leq p_2-1)\quad &\geq\frac{3}{N}(p_2-1)>0\,.
\end{aligned}
$$

Therefore, $\Delta_4(G_2)<f(N)$.

(*b*) Since $\frac{1}{p_i}+\frac{p_i}{N}\leq\frac{1}{p_1}+\frac{1}{N}$, we obtain $\Delta_4(G_2)\leq 2\left(\frac{1}{p_1}+\frac{1}{N}\right)=f(N)$.

(*c*) In this case, we have $\Delta_4(G_2)\leq f(N)$ as in case (*a*) putting $i=2$.

(*d*) It holds that $\Delta_4(G_2)=\frac{1}{p_1}+\frac{1}{N}+\frac{1}{p_2}+\frac{p_1}{N}<2\left(\frac{1}{p_1}+\frac{p_1}{N}\right)\leq f(N)$.

We have proven that $\Delta_4\leq f(N)$, which completes the proof of Theorem 3.2. ■

Hence $g_{\min}(N)$ does not exceed the expected value for it when $n=4$ and $\gamma=0$ also.

(V) $n\geq 5$: We shall prove that any genus g in this case is larger than the expected value for $g_{\min}(N)$. Suppose to the contrary that the genus for a $\boldsymbol{Z}_N$-cyclic orbifold $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ where $n\geq 5$ is smaller than or equal to the expected value. Then, together with Remark 3.3, we have $\frac{2g-2}{N}=2\gamma-2+\sum_{i=1}^{n}\left(1-\frac{1}{m_i}\right)\leq 2-\frac{2}{p_1}-\frac{2}{N}$. This inequality fails when $\gamma>1$. Now suppose that $\gamma=1$. Then, $\sum_{i=1}^{n}\left(1-\frac{1}{m_i}\right)\leq 2-\frac{2}{p_1}-\frac{2}{N}$, which is impossible since $m_i\geq p_1$ and $n\geq 5$. Therefore, the only possibility is $\gamma=0$. Thus,

$$
\sum_{i=1}^{n}\frac{1}{m_i}\geq n-4+\frac{2}{p_1}+\frac{2}{N}\,. \tag{3.2}
$$

On the other hand, each $m_i\geq p_1$, and at least two of the m_i's≥ 3 since the m_i's must satisfy the *lcm*-condition and $N\geq 3$. Therefore, we have $\frac{n-2}{p_1}+\frac{2}{3}\geq \sum_{i=1}^{n}\frac{1}{m_i}\geq n-4+\frac{2}{p_1}+\frac{2}{N}$.

If $p_1\geq 3$, we obtain $n\leq 5-\frac{3}{N}$, which contradicts to $n\geq 5$. Therefore, the only possibility is $p_1=2$, and then $n=5$. Even in this case, at least three of the m_i's must be equal to 2. Then, the cyclic condition for $S^2(2, 2, 2, m_4, m_5)$ together with $N\geq 3$ implies $\frac{1}{m_4}+\frac{1}{m_5}\leq\frac{1}{2}$, which contradicts to (3.2).

By now, we have proved that $g_{\min}(N)$ is not less than the expected value for it. Furthermore, the examples given in Table 2 assure that they certainly coincide. This completes the proof of Theorem 3.1. ■

REMARK 3.5. In cases (i) and (iii) of Theorem 3.1, with some exceptions, the above result is just the twice of the minimum genus for periodic but not necessarily reducible automorphism of the same order ([3]). The only exceptions are $N=2, 3, 4, 6$. The exceptions $N=3, 4, 6$ occur since we are concerned with $g\geq 2$ as same as [3]. The exception $N=2$ occurs since $S^2(2, 2, 2)$ is not cyclic (the cyclic condition (iv) fails).

4. Orders of periodic automorphisms

In this section, we show

Theorem 4.1. *Let* $\mathfrak{f}\in\mathcal{M}_g$ *be a periodic automorphism of order* N. *Then, the followings hold*:

(I) *if* $\mathfrak{f}$ *is irreducible, then* $N\geq 2g+1$,

(II) *if* $\mathfrak{f}$ *is reducible, then* $N\leq 2g+2$;

furthermore, if the genus g *is odd, then* $N\leq 2g$.

All the inequalities are best possible. That is to say, there certainly exists a periodic automorphism of Σ_g *having as order the value of the right-hand term of each inequality, with required reducibility equipped.*

REMARK 4.2. By this result, the periodic automorphism of Σ_g of the maximum order $4g+2$ (see [3]) must be irreducible.

Proof. The result (I) is a direct consequence of Proposition 1.1 as shown bellow. Let $\mathfrak{f}\in\mathcal{M}_g$ be a periodic and irreducible automorphism of order N. Then, by Proposition 1.1, the quotient orbifold $O_{\mathfrak{f}}$ is of type $S^2(m_1, m_2, m_3)$ with each $m_i\leq N$. Applying $m_i\leq N$ to the Riemann-Hurwitz formula (1.1), we directly have $N\geq 2g+1$. Example 2.3 assures the best possibility.

Now, we shall prove (II). Let N be the order of a periodic and reducible automorphism of Σ_g, with the same prime decomposition in Theorem 3.1. The proof of the first part is based on the observation $g\geq g_{\min}(N)$ and divided into the same subcases of Theorem 3.1.

(i) $r_1>1$ or N is prime: By Theorem 3.1, we obtain $g\geq\frac{N}{p_1}(p_1-1)$. Therefore, $N\leq\frac{g}{p_1-1}+g$. Since $p_1\geq 2$, we obtain $N\leq 2g$.

(ii) $N=p_1p_2p_3$ and $p_3\leq\frac{p_1p_2-2p_1+1}{p_2-p_1}$: By Theorem 3.1, we obtain $g\geq N-\frac{1}{2}\left(\frac{N}{p_1}+\frac{N}{p_2}+\frac{N}{p_3}-1\right)$. Since $\frac{\partial}{\partial p_2}\left(\frac{p_1p_2-2p_1+1}{p_2-p_1}\right)<0$, and $p_2<p_3\leq\frac{p_1p_2-2p_1+1}{p_2-p_1}$, we have $p_1\geq 5$. Therefore,

$$g\geq\frac{7}{10}N+\frac{1}{2}\,. \tag{4.1}$$

Hence, we have $N\leq\frac{10}{7}\left(g-\frac{1}{2}\right)<2g$.

(iii) otherwise: By Theorem 3.1, we obtain $g\geq(p_1-1)\left(\frac{N}{p_1}-1\right)$. Therefore $N\leq\frac{p_1g}{p_1-1}+p_1$. We shall see this implies $N\leq 2g+2$. Suppose to the contrary that $N>2g+2$. Then we have $(2-p_1)\left(\frac{g}{p_1-1}-1\right)>0$. If $p_1=2$, then clearly this inequality fails. If $p_1>2$, then we obtain $\frac{g}{p_1-1}-1<0$ since $2-p_1<0$. Hence $g<p_1-1$. On the other hand, by Theorem 3.1, we have $g\geq(p_1-1)\left(\frac{N}{p_1}-1\right)$, which yields a contradiction since N is not a prime. This completes the first part of (II). The best possibility for even genus is assured by Example 2.5.

We next remark that no closed surface Σ_g admits any periodic and reducible automorphism of order $2g+1$. This can be seen by estimating the minimum genus $g_{\min}(2g+1)$, by Theorem 3.1, to be greater than g. We omit the detailed calculation since it is similar to the one given above to show $N\leq 2g+2$.

We shall next see that Σ_g admits no periodic and reducible automorphisms of order $2g+2$ if the genus g is odd. In fact, if g is odd, the minimum genus $g_{\min}(2g+2)$ is $g+1$ by Theorem 3.1 since $2^2|2g+2$. This completes the last part of (ii). The best possibility is assured by Example 2.2. ■

REMARK 4.3. Theorem 4.1 shows that given a periodic automorphism of Σ_g, its order almost determines its reducibility. In fact, the determination by order is complete if the genus g is odd. Even if g is even, it is only the order $2g+2$ that fails to determine the reducibility because the order $2g+1$ does not occur in reducible case by the argument in the proof above. It is seen that the failure does always occur, however, in view of both Examples 2.4 and 2.5.

Appendix—the case $g=1$.

The main Theorem 4.1 holds for the case $g=1$, which might be well known. This is to be checked in view of Table 3, which lists up the whole quotient orbifolds of the actions on Σ_1 of finite cyclic groups which descend injectively into $\mathcal{M}_1$ (note that Proposition 1.1 still holds for the case $g=1$).

Table 3

Cyclic quotients of Σ_1

Order	Quotient orbifold
2	$S^2(2, 2, 2, 2)$
3	$S^2(3, 3, 3)$
4	$S^2(2, 4, 4)$
6	$S^2(2, 3, 6)$

Table 3 was obtained through a computer-aided calculation based on Harvey's cyclic condition (given in Proposition 2.1). We remark that Theorem 4.1 itself was first observed for the cases $g=2$-44 through such computer-aided calculations.

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Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro
Tokyo 152
Japan