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<th><strong>Title</strong></th>
<th>Pseudorank functions on crossed products of finite groups over regular rings</th>
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<tr>
<td><strong>Author(s)</strong></td>
<td>Kado, Jiro</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 22(4) P.821–P.833</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1985</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/4771">https://doi.org/10.18910/4771</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/4771</td>
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Let $R$ be a regular ring with a pseudo-rank function. The collection of all pseudo-rank functions of $R$ (see [2, Ch. 17]) is denoted by $P(R)$ which is a compact convex set, and the extreme boundary of $P(R)$ is denoted by $\partial P(R)$. Our main objective is to study a crossed product $R*G$ of a finite multiplicative group $G$ over a regular ring $R$. A crossed product $R*G$ of $G$ over $R$ is an associative ring which is a free left $R$-module containing an element $x \in R*G$ for each $x \in G$ and the set generated by the symbols $\{x : x \in G\}$ is a basis of $R*G$ as a left $R$-module. Hence every element $\alpha \in R*G$ can be uniquely written as a sum $\alpha = \sum_{x \in G} r_x x$ with $r_x \in R$. The addition in $R*G$ is the obvious one and the multiplication is given by the formulas

$$x \bar{y} = t(x, y)x \bar{y} \quad r \bar{x} = x \bar{r}$$

for all $x, y \in G$ and $r \in R$. Here the twisting $t: G \times G \rightarrow U(R)$ is a map from $G \times G$ to the group of units of $R$ and for fixed $x \in G$, the map $\bar{x}: r \mapsto r \bar{x}$ is an automorphism of $R$. We assume throughout this note that the order $|G|$ of $G$ is invertible in $R$. The Lemma 1.1 of [9] implies that $R*G$ is also a regular ring. First we will study the question whether a pseudo-rank function $P$ of $R$ can be extended to one of $R*G$. We shall show that $P$ is extensible to $R*G$ if and only if $P$ is $G$-invariant, i.e., $P(r) = P(r \bar{x})$ for all $r \in R$ and $x \in G$. More precisely for a $G$-invariant pseudo-rank function $P$, put

$$P^G(\alpha) = |G|^{-1} \sum_{i} P(r_i)$$

for $\alpha \in R*G$ if $\delta(R*G \alpha) = \bigoplus_{i} Rr_i$, where $r_i \in R$. Then $P^G$ is a desired one of $P$.

$R$ admits a pseudo-metric topology induced by each $P \in P(R)$. In [2, Ch. 19], K.R. Goodearl has studied the structure of the completion of $R$ with respect to $P$-metric. Let $\bar{R}$ be the $P$-completion of $R$, let $\bar{P}$ be the extension of $P$ on $\bar{R}$ and let $\phi: R \rightarrow \bar{R}$ be the natural ring map, Our theorems are following:

(1) There exists a crossed product $\bar{R}*G$ and a ring map $\bar{\phi}: R*G \rightarrow \bar{R}*G$ such that the following diagram commute
and \( \bar{P} \) is also \( G \)-invariant and we have \( P^g = (\bar{P})^g \bar{\phi} \).

(2) If \( P \) is in \( \partial G P(R) \), then \( \bar{R}^G \) is a \( P^g \)-completion of \( R^G \) and \( (P)^G \) is an extension of \( P^g \). We have that \( P^g = \sum_i \alpha_i N_i \), where \( N_i \in \partial G P(R^G) \) and \( 0 < \alpha_i < 1 \) and \( \sum_i \alpha_i = 1 \).

Let \( \theta: P(R^G) \rightarrow P(R) \) be the natural restriction-map and we use \( N|_R \) to denote the image of \( N \in P(R^G) \) by \( \theta \). We shall show that for any \( N \in \partial G P(R^G) \), there exists some positive real number \( \alpha \leq 1 \) and some \( N' \in P(R^G) \) such that \( (N|_R)^g = \alpha N + (1-\alpha) N' \).

In the second section we study types of crossed products of finite groups \( G \) over directly finite, left self-injective, regular rings \( R \). We shall show that \( R^G \) is of Type II\( _f \) if and only if \( R \) is of Type II\( _f \).

In the final section we study the fixed ring of a finite group of automorphisms of a regular ring. We shall show that for any \( P \in \partial G P(R) \), \( P|_{R^G} \) is a finite convex combination of distinct extremal elements in \( \partial G P(R^G) \). Under the assumption that \( R \) is a finitely generated projective right \( R^G \)-module, we shall show that for any extremal element \( Q \in \partial G P(R^G) \), there exist some \( P \in P(R) \) some \( Q' \in P(R^G) \) and some real number \( 0 < \alpha \leq 1 \) such that \( P|_{R^G} = \alpha Q + (1-\alpha) Q' \).

1. **Extensions of pseudo-rank functions**

Let \( R \) be a regular ring and we use \( FP(R) \) to denote the set of all finitely generated projective left \( R \)-modules. For modules \( A, B \), \( A \leq B \) implies that \( A \) is isomorphic to a submodule of \( B \).

**Definition [2, p. 226].** A pseudo-rank function on \( R \) is a map \( N: R \rightarrow [0, 1] \) such that

1. \( N(1) = 1 \).
2. \( N(rs) \leq N(r) \) and \( N(rs) \leq N(s) \) for all \( r, s \in R \).
3. \( N(e+f) = N(e) + N(f) \) for all orthogonal idempotents \( e, f \in R \).
4. \( N(r) > 0 \) for all non-zero \( r \in R \),

then \( N \) is called a rank function. We use \( B(R) \) to denote the set of all pseudo-rank functions on \( R \).

**Definition [2, p. 232].** A dimension function on \( FP(R) \) is a map \( D: FP(R) \rightarrow R^+ \) such that

\[
\begin{array}{ccc}
R & \xrightarrow{\phi} & \bar{R} \\
\downarrow & & \downarrow \\
R^G & \xrightarrow{\bar{\phi}} & \bar{R}^G
\end{array}
\]
PSEUDO-RANK FUNCTION

(1) \( D(R) = 1 \)

(2) If \( A, B \in FP(R) \) and \( A \leq B \), then \( D(A) \leq D(B) \).

(3) \( D(A + B) = D(A) + D(B) \) for all \( A, B \in FP(R) \).

Let \( D(R) \) denote the set of all dimension functions on \( FP(R) \).

Pseudo-rank functions on \( R \) and dimension functions on \( FP(R) \) are equivalent functions as follows.

**Lemma 1** [2, Prop. 16.8]. There is a bijection \( \Gamma_R : P(R) \rightarrow D(R) \) such that \( \Gamma_R(P)(Rr) = P(r) \) for all \( P \in P(R) \) and \( r \in R \).

We always view \( R \) as a subring \( R*G \) via the embedding \( r \rightarrow r1 \). Then there exists a restriction-map \( \theta : P(R*G) \rightarrow P(R) \). We consider the same connections between \( D(R*G) \) and \( D(R) \). For all \( D \in D(R*G) \) and \( A \in FP(R) \), define \( (D|R)(A) = D(R*G \otimes_R A) \). We can easily see that \( D|R \) is a dimension function on \( FP(R) \) and \( \Gamma_{R*G}(N)|_R = \Gamma_R(N|_R) \).

**Lemma 2.** Let \( N \) be in \( P(R*G) \) and \( D \) be in \( D(R*G) \). Then we have that \( (N|R)(r) = (N|(r\tilde{r})) \) and that \( (D|R)(Rr) = (D|(r\tilde{r})) \) for all \( r \in R \) and all \( x \in G \).

Proof. Since \( R*G \otimes_R Rr \approx R*Gr \approx R*Gx^{-1}rx = R*Grx \approx R*G \otimes_R R\tilde{r} \), we have \( (D|R)(Rr) = (D|(R\tilde{r})) \) and \( (N|R)(r) = (N|(r\tilde{r})) \).

Now we shall define an extended dimension function on \( R*G \) for a \( G \)-invariant \( D \in D(R) \). Note that for \( A \in FP(R*G) \), \( _eA \in FP(R) \).

**Proposition 3.** Let \( D \) be a \( G \)-invariant dimension function on \( FP(R) \). Put \( D^G(A) = |G|^{-1}D(\Lambda A) \) for all \( A \in FP(R*G) \). Then \( D^G \) is a dimension function on \( FP(R*G) \) and \( D^G|_R = D \).

Proof. Since \( \_e(R*G) \) isomorphic to \( |G| \) copies of \( R \), \( D^G(R*G) = 1 \). We can easily check that \( D^G \) satisfies the properties (2) and (3). Since \( _e(R*Gr) \approx \oplus_{x \in G} Rx^2 \) and \( D \) is \( G \)-invariant, then we have \( D^G(R*Gr) = |G|^{-1} \sum_{x \in G} D(Rx^2) = D(Rr) \) for all \( r \in R \). Every \( A \in FP(R) \) is isomorphic to a finite direct sum of cyclic left ideals of \( R \). Therefore we have \( (D^G|R)(A) = D(A) \) for all \( A \in FP(R) \).

**Corollary 4.** Let \( P \) be a \( G \)-invariant pseudo-rank function on \( R \). Define \( P^G(\alpha) = (\Gamma_R(P)(R*G\alpha)) \) for all \( \alpha \in R*G \), then

(1) \( P^G \) is a pseudo-rank function on \( R*G \) and \( P^G|_R = P \)

(2) We have \( P^G(\alpha) = |G|^{-1} \sum_i P(r_i) \), if \( _e(R*G\alpha) \approx \oplus_i Rr_i \), where \( r_i \in R \).

Proof. (1) is clear by lemma 1 and Proposition 3. Recall that \( \Gamma_R(P) \) is \( G \)-invariant dimension function on \( FP(R) \) by Lemma 1. Since \( P^G(\alpha) = |G|^{-1} \Gamma_R(P)(R*G\alpha) = |G|^{-1} \sum_i \Gamma_R(P)(Rr_i) = |G|^{-1} \sum_i P(r_i) \), we have completed the proof.
Lemma 5. Let \( N \) be a pseudo-rank function on \( R^*G \). Then we have that \( N(\alpha) \leq |G| (N|_R)^G(\alpha) \) for all \( \alpha \in R^*G \).

Proof. Put \( N|_R = P \). Since \( \Gamma^R_G(N)|_R = \Gamma^R_G(P) \), then we have
\[
\Gamma^R_G(P)(R^*G\alpha) = (\Gamma^R_G(N)|_R)((R^*G\alpha)) = \Gamma^R_G(N)(R^*G \otimes_R R^*G\alpha).
\]
On the other hand, there exists a natural epimorphism \( (R^*G \otimes_R R^*G) \to R^*G \alpha \). Since this map splits, we have \( N(\alpha) = \Gamma^R_G(N)(R^*G\alpha) \leq \Gamma^R_G(N)(R^*G \otimes_R R^*G\alpha) \). We have obtained that \( N(\alpha) \leq |G| P^G(\alpha) \) by Corollary 4.

Definition [2, Ch. 19]. Let \( P \) be in \( P(R) \). \( R \) admits a pseudo-metric \( \delta \) by the rule: \( \delta(r, s) = P(r-s) \). Note that \( \delta \) is a metric if and only if \( P \) is a rank function. We call \( \delta \) the \( P \)-metric. Let \( \bar{R} \) be the completion of \( R \) with respect to \( \delta \) and we call it the \( P \)-completion of \( R \). \( \bar{R} \) is a unit-regular, left and right self-injective ring by [2, Th. 19.7]. There exists a natural ring map \( \phi: R \to \bar{R} \) and a continuous map \( P: \bar{R} \to [0, 1] \) such that \( P\phi = P \). By [23, Th. 19.6], \( P \) is a rank function on \( \bar{R} \). Put \( \ker P = \{ r \in R : P(r) = 0 \} \), which is a two-sided ideal. \( P \) induces the rank function \( \bar{P} \) on \( R/\ker P \). Then \( R \) is equal to the \( \bar{P} \)-completion of \( R/\ker P \) and ker \( \phi = \ker P \).

Now let \( R^*G \) be a given crossed product of a finite group \( G \) over a regular ring \( R \) and let \( P \) be a \( G \)-invariant pseudo-rank function. Since \( P \) is \( G \)-invariant, \( \ker P \) is \( G \)-invariant ideal and therefore each automorphism \( \tilde{x} \) of \( R/\ker P \) and \( \tilde{x} \) is uniformly continuous with respect to the induced metric. Consequently we have an automorphism of \( \bar{R} \), which is again denoted by \( \tilde{x} \), such that \( \phi(r) = \phi(r) \tilde{x} \) for all \( r \in R \). Let a map \( t': G \times G \to U(\bar{R}) \) be \( t'(x, y) = \phi(t(x, y)) \) for all \( x, y \in G \). Here of course \( t: G \times G \to U(\bar{R}) \) is the given map for \( R^*G \). We define a crossed product \( \bar{R}^*G \) of \( G \) over \( \bar{R} \) using multiplication formula \( (a\tilde{x})(b\tilde{y}) = (ab^{\tilde{x}-1}t'(x, y))\tilde{xy} \) for \( a, b \in R \) and \( x, y \in G \), and define a map \( \bar{\phi}: R^*G \to \bar{R}^*G \) by the rule: \( \bar{\phi}(\sum_{x \in G} \tilde{x} \tilde{y}) = \sum_{x \in G} \phi(r) \tilde{x} \). Then \( \bar{\phi} \) is a ring homomorphism and the following diagram is commutative
\[
\begin{array}{c}
R \quad \xrightarrow{\phi} \quad \bar{R} \\
\downarrow \quad \downarrow \\
R^*G \quad \xrightarrow{\bar{\phi}} \quad \bar{R}^*G
\end{array}
\]

Proposition 6. Let \( P \) be a \( G \)-invariant pseudo-rank function on \( R \), let \( \bar{R} \) be a \( P \)-completion, let \( \bar{P} \) be a continuous extension of \( P \) and let \( \phi: R \to \bar{R} \) the natural map. Then we have the relationship between \( P^G \) and \( \bar{P}^G \) such that the following diagram is commutative
\[
\begin{array}{c}
R^*G \quad \xrightarrow{P^G} \quad [0, 1] \\
\downarrow \quad \bar{\phi} \\
\bar{R}^*G \quad \xrightarrow{\bar{P}^G} \quad [0, 1]
\end{array}
\]
Proof. For \( \alpha \in R^*G \), we assume that \( \mathbb{R}(R^*G\alpha) = \bigoplus_i Rr_i \), where \( r_i \in R \). We have

\[
\Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R} \otimes_R (R^*G\alpha)) = \Gamma_{\mathbb{R}}(\mathbf{P})(\bigoplus_i \mathbb{R}\phi(r_i)) = \sum_i \Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R}\phi(r_i)) = \Gamma_{\mathbb{R}}(\mathbf{P})(Rr_i) = \Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R}(R^*G\alpha)) \cdots (*)
\]

Consider the natural map \( \psi : \mathbb{R} \otimes_R (R^*G\alpha) \rightarrow \mathbb{R}\phi(R^*G\alpha) = (\mathbb{R}^*G)\phi(\alpha) \). Since \( \psi \) is an epimorphism as a \( \mathbb{R} \)-module, we have

\[
\mathbb{R}((\mathbb{R}^*G)\phi(\alpha)) \subseteq \mathbb{R} \otimes_R (R^*G\alpha) .
\]

Therefore we have

\[
(P)^G(\phi(\alpha)) = (\Gamma_{\mathbb{R}}(\mathbf{P}))^G((\mathbb{R}^*G)\phi(\alpha)) = |G|^{-1} \Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R}^*G)\phi(\alpha) \leq |G|^{-1} \Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R} \otimes_R (R^*G\alpha)) = |G|^{-1} \Gamma_{\mathbb{R}}(\mathbf{P})(\mathbb{R}(R^*G\alpha)) \cdots (by \ (*) )
\]

\[
P^G(\alpha) .
\]

Since \((P)^G(\phi(\alpha)) \leq P^G(\alpha)\) for all \( \alpha \in R^*G \), we have \((P)^G\phi = P^G\) by [2, Lemma 16.13].

**Definition** [2, Ch. 16 and Appendix]. For a regular ring \( R \), we view \( P(R) \) as a subset of the real vector space \( \mathbb{R}^g \), which we equip with the product topology. Then \( P(R) \) is a compact convex subset of \( \mathbb{R}^g \) by [2, Prop. 16.17]. A **extreme point** of \( P(R) \) is a point \( P \in P(R) \) which cannot be expressed as a positive convex combination of distinct two points of \( P(R) \). We use \( \partial P(R) \) to denote the set of all extreme points of \( P(R) \). The important result is that \( P(R) \) is equal to the closure of the convex hull of \( \partial P(R) \) by Krein-Milman Theorem.

**Theorem 7.** Let \( R^*G \) be a crossed product of a finite group \( G \) over a regular ring \( R \) with \( |G|^{-1} \in R \). Let \( P \) be a \( G \)-invariant extreme point of \( P(R) \), let \( \bar{R} \) be the \( P \)-completion of \( R \), let \( \phi : R \rightarrow \bar{R} \) be the natural ring map and let \( \bar{P} \) be the continuous extension of \( P \) over \( \bar{R} \).

1. The crossed product \( \bar{R}^*G \) of \( G \) over \( \bar{R} \) defined above, is the completion of \( R^*G \) with respect to \( P^G \)-metric.

2. The extension \( P^G \) can be expressed as a positive convex combination of finite distinct elements in \( \partial_{\mathbb{R}}(R^*G) \), i.e., \( P^G = \sum_i \alpha_i N_i \), where \( N_i \in \partial_{\mathbb{R}} P(R^*G) \), \( 0 < \alpha_i < 1 \) and \( \sum_i \alpha_i = 1 \).

Proof. Since \( P \in \partial_{\mathbb{R}} P(R) \), \( \bar{R} \) is a simple, left and right self-injective, regular
ring by [2, Th. 19.2 and Th. 19.14]. Since \(|G|\) is invertible in \(\bar{R}\), we can easily check that \(\bar{R}^*G\) is self-injective on both sides by the routin way. Since \(\bar{R}\) is a simple ring, \(\bar{R}^*G\) is a finite direct product of simple rings by [8, Cor. 3.10]. Therefore, by [2, Cor. 21.12 and Th. 21.13], \(R^*G\) is complete with respect to the metric induced by any rank function and so is especially with respect to the \((\bar{P})^G\)-metric. We have already shown that \((\bar{P})^G\phi = P^G\) by Proposition 6. Finally we shall show that \(\text{Im} \phi\) is dense in \(R^*G\) with respect to \((\bar{P})^G\)-metric. For any \(\alpha = \sum_{x \in G} a_x x \in R^*G\) and any \(\varepsilon > 0\), there exist \(r_x \in R\) for each \(a_x\) such that \(\bar{P}(a_x - \phi(r_x)) < \varepsilon |G|^{-1}\). Put \(\beta = \sum_{x \in G} r_x x\). Then we have that

\[
(\bar{P})^G(\alpha - \phi(\beta)) = (\bar{P})^G(\sum_{x \in G} (a_x - \phi(r_x)) x) \\
\leq \sum_{x \in G} (\bar{P})^G((a_x - \phi(r_x)) x) \\
\leq \sum_{x \in G} (\bar{P})^G((a_x - \phi(r_x)) x) < \varepsilon.
\]

Thus we have completed the proof of (1). Since the \(P^G\)-completion \(\bar{R}^*G\) of \(R^*G\) is a finite direct product of simple rings, \(P^G\) is a positive convex combination of finite distinct extreme points in \(P(R^G)\) by [2, Th. 19.19].

A simple, left and right self-injective, regular ring \(R\) has a unique rank function \(N\) and it is complete with respect to \(N\)-metric and these rings are classified into two types according to the range of \(N\), namely

(1) \(R\) is artinian if and only if the range of \(N\) is a finite set.

(2) \(R\) is non-artinian if and only if the range of \(N\) equal to \([0, 1]\) ([4]).

For a given \(Q \subseteq \partial P(R)\), the \(Q\)-completion \(\bar{R}\) of a regular ring \(R\) is a simple, left and right self-injective, regular ring by [2, Th. 19.14]. Hence we call \(Q\) to be discrete if \(\bar{R}\) is artinian and to be continuous if \(\bar{R}\) is non-artinian.

**DEFINITION.** Let \(P\) be a \(G\)-invariant pseudo-rank function on \(R\). If \(P^G = \sum_i \alpha_i N_i\), where \(N_i \in \partial P(R^*G)\), \(0 < \alpha_i < 1\) and \(\sum_i \alpha = 1\), then we call \(N_i\), \(\ldots\), \(N_i\) to be associated with \(P\).

**Proposition 8.** For a given crossed product \(R^*G\), let \(P\) be a \(G\)-invariant extremal pseudo-rank function on \(R\) and let \(N_1, \ldots, N_t\) be extremal pseudo-rank functions associated with \(P\). Then the following conditions are equivalent:

(1) \(P\) is discrete.

(2) \(N_i\) is discrete for some \(i\).

(3) \(N_j\) is discrete for all \(j = 1, \ldots, t\).

Consequently the following conditions are also equivalent:

(1) \(P\) is continuous.

(2) \(N_i\) is continuous for some \(i\).

(3) \(N_j\) is continuous for all \(j = 1, \ldots, t\).
Proof. Let \( \bar{R} \) be the \( P \)-completion of \( R \) and let \( \bar{P} \) be the extension of \( P \) on \( R \). By Theorem 7, the crossed product \( \bar{R}^*G \) is the \( P^G \)-completion of \( R^*G \) and \( (\bar{P})^G \) is the extension of \( P^G \). Let \( \bar{N}_i \) be the continuous extension of \( N_i \) on \( \bar{R}^*G \). The each \( \ker \bar{N}_i \) is a maximal two-sided ideal and each \( \bar{R}^*G/\ker \bar{N}_i \) is a regular, left and right self-injective ring by \([2, \text{Th. 9.13}]\). Since \( 0=\ker(\bar{P})^G = \bigcap_{i=1}^n \ker \bar{N}_i \), then we have \( \bar{R}^*G = \prod_{i=1}^n \bar{R}^*G/\ker \bar{N}_i \).

And \( \bar{R}^*G/\ker \bar{N}_i \) is isomorphic to the \( N_i \)-completion of \( R^*G \). We assume that \( P \) is discrete. So \( \bar{R} = R/\ker P \) is a simple artinian ring. Then the crossed product \( \bar{R}^*G \) is semi-simple by \([9, \text{Lemma 1.1}]\). In particular each \( \bar{R}^*G/\ker \bar{N}_j \) is an artinian ring, and thus \( N_j \) is discrete for all \( j \). Next we assume that some \( N_i \) (say \( i=1 \)) is discrete. Let \( \bar{N}_i \) be the induced rank function on \( \bar{R}^*G/\ker \bar{N}_i \) by \( \bar{N}_i \), and let \( \pi: \bar{R} \to \bar{R}^*G/\ker \bar{N}_i \) be the map obtained by compositing \( \bar{R} \to \bar{R}^*G \to \bar{R}^*G/\ker \bar{N}_i \). Then \( \pi \) is monomorphism and we have \( \bar{N}_i \pi = P \). By the assumption, the range of \( \bar{N}_i \) is a finite set and so is the range of \( \bar{P} \). Then \( P \) is discrete. Since each extremal pseudo-rank function is either discrete or continuous, latter assertion is clear.

For \( N \in \partial_\epsilon P(R^*G) \), we have the following relationship between \( N \) and \( (N|_R)^G \).

**Theorem 9.** Let \( R^*G \) be a crossed product of a finite group \( G \) over a regular ring \( R \) with \( |G|^{-1} \in R \) and let \( N \) be extremal pseudo-rank function on \( R^*G \). Then we have \( (N|_R)^G = \alpha N + (1-\alpha)N' \) for some \( N' \in P(R^*G) \) and some positive real number \( \alpha \leq 1 \).

Proof. Put \( N|_R = P \), then \( P \) is \( G \)-invariant by Lemma 2. Let \( T \) be the \( P^G \)-completion on \( R^*G \) and let \( \bar{P}^G \) be the extension of \( P^G \) on \( T \). Since \( N \) is uniformly continuous with respect to \( P^G \)-metric by Lemma 5, we have the continuous extension \( \bar{N} \) of \( N \) on \( T \). By \([2, \text{Th. 19.22}]\), there exists a non-zero central idempotent \( e \in T \) such that \( \ker \bar{N} = (1-e)T \) and \( Te \) is a simple ring. Since \( Te \) has the unique rank function, \( \bar{P}^G \) and \( \bar{N} \) induce the same rank function \( Q \) on \( Te \), i.e., \( Q(te) = \bar{P}^G(e)^{-1} \bar{P}^G(te) = \bar{N}(te) = \bar{N}(t) \) for all \( te \in Te \). Put \( L(t) = \bar{P}^G(1-e)^{-1} \bar{P}^G(t(1-e)) \) for any \( t \in T \), then \( L \) is a pseudo-rank function on \( T \). We have \( \bar{P}^G = \bar{P}^G(e)N + \bar{P}^G(1-e)L \) by investigating the decomposition \( T = Te \oplus T(1-e) \). Let \( \phi: R^*G \to T \) be the natural map and let \( N' = L \phi \) and let \( \alpha = \bar{P}^G(e) \). Then we have that \( (N|_R)^G = \alpha N + (1-\alpha)N' \).

**Remark.** For a \( G \)-invariant element \( P \in \partial_\epsilon P(R) \), let \( N_1, \ldots, N_n \) be elements in \( \partial_\epsilon P(R^*G) \) associative with \( P \). We can easily prove that \( \{N_1, \ldots, N_n\} \) is
equal to the set \( \{N \in \partial_e P(R^*G) : \theta(N) = N |_R = P \} \), where \( \theta : P(R^*G) \to P(R) \), by theorem 7 and Theorem 9. Unfortunately we don't know whether \( N |_R \) is always extremal for any extremal pseudo-rank function \( N \) on \( R^*G \) or not.

Now we consider a pseudo-rank function \( P \) which is not necessarily \( G \)-invariant. For each \( x \in G \), put \( P^x(r) = P(r^{x^{-1}}) \) for all \( r \in R \). Then \( P^x \) is also a pseudo-rank function and \( \ker P^x = (\ker P)^x \). Put \( t(P) = \sum_{x \in G} |G|^{-1} P^x \), then \( t(P) \) is \( G \)-invariant pseudo-rank function with \( P \leq |G| |t(P)\). We call \( t(P) \) to the trace of \( P \).

**Proposition 10.** Let \( R^*G \) be a crossed product of a finite group \( G \) over a regular ring \( R \) whit \( |G|^{-1} \in R \). Let \( P \) be in \( \partial_e P(R) \) which is not necessarily \( G \)-invariant and let \( t(P) \) be the trace of \( P \). Then the extension \( t(P) \) can be expressed as a positive convex combination of finite distinct elements in \( \partial_e P(R^*G) \).

**Proof.** Let \( \overline{R} \) be the \( t(P) \)-completion of \( R \). Since \( t(P) \) is a finite convex combination of extreme points in \( P(R) \), \( \overline{R} \) is a finite direct product of simple regular self-injective rings by [2, Th. 19.19], \( R^*G \) is also a finite direct product of simple regular self-injective rings. In the same way as in the proof of Theorem 7, we can prove that \( \overline{R}^*G \) is the \( t(P) \)-completion of \( R^*G \) and that \( \overline{t}(P)^G = \sum_1^t \alpha_i \overline{N}_i \), where \( \overline{N}_i \in \partial_e P(\overline{R^*G}) \), \( 0 < \alpha_i < 1 \) and \( \sum \alpha_i = 1 \).

**Corollary 11.** Let \( R^*G \) be a crossed product of a finite group \( G \) over a regular ring \( R \) with \( |G|^{-1} \in R \). If \( \partial_e P(R) \) is a finite set, then \( \partial_e P(R^*G) \) is also a finite set.

**Proof.** Let \( \partial_e P(R) = \{P_1, \ldots, P_t\} \) and let \( \{N_{ij} : j = 1, \ldots, s(i)\} \) be extremal pseudo-rank functions associated with \( t(P_i) \) for each \( i = 1, \ldots, t \) by Proposition 10. We shall show that \( \partial_e P(R^*G) = \{N_{ij} : i = 1, \ldots, t; j = 1, \ldots, s(i)\} \). We choose \( N \in \partial_e P(R^*G) \) and put \( P = N |_R \). Since \( P(R) \) is equal to the convex-hull of \( \{P_1, \ldots, P_t\} \) by [2, A.6], \( P = \sum_1^t \alpha_i P_i \), for some \( 0 < \alpha_i < 1 \) and \( \sum \alpha_i = 1 \). Put \( Q = \sum_1^t \alpha_i t(P_i) \), then \( Q \) is \( G \)-invariant and \( Q^G = \sum_1^t \alpha_i t(P_i)^G \). Since \( P_i \leq |G| |t(P_i) \) for each \( i = 1, \ldots, t \), \( P \leq |G| Q \) and so \( P^G \leq |G| Q^G \). Let \( T \) be the \( Q^G \)-completion of \( R^*G \) and \( \overline{Q^G} \) (resp. \( \overline{P^G} \)) be the extension of \( Q^G \) (resp. \( P^G \)) on \( T \). Since \( N \leq |G| P^G \) on \( R^*G \) by Lemma 5, \( \overline{N} \leq |G| \overline{P^G} \) on \( T \), \( \ker \overline{Q^G} \subseteq \ker \overline{N} \). Let \( \overline{N}_{ij} \) be the extension of \( N_{ij} \) on \( T \) for each \( i, j \). Since \( Q^G \) is a convex combination of \( \{N_{ij} : i = 1, \ldots, s(i)\} \in \partial_e P(R) \), \( \overline{Q^G} \) is a convex combination of \( \{\overline{N}_{ij} : i = 1, \ldots, s(i)\} \in P(T) \). Then we have \( \cap_{i,j} \ker \overline{N}_{ij} = \ker \overline{Q^G} \) and therefore \( \ker \overline{N}_{ij} \subseteq \ker \overline{N} \) for some \( i, j \) by primeness of \( \ker \overline{N} \). Since \( \ker \overline{N}_{ij} \) is also a maximal ideal by [2, Th. 19.22], \( \ker \overline{N}_{ij} = \ker \overline{N} \). Consequently we have \( \overline{N}_{ij} = \overline{N} \) by [5, Prop. II. 14.5] and hence \( N_{ij} = N \).
2. Directly finite left self-injective regular rings

In this section, we consider a crossed product of a finite group $G$ over a directly finite, left self-injective, regular ring $R$ with $|G|^{-1} \in R$. K.R. Goodearl has constructed a structure theory on self-injective regular rings. Now we refer to [3, Ch. 10] for definitions and notations. We study types of crossed products $R^*G$. We begin with the following lemma.

**Lemma 12** [4, II. 14.5]. Let $R$ be a directly finite, left self-injective, regular ring. We define a map $v: \partial, P(R) \rightarrow \text{Max}(R)$ by the rule: $v(P) = \ker P$. Then $v$ is a bijection.

**Theorem 13.** Let $R$ be a directly finite, left self-injective, regular ring and let $G$ be a finite group such that $|G|^{-1} \in R$. Then the following conditions are equivalent.

1. A crossed product $R^*G$ of $G$ over $R$ is of Type II$_f$.
2. $R$ is of Type II$_f$.

**Proof.** We know that $R^*G$ is a directly finite, left self-injective, regular ring.

(1) $\Rightarrow$ (2). It suffices to prove that $R$ has no simple artinian homomorphic images, by [3, Th. 7.10 and Th. 10.24]. Assume that there exists $M \in \text{Max}(R)$ such that $R/M$ is artinian. By Lemma 12, we have $P \in \partial, P(R)$ such that $\ker P = M$. Let $H$ be the stabilizer of $M$ in $G$ and let $\Lambda$ be a transversal for $H$ in $G$ with $1 \in \Lambda$. Let $J = \cap_{y \in \Lambda} M^y$, then $J$ is $G$-invariant and $R/J \simeq \Pi_{y \in \Lambda} R/M^y$. Since each $y$ induces an automorphism on $R/J$, there gives rise to a crossed product $(R/J)^*G$ of $G$ over $R/J$ with the natural map $\psi: R^*G \rightarrow (R/J)^*G$. Since $R/J$ is a semi-simple ring, so is the crossed product $(R/J)^*G$ by [9, Lemma 1]. Since $\psi$ is epimorphism, $R^*G$ has a simple artinian homomorphic image. This contradicts that $R^*G$ is of Type II$_f$ by [2, Th. 10.29].

(2) $\Rightarrow$ (1). Assume that there exists $N \in \partial, P(R^*G)$ such that $R^*G/\ker N$ is artinian. Put $P = N|_G$. Since $\ker P = \ker N \cap R$ and $\ker N$ is a maximal ideal of $R^*G$, $\ker P = \cap_{y \in \Lambda} I^y$, where $I$ is a maximal ideal of $R$ by [7, p. 295]. Let $K$ be the stabilizer of $I$ in $G$ and let $\Lambda$ be a transversal for $K$ in $G$ with $1 \in \Lambda$. Then $R/\ker P \simeq \Pi_{y \in \Lambda} R/I^y$, where $R/I^y$ is simple, left and right self-injective, regular ring. We claim that all $R/I^y$ is artinian. Since $\ker P$ is a $G$-invariant ideal, there exists a crossed product $(R/\ker P)^*G$ of $G$ over $R/\ker P$ such that $R^*G/(\ker P)^*G \simeq (R/\ker P)^*G$. By [8, Cor. 3.10], $R^*G/(\ker P)^*G$ is a finite direct of simple, left and right self-injective, regular rings. Since $(\ker P)^*G \subset \ker P \subset \ker N$, $R^*G/\ker N$ is isomorphic to a simple component of $R^*G/(\ker P)^*G \simeq (R/\ker P)^*G$. By considering $\Pi_{y \in \Lambda} R/I^y \simeq R/\ker P \subset (R/\ker P)^*G$, we find a ring homomorphism $f: \Pi_{y \in \Lambda} R/I^y \rightarrow R^*G/\ker N$. Then we have a ring-monomorphism $f': T = \Pi_{y \in \Lambda} R/I^y \rightarrow R^*G/\ker N$ for some
\( \Lambda' \subset \Lambda \). Let \( N \) be the unique rank function of \( R^*G/\ker N \) and let \( P \) be the unique rank function of \( R/I \) and put \( Q = Nf' \). This is a rank function on \( T \). Let \( e \) be a central idempotent of \( T \) which is identity element for \( R/I \). By the uniqueness of rank function on \( R/I \), we have \( P(\alpha) = Q(e_\alpha)^{-1}Q(\alpha) \) for all \( \alpha \in R/I \). By our assumption, the range of \( N \) is a finite set and so is the range of \( Q \). Consequently the range of \( P \) is a finite set. Therefore \( R/I \) is a simple artinian ring by [4]. This is a contradiction by [2, Th. 10.29].

Even when \( R \) is a self-injective regular ring, \( N|_R \) is not necessarily extremal for \( N \in \partial eP(R^*G) \). If each maximal ideal of \( R \) is \( G \)-invariant, then \( N|_R \) is extremal. In fact, since \( \ker(N|_R) = \ker N \cap R \) is a maximal ideal by [7, p. 295], \( N|_R \) is extremal by Lemma 12. Hence we shall consider the map \( \theta: \partial eP(R^*G) \rightarrow \partial eP(R) \). We denote the set of all central idempotents of \( R \) by \( B(R) \).

**Lemma 14.** Let \( R \) be a directly finite, left self-injective, regular ring and let \( G \) be a finite group of automorphisms of \( R \). The following conditions are equivalent:

1. Every maximal ideal of \( R \) is \( G \)-invariant.
2. Every extremal pseudo-rank function on \( R \) is \( G \)-invariant.
3. Every central idempotent of \( R \) is \( G \)-invariant.

**Proof.** (1) \( \Rightarrow \) (2) It is clear by Lemma 12.

(1) \( \Rightarrow \) (3) Take \( e \in B(R) \) and \( g \in G \). For \( M \in \text{Max}(R) \), we have \( e \in M \) or \( 1 - e \in M \) by [3, Th. 8.20]. Since \( e - e^g = (1 - e^g)^* - (1 - e) \), \( e - e^g \in \cap \{ M : M \in \text{Max}(R) \} \). By [3, Cor. 8.19], we conclude \( e = e^g \).

(3) \( \Rightarrow \) (1). Let \( M \) be any maximal ideal of \( R \) and let \( g \) be any element in \( G \). By [3, Th. 8.20 and Cor. 8.22], \( B(R) \cap M \) is a \( G \)-invariant, minimal prime ideal. Since any minimal prime ideal of \( R \) is contained in a unique maximal ideal by [3, Cor. 8.23], \( M = M^g \).

In [4], the Grothendieck group \( K_d(R) \) of a regular ring \( R \) is investigated as a partially ordered abelian group with order-unit. We refer to [4, 8] for the terminologies of partially ordered abelian groups.

We shall study conditions under which \( \theta \) is a homeomorphism.

**Theorem 15.** Let \( R \) be a left self-injective, regular ring of Type II, and \( R^*G \) be a crossed product of a finite group \( G \) over \( R \) with \( |G|^{-1} \in R \). We assume any \( M \in \text{Max}(R) \) is \( G \)-invariant. Let \( \theta: \partial eP(R^*G) \rightarrow \partial eP(R) \) be a natural restriction map. Then the following conditions are equivalent:

1. \( \theta \) is a homeomorphism.
2. The natural map \( f: K_d(R) \rightarrow K_d(R^*G) \), defined by \( f([A]) = [R^*G \otimes_R A] \) for \( A \in FP(R) \), is an isomorphism as a partially ordered abelian group with order-unit.
(3) \( B(R) = B(R^*G) \).

Proof. We know that \( R^*G \) is a left self-injective regular ring of Type II, by Theorem 13.

(1) \(\Rightarrow\) (2). By Lemma 12, \( \partial_e P(R) \) and \( \partial_e P(R^*G) \) are compact. Combining [8, Th. 3.6] with [9, Prop. II. 3.13], we see that \((K_\theta(R), [R]) \simeq (C(\partial_e P(R), R), 1)\) and \((K_\theta(R^*G), [R^*G]) \simeq (C(\partial_e P(R^*G), R), 1)\), where 1 is the constant function with value 1. Therefore we have that \( f: (K_\theta(R), [R]) \simeq (K_\theta(R^*G), [R^*G]) \) is an isomorphism.

(2) \(\Rightarrow\) (3). Let \( e \) be any element in \( B(R^*G) \). For the element \([R^*Ge] \subseteq K_\theta(R^*G)\), we choose an element \([A] \subseteq K_\theta(R)\), such that \( f([A]) = [R^*Ge] \), where \( A \subseteq FP(R) \). First we shall show that \( A \subseteq R \). In fact, since \([R^*G \otimes_R A] = [R^*Ge], R^*G \otimes_R A \simeq R^*Ge \) by [3, Prop. 15.2]. Let \( A \simeq \bigoplus_{i} Rr_i \), where \( r_i \subseteq R \). For any \( P \in \partial_e P(R) \)
\[
\sum_{i} P(r_i) = \sum_{i} \Gamma_{R^*G}(P)(r_i)
= \Gamma_{R^*G}(P)(A)
= \Gamma_{R^*G}(P)(R^*Ge)
\leq 1
\]

Then we have \( A \subseteq R \) by [8, Cor. 2.7]. We may assume that \( R^*Ge = R^*Gh \) for some idempotent \( h \subseteq R \). As \( e \) is central, we have \( e = h \). On the other hand, since any \( h' \subseteq B(R) \) is \( G \)-invariant by Lemma 14, \( h' \) is central in \( R^*G \).

(3) \(\Rightarrow\) (1). In general, \( \theta \) is a continuous epimorphism. We shall that \( \theta \) is a monomorphism. Assume that there exist \( N_1 \neq N_2 \subseteq \partial_e P(R^*G) \) such that \( \theta(N_1) = \theta(N_2) \). By Lemma 12, \( \ker N_1 \neq \ker N_2 \) and so \( B(R^*G) \cap \ker N_1 \neq B(R^*G) \cap \ker N_2 \) by [3, Th. 8.25]. Then there exists \( e \in B(R^*G) \) such that \( N_1(e) = 0 \) and \( N_2(e) = 1 \). However since \( e \in B(R) \) and \( \theta(N_1) = \theta(N_2) \), we have a contradiction. Hence \( \theta \) is a monomorphism. Next let \( W \) be any clopen set in \( \partial_e P(R^*G) \). Then \( W = \{ N \in \partial_e P(R^*G) : N(e) = 0 \} \) for some \( e \in B(R^*G) \). Now it is easy to see that \( \theta(W) = \{ P \in \partial_e P(R) : P(e) = 0 \} \). Therefore \( \theta(W) \) is an also clopen set in \( \partial_e P(R) \) and so \( \theta \) is a homeomorphism.

3. Fixed subrings of a finite group of automorphisms

In this section, let \( R \) be a regular ring and let \( G \) be a finite group of automorphisms of \( R \) with \( |G|^{-1} \subseteq R \). We shall consider a relationship between \( P(R) \) and \( P(R^G) \). For any \( P \subseteq P(R) \), the restriction of \( P \) on \( R^G \), which is denoted by \( P \big|_{R^G} \), is also a pseudo-rank function on \( R^G \). If \( P \) is extremal, then we have the following result.
Proposition 16. For \( P \in \partial_pP(R) \), \( P|_{R^G} \) can be the expressed as a positive convex combination of finite distinct elements in \( \partial_pP(R^G) \).

Proof. Since \( P \) is not necessarily \( G \)-invariant, we consider the trace \( t(P) \) of \( P \) instead of \( P \). Let \( \bar{R} \) be the \( t(P) \)-completion of \( R \). Since \( t(P) \) is a finite convex combination of extreme points in \( P(R) \), \( \bar{R} \) is a finite direct product of simple regular self-injective rings by [2, Th. 19.19]. Let \( t(P) \) be the extension of \( t(P) \) on \( \bar{R} \). Since \( P|_{R^G} = t(P)|_{R^G} \) on \( R^G \), \( (\bar{R})^G \) is the \( P|_{R^G} \)-completion of \( R^G \). By [8, Cor. 3.10], \( (\bar{R})^G \) is also a direct product of simple regular self-injective rings. Therefore \( P|_{R^G} \) can be the expressed as a positive convex combination of finite distinct elements in \( \partial_pP(R^G) \) by [2, Th. 19.19].

In this section, \( R^*G \) implies the skew group ring of \( G \) over \( R \). Put \( e = [G]^{-1} \sum_{g \in G} g \) in \( R^*G \), then \( e \) is an idempotent. Between \( eR^*Ge \) and \( R^G \), there exists an isomorphism by the rule: \( a \rightarrow ea \). Put \( X = eR^*G \), then \( X \) is a \( (R^G, R^*G) \)-bimodule. Throughout this section, we assume \((*)\), \( R \) is a finitely generated projective right \( R^G \)-module.

Since \( \text{Hom}_{R^G}(X, R^*G) \cong R^*Ge \cong R \) as a right \( R^G \)-module, \( \text{Hom}_{R^G}(X, A) \) is a finitely generated projective right \( R^G \)-module for all \( A \in FP(R^*G) \). Therefore, for \( D \in D(R^G) \), \( D(\text{Hom}_{R^G}(X, A)) \) gives an unnormalized dimension function on \( FP(R^*G) \). We note that \( D(R^G) \geq 1 \), because \( R^G \supseteq R^G \). We define

\[
D_{R^*G}(A) = D(R^G)^{-1}D(\text{Hom}(X, A)) \quad \text{for } A \in FP(R^*G),
\]

then \( D_{R^*G} \) is a dimension function on \( FP(R^*G) \). For a given pseudo-rank function \( Q \) on \( R^G \), put \( D_Q = \Gamma_{R^G}(Q) \). We define

\[
N_Q(x) = D_Q(R^G)^{-1}D_Q(\text{Hom}(X, xR^*G)) \quad \text{for } x \in R^*G.
\]

Then by Lemma 1, \( N_Q \) is a pseudo-rank function on \( R^*G \). Especially for an idempotent \( x \in R^*G \), we have

\[
N_Q(x) = D_Q(R^G)^{-1}D_Q((xR^*Ge)_R^G),
\]

because \( \text{Hom}_{R^G}(X, xR^*G) \cong xR^*Ge \) as a right \( R^G \)-module. For the induced pseudo-rank function \( N_Q \in P(R^*G) \) by \( Q \in P(R^G) \), the restriction-function on \( R \), denoted by \( P_Q \), is also a pseudo-rank function on \( R \). \( P_Q|_{R^G} \) is not necessarily equal to \( Q \), but we have the following relations between them.

Lemma 17. Let \( R \) be a regular ring, let \( G \) be a finite group of automorphisms of \( R \) with \( |G|^{-1} \in R \) and let \( R^*G \) be a skew group ring of \( G \) over \( R \). We assume that \( R \) satisfies the condition \((*)\). Then for a given \( Q \in P(R^G) \), we have the following relation;

\[
Q(a) \leq D_Q(R^G)(P_Q|_{R^G})(a) \quad \text{for all } a \in R^G.
\]
Proof. For any idempotent $b \in R^e$,
\[ Q(b) = D_\varphi(bR^e) = D_\varphi(beR^*Ge) = D_\varphi(ebR^*Ge). \]
Since there exists a natural epimorphism $bR^*Ge \rightarrow ebR^*Ge$ as a $R^e$-module, we have $ebR^*Ge \lesssim bR^*Ge$. Then we have
\[ Q(b) \leq D_\varphi(bR^*Ge) = D_\varphi(R^e_{\varphi^e})(P_\varphi|_{\varphi^e})(b). \]

**Proposition 18.** Let $R$ be a regular ring and let $G$ be a finite group of automorphisms of $R$ with $|G|^{-1} \in R$. We assume that $R$ satisfies the condition $(*)$. Then, for a given extremal pseudo-rank function $Q$ on $R^e$, we have
\[ P_\varphi|_{\varphi^e} = \alpha Q + (1 - \alpha)Q' \]
for some $Q' \in P(R^e)$ and some $0 < \alpha \leq 1$.

Proof. We consider $R$ as a ring with $P_\varphi|_{\varphi^e}$-metric. By Lemma 17, $Q$ is continuous with respect to the metric. Therefore there exist some $Q' \in P(R^e)$ and some real number $0 < \alpha \leq 1$ such that $P_\varphi|_{\varphi^e} = \alpha Q + (1 - \alpha)Q'$, using the same way as Theorem 9.

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**References**


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