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## DENSITY PROPERTIES OF COMPLEX LIE GROUPS

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### 0. Introduction

Let  $G$  be a locally compact group. A subgroup  $H$  of  $G$  is called of *finite covolume* if  $H$  is closed and  $G/H$  has a finite  $G$ -invariant Radon measure.

A. Borel studied properties of subgroups of finite covolume in semi-simple Lie groups without compact factors [1] and proved:

**Theorem** (Borel's density theorem). *Let  $G$  be a semi-simple Lie group without compact factors and  $H$  a subgroup of finite covolume in  $G$ . Let  $f$  be a finite dimensional linear representation of  $G$ . Then every  $f(H)$ -invariant vector subspace is  $f(G)$ -invariant.*

H. Furstenberg showed that Borel's theorem holds for minimally almost periodic groups and those subgroups of finite covolume [2].

In this paper, applying Furstenberg's idea to some more general situations, we shall prove complex Lie group version of Borel's theorem, that is:

**Theorem.** *Let  $G$  be a complex analytic group and  $H$  a subgroup of finite covolume in  $G$ . Let  $f$  be a holomorphic representation of  $G$  on a finite dimensional complex vector space. Then every  $f(H)$ -invariant vector subspace is  $f(G)$ -invariant.*

Using this theorem, we obtain properties of subgroups of finite covolume in a complex analytic group [see Section 3].

### 1. Preliminary results

Let  $G$  be a locally compact group and  $V$  a finite dimensional vector space over the field  $K$ , where  $K$  is the real number field  $R$  or the complex number field  $C$ . Let  $f$  be a continuous representation of  $G$  on  $V$ .

DEFINITION 1.  $(G, f)$  is said to have *property (A)* if the following conditions are satisfied:

- (1)  $G$  has no closed subgroup of finite index.
- (2) For any  $f(G)$ -invariant subspace  $W$  of  $V$ ,  $f(G)|_W \subset K \cdot 1_W$  or

$$\{|\det f(g)|_W|^{-1/\dim W} \cdot f(g)|_W; g \in G\}$$

is unbounded in  $GL(W)$ , where  $f(g)|_W$  and  $\det f(g)|_W$  are the restriction of  $f(g)$  to  $W$  and its determinant, respectively.

Let  $P(V)$  denote the projective space corresponding to  $V$ . For a subset  $A \subset V$ ,  $\bar{A}$  denotes the canonical image of  $A$  in  $P(V)$ . For a vector subspace  $W \subset V$ ,  $\bar{W}$  is called a linear subvariety. Following Furstenberg's terminology, we call a finite union of linear subvarieties a *quasi-linear subvariety*. For simplicity we denote a quasi-linear subvariety by "*q.l.v.*". By the descending chain condition for all the algebraic sets, we have that for any subset  $B \subset P(V)$  there exists a minimal *q.l.v.* containing  $B$ . In this case this *q.l.v.* is determined uniquely. We denote it by  $q(B)$ . For a linear map  $t$  of a subspace  $W \subset V$  to  $V$ ,  $\bar{t}$  denotes the map of  $\overline{W \setminus \ker t}$  to  $P(V)$  corresponding to  $t$ . The following lemma is essentially due to Furstenberg.

**Lemma 1.** *Let  $\{t_k\}_{k=1}^\infty$  be in  $GL(V)$  such that*

$$|\det t_k| / \|t_k\|^n \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

where  $n = \dim W$  and  $\|\cdot\|$  is a suitable norm on  $\text{End}(V)$ .

Then there exist a transformation  $T$  on  $P(V)$  whose range is a proper *q.l.v.*  $\subseteq P(V)$  and a suitable subsequence  $\{t'_k\}_{k=1}^\infty$  of  $\{t_k\}_{k=1}^\infty$  such that

$$\bar{t}'_k(x) \rightarrow T(x) \quad \text{as } k \rightarrow \infty$$

for any  $x \in P(V)$ .

*Proof.* Let  $W$  be a subspace of  $V$ . Passing to a subsequence and taking suitable constants  $a_k$ 's such that  $\|a_k \cdot t_k\| = 1$  (where  $\|\cdot\|$  is a suitable norm on  $\text{Hom}(W, V)$ ), we may assume that  $a_k \cdot t_k$  converges to a nonzero linear map  $h$  of  $W$  into  $V$  with respect to the natural topology in  $\text{Hom}(W, V)$ . We note that for  $v \in W \setminus \ker h$

$$\bar{t}_k(v) \rightarrow \bar{h}(v) \quad \text{as } k \rightarrow \infty.$$

Set  $W_0 = V$ . There exist a subsequence  $\{t_k(0)\}$  of  $\{t_k\}$  and a linear map  $h_0$  of  $W_0 = V$  to  $V$  such that  $\{\bar{t}_k(0)\}$  converges pointwise to  $\bar{h}_0$  on  $P(V) \setminus \ker \bar{h}_0$ .

We shall define for  $i = 1, 2, 3, \dots$ , subspaces  $W_i$ , subsequences  $\{t_k(i)\}$  of  $\{t_k\}$  and linear maps  $h_i$  of  $W_i$  to  $V$ , inductively. Set  $W_i = \ker h_{i-1}$ . Take a subsequence  $\{t_k(i)\}$  of  $\{t_k(i-1)\}$  and a linear map  $h_i$  such that for  $v \in W_i \setminus \ker h_i$

$$\bar{t}_k(v)(v) \rightarrow \bar{h}_i(v) \quad \text{as } k \rightarrow \infty.$$

Since  $\dim V < +\infty$ , there exists an integer  $m$  such that  $\dim W_{m+1} = \dim \ker h_m = 0$ .

Set  $T(v) = \bar{h}_i(v)$  for  $v \in W_i \setminus W_{i+1}$ , where  $i = 0, 1, 2, \dots$ . Let  $\{t'_k\} = \{t_k(m)\}$ .

Then the range of  $T$  is  $\cup \bar{h}_i(\bar{W}_i)$  and  $\{\bar{t}_k\}$  converges pointwise to  $T$  on  $P(V)$ . In order to show that the range of  $T$  is proper, it is sufficient to prove  $\det h_0 = 0$ . By the assumption we have that

$$\begin{aligned} |\det h_0| &= |\det \lim a_k \cdot t_k(0)| \\ &= \lim |\det a_k \cdot t_k(0)| / \|a_k \cdot t_k(0)\|^n \\ &= \lim |\det t_k| / \|t_k\|^n = 0 \end{aligned}$$

where  $\| \cdot \|$  is a norm on  $\text{End}(V) = \text{Hom}(W_0, V)$  and  $a_k$  is a scalar constant. q.e.d.

**Lemma 2.** *Let  $W_i$  for  $i=1, 2, 3, \dots, k$ , be a subspace of  $V$ . If a subspace  $W \subset V$  is contained in  $\cup_{i=1}^k W_i$ , there exists an integer  $1 \leq i' \leq k$  such that  $W \subset W_{i'}$ .*

Proof. Suppose that  $W$  is not contained in any  $W_i$ . Then for every  $i=1, 2, 3, \dots, k$ , there exists a nonzero vector  $v_i \in W$  which is not contained in  $W_i$ .

We shall prove that for  $j=1, 2, 3, \dots, k$ , there exist  $j$  real numbers  $t_i$ 's such that  $\sum_{i=1}^j t_i \cdot v_i$  is not contained in  $\cup_{i=1}^j W_i$ , by induction on  $j$ . By the assumption of induction we can find  $(j-1)$  real numbers  $t_i$ 's such that  $u = \sum_{i=1}^{j-1} t_i \cdot v_i$  is not contained in  $\cup_{i=1}^{j-1} W_i$ . If  $u \notin W_j$ , set  $t_j = 0$ . Assume that  $u \in W_j$ . Since  $\cup_{i=1}^{j-1} W_i$  is closed, we can find a sufficiently small number  $t_j$  such that  $u + t_j \cdot v_j \notin \cup_{i=1}^{j-1} W_i$ . Since  $u \in W_j$  and  $t_j \cdot v_j \in W_j$ , we have that  $\sum_{i=1}^j t_i \cdot v_i = u + t_j \cdot v_j \in W_j$ . Consequently we can find  $k$  real numbers  $t_i$ 's such that  $\sum_{i=1}^k t_i \cdot v_i \in \cup_{i=1}^k W_i$ .

However  $\sum_{i=1}^k t_i \cdot v_i \in W \subset \cup_{i=1}^k W_i$  leads to a contradiction. q.e.d.

**Lemma 3.** *Assume that  $(G, f)$  has a property (A). Let  $\bar{f}$  be a representation of  $G$  on  $P(V)$  induced by  $f$  and  $\mu$  a finite  $\bar{f}(G)$ -invariant Radon measure on  $P(V)$ . Then the support of  $\mu$  consists of  $\bar{f}(G)$ -fixed points.*

Proof. If  $f(G) \subset K \cdot 1_V$ , there is nothing to prove. Hence we may assume that there exists a sequence  $\{g_k\} \subset G$  such that

$$\{|\det f(g_k)|^{-1/\dim V} \cdot f(g_k); k = 1, 2, 3, \dots\}$$

is unbounded in  $\text{End}(V)$ . If necessary taking a subsequence we may assume, by Lemma 1, that there exists a transformation  $T$  on  $P(V)$  whose range is a proper *q.l.v.*  $Q$  and that  $\bar{f}(g_k)$  converges pointwise to  $T$ .

Let  $D(x)$  be the distance from  $x$  to  $Q$  for some metric on  $P(V)$ . By the bounded convergence theorem, we have that for any  $x \in P(V)$

$$\begin{aligned} 0 &= \int_{P(V)} D(T(x)) d\mu \\ &= \int_{P(V)} \lim D(\bar{f}(g_k)(x)) d\mu \end{aligned}$$

$$\begin{aligned} &= \lim \int_{P(V)} D(\bar{f}(g_k)(x)) d\mu \\ &= \int_{P(V)} D(x) d\mu \end{aligned}$$

This implies that  $\text{supp } \mu \subset Q$ .

Let  $X$  be the unique minimal  $q.l.v.$  containing  $\text{supp } \mu$ .  $X$  can be denoted by  $X = \cup_{i=1}^m \bar{W}_i$  where  $W_i$  is a subspace of  $V$ . We may assume that there is no inclusion relation among  $W_i$ 's. Remark that  $X$  is also proper in  $P(V)$ . Since  $\text{supp } \mu$  is  $\bar{f}(G)$ -invariant and  $X$  is the smallest  $q.l.v.$  containing  $\text{supp } \mu$ ,  $X$  is  $\bar{f}(G)$ -invariant. For every  $g \in G$ ,  $f(g)W_i \subset \cup_{i=1}^m W_i$ . By Lemma 2, there exists for every  $i=1, 2, 3, \dots, m$ , there exists an integer  $s(i)$  such that  $f(g)W_i \subset W_{s(i)}$ . Since there is no inclusion relation among  $W_i$ 's,  $\cup_{i=1}^m W_i = f(g) \cup_{i=1}^m W_i \subset \cup_{i=1}^m W_{s(i)} \subset \cup_{i=1}^m W_i$  implies that  $f(g)W_i = W_{s(i)}$  for  $i=1, 2, 3, \dots, m$ . Thus  $G$  permutes  $W_i$ 's. Since  $G$  has no closed subgroup of finite index,  $G$  leaves each  $W_i$  invariant.

We shall show that  $f(G)|_{W_i} \subset K \cdot 1_{W_i}$  for  $i=1, 2, 3, \dots, m$ .

Suppose that there exists  $W_{i'}$  such that  $f(G)|_{W_{i'}} \not\subset K \cdot 1_{W_{i'}}$ . The same argument as above with respect to  $\bar{W}_{i'}$ ,  $f|_{W_{i'}}$ , and  $\mu|_{\bar{W}_{i'}}$  shows that there exists a  $q.l.v.$   $X'$  contained properly in  $\bar{W}_{i'}$  such that  $\text{supp } \mu|_{\bar{W}_{i'}} \subset X'$ . Thus  $X$  contains  $(\cup_{i \neq i'} W_i) \cup X'$  properly. This contradicts the definition of  $X$ .

Therefore  $f(G)|_{W_i} \subset K \cdot 1_{W_i}$  for  $i=1, 2, 3, \dots, m$ . q.e.d.

### 2. Main theorem

Let  $G$  be a locally compact group and  $f$  a continuous representation of  $G$  on a finite dimensional vector space  $V$  over  $K$ .

**Lemma 4.** *Assume that  $(G, f)$  has property (A). Let  $H$  be a subgroup of finite covolume in  $G$ . Then for 1-dimensional subspace  $W$  of  $V$ ,  $W$  is  $f(G)$ -invariant if and only if  $W$  is  $f(H)$ -invariant.*

Proof. In order to prove the lemma it is sufficient to show "if" part. Set  $p = \bar{W} \in P(V)$ . Define the map  $\pi$  of  $G/H$  to  $P(V)$  by

$$\pi: G/H \ni gH \mapsto \overline{f(g)p} \in P(V).$$

Then  $\pi$  carries a finite  $G$ -invariant measure on  $G/H$  to a finite  $\bar{f}(G)$ -invariant measure on  $P(V)$ . Since  $p$  is contained in the support of this measure, by Lemma 3,  $p$  is a  $\bar{f}(G)$ -fixed point. q.e.d.

For a representation  $f$  of  $G$  on a vector space  $V$ , the  $k$ -th exterior product representation  $\Lambda_k f$  of  $f$  on  $\Lambda_k V$  is defined by

$$\begin{aligned} \Lambda_k f(g) (v_1 \wedge v_2 \wedge v_3 \wedge \dots \wedge v_k) \\ = f(g)v_1 \wedge f(g)v_2 \wedge f(g)v_3 \wedge \dots \wedge f(g)v_k \end{aligned}$$

where  $g \in G$  and  $v_i \in V$ .

**DEFINITION 2.**  $(G, f)$  is said to have *property (B)* if for  $k=1, 2, 3, \dots$ ,  $\dim V, (G, \Lambda_k f)$  has property (A).

**Theorem 1.** *Assume that  $(G, f)$  has property (B). Let  $H$  be a subgroup of finite covolume in  $G$ . Then for any subspace  $W$  of  $V$ ,  $W$  is  $f(G)$ -invariant if and only if  $W$  is  $f(H)$ -invariant.*

*Proof.* In order to prove the theorem it is sufficient to show “if” part. Let  $k$  be  $\dim W$ . Taking  $k$ -th exterior product of  $f$ , we can reduce the proof to Lemma 4.

**Proposition 1.** *Let  $G$  be a complex analytic group and  $f$  a holomorphic representation of  $G$  on a finite dimensional complex vector space  $V$ . Then  $(G, f)$  has property (B).*

*Proof.* If  $f$  is holomorphic, so is  $\Lambda_k f$ . Thus, in order to prove the proposition, it is sufficient to show that  $(G, f)$  has property (A). Since  $G$  is connected  $G$  has no closed subgroup of finite index. In order to show the condition (2) of Definition 1 we may assume that  $W=V$ , for the restriction  $f|_W$  of  $f$  to a invariant subspace  $W$  is also holomorphic.

Let  $G'$  denote the complex linear group  $f(G)$  and  $\hat{G}'$  its Lie algebra. For  $A=(a_{ij}) \in \text{End}(V)$ , we define the norm of  $A$  by  $\|A\|=(\sum_{i,j} |a_{ij}|^2)^{1/2}$ .

For nonzero  $X \in \hat{G}'$ , set

$$f_X(z) = \|\exp n z X\| / |\det \exp z X|$$

where  $n=\dim V$  and  $z \in C$ . We note that  $f_X(z)$  can be written the form;

$$f_X(z) = (|f_1(z)|^2 + |f_2(z)|^2 + \dots + |f_m(z)|^2)$$

where  $m=n^2$  and  $f_i(z)$  is a holomorphic function of  $z$ . Thus there exist only two possible cases:

Case 1. There exists a nonzero  $X \in \hat{G}'$  such that  $f_X(z)$  is unbounded. Since  $f_X(z) = \|\exp n z X\| / |\det \exp z X| \leq \|\exp z X\|^n / |\det \exp z X|$ , we have that

$$\{|\det \exp z X|^{-1/n} \cdot \exp z X; z \in C\}$$

is unbounded.

Case 2. For every  $X \in \hat{G}'$ ,  $f_X(z)$  is constant. In this case each element of the matrix  $(\exp n z X) / (\det \exp z X)$  is constant. Substituting 0 for  $z$ , we have that

$$(\exp n z X) / (\det \exp z X) = 1_V$$

for all  $z \in C$  and all  $X \in \hat{G}$ . Consequently we have that

$$\exp z X \in C \cdot 1_V \quad \text{for all } X \in \hat{G}.$$

Since  $f(G) = G'$  is connected,  $f(G) \subset C \cdot 1_V$ . q.e.d.

From Theorem 1 and Proposition 1, it follows that:

**Theorem 2.** *Let  $G$  be a complex analytic group and  $H$  a subgroup of finite covolume in  $G$ . Let  $f$  be a holomorphic representation of  $G$  on a finite dimensional complex vector space. Then every  $f(H)$ -invariant subspace is  $f(G)$ -invariant.*

REMARK. There are several other cases in which property (B) holds. If  $G$  is minimally almost periodic and  $f$  is an arbitrary representation, or if  $G$  is an analytic group and  $f$  is a unipotent representation  $(G, f)$  has property (B). In both the cases Theorem 1 holds [2, 4].

### 3. Density properties

In this section  $G$  always denotes a complex analytic group,  $H$  a subgroup of finite covolume in  $G$  and  $f$  a holomorphic representation of  $G$  on a finite dimensional complex vector space  $V$ .

**Corollary 1.** *Every element of  $f(G)$  is a linear combination of elements of  $f(H)$ .*

Proof. Let  $W$  be the subspace spanned by the elements of  $f(H)$  in  $\text{End}(V)$ . The action of  $G$  on  $\text{End}(V)$

$$G \times \text{End}(V) \ni (g, A) \mapsto f(g) \circ A \in \text{End}(V)$$

defines a holomorphic representation of  $G$  on  $\text{End}(V)$ . Since  $W$  is  $H$ -invariant under this action Theorem 2 concludes that  $W$  is  $G$ -invariant. Thus we have that for every  $g \in G$

$$f(g) = f(g) \circ 1_V \in W. \quad \text{q.e.d.}$$

From Corollary 1, it follows immediately that:

**Corollary 2.** *The centralizer of  $f(H)$  in  $\text{GL}(V)$  coincides with the centralizer of  $f(G)$ .*

**Corollary 3.**  *$f(G)$  and  $f(H)$  has the same Zariski closure in  $\text{GL}(V)$ .*

Proof. Let  $G'$  and  $H'$  be the Zariski closures of  $f(G)$  and  $f(H)$ , respectively. Clearly  $H' \subset G'$ . By Chevalley's theorem we can find a rational representation  $r$  of  $\text{GL}(V)$  on a complex vector space  $E$  and a nonzero vector  $v \in E$  such that

$$H' = \{x \in \text{GL}(V); r(x)v \in C \cdot v\}.$$

Since  $\rho f$  is a holomorphic representation of  $G$  and  $C \cdot v$  is  $\rho f(H)$ -invariant, by Theorem 2,  $C \cdot v$  is  $\rho f(G)$ -invariant. Thus we have that  $f(G) \subset H'$ . q.e.d.

**Appendix.** Professor Goto pointed a criterion for property (A). This criterion seems to make the meaning of property (A) clear.

Let  $V$  be a finite dimensional complex vector space. An endomorphism  $A$  on  $V$  is called *conformal* if  $A$  is semi-simple and the real part of every eigen value of  $A$  is equal to each other. Let the totality of the conformal endomorphisms on  $V$  be denoted by  $c(V)$ . For a representation  $f$  of a Lie group by  $df$  we denote the associated representation of its Lie algebra.

**Proposition** (M. Goto). *Let  $G$  be an analytic group and  $\hat{G}$  its Lie algebra. Let  $f$  be a representation of  $G$  on a finite dimensional complex vector space  $V$ . Assume that for every  $f(G)$ -invariant subspace  $W$  of  $V$   $df(\hat{G})|_W \subset c(W)$  implies that  $df(\hat{G})|_W \subset C \cdot 1_W$ . Then  $(G, f)$  has property (A).*

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