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LEVEL-TWO-STRUCTURES AND HYPERELLIPTIC CURVES

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1. Introduction

The main idea in this paper is to apply the results in ([31],$\cdots$,[34]) to geometry. The paper contains some results on Specht modules due to several exceptional isomorphisms of subgroups of the symplectic groups to symmetric groups. The geometry of the Igusa desingularization in genus 2 and level 2 is described.

We always choose as coordinates the theta constants of second kind. The ordianry theta constants are used for auxiliary purposes. They give the equations of the Humbert surfaces in genus two or of the hyperelliptic points in genus three.

2. Notations and first results

Throughout the paper we will use the following notations in accordance with [31], [32]. General references are [9], [24], [26], [27], [29] and [37].

$$\begin{aligned}
&\boldsymbol{H}_g=\{\tau\in \mathrm{Mat}_{g\times g}(\boldsymbol{C})\mid \tau \textit{ symmetric},\ \mathrm{Im}(\tau)>0\},\\
&\Gamma_g=Sp(2g,\boldsymbol{Z}),\\
&\Gamma_g(n)=\mathrm{Ker}(\Gamma_g\to Sp(2g,\boldsymbol{Z}/n)).
\end{aligned}$$

For a subgroup Γ of finite index in Γ_g we denote by $A(\Gamma)=\oplus_k[\Gamma,k]$ the ring of modular forms for Γ and by $\mathscr{A}_g(\Gamma)=\mathrm{Proj}(A(\Gamma))$ the corresponding Satake compactification. (This is *not* the standard notation.) The variety $\mathscr{A}_g(\Gamma)$ contains $\boldsymbol{H}_g/\Gamma$ as an open dense subset. $\boldsymbol{H}_g/\Gamma$ is a coarse moduli space for principally polarized abelian varieties (ppav) with level-Γ-structure. The ring $A(\Gamma)$ is a normal graded integral domain finitely generated as an algebra over $\boldsymbol{C}=[\Gamma,0]$. For general facts about such graded rings see [30].

The thetas (of second kind) are given by (we use Mumford's notation f_a)

$$f_a(\tau)=\theta\begin{bmatrix}a\\0\end{bmatrix}(2\tau)=\sum_{x\in \boldsymbol{Z}^g}\exp 2\pi i\left(\tau\left[x+\frac{1}{2}a\right]\right)$$

for $a \in \mathbf{Z}^g$. The functions $f_a(\tau)$ only depend on $a \bmod 2$ hence a is regarded as element in $\mathbf{F}_2^g$.

The squares of the ordinary thetas may be defined as (Veronese formula)

$$\theta^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)=\sum_{x\in \mathbf{F}_2^g}(-1)^{\langle x,\beta\rangle} f_x(\tau) f_{\alpha+x}(\tau).$$

The group $Sp(2g,\mathbf{R})$ acts on $\mathbf{H}_g$ by

$$\sigma\langle\tau\rangle=(A\tau+B)(C\tau+D)^{-1} \quad \text{for} \quad \sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(2g,\mathbf{R}) \quad \text{and} \quad \tau\in\mathbf{H}_g.$$

This action induces for any $k\in\mathbf{Z}$ a (right) group action on the algebra of holomorphic functions $\{f:\mathbf{H}_g\to\mathbf{C}\}$ by

$$f|_k\sigma(\tau)=\det(C\tau+D)^{-k}f(\sigma\langle\tau\rangle).$$

A holomorohic function f on $\mathbf{H}_g$ is a modular form of weight k and level Γ, or in short $f\in[\Gamma,k]$, iff $f|_k\sigma=f$ for all $\sigma\in\Gamma$. In genus $g=1$ one has to add a condition for the cusps.

It is well known that the group Γ_g is generated by $J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $\sigma_S=\begin{pmatrix}1 & S\\ 0 & 1\end{pmatrix}$ where S runs over all symmetric $g\times g$-matrices and $1\in Gl(g,\mathbf{Z})$ ([9]). If we allow for a moment half-integral weights, the modular group Γ_g acts on the thetas by

$$f_a|_{1/2}\sigma_S=i^{S[a]}f_a$$

$$f_a|_{1/2}J=\pm\sum_{b\in\mathbf{F}_2^g}(T_g)_{a,b}f_b$$

where $T_g\in Gl(2^g,\mathbf{C})$ is the matrix

$$T_g=\left(\frac{1+i}{2}\right)^g((-1)^{\langle a,b\rangle})_{a,b\in\mathbf{F}_2^g}.$$

Because of its meaning in coding theory we call T_g MacWilliams identity. The equation for J depends of the choice of the square root $\sqrt{\det(-\tau)}$. However, on the 2-ring, i.e. on the ring $\mathbf{C}[f_a(\tau)]_{(2)}=\{f\in\mathbf{C}[f_a(\tau)] \text{ with } 2\,|\deg(f)\}$ the action is independent of the chosen sign.

Take $D_S=\operatorname{diag}(i^{S[a]}$ for $a\in\mathbf{F}_2^g)$ and let

$$\mathcal{H}_g=\langle T_g,\ D_S \text{ for symmetric } S\in M_g(\mathbf{Z})\rangle$$

be the finite subgroup of $Gl(2^g,\mathbf{C})$ generated by the elements T_g and all the D_S. If we map J to T_g and σ_S to D_S we get the theta representation

$$\rho_{theta}: \Gamma_g \to H_g/(\pm 1).$$

The kernel, denoted by $\Gamma_g^*(2,4)$, is described in [31].

We recall from [31] that the ring of modular forms of even weight is given by

$$A(\Gamma_g)_{(2)} = \bigoplus_{2|k} [\Gamma_g, k] = (\boldsymbol{C}[f_a(\tau)]^{H_g})^N.$$

Here N denotes the normalization (in its field of fractions). Moreover, $A(\Gamma_1) = \boldsymbol{C}[f_a(\tau)]^{H_1}$, $A(\Gamma_2)_{(2)} = \boldsymbol{C}[f_a(\tau)]^{H_2}$ and $A(\Gamma_3) = \boldsymbol{C}[f_a(\tau)]^{H_3}$. We use binary numbers to index the thetas, i.e. (in genus $g=2$) $f_0 = f_{\substack{0\\0}}, f_1 = f_{\substack{1\\0}}, f_2 = f_{\substack{0\\1}}$ and $f_3 = f_{\substack{1\\1}}$.

The Siegel Φ-operator may be defined as follows. Siegel modular forms in even weight are always rational functions in the theta constants of second kind. On them, the Φ-operator is given by

$$\Phi(f_{\substack{a\\0}}) = f_a \quad \text{and} \quad \Phi(f_{\substack{a\\1}}) = 0.$$

(Here a is considered as element in $\boldsymbol{F}_2^g$ and ${}^{a}_{*}$ as an element in $\boldsymbol{F}_2^{g+1}$.)

3. Azygetic sets and cosets

On $\boldsymbol{F}_2^g$ one has the standard scalar product with values in $\boldsymbol{F}_2 = \{0,1\}$ and for $m = [{}^{\alpha}_{\beta}] \in \boldsymbol{F}_2^{2g}$ we define:

$$|m| = \langle \alpha, \beta \rangle,$$
$$\left\{ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \right\} = \langle \alpha, \delta \rangle + \langle \beta, \gamma \rangle,$$
$$\{m,p,q\} = \{m,p\} + \{m,q\} + \{p,q\}.$$

Usually one takes values ± 1 but we prefer the above notation. $|m|$ is called the characteristic of $m = [{}^{\alpha}_{\beta}]$.

We refer to the book of Rauch and Farkas [29] for proofs of some facts which are only given as additional information. Let us start with some easy considerations of characteristics and the bilinear form on $\boldsymbol{F}_2^{2g}$.

Lemma 3.1. 1. $|m+n| = |m| + |n| + \{m,n\}$

2. $|m+p+q| = |m| + |m+p| + |m+q| + \{p,q\}$

3. $\{m,p,q\} = |m| + |p| + |q| + |m+p+q|$
$= \{m+p, m+q\}$.

Proof. This follows easily from the definitions. $\square$

As usual a characteristic $m=[{\alpha \atop \beta}]$ is called even or odd, iff $|m|$ is 0 or 1 respectively. Two characteristics m and n are called azygetic iff $\{m,n\}=1$ and three characteristics m, n and p are azygetic iff $\{m,n,p\}=1$. A set of characteristics is called azygetic iff every tripel of elements is azygetic.

By the lemma $|\ |$ is a homomorphism on any isotropic subspace of F_2^{2g}. Because the bilinear form is the standard non-degenerated alternating form, any linear form on a subspace is given by a uniquely (modulo the orthogonal complement) defined element via the bilinear form ($|x|=\{x,a\}$ for some a). Furthermore the group $Sp(2g,F_2)$ is operating transitively on the isotropic subspaces of equal dimension. It follows that for any r-dimensional isotropic subspace U there is an element a_U such that a_U+U is uniform.

(Let us recall the following definitions $S=a_U+U$ is uniform iff $|s_1|=|s_2| \forall s_1,s_2 \in S$ and even iff $|s|=0$ $\forall s \in S$.)

If $S=a_U+U$ is even, it follows easily from the lemma that $\lambda+S$ is uniform iff λ is orthogonal to U. It is easy to see that there are $2^{g-r-1}(2^{g-r}+1)$ even sets of type $S=a_U+U$ for any r-dimensional isotropic subspace U (Rauch / Farkas, p.40). All other sets of type $S=a_U+U$ contain only odd characteristics or half odd and half even ones.

We recall from [29] the following:

Lemma 3.2. *Let $S=\{x_1,\cdots,x_n\}$ be a set of in pairs azygetic elements in F_2^{2g} (i.e. $\{x_i,x_j\}=1$ for $i\neq j$), then we have* $(1)\Rightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)\Rightarrow(5)$ *for the following conditions*:

(1) $\#S=2g+1$
(2) *S maximal*
(3) $\Sigma x_i=0$
(4) *S linearly dependent*
(5) *S odd.*

Proof. $(3)\Rightarrow(5)$:
$0=\{x_n,x_n\}=\Sigma_{i=1..n-1}\{x_n,x_i\}=n-1$
$(3)\Rightarrow(2)$:
If $S\cup\{x\}$ is in pairs azygetic, then $\{x,x_i\}=1$, hence $0=\{0,x\}=\{\Sigma x_i,x\}=\#S=1$ contradiction.
$(4)\Rightarrow(3)$:
If $S'\subset S$ and $\Sigma_{x_i\in S'}x_i=0$, then S' maximal.
$(3)\Rightarrow(4)$ is trivial.
$(2)\Rightarrow(3)$:
If $\Sigma x_i\neq 0$, then S is linear independent, hence $\#S\leq 2g$. If $\#S=2g$, then let $x=\Sigma x_i$ and $S\cup\{x\}$ is again in pairs azygetic, contradiction. Hence there exists an element y in F_2^{2g} with $S\cup\{y\}$ linear independent. Hence one may solve the equations $\{x,y\}=\{x,x_i\}=1$ to get a bigger in pairs azygetic set $S\cup\{x\}$. □

The proof secures the existence of in pairs azygetic sets of cardinality $2g+1$.

Corollary 3.3. *Let $B \subset F_2^{2g}$ with $\#B = 2g+1$, $\{x,y\} = 1$ for $x \neq y \in B$ an in pairs azygetic set. Then*

$$\{T \subset B \quad \text{with} \quad \#T = 1(2)\} \to F_2^{2g}$$
$$T \mapsto \Sigma T$$

is a bijection.

Proof. Injectivity follows from 3.2. The surjectivity follows from the formula (see [29], p.36)

$$2^{2g} = \frac{(1+1)^{2g+1} - (1-1)^{2g+1}}{2} = \sum_{i=1..g} \binom{2g+1}{2i+1}.$$

□

Lemma 3.4. *Let $B \subset F_2^{2g}$ with $\#B = 2g+1$, $\{x,y\} = 1$ for $x \neq y \in B$ an in pairs azygetic set. Let T_1 and T_2 be subsets of B. Then*

$$\{\Sigma T_1, \Sigma T_2\} = \#(T_1)\#(T_2) + \#(T_1 \cap T_2).$$

Proof. Let $m = \Sigma(T_1 \cap T_2)$, $p = \Sigma(T_1 \backslash T_2)$ and $q = \Sigma(T_2 \backslash T_1)$. Then

$$\begin{aligned}
\{\Sigma T_1, \Sigma T_2\} &= \{m+p, m+q\} \\
&= \{m,p\} + \{m,q\} + \{p,q\} \\
&= (\#m)(\#p) + (\#m)(\#q) + (\#p)(\#q) \\
&= (\#m + \#p)(\#m + \#q) + (\#m)^2 \\
&= (\#T_1)(\#T_2) + \#(T_1 \cap T_2).
\end{aligned}$$

□

Corollary 3.5. *An in pairs azygetic set is azygetic.*

Proof. Regard $T_1 = \{x_i, x_j\}$ and $T_2 = \{x_i, x_k\}$, Then $\{x_i, x_j, x_k\} = \{x_i + x_j, x_i + x_k\} = 1$. □

Proposition 3.6. *Let $B \subset F_2^{2g}$, $\#B = 2g+1$, $\{x,y\} = 1$ for $x \neq y \in B$ an in pairs azygetic set. Let*

$$U = \{x \in B \quad \text{with} \quad |x| = 0\}$$
$$V = \{x \in B \quad \text{with} \quad |x| = 1\}$$

be the even and odd subset of B and T an arbitrary subset of B. Then

1. $\Sigma U = \Sigma V$
2. $\sharp U \equiv g+1(4)$ *and* $\sharp V \equiv g(4)$.
3. $|\Sigma U| = |\Sigma V| = \begin{cases} 0 & \textit{if } g \equiv 0,3(4) \\ 1 & \textit{else} \end{cases}$
4. $|\Sigma T| = \begin{cases} \sharp(V \cap T) & \textit{if } \sharp T \equiv 0,1(4) \\ \sharp(V \cap T)+1 & \textit{else} \end{cases}$
5. $|\Sigma U + \Sigma T| = \begin{cases} 0 & \textit{if } \sharp T \equiv g,\ g+1(4) \\ 1 & \textit{else}. \end{cases}$

Proof. For any $x \in B$ the set $B \backslash \{x\}$ is a basis of the F_2-vectorspace F_2^{2g}. The map $s \mapsto |s|+1$ is additive for $s_1, s_2 \in B \backslash \{x\}$ hence

$$|s|+1 = \{y,s\} \quad \text{for} \quad s \in B \backslash \{x\}$$

for some $y \in F_2^{2g}$. Hence

$$|s+y| = |s| + |y| + \{s,y\} = |y|+1$$

does not depend on $s \in B \backslash \{x\}$. By lemma 3.1. we get for $S_0 \subset B$ with $\sharp S_0 = 0(2)$

$$\begin{aligned} |y+\Sigma S_0 + s_1 + s_2 + s_3| &= |y + \Sigma S_0 + s_1| + |y + \Sigma S_0 + s_2| \\ &\quad + |y + \Sigma S_0 + s_3| + \{s_1, s_2, s_3\} \\ &= |y + \Sigma S_0 + s_1| + 1 \end{aligned}$$

by induction. Hence

$$|y + \Sigma T| = |y| + \frac{\sharp T + 1}{2} \quad \text{for} \quad \sharp T \equiv 1(2) \quad \text{and} \quad x \notin T.$$

For two elements we get

$$\begin{aligned} |y + s_1 + s_2| &= |y + s_1| + |s_2| + \{y + s_1, s_2\} \\ &= |y| + 1 + |s_2| + \{y, s_2\} + 1 \\ &= |y| + 1 \end{aligned}$$

hence we get altogether the formula

$$|y + \Sigma T| = \begin{cases} |y| & \text{if } \sharp T \equiv 0,3(4) \\ |y|+1 & \text{else} \end{cases}$$

for $x \notin T \subset B$. In particular, for $B \backslash \{x\}$ we get

$$|y+x| = |y+\Sigma_{s\in B\backslash\{x\}} s| = |y| + g.$$

Now we set $y = \Sigma P$ with $P \subset B$ and $\sharp P \equiv 1(2)$. Then for $s \neq x$ we get

$$|s| + 1 = \{y, s\} = 1 + \sharp(P \cap \{s\})$$

hence the set P contains all the elements in V different from x and no element of U except possibly x. Hence

$$y = \Sigma V + (\sharp V + 1)x$$

and

$$g \equiv |y| + |y+x| = |x| + \{x, \Sigma V\}.$$

If we choose x in U or V we get

$$\sharp V \equiv g \mod (2).$$

An easy consideration of the possible cases yields now the formula in 4, 5 and 3 and by regarding the different cases for the genus g we get 2. □

With the notation of the proof for odd genus $y = \Sigma V$ is independent of the element x and

$$|y+s| = |y| + 1 = \begin{cases} 1 & \text{for } g \equiv 3(4) \\ 0 & \text{else} \end{cases}$$

but for even genus g we have $y = \Sigma V + x$ and

$$|y+s| = |\Sigma V| = \begin{cases} 1 & \text{for } g \equiv 0(4) \\ 0 & \text{else} \end{cases}$$

hence $P = \{\Sigma V + s \text{ with } s \in B\}$ is a *principal set* (Hauptreihe) in F_2^{2g}. The characteristic of every member in a principal set is even for $g \equiv 0$, 1 mod (4) and odd for $g \equiv 2$, 3 mod (4).

The result in 3.6, 2) is the best possible result. If $B = \{x_1, \cdots, x_{2g+1}\}$ is an in pairs azygetic set and $|x_i| = |x_j|$ for $i, j \in \{1, \cdots, 4\}$ then $B' = \{x_1 + x_2 + x_3, x_1 + x_2 + x_4, x_1 + x_3 + x_4, x_2 + x_3 + x_4, x_5, \cdots, x_{2g+1}\}$ is again an in pairs azygetic set, but the number of odd and even elements changed by four. Moreover any in pairs azygetic set can be reached by several application of this operation.

One may regard the above proposition as a cohomological result. Let denote by $\phi_0(x) = |x| + 1$ and by $\phi_{i,j}(x) = |x + x_i + x_j|$ and $\phi_i(x) = |x + x_i|$, then

$$\phi_i+\phi_j=\phi_{i,j}+\phi_0 \quad \text{and} \quad \phi_{i,j}+\phi_{j,k}+\phi_{i,k}=\phi_0$$

as elements in $Hom(\boldsymbol{F}_2^{2g},\boldsymbol{F}_2)$.

The next point is a generalization of 3.5. We denote by $\mathscr{M}$ the set of azygetic sets of cardinality $2g+2$. Moreover we call two such sets M_1 and M_2 equivalent if $M_1=M_2+v$ for some v. Any azygetic set remains azygetic if one adds a fixed vector. One could denote such sets as cosets and may normalize azygetic sets by the assumption of containing 0. For any in pairs azygetic set B of cardinality $2g+1$ we get with $B\cup\{0\}$ a set in $\mathscr{M}$. On the other hand for any azygetic (co)set $M\in\mathscr{M}$ and any $x\in M$ we get with

$$M_x=\{y+x \quad \text{with} \quad y\in M\backslash\{x\}\}$$

an in pairs azygetic set of cardinality $2g+1$. The number of equivalence classes of in pairs azygetic sets of cardinality $2g+1$ is for $g\geq 2$ just

$$[Sp(2g,\boldsymbol{F}_2):S_{2g+2}]=\frac{2^{g^2}(4^g-1)(4^{g-1}-1)\cdots(4^1-1)}{(2g+2)!}$$

There is a classical discussion Stahl / Frobenius, whether it is more natural to regard in pairs azygetic sets of cardinality $2g+1$ or azygetic sets of cardinality $2g+2$.

There is a standard way to get the symplectic form on $\boldsymbol{F}_2^{2g}$. We regard in the $\boldsymbol{F}_2$-vectorspace $\boldsymbol{F}_2^{2g+2}$ the subspaces $\{0,1\}$ and $H=\{x\in\boldsymbol{F}_2^{2g+2} \text{ with } |x|=0\}$. The *weight* $|x|$ is just the number of entries 1 and is used as a number in $\boldsymbol{Z}$ or in $\boldsymbol{F}_2$. The element 1 is just the vector with all entries 1. One has the component-wise product of elements and on $\boldsymbol{F}_2^{2g}\cong H/\{0,1\}$ one has the standard symplectic form

$$\{[x],[y]\}=|xy|.$$

The symmetric group S_{2g+2} is acting on $\boldsymbol{F}_2^{2g+2}$ by permuting the coordinates. This action induces a map from $S_{2g+2}\to Sp(2g,\boldsymbol{F}_2)$ which is an injection for $g\geq 2$. For genus 1 we get a surjection $S_4\to Sp(2,\boldsymbol{F}_2)\cong S_3$, for genus 2 we get an isomorphism $S_6\cong Sp(4,\boldsymbol{F}_2)$. For even genus S_{2g+2} is moreover a maximal subgroup of $Sp(2g,\boldsymbol{F}_2)$. For odd genus the action of the symmetric group preserves a (unique) quadratic form and this gives an inclusion of S_{2g+2} as a maximal subgrop of $O^+(2g,\boldsymbol{F}_2)$ for $g\equiv 3(4)$ and $O^-(2g,\boldsymbol{F}_2)$ for $g\equiv 1(4)$. For small genus we have some exceptional isomorphisms

$$O^+(2,\boldsymbol{F}_2)\cong S_2\subset S_3\cong O^-(2,\boldsymbol{F}_2)\cong Sp(2,\boldsymbol{F}_2)$$

and

$$O^-(4,\boldsymbol{F}_2)\cong S_5\subset Sp(4,\boldsymbol{F}_2)\cong S_6$$

and

$$O^+(6,\boldsymbol{F}_2)\cong S_8 \subset Sp(6,\boldsymbol{F}_2),$$

but for higher genus we get proper maximal subgroups.[1] The orthogonal groups are maximal subgroups in the symplectic groups. Hence for genus 3 S_8 is maximal in $Sp(6,\boldsymbol{F}_2)$. For these results see [6]. For genus $g\geq 3$ the inclusion $S_{2g+2} \subset Sp(2g,\boldsymbol{F}_2)$ is unique up to conjutgation. For genus 2 there are exactly two isomorphisms for $S_6\cong Sp(4,\boldsymbol{F}_2)$ up to conjugation. There is the extraordinary outer automorphism of S_6. This automorphism permutes the transpositions with the triple transpositions. For $n>6$ there are no outer automorphisms ($S_n\cong \mathrm{Aut}(S_n)$). For these results see [36] and [28].

In $\boldsymbol{F}_2^{2g+2}$ we have fixed the hyperplane H of even elements. We fix for odd genus g some $v\in H$ and for even genus g some $v\notin H$. Then the set

$$T=\{v+x \quad \text{with} \quad |x|=g+1\}$$

is a subset of H. We fix the action of S_{2g+2} by permuting the entries of x. This action is conjugated to the standard one. By abuse of notation T is also regarded as an subset of $\boldsymbol{F}_2^{2g}\cong H/\{0,1\}$. Moreover the subset T is stable under S_{2g+2}. Due to Sasaki [35] it holds moreover

Lemma 3.7. *The symmetric group S_{2g+2} is the stabilizer of T.*

Proof. (Sasaki)
Let $\sigma\in Sp(2g,\boldsymbol{F}_2)$ with $\sigma(T)=T$.
First step: $|x|=2 \Rightarrow |\sigma(x)|=2$ or $2g$.
Regard the following sets:

$$W=\{t\in T \quad \text{with} \quad \exists t' \quad \text{and} \quad t+t'=x\}$$

$$W_\sigma=\{t\in T \quad \text{with} \quad \exists t' \quad \text{and} \quad t+t'=\sigma(x)\}.$$

Remark that $T+1=T$, $W+1=W$, $W_\sigma+1=W_\sigma$. Let $|\sigma(x)|=2k$. An easy consideration shows

$$\# W=\frac{1}{2}\binom{2}{1}\binom{2g}{g}$$

and

$$\# W_\sigma=\frac{1}{2}\binom{2k}{k}\binom{2g+2-2k}{g+1-k}$$

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[1] Remark the difference of this notation and the atlas notation [1], where $O_{2g}^*(2)$ denotes (for $g\geq 2$) the simple group of index two in $O^*(2g,\boldsymbol{F}_2)$. Then one gets the exceptional isomorphisms $O_4^-(2)\cong A_5$ and $O_6^+(2)\cong A_8$.

The map $t \mapsto \sigma(t)$ is an injection of W in W_σ, but for $g>k>1$ we have $\sharp W_\sigma < \sharp W$.

Second step:

We identify $1=(1,0,\cdots)$, $2=(0,1,0,\cdots)$,$\cdots$ the numbers $\{1,\cdots,2g+2\}$ with the corresponding standard basis vector. It is enough to show, that up to the action of S_{2g+2} one may arrange that $\sigma(1+k)=1+k$.

The proof is by induction. The first step $\sigma(1+2)=1+2$ is trivial. Let $\sigma(1+3)=a+b$, then $\sigma(2+3)=1+2+a+b$. Hence a or b has to be 1 or 2, but not both cases can occur. Hence after a permutation one may assume that $\sigma(1+2)=1+2$, $\sigma(1+3)=1+3$, $\sigma(2+3)=2+3$ and by induction one gets the result. $\square$

A second proof is as follows: We may suppose $g\geq 3$. The cardinality of T is just $\frac{1}{2}\binom{2g+2}{g+1}=\binom{2g+1}{g}$. Moreover by 3.6. one may arrange up to conjugation for odd genus that $v=\Sigma V$ and $S_{2g+2}\subset O^*(2g,F_2)=\mathrm{Stab}(v)$ (for $g\equiv 1(4)$ one has $*=-$, for $g\equiv 3(4)$ one has $*=+$ in accordance with the characteristic of v). In any case the symmetric group is maximal in the orthogonal or symplectic group, which are acting transitively on F_2^{2g} or the odd or even elements of F_2^{2g}. For higher genus the cardinality of even or odd or all elements of F_2^{2g} is much bigger than $\binom{2g+1}{g}$, hence the result.

4. The moduli space of hyperelliptic curves

Following the notation in [33] we regard the moduli space of hyperelliptic curves for genus $g\geq 2$ as a closed subscheme in the corresponding Satake compactification $\mathscr{A}_g=\mathscr{A}(\Gamma_g)$ (maybe with some level structure). The result of Thomae may be stated as follows ([27], II, 8.13):

Theorem 4.1. *The moduli space of hyperelliptic curves with level-2-structure is given by*

$$\mathscr{H}yp_g(2)=\cup_{B\in\mathscr{B}}Z(B)$$

where B runs over in pairs azygetic sets of cardinality $2g+1$ and

$$Z(B)=\mathrm{Proj}(C[\theta^4[\Sigma V+\Sigma T] \quad \textit{with} \quad \sharp T=g,\ g+1 \quad \textit{and} \quad T\subset B])$$

where B is an in pairs azygetic set of cardinality $2g+1$ with $g+1$ even and g odd characteristics.

The number of irreducible components in $\mathscr{H}yp_g(2)$ is just $[Sp(2g,F_2):S_{2g+2}]$. This is stated for example in [5], p.145. The proof, however, is not correct. Dolgachev / Ortland assert that it is known, that S_{2g+2} is a maximal subgroup of $Sp(2g,F_2)$. But this is true only for even genus and genus 3. In general one has to argue as in the last chapter.

It follows from the smoothness of the locus of indecomposable abelian varieties ([31]) and lemma 3.7. that the irreducible components of $\mathscr{H}yp_g(2)$ only have nonempty intersection in boundary points. Or, in other words, the irreducible components of $\mathscr{H}yp_g(2) \cap H_g/\Gamma_g(2)$ are the connected components.

Proposition 4.2. *An irreducible component of $\mathscr{H}yp_g(2)$ corresponds to an in pairs azygetic set B of cardinality $2g+1$. To any such set B with $g+1$ even and g odd characteristics we get a set of defining equtions in P^{2^g-1}/N_g given by $\theta^2[\Sigma V+\Sigma T](\tau)=0$ for $\sharp T \neq g, g+1$, regarded as equations in the mixed $f_a f_b$ by the Veronese formula.*

See [33] for details. The group N_g is an extension of a finite Heisenberg group acting on the projective space by the Schrödinger representation. This is a translation of Mumfords theorem 9.1. in [27], II via the following lemma.

Lemma 4.3. *It is equivalent to have the following data:*

(1) *Let $U \subset M \subset F_2^{2g}$ with $\sharp U = g+1$ and $\sharp M = 2g+2$ together with an isomorphism*

$$\eta: \begin{Bmatrix} \text{group of subsets } T \subset M \\ \sharp T \text{ even} \bmod T \sim M \backslash T \end{Bmatrix} \overset{\approx}{\to} \mathrm{F}_2^{2g}$$

satisfying

$$\sharp(T_1 \cap T_2) = \{\eta(T_1), \eta(T_2)\}$$

and

$$\frac{\sharp(T \bullet U) - (g+1)}{2} = |\eta(T)|$$

and

(2) *An in pairs azygetic set $B = U \cup V$ of cardinality $2g+1$ with even subset U and odd subset V and $\sharp U = g+1$.*

Proof. One fixes an element $\infty \in M \backslash U$ which is mapped to 0. Then one may write the isomorphism additively as

$$\eta(T) = \sum_{t \in T} \eta(t)$$

and η becomes a homomorphism with respect to the symmetric difference

$$\eta(T_1 \bullet T_2) = \eta(T_1) + \eta(T_2).$$

We have

$$\#(T \bullet U) = \#T + \#U - 2\#(T \cap U)$$

hence the second condition of (1) is equivalent to

$$|\eta(T)| + \frac{1}{2}\#T + \#(T \cap U) = 0 \quad \text{for} \quad \#T \equiv 0(2).$$

In particular, the elements of U are mapped to even elements. If one regards $T_1 = \{\infty, x\}$ and $T_2 = \{\infty, y\}$, the first condition of (1) is equivalent to in pairs azygetic. The rest is clear. □

EXAMPLE 4.4: The genus one case

It is well known that

$$\mathscr{A}_1(2) = \mathscr{A}(\Gamma_1(2)) = \mathrm{Proj}(\boldsymbol{C}[X, Y]) \cong \boldsymbol{P}^1 / N_1 \cong \boldsymbol{P}^1$$

with

$$X = (4) = f_0^4 + f_1^4$$
$$Y = 2(2,2) = 2f_0^2 f_1^2$$

(see for example [32]). The cusps correspond to the nontrivial involutions of the extended Heisenberg group N_1 of order $2^4 = 16$. A cusp is called even/odd iff the square of the corresponding involution is $+1/-1$ on the ring of modular forms (of even weight), which is our fixed homogeneous coordinate ring of the Satake compactification. There are exactly $4^g - 1 = 3$ nontrivial involutions (see [33]).

$R(0,1)$.

The involution is given by

$$f_0 \mapsto f_0$$
$$f_1 \mapsto -f_1 .$$

The cusp is even and the eigenspace condition is $f_1 = 0$. Hence $X = f_0^4$ and $Y = 0$. $R(0,1) = [1:0] = \infty$.

$R(1,0)$

The involution is given by

$$f_0 \mapsto f_1$$

$$f_1 \mapsto f_0 .$$

The cusp is even and the eigenspace condition is $f_0 = f_1$. Hence $X = 2f_0^4 = Y$. $R(1,0) = [1:1] = 1$.

$R(1,1)$.

The involution is given by

$$f_0 \mapsto f_1$$

$$f_1 \mapsto -f_0 .$$

The cusp is odd and the eigenspace condition is $f_1 = if_0$. Hence $X = 2f_0^4$ and $Y = -2f_0^4$, hence $X + Y = 0$. $R(1,1) = [-1:1] = -1$.

The computation corresponds to the usual picture of a fundamental domain for $\Gamma_1(2)$. In particular, we get

$$\boldsymbol{H}_1 / \Gamma_1(2) \cong \boldsymbol{P}^1 \backslash \{1, -1, \infty\}.$$

The group $Sp(2, \boldsymbol{F}_2) \cong S_3$ is acting on $\boldsymbol{P}^1 \cong \mathscr{A}_1(2)$ and permutes the cusps, hence one may ask for the structure as a Specht module. The character table for S_3 is the following one:

	(3)	(2,1)	(1,1,1)
(3)	1	1	1
(2,1)	-1	0	2
(1,1,1)	1	-1	1 .

The columns correspond to the conjugacy classes given by the partition, the rows correspond to the characters of the Specht modules to the partition. In particular, the first row is the trivial representation, the last row is the signum representation and the last column yields the dimension of the corresponding representation (or Specht module).

To identify the action we compute the action of the MacWilliams identity.

$$X \mapsto -\frac{1}{4}((f_0 + f_1)^4 + (f_0 - f_1)^4) = \frac{1}{2}(-X - 3Y)$$

$$Y \mapsto -\frac{1}{4} 2((f_0 + f_1)^2 (f_0 - f_1)^2) = \frac{1}{2}(-X + Y).$$

Hence we get trace 0 for involution on $[\Gamma_1(2), 2]$, which yields

$$[\Gamma_1(2), 2] \cong S^{2,1}$$

as a S_3-Specht module.

EXAMPLE 4.5: The genus two case

It is well known that

$$\mathscr{A}_2(2)=\mathscr{A}(\Gamma_2(2))=\mathrm{Proj}(\boldsymbol{C}[A,B,C,D,E]/f)\cong \boldsymbol{P}^3/N_2\subset \boldsymbol{P}^4$$

with

$$\begin{aligned}
A&=(4)=f_0^4+f_1^4+f_2^4+f_3^4\\
B&=4(1,1,1,1)=4f_0f_1f_2f_3\\
C&=2(2,2,0,0)=2(f_0^2f_1^2+f_2^2f_3^2)\\
D&=2(2,0,2,0)=2(f_0^2f_2^2+f_1^2f_3^2)\\
E&=2(2,0,0,2)=2(f_0^2f_3^2+f_1^2f_2^2)
\end{aligned}$$

and

$$f=B^4+A^2B^2+(C^2D^2+C^2E^2+D^2E^2)-2ACDE-B^2(C^2+D^2+E^2)$$

is a quartic in $\boldsymbol{P}^4$ (see for example [32], the constant factors make the computation easier). The translation to Igusas notation in [19] is given by

$$\begin{aligned}
y_0&=A\\
y_1&=2C+2B\\
y_2&=A+C+D+E\\
y_3&=-2C-2D\\
y_4&=-2E-2D.
\end{aligned}$$

The cusps correspond to the nontrivial involutions of the extended Heisenberg group N_2 of order $2^6=64$. There are exactly $4^g-1=15$ nontrivial involutions (see [33]).

The same computation as in the genus one case yields the following tabel.

cusp	defining equations
01	$0=B=C=E$
02	$0=B=D=E$
03	$0=B=C=D$
10	$A=C,\ B=D=E$
11	$A=-C,\ B=E=-D$
12	$A=C,\ B=-D=-E$
13	$A=-C,\ B=D=-E$
20	$A=D,\ B=C=E$
21	$A=D,\ B=-C=-E$

22	$A=-D,\ B=E=-C$
23	$A=-D,\ B=C=-E$
30	$A=E,\ B=C=D$
31	$A=-E,\ B=C=-D$
32	$A=-E,\ B=D=-C$
33	$A=E,\ B=-C=-D.$

Hence all the 15 boundary components (isomorphic to $\boldsymbol{P}^1$) are given as lines in $\boldsymbol{P}^4$.

The group $Sp(4,\boldsymbol{F}_2)\cong S_6$ is acting on $\mathscr{A}_2(2)\subset \boldsymbol{P}^4$, hence one may ask again for the structure as a Specht module. The interesting part of the character table for S_6 is the following one:

	(2,2,2)	(2,2,1,1)	(2,1,1,1,1)	(1^6)
(6)	1	1	1	1
(5,1)	-1	1	3	5
(3,3)	-3	1	1	5
(2,2,2)	3	1	-1	5
$(2,1^4)$	1	1	-3	5
(1^6)	-1	1	-1	1 .

The columns correspond to the conjugacy classes of the involutions given by the partition, the rows correspond to the characters of the Specht modules to the partition of dimension up to 5. In particular, the first row is the trivial representation, the last row is the signum representation and the last column yields the dimension of the corresponding representation (or Specht module).

To identify the action we compute the action of the MacWilliams identity. In the basis (of the polynomial ring) A, B, C, D, E as above we get the matrix

$$\frac{1}{2}\begin{matrix} -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \end{matrix}\ .$$

Hence the trace is -1. Regarding the character table we conclude, that as a Specht module $[\Gamma_2(2),2]$ is isomorphic to $S^{2,2,2}$ or $S^{5,1}$. But as pointed out in the last chapter there is an (unique up to conjugation) outer automorphism of S_6, which permutes the transpositions with the triple transpositions. Hence the structure as a S_6-Specht module depends on the choice of the isomorphism $S_6\cong Sp(4,\boldsymbol{F}_2)$. Hence we get the following result.

Proposition 4.6. *The vectorspace* $[\Gamma_2(2),2]$ *has two (nonisomorphic) structures as* S_6*-Specht module, which belong to the partition* (2,2,2) *or* (5,1).

REMARK 4.7. In Igusas choice of the isomorphism $S_6 \cong Sp(4,F_2)$ one gets the partition (2,2,2), see [19]. However, this partition is not uniquely determined.

REMARK 4.8. In [13] the proof of (5.2.) is not correct. It is impossible to change the Specht module structure by an outer automorphism to get a (3,3)-module structure on a (2,2,2)-module. Also chapter 4 in [13] contains the same mistake. The action of S_6 by permutation on a 6-dimensional vectorspace with basis $\{x_1,\cdots,x_6\}$ induces on $V=\{x \text{ with } \Sigma x_i=0\}$ automatically the structure as a (5,1)-module and not a (3,3)-module.

The action of the other generators of $Sp(4,F_2)$ is given as follows. The subgroup $Gl(2,F_2)\cong S_3$ is acting by permuting C, D, E. The diagonal elements D_S are acting by $B \mapsto -B$ or by multiplication of two of the three elements C, D, E by (-1). If one regards $[\Gamma_2(2),2]$ as $S_3 \cong Gl(2,F_2)$-module one gets

$$[\Gamma_2(2),2] \cong S^3 \oplus S^3 \oplus S^3 \oplus S^{2,1}.$$

For the restriction of a Specht modules with respect to some smaller S_n there is an easy rule (see [25], p.34), which is given in this special case as

$$S^{2,2,2} \downarrow S_3 \cong S^{2,1} \oplus S^{2,1} \oplus S^{1,1,1}$$

and

$$S^{5,1} \downarrow S_3 \cong S^3 \oplus S^3 \oplus S^3 \oplus S^{2,1}.$$

Hence we have the (5,1)-structure.

If one chooses the following bases

$$x_1 = 2E+2B$$

$$x_2 = 2D+2B$$

$$x_3 = A+C+D+E$$

$$x_4 = 2E-2B$$

$$x_5 = 2C+2B$$

one may identify this basis with tableaus

$$x_1 = (12)(34)(56)$$

$$x_2 = (13)(24)(56)$$

$$x_3 = (13)(25)(46)$$
$$x_4 = (14)(25)(36)$$
$$x_5 = (12)(35)(46)$$

and gets an isomorphism $S_6 \cong Sp(4,F_2)$ with $M = (12)$ acting via the canonical representation belonging to (2,2,2), i.e.

$$M = \begin{pmatrix} 1 & & & & \\ -1 & -1 & & & \\ & & -1 & & -1 \\ -1 & & & -1 & 1 \\ & & & & 1 \end{pmatrix}$$

and some other generators correspond to

$$(E \leftrightarrow D)(B \to -B) = (12)(35)$$
$$(E \leftrightarrow C)(B \to -B) = (24)(56)$$
$$(B \to -B)(C \to -C)(E \to -E) = (13).$$

There is a special property of the genus two case, namely that there is only one type of decomposable points in H_2. The locus of decomposable points is just described by the vanishing of

$$\theta^2\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \sum_{a \in F_2^2} (-1)^{\langle \alpha, \beta \rangle} f_{a+\alpha} f_a$$

and its conjugates under $Sp(4,F_2)$, hence by the usual theta squares for all even characteristics, see for example [32].

The irreducible components after intersecting with the moduli space are Humbert surfaces with discriminant 1 and numbered by the even characteristics. The following table provides the equations of the 10 Humbert surfaces.

(α,β)	defining equation
00	$A+C+D+E=0$
01	$A+D-C-E=0$
02	$A+C-D-E=0$
03	$A+E-C-D=0$
10	$C+B=0$
12	$C-B=0$
20	$D+B=0$
21	$D-B=0$

30	$E+B=0$
33	$E-B=0$.

Hence all the Humbert surfaces are hyperplane sections. Substituting this equations into the quartic equation f one gets a complete square. Hence all the Humbert surfaces are quadrics in $\boldsymbol{P}^3$ counted with multiplicity 2. The quadric equations are as follows:

(α,β)	defining equation
00	$B^2+CD+CE+DE$
01	$B^2+CE-DC-DE$
02	$B^2+DE-CD-CE$
03	$B^2+CD-CE-DE$
10	$AB+DE$
12	$AB-DE$
20	$AB+CE$
21	$AB-CE$
30	$AB+CD$
33	$AB-CD$.

Hence all the Humbert surfaces are nonsingular quadrics isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$. The group $S_6\cong Sp(4,\boldsymbol{F}_2)$ is permuting the Humbert surfaces. The quartic $\mathscr{A}_2(2)$ has a beautiful geometry, which is described in the next chapter.

EXAMPLE 4.9: The genus three case

It is known that $\mathscr{A}_3(2)$ is the quotient of a hypersurface in $\boldsymbol{P}^7$ by a group N_3 (see [32]). But the variety is no longer a complete intersection, although it is normal and Cohen-Macaulay.

Again the cusps correspond to the nontrivial involutions of the extended Heisenberg group N_3 of order $2^8=256$. There are exactly $4^g-1=63$ nontrivial involutions (see [33]).

The group $O^+(6,\boldsymbol{F}_2)\cong S_8$ is acting on $\mathscr{A}_3(2)=\mathscr{A}(\Gamma_3(2)$, hence one may ask (for every weight) again for the structure as a Specht module. The interesting part of the character table for S_8 is the following one:

	(2,2,2,2)	(2,2,2,1,1)	(2,2,1,1,1,1)	$(2,1^6)$	(1^8)
(8)	1	1	1	1	1
(7,1)	-1	1	3	5	7
$(2,1^6)$	-1	-1	3	-5	7
(4,4)	6	0	2	4	14
(2,2,2,2)	6	0	2	-4	14
(1^8)	1	-1	1	-1	1

The columns correspond to the conjugacy classes of the involutions given by the partitions, the rows correspond to the characters of the Specht modules to the partition of dimension up to 15. In particular, the first row is the trivial representation, the last row is the signum representation and the last column yields the dimension of the corresponding representation (or Specht module).

To identify the action we compute the action of the MacWilliams identity. In the basis (given in [32]) we get the matrix

$$-\frac{1}{4}\begin{pmatrix}
2 & 1 & & & & & & & & & & & & & \\
12 & -2 & & & & & & & & & & & & & \\
 & & 2 & 2 & & & & & 1 & & & & & & \\
 & & 2 & 2 & & & & & -1 & & & & & & \\
 & & & & 2 & 2 & & & & 1 & & & & & \\
 & & & & 2 & 2 & & & & -1 & & & & & \\
 & & & & & & 2 & 2 & & & 1 & & & & \\
 & & & & & & 2 & 2 & & & -1 & & & & \\
 & & 8 & -8 & & & & & & & & & & & \\
 & & & & 8 & -8 & & & & & & & & & \\
 & & & & & & 8 & -8 & & & & & & & \\
 & & & & & & & & & & & 2 & -2 & 2 & -2 \\
 & & & & & & & & & & & -2 & 2 & 2 & -2 \\
 & & & & & & & & & & & 2 & 2 & 2 & 2 \\
 & & & & & & & & & & & -2 & -2 & 2 & 2
\end{pmatrix}$$

Hence the trace is -5. Regarding the character table we conclude, that as a Specht module $[\Gamma_3(2),2]$ is isomorphic to $S^{2,2,2,2} \oplus S^{1,1,1,1,1,1,1,1}$.

Proposition 4.10. *The vectorspace of modular forms for level* 2 *in weight* 2 ($[\Gamma_3(2),2]$) *is the unique irreducible* $Sp(6,F_2)$*-module of dimension* 15 *and has as* S_8*-module the structure* $S^{2,2,2,2} \oplus S^{1,1,1,1,1,1,1,1}$.

Proof. The first part follows from [11] and [7] and the atlas of the finite simple groups [1]. The second part follows from that and the character table. □

Igusa proved in [22] the following:

Proposition 4.11. *The closure of the locus of hyperelliptic points is defined by the exact vanishing of one of the* 36 *theta squares.*

Proof. We just have to check the criterion in 4.1 or 4.2. For any in pairs azygetic set of cardinality 7 with 3 odd and 4 even characteristics we have to regard subsets T with $\sharp T \equiv 3,4(4)$ and $\sharp T \neq 3,4$. Hence $T=B$ or $T=\emptyset$ and for the vanishing condition we get that some theta constant is vanishing. On the other hand for any concrete hyperelliptic τ $\binom{2g+1}{g}=35$ theta constants are different from zero. Hence any hyperelliptic point lies on exactly one hypersurface

$$\theta^2\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\sum_{a\in F_2^3}(-1)^{\langle a,\beta\rangle}f_{a+\alpha}f_a=0$$

in $\boldsymbol{P}^7$. □

Hence from the geometric point of view the Humbert surfaces in genus 2 correspond to the locus of hyperelliptic points in genus 3. In level two the equation corresponds to a hyperplane section in $\boldsymbol{P}^{14}\cong \boldsymbol{P}([\Gamma_3(2),2])$. The union of the hyperelliptic components is given as the divisor of a modular form of weight 18 called

$$\chi_{18}=\prod_{even\ m}\theta[m].$$

In the appendix we give a formula for χ_{18} as a polynomial in the f_a following lemma 3.3 and 3.4 in [31]. We use the identities

$$\theta_0\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7=\frac{1}{2}((\theta_0\theta_1\theta_2\theta_3)^2+(\theta_4\theta_5\theta_6\theta_7)^2-(\theta_{10}\theta_{12}\theta_{14}\theta_{16})^2)$$

$$=(8)-2(4,4)+8(2,2,2,2)-64(1,1,1,1,1,1,1,1)$$

(as a polynomial in f_a) and

$$(\theta_{10}\theta_{12}\theta_{14}\theta_{16})=((\theta_0\theta_2\theta_4\theta_6)-(\theta_1\theta_3\theta_5\theta_7))$$
$$(\theta_{20}\theta_{21}\theta_{24}\theta_{25})=((\theta_0\theta_1\theta_4\theta_5)-(\theta_2\theta_3\theta_6\theta_7))$$
$$(\theta_{30}\theta_{33}\theta_{34}\theta_{37})=((\theta_0\theta_3\theta_4\theta_7)-(\theta_1\theta_2\theta_5\theta_6))$$
$$(\theta_{40}\theta_{41}\theta_{42}\theta_{43})=((\theta_0\theta_1\theta_2\theta_3)-(\theta_4\theta_5\theta_6\theta_7))$$
$$(\theta_{50}\theta_{52}\theta_{55}\theta_{57})=((\theta_0\theta_2\theta_5\theta_7)-(\theta_1\theta_3\theta_4\theta_6))$$
$$(\theta_{60}\theta_{61}\theta_{66}\theta_{67})=((\theta_0\theta_1\theta_6\theta_7)-(\theta_2\theta_3\theta_4\theta_5))$$
$$(\theta_{70}\theta_{73}\theta_{75}\theta_{76})=((\theta_0\theta_3\theta_5\theta_6)-(\theta_1\theta_2\theta_4\theta_7))$$

to express $\chi_{18}=P+Q\theta_0\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7$, where P and Q are polynomials (with 64 terms each) in the theta squares of θ_0, θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, θ_7 and then we use the Veronese formula. The writing is not unique due to the relation $\theta_{16}-\theta_8^2=\phi$ in the notation of [31]. (With the help of a computer one regards specializations as a polynomial in three variables.)

REMARK 4.12. In genus 3 the group $S_8 \cong O^+(6, F_2)$ is up to a root of unity just the image of $\Gamma_3(1,2)$. For details see [34].

5. The geometry of $\mathscr{A}_2(2)$

In this chapter we continue to describe the geometry of $\mathscr{A}_2(2)$. For a hypersurface it is very easy to describe the singular locus and the blowing up of the singular locus, which is just the desingularization first studied by Igusa in [23].

We recall the hypersurface equation.

$$f = B^4 + A^2B^2 + (C^2D^2 + C^2E^2 + D^2E^2) - 2ACDE - B^2(C^2 + D^2 + E^2).$$

This equation is S_6-invariant (the action is described in the last chapter). For the singular locus we first remark that the indecomposable points are smooth (see [31]). Hence we may assume that the singular point lies on one of the Humbert surfaces, so after a S_6-action we may assume

$$E - B = 0$$

which is the equation of $H(3,3)$. Moreover we get

$$AB - CD = 0$$

from the quartic. We shall use the well-known Jacobian criterion. The conditions for a singular point are as follows:

$$\begin{aligned}
2AB^2 &= 2CDE \\
4B^3 + 2A^2B &= 2B(C^2 + D^2 + E^2) \\
2C(D^2 + E^2) &= 2ADE + 2CB^2 \\
2D(C^2 + D^2) &= 2ACE + 2DB^2 \\
2E(C^2 + D^2) &= 2ACD + 2EB^2
\end{aligned}$$

hence only the following equations remain:

$$AB = CD \quad \text{and} \quad B^3 + A^2B = B(C^2 + D^2).$$

If $B = 0$ then $CD = 0$ and we get as solution the boundary lines $R(01)$ and $R(02)$. If $B \neq 0$ we get two nonsingular quadrics

$$B^2 + A^2 = C^2 + D^2 \quad \text{and} \quad AB = CD.$$

We have the lines $R(10)$, $R(20)$, $R(22)$ and $R(11)$ in the intersection, hence (by [16] I.7.7.) we get

$$\mathscr{A}_2(2)^{sing} \cap H(3,3) = \cup R(ij)$$

for exactly 6 lines $R(m)$.

It is easy to check the following statements from the explicit description of the hyperplanes $H(q)$ and the lines $R(p)$.

$$R(p) \cong \boldsymbol{P}^1, \; H(q) \cong \boldsymbol{P}^3, \; \mathscr{A}_2(2) \cap H(q) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$$

$$R(p) \subset H(q) \Leftrightarrow |p+q| = 1$$

$$R(p_1) \cap R(p_2) \neq \emptyset \Leftrightarrow \{p_1, p_2\} = 0$$

$$\mathscr{A}_2(2)^{sing} \cap H(q) = \bigcup_{|p+q|=1} R(p)$$

$$\mathscr{A}_2(2)^{sing} = \bigcup R(p).$$

Every line $R(p)$ intersects six other lines in three points.
Any intersection point lies on exactly 3 $R(p_i)$ and it holds

$$p_1 + p_2 + p_3 = 0 \quad \text{and} \quad |p_1| + |p_2| + |p_3| = 0.$$

There are exactly 15 lines $R(p)$ and 15 intersection points.
On any Humbert surface $H(q) \cap \mathscr{A}_2(2)$ are exactly six lines $R(p)$.
Any line $R(p)$ lies on exactly four Humbert surfaces $H(q) \cap \mathscr{A}_2(2)$.
If $R(p_i) \subset H(q) \cap \mathscr{A}_2(2)$ for $i = 1..6$, then $|p_i| = 1$ for all i iff $q = 0$ or $|p_i| = 1$ for exactly two lines. Moreover

$$\sum p_i = 0 \quad \text{and} \quad \sum_{|p_i|=0} p_i = q = \sum_{|p_i|=1} p_i.$$

REMARK 5.1. Some of these results are contained in [13]. Van der Geer studied Humbert surfaces of arbitrary discriminant, which yields more geometry on $\mathscr{A}_2(2)$. Moreover, he computed the Chern numbers of the desingularization $\widetilde{\mathscr{A}_2(2)}$. However, the Chern number $c_3(\widetilde{\mathscr{A}_2(2)})$ is not correct [38].

REMARK 5.2. Some of these results are contained in [5]. For example theorem 5 on page 157. But the proof is not correct. Dolgachev/Ortland are pointing out, that it would be known, that $\Gamma_2(2)$ has no torsion element and acts freely on the Siegel half space, which would indeed imply, that $\boldsymbol{H}_2/\Gamma_2(2)$ is nonsingular. However, Gottschling computed the torsion [14, 15]. It is known that $\Gamma_g(n)$ acts freely for $n \geq 3$, but $\Gamma_g(2)$ always contains torsion elements, see [9]. Nevertheless, the above computation proves that $\boldsymbol{H}_2/\Gamma_2(2)$ is nonsingular, but the proof is quite different.

The next point is the following easy observation. Instead of regarding the theta nullwerte, one may have regarded the theta functions

$$f_a(\tau,z)=\sum_{x\in \boldsymbol{Z}^g} e\left(\tau\left[x+\frac{\alpha}{2}\right]+\left\langle x+\frac{\alpha}{2},2z\right\rangle\right)$$

as analytic functions on $\boldsymbol{H}_g\times \boldsymbol{C}^g$. The finite group H_g is acting on the theta functions in the same way as on the theta constants. On $\boldsymbol{H}_g\times \boldsymbol{C}^g$ we have the action of the semidirect product $\boldsymbol{Z}^{2g}\times\Gamma_g$ by the following formulae:

$$M=\begin{pmatrix} A & B\\ C & D\end{pmatrix} \text{ is acting by } M(\tau,z)=(M\langle\tau\rangle,\ {}^t(C\tau+D)^{-1}Z)$$

and

$$\boldsymbol{Z}^{2g} \text{ is acting by } (x,y)(\tau,z)=(\tau,z+\tau x+y).$$

It is easy to check that one gets an action of the semidirect product. After dividing out the action of $\boldsymbol{Z}^{2g}$ one gets the universal family of principally polarized abelian varieties together with the projection to $\boldsymbol{H}_g$. If one moreover divides out the action of $\Gamma_g(2)$, one gets the universal Kummer variety. The theta functions f_a just induce a map of this universal Kummer variety to $\boldsymbol{P}^{2^g-1}/N_g$. By fixing τ on gets the embedding of the Kummer variety $K(\tau)$ into the projective space divided out by the Heisenberg group N_g (see for example [12], [11]). Moreover the image of $(\tau,0)$ is just the moduli point of the Kummer variety $K(\tau)$.

Now lets come back to the genus two case. The quartic equation remains the same, if one interprets the f_a as $f_a(\tau,z)$ in the definition of A, B, C, D, E. Hence the intersection

$$T_\tau(\mathscr{A}_2(2))\cap\mathscr{A}_2(2)\subset \boldsymbol{P}^4$$

is just the Kummer variety by dimension reasons. (It has to be a hyperplane section and the image of τ must be a singular point, hence the hyperphlane is the tangent plane.) This is theorem 6.1 in [13]. In higher genus we get only an embedding.

In genus one the theta map just fixes an isomorphism of the Kummer curve $K(\tau)$ with $\boldsymbol{P}^1$ together with fixing 4 points $\{1,-1,\infty,\tau\}$.

REMARK 5.3. The next general point is the Prym map. It is given by the Schottky substitution on the graded rings of modular forms in level 2. For details we refer to [33]. In small genus the Prym map is just invers to the Siegel Φ-operator, which induces the embedding of a cusp. The finite group H_g permutes the cusps, so it is enough to regard the cusp at infinity.

Although the Schottky substitution is a morphism of graded rings, some problems come from the fact, that it is given only for weight divisible by 4.

In the genus one case the computation is as follows:

$$\begin{aligned}
S(X) &= S(\frac{1}{2}(\theta^4\begin{bmatrix}0\\0\end{bmatrix}+\theta^4\begin{bmatrix}0\\1\end{bmatrix}))\\
&= \frac{1}{2}(\theta^2\begin{bmatrix}00\\00\end{bmatrix}\theta^2\begin{bmatrix}00\\01\end{bmatrix}+\theta^2\begin{bmatrix}00\\10\end{bmatrix}\theta^2\begin{bmatrix}00\\11\end{bmatrix})\\
&= f_0^4-f_1^4+f_2^4-f_3^4\\
S(Y) &= S(\frac{1}{2}\theta^4\begin{bmatrix}1\\0\end{bmatrix})\\
&= \frac{1}{2}\theta^2\begin{bmatrix}10\\00\end{bmatrix}\theta^2\begin{bmatrix}10\\01\end{bmatrix}\\
&= 2(f_0^2f_2^2-f_1^2f_3^2).
\end{aligned}$$

Hence we get the following map

$$\begin{aligned}
S(X^2) &= A^2-C^2-E^2+B^2\\
S(XY) &= AD-CE\\
S(Y^2) &= D^2-B^2.
\end{aligned}$$

It is easy to check that this induces a ring homomorphism (due to the quartic equation) which is invers to the Φ-operator.

In genus 0 one has the computation:

$$\begin{aligned}
S(T) &= (\theta^2\begin{bmatrix}0\\0\end{bmatrix}\theta^2\begin{bmatrix}0\\1\end{bmatrix})\\
&= f_0^4-f_1^4\\
S(T^2) &= (4)-(2(2,2))^2\\
&= X^2-Y^2.
\end{aligned}$$

The ring homomorphism S is obviously not defined on the whole polynomial ring, but for the projective schemes it is enough to have a homomorphism on the subring of weight divisible by 4.

In [33] we proved an idealtheoretical version of the result of Donagi in [2]. Moreover the Prym map

$$\mathcal{M}_g(2)=\mathcal{M}_g\times_{\mathcal{A}_g}\mathcal{A}_g(2)\rightarrow\mathcal{A}_{g-1}(2)$$

extends to a morphism

$$\mathscr{A}_g(2) \to \boldsymbol{P}(g) = \boldsymbol{P}^{2^g-1}/N_g$$

together with a diagram

$$\begin{tikzcd}
\mathscr{M}_g(2) \arrow[r, "Torelli"] \arrow[dr, "Prym_g"] & \mathscr{A}_g(2) \arrow[r, "Th"] \arrow[dr, "\text{“}Prym\text{”}_g"] & \boldsymbol{P}(g) \\
\mathscr{M}_{g-1}(2) \arrow[r, "Torelli"] & \mathscr{A}_{g-1}(2) \arrow[r, "Th"] \arrow[u] & \boldsymbol{P}(g-1). \arrow[u]
\end{tikzcd}$$

It follows immediately from this diagram that the Prym maps are proper, which is a theorem in [3]. For small genus the dimensions are

$$\begin{tikzcd}
12 \arrow[r, "Torelli"] \arrow[dr, "Prym_5"] & 15 \arrow[r, "Th"] \arrow[dr, "\text{“}Prym\text{”}_5"] & 31 \\
9 \arrow[r, "Torelli"] \arrow[dr, "Prym_4"] & 10 \arrow[r, "Th"] \arrow[u] \arrow[dr, "\text{“}Prym\text{”}_4"] & 15 \arrow[u] \\
6 \arrow[r, "Torelli"] \arrow[dr, "Prym_3"] & 6 \arrow[r, "Th"] \arrow[u] \arrow[dr, "\text{“}Prym\text{”}_3"] & 7 \arrow[u] \\
3 \arrow[r, "Torelli"] & 3 \arrow[r, "Th"] \arrow[u] & 3. \arrow[u]
\end{tikzcd}$$

Now we come to Igusas desingularization, as described in [23]. It is nothing but just the blowing up of $\mathscr{A}_2(2)$ along its singular locus $\mathscr{A}_2(2)^{sing}$ which is an arrangement of 15 lines. We denote the blowing up by $\widetilde{\mathscr{A}_2(2)}$.

Proposition 5.4. *The blowing up $\widetilde{\mathscr{A}_2(2)}$ is nonsingular.*

Proof. The question is local and only unclear for points on the strict transform of some line in the boundary. Because of the group action we may assume, that P is some point on the strict transform of $R(01)$, different from $\{1, -1\}$. Hence we may assume

$$B = C = E = 0 \quad \text{and} \quad A^2 \neq D^2.$$

The blowing up with respect to the ideal (B, C, E) is locally given in $\boldsymbol{P}^4 \times \boldsymbol{P}^2$ by the equations

$$bC = Be,\ eC = cE,\ bE = eB,\ f.$$

Up to permutation we may assume $b=1$, hence $C=Bc$, $E=eB$. The quartic equation yields for the strict transform the equation

$$B^2(1+c^2e^2-c^2-e^2)+A^2+D^2(c^2+e^2-1)=2ADce.$$

The singularity condition is

$$\begin{aligned} 2A &= 2Dce \\ 2B(1+c^2e^2-c^2-e^2) &= 0 \\ 2D(c^2+e^2-1) &= 2Aec \\ B^2(-2c+2ce^2)+D^2(2c) &= 2ADe \\ B^2(-2e+2ec^2)+D^2(2e) &= 2ADc. \end{aligned}$$

The blowing down of P has coordinates $(A:0:0:D:0)$, hence we get the conditions:

$$\begin{aligned} A &= Dce \\ D(c^2+e^2-1) &= Aec = Dc^2e^2 \\ D(cD-Ae) &= 0 \\ D(eD-Ac) &= 0. \end{aligned}$$

If $D=0$ it would follow $A=0$, which is impossible, also $ce \neq 0$. Hence we get

$$c = \frac{A}{D}e = e\frac{D}{A}$$

which implies $A^2 = D^2$, contradiciton. □

REMARK 5.5. Igusa proved more generally, that the blowing up is a desingularization for $g \leq 3$. For arbitrary genus this may be regarded as a special toroidal desingularization, but for genus two the blowing up is more elementary.

REMARK 5.6. It is easy to see that the strict transform of the Humbert surfaces are double lines. By blowing them down, one gets the dual hypersurface in $\boldsymbol{P}^4$, which is known as Segres cubic primal, see [5] for more details. In the coordinates of 4.8. the cubic primal is given by

$$A^3 - 9A(C^2+D^2+E^2) + 36AB^2 + 54CDE = 0,$$

which is the unique S_6-invariant cubic. It has 10 singular points

$$[3:0:1:1:1]$$
$$[3:0:1:-1:-1]$$
$$[3:0:-1:1:-1]$$
$$[3:0:-1:-1:1]$$
$$[0:1:0:0:\sqrt{8}]$$
$$[0:1:0:0:-\sqrt{8}]$$
$$[0:1:0:\sqrt{8}:0]$$
$$[0:1:0:-\sqrt{8}:0]$$
$$[0:1:\sqrt{8}:0:0]$$
$$[0:1:-\sqrt{8}:0:0]$$

and also a beautiful geometry. Hence $\widetilde{\mathcal{A}_2(2)}$ is the common desingularization of a quartic and a cubic hypersurface in $\boldsymbol{P}^4$.

Appendix

In this appendix we give a formula for χ_{18} as a polynomial in the theta constants of second order. The way of computation is described after 4.11.. The notation is according the conventions in [31,32]. In weight 18 (degree 36) there are 190 admissible monomials for polynomial cusp forms (i.e.genus 3 as monomial). Unfortunately 185 coefficients are nonzero. The result is

$$\chi_{18} = 2048*($$

2	(29,1,1,1,1,1,1,1)	−10	(17,13,1,1,1,1,1,1)
−1	(28,2,2,0,2,0,0,2)	−22	(17,9,5,1,1,1,1,1)
1	(26,4,2,0,2,0,2,0)	160	(17,7,3,1,3,1,1,3)
−10	(25,5,1,1,1,1,1,1)	108	(17,5,5,5,1,1,1,1)
4	(24,6,2,0,2,0,0,2)	−84	(17,5,5,1,5,1,1,1)
−1	(24,4,4,0,4,0,0,0)	240	(17,5,3,3,3,3,1,1)
1	(24,4,2,2,2,2,0,0)	−2	(16,12,4,0,4,0,0,0)
32	(23,3,3,3,1,1,1,1)	−3	(16,12,2,2,2,2,0,0)
−4	(22,8,2,0,2,0,2,0)	8	(16,10,6,0,2,0,0,2)
−2	(22,6,4,0,2,2,0,0)	−17	(16,10,4,2,2,0,2,0)
−16	(22,4,4,2,2,0,0,2)	−4	(16,8,8,0,4,0,0,0)
−56	(22,2,2,2,2,2,2,2)	−4	(16,8,6,2,2,2,0,0)
18	(21,9,1,1,1,1,1,1)	−7	(16,8,4,4,4,0,0,0)

28	(21,5,5,1,1,1,1,1)	−9	(16,8,4,0,4,0,4,0)
−80	(21,3,3,1,3,1,1,3)	−62	(16,8,4,0,4,0,0,4)
−5	(20,10,2,0,2,0,0,2)	−65	(16,8,4,0,2,2,2,2)
3	(20,8,4,0,4,0,0,0)	−34	(16,6,6,4,2,0,0,2)
3	(20,8,2,2,2,2,0,0)	56	(16,6,6,0,6,0,0,2)
−10	(20,6,6,0,2,0,0,2)	62	(16,6,6,0,4,2,2,0)
8	(20,6,4,2,2,0,2,0)	40	(16,6,4,2,4,2,0,2)
16	(20,4,4,4,4,0,0,0)	12	(16,4,4,4,4,4,0,0)
44	(20,4,4,0,4,0,0,4)	342	(16,4,4,4,2,2,2,2)
94	(20,4,4,0,2,2,2,2)	64	(15,11,3,3,1,1,1,1)
−96	(19,7,3,3,1,1,1,1)	−64	(15,9,3,1,3,1,3,1)
−16	(19,5,3,1,3,1,3,1)	128	(15,7,7,3,1,1,1,1)
5	(18,12,2,0,2,0,2,0)	64	(15,7,5,1,3,3,1,1)
−1	(18,10,4,0,2,2,0,0)	−384	(15,5,5,3,3,1,1,3)
4	(18,8,6,0,2,0,2,0)	−2048	(15,3,3,3,3,3,3,3)
31	(18,8,4,2,2,0,0,2)	4	(14,14,4,0,2,2,0,0)
−10	(18,6,6,2,4,0,0,0)	−4	(14,12,6,0,2,0,2,0)
−32	(18,6,4,0,4,0,2,2)	−8	(14,12,4,2,2,0,0,2)
−152	(18,6,2,2,2,2,2,2)	12	(14,10,6,2,4,0,0,0)
−130	(18,4,4,2,4,2,2,0)	16	(14,10,4,4,2,2,0,0)
48	(14,10,4,0,4,0,2,2)	44	(12,8,6,6,2,2,0,0)
208	(14,10,2,2,2,2,2,2)	−52	(12,8,6,2,6,0,0)
−48	(14,8,8,2,2,0,0,2)	80	(12,8,6,2,4,0,2,2)
−12	(14,8,6,4,2,0,2,0)	−56	(12,8,4,4,4,4,0,0)
−8	(14,8,6,0,6,0,2,0)	68	(12,8,4,4,4,0,4,0)
−44	(14,8,6,0,4,2,0,2)	−52	(12,8,4,4,2,2,2,2)
−40	(14,8,4,2,4,2,2,0)	−112	(12,6,6,4,6,0,0,2)
24	(14,6,6,6,4,0,0,0)	40	(12,6,6,4,4,2,2,0)
−16	(14,6,6,2,4,4,0,0)	−264	(12,6,6,0,6,0,0,6)
112	(14,6,6,2,4,0,0,4)	−344	(12,6,6,0,4,2,2,4)
32	(14,6,6,2,2,2,2,2)	−224	(12,6,4,2,4,2,4,2)
40	(14,6,4,4,4,0,2,2)	−112	(12,4,4,4,4,4,4,0)
−160	(14,4,4,2,4,2,2,4)	−64	(11,11,7,3,1,1,1,1)
8	(13,13,5,1,1,1,1,1)	−96	(11,11,5,1,3,3,1,1)
−32	(13,11,3,1,3,1,1,3)	64	(11,9,7,1,3,1,3,1)
4	(13,9,9,1,1,1,1,1)	288	(11,9,5,3,3,1,1,3)
−136	(13,9,5,5,1,1,1,1)	−128	(11,7,7,7,1,1,1,1)
56	(13,9,5,1,5,1,1,1)	−64	(11,7,7,3,5,1,1,1)
−160	(13,9,3,3,3,3,1,1)	384	(11,7,5,5,3,3,1,1)
−192	(13,7,7,1,3,1,1,3)	−128	(11,7,5,1,5,1,3,3)
−128	(13,7,5,3,3,1,3,1)	−832	(11,5,5,3,5,3,3,1)
144	(13,5,5,5,5,1,1,1)	−22	(10,10,10,2,4,0,0,0)

528	(13,5,5,1,5,1,1,5)	−38	(10,10,8,4,2,2,0,0)
1088	(13,5,5,1,3,3,3,3)	8	(10,10,8,0,6,2,0,0)
2	(12,12,8,0,4,0,0,0)	−38	(10,10,8,0,4,0,2,2)
−16	(12,12,6,2,2,2,0,0)	−28	(10,10,6,6,4,0,0,0)
−16	(12,12,4,4,4,0,0,0)	96	(10,10,6,2,4,4,0,0)
12	(12,12,4,0,4,0,4,0)	−192	(10,10,6,2,4,0,4,0)
−60	(12,12,4,0,2,2,2,2)	−272	(10,10,6,2,2,2,2,2)
−2	(12,10,10,0,2,0,0,2)	−204	(10,10,4,4,4,0,2,2)
22	(12,10,8,2,2,0,2,0)	−32	(10,8,8,6,2,0,0,2)
40	(12,10,6,4,2,0,0,2)	96	(10,8,8,2,6,0,0,2)
−28	(12,10,6,0,6,0,0,2)	−6	(10,8,8,2,4,2,2,0)
8	(12,10,6,0,4,2,2,0)	76	(10,8,6,4,6,0,2,0)
52	(12,10,4,2,4,2,0,2)	−88	(10,8,6,4,4,2,0,2)
4	(12,8,8,4,4,0,0,0)	120	(10,8,6,0,6,0,6,0)
4	(12,8,8,0,8,0,0,0)	108	(10,8,6,0,6,0,2,4)
6	(12,8,8,0,4,4,0,0)	200	(10,8,6,0,4,2,4,2)
92	(12,8,8,0,4,0,0,4)	236	(10,8,4,2,4,2,2,4)
126	(12,8,8,0,2,2,2,2)	−80	(10,6,6,6,4,4,0,0)
−416	(10,6,6,6,2,2,2,2)	−408	(8,8,8,0,6,2,2,2)
224	(10,6,6,2,6,2,2,2)	−282	(8,8,8,0,4,4,4,0)
88	(10,6,6,2,4,4,4,0)	−144	(8,8,6,6,6,2,0,0)
352	(10,6,4,4,4,4,2,2)	180	(8,8,6,6,4,0,2,2)
316	(9,9,9,5,1,1,1,1)	20	(8,8,6,2,6,2,4,0)
−68	(9,9,9,1,5,1,1,1)	−144	(8,8,6,2,4,4,2,2)
−64	(9,9,7,3,3,3,1,1)	8	(8,8,4,4,4,4,4,0)
−120	(9,9,5,5,5,1,1,1)	−48	(8,6,6,4,6,4,0,2)
−184	(9,9,5,1,5,1,5,1)	104	(8,6,6,4,6,0,0,6)
−608	(9,9,5,1,3,3,3,3)	−232	(8,6,6,4,4,2,2,4)
−320	(9,7,7,5,3,1,1,3)	−1008	(8,4,4,4,4,4,4,4)
128	(9,7,7,1,7,1,1,3)	640	(7,7,7,7,5,1,1,1)
192	(9,7,7,1,5,3,3,1)	−256	(7,7,7,3,5,5,1,1)
128	(9,7,5,3,5,3,1,3)	128	(7,7,5,5,5,1,3,3)
240	(9,5,5,5,5,5,1,1)	−256	(7,5,5,3,5,3,3,5)
448	(9,5,5,5,3,3,3,3)	192	(6,6,6,6,6,2,2,2)
80	(8,8,8,8,4,0,0,0)	−48	(6,6,6,6,4,4,4,0)
−140	(8,8,8,4,8,0,0,0)	512	(6,6,6,2,4,4,4,4)
132	(8,8,8,4,4,4,0,0)	−992	(5,5,5,5,5,5,5,1)
746	(8,8,8,4,2,2,2,2)		

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