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Osaka University
On Kahler submanifolds of a complex projective space, J-I Hano [5] has studied complete intersections of hypersurfaces in a complex projective space and proved that if a complete intersection \( M \) of hypersurfaces is an Einstein manifold with respect to the induced metric then \( M \) is a complex projective space or a complex quadric. The purpose of this note is to investigate hypersurfaces of a complex Grassmann manifold by using Hano's method. Let \( G_{m+n,n}(\mathbb{C}) \) denote the complex Grassmann manifold of \( n \)-planes in \( \mathbb{C}^{m+n} \). Let \( X \) be a compact complex hypersurface of \( G_{m+n,n}(\mathbb{C}) \). Then \( X \) defines a positive divisor on \( G_{m+n,n}(\mathbb{C}) \) and hence a holomorphic line bundle \( \{X\} \) on \( G_{m+n,n}(\mathbb{C}) \). We denote by \( c(X) \) the Chern class of the line bundle \( \{X\} \). Since the second cohomology group \( H^2(G_{m+n,n}(\mathbb{C}), \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \), we can write \( c(X) = a(X) - \sigma \), where \( a(X) \in N \) and \( \sigma \) is a generator of \( H^2(G_{m+n,n}(\mathbb{C}), \mathbb{Z}) \). We call \( a(X) \) the degree of \( X \). We equip an hermitian inner product on \( \mathbb{C}^{m+n} \). The complex Grassmann manifold \( G_{m+n,n}(\mathbb{C}) \) has a Kahler metric invariant under the action of the unitary group \( U(m+n) \). Moreover we may assume that \( m \geq n \). Under these notations, we have a following Theorem.

**Theorem.** Let \( X \) be a compact complex hypersurface of a complex Grassmann manifold \( G_{m+n,n}(\mathbb{C}) \) and \( a(X) \) the degree of \( X \). If \( a(X) \geq r + 2 \), where \( r = \binom{m+n}{n} - mn - 1 \) and \( n \geq 2 \), \( X \) is not an Einstein manifold with respect to the induced metric.

1. Preliminaries

Let \( G_{m+n,n}(\mathbb{C}) \) be the complex Grassmann manifold of \( n \)-planes in \( \mathbb{C}^{m+n} \). An element of \( G_{m+n,n}(\mathbb{C}) \) can be given by a non-zero decomposable \( n \)-vector \( \Lambda = X_1 \wedge \cdots \wedge X_n \neq 0 \) defined up to a constant factor. If \( (e_1, \cdots, e_{m+n}) \) denotes a fixed frame in \( \mathbb{C}^{m+n} \), we can write

\[
\Lambda = \sum_{i_1, \cdots, i_n} p_{i_1 \cdots i_n} e_{i_1} \wedge \cdots \wedge e_{i_n} \quad (1 \leq i_1, \cdots, i_n \leq m+n)
\]

where the \( p_{i_1 \cdots i_n} \)'s are skew-symmetric in their indices. The \( p_{i_1 \cdots i_n} \) are called the
Plücker coordinates in $G_{m+n,n}(C)$. By considering $p_{i_1 \cdots i_n}$ as the homogeneous coordinates of the complex projective space $P^n(C)$ of dimension $\mu = \binom{m+n}{n} - 1$, we get an imbedding $j: G_{m+n,n}(C) \to P^n(C)$.

We equip an hermitian inner product in $C^{m+n}$. Then we can define a Kähler metric on $G_{m+n,n}(C)$ which is invariant under the action of the unitary group $U(m+n)$. We also have the Fubini-Study metric on the complex projective space $P^n(C)$ induced from the hermitian inner product in the $n$-th exterior product $\Lambda^n C^{m+n}$ of $C^{m+n}$. Then the imbedding $j$ is isometric with respect to these Kähler metrics (cf. for example [3] §8).

From now on we identify $G_{m+n,n}(C)$ with the image of the imbedding $j$. Let $I(V)$ denote the ideal associated to a subvariety $V$ of $P^n(C)$. We recall the generators of the ideal $I(G_{m+n,n}(C))$. Let $i_1, \cdots, i_{n-1}$ be $n-1$ distinct numbers which are chosen from a set $\{1, \cdots, m+n\}$ and let $j_0, \cdots, j_n$ be $n+1$ distinct numbers chosen from the same set. We define homogeneous polynomials $Q(i_1 \cdots i_{n-1} j_0 \cdots j_n)$ of degree 2 on $C^{m+1}$ by

$$(1.2) \quad Q(i_1 \cdots i_{n-1} j_0 \cdots j_n) = \sum_{k=0}^{n} (-1)^k p_{i_1 \cdots i_{n-1} j_k} p_{j_0 \cdots j_k \cdots j_n}.$$ 

Then it is known that $Q(i_1 \cdots i_{n-1} j_0 \cdots j_n) = 0$ are the generators of the ideal $I(G_{m+n,n}(C))$ (See [7] Chapter 7 §6 Theorem 2 and §7 Theorem 1). The relations $Q(i_1 \cdots i_{n-1} j_0 \cdots j_n) = 0$ are called the quadratic $p$-relations.

Let $\pi$ denote the canonical projection of $C^{m+1} - (0)$ onto the complex projective space $P^n(C)$. The triple $(C^{m+1} - (0), \pi, P^n(C))$ is a principal $C^*$-bundle over $P^n(C)$. Let $E$ be the standard line bundle over $P^n(C)$ associated to the above principal bundle. We denote by $H(M, \theta^*)$ the group of all equivalent classes of holomorphic line bundles over a compact complex manifold $M$. On the line bundles over a Grassmann manifold $G_{m+n,n}(C)$, the following propositions are known.

**Proposition 1.1.** Let $H$ denote the dual bundle of $E$ over $P^n(C)$. Then, for any integer $k > 0$, the inclusion map $j: G_{m+n,n}(C) \to P^n(C)$ induces the surjective map $j^*: H^0(P^n(C), H^k) \to H^0(G_{m+n,n}(C), j^*H^k)$, that is, every holomorphic section of the line bundle $j^*H^k$ is given by the restriction of a section of the line bundle $H^k$ on $P^n(C)$.

**Proposition 1.2.** The inclusion map $j: G_{m+n,n}(C) \to P^n(C)$ induces the canonical isomorphism $j^*: H^1(P^n(C), \theta^*) \to H^1(G_{m+n,n}(C), \theta^*)$. Moreover each positive divisor $X$ of $G_{m+n,n}(C)$ is the complete intersection of $G_{m+n,n}(C)$ and a subvariety $Y$ of codimension 1 of $P^n(C)$. Furthermore, for an irreducible subvariety $X$ of codimension 1 in $G_{m+n,n}(C)$, $I(X) = I(G_{m+n,n}(C)) + (F)$ where $F$ is an irreducible homogeneous polynomial on $C^{n+1}$.

For a compact connected complex submanifold $X$ of codimension 1 in $G_{m+n,n}(C)$, let $[X]$ denote the positive divisor defined by $X$ and $c(X)$ the Chern class of the line bundle $\{X\}$ defined by $[X]$. Since $H^2(G_{m+n,n}(C), Z) \cong Z$, $c(X) = a(X)\sigma$ where $a(X) \in N$ and $\sigma$ is a generator of $H^2(G_{m+n,n}(C), Z)$. We call $a(X)$ the degree of $X$. Note that the degree of an irreducible subvariety $Y$ of codimension 1 of $P^n(C)$ corresponding to $X$ is given by $a(X).

2. The canonical line bundle

With respect to the hermitian inner product on $C^{m+1}$ induced from the hermitian inner product on $C^{m+n}$, the square of the norm $||z||$ is given by

$$\sum_{i<\cdots<i_n} |p_i \cdots i_n(z)|^2$$

for an orthonormal frame $(e_1, \cdots, e_{m+n})$ of $C^{m+n}$. The function $||z||^2$ can be regarded as a hermitian fiber metric on the standard line bundle $E$ on $P^n(C)$. A unique connection of type $(1, 0)$ on $E$ is determined by the fiber metric $||z||^2$ on $E$ and gives rise to the curvature form $-\Omega$ on $P^n(C)$. The form $\Omega$ is the associated $(1, 1)$-form of the Fubini-Study metric on $P^n(C); \pi^*\Omega = \frac{\sqrt{-1}}{2\pi} d'd'' \log ||z||^2$.

Let $K, K(G_{m+n,n}(C))$ and $K(X)$ be the canonical line bundle of $P^n(C)$, $G_{m+n,n}(C)$ and $X$ respectively. The normal bundle of $X$ in $P^n(C)$ is a holomorphic vector bundle over $X$ whose fiber dimension is $r+1 = mn+1$. We denote by $N$ the $(r+1)$-th exterior product of the dual bundle of the normal bundle of $X$ in $P^n(C)$. Denoting by $\iota$ the inclusion $X \subset P^n(C)$, we have

$$\iota^* K = K(X) \cdot N.$$

Let $U_{i_1 \cdots i_n}$ denote an open subset of $P^n(C)$ given by \{$(z) \in P^n(C) | p_{i_1 \cdots i_n}(z) \neq 0$\}. The functions $u_{i_1 \cdots i_n} : p_{i_1 \cdots i_n}((\beta_1, \cdots, \beta_n) = (i_1, \cdots, i_n), \beta_1 < \cdots < \beta_n)$ form a holomorphic coordinates system on $U_{i_1 \cdots i_n}$. We arrange the Plücker coordinates in the lexicographical order. Let $p_{i_1 \cdots i_n}$ be the $\sigma(j_1, \cdots, j_n)$-th component of the Plücker coordinates in above order. The map $s_{i_1 \cdots i_n} : U_{i_1 \cdots i_n} \to C^{m+1}-0$ defined by

$$s_{i_1 \cdots i_n}(y) = (u_{i_1 \cdots i_n, 1 \cdots m}(y), \cdots, 1, \cdots, u_{i_1 \cdots i_n, m+1 \cdots n}(y)) \ (y \in U_{i_1 \cdots i_n})$$

is a holomorphic section on $U_{i_1 \cdots i_n}$ of the principal $C^*$-bundle $(C^{m+1}-0, \pi, P^n(C))$. We put

$$g_{i_1 \cdots i_n, j_1 \cdots j_n} = p_{i_1 \cdots i_n} | p_{j_1 \cdots j_n}$$

on $U_{i_1 \cdots i_n} \cap U_{j_1 \cdots j_n}$. Then $(g_{i_1 \cdots i_n, j_1 \cdots j_n})$ is the system of transition functions of the principal bundle associated to the holomorphic local trivialization $(U_{i_1 \cdots i_n}, s_{i_1 \cdots i_n})$ of the bundle. Let $V_{i_1 \cdots i_n}$ denote the connected open set of $G_{m+n,n}(C)$ given by
Now we shall consider the structure of the holomorphic line bundle $N$ on $X$. Let $Q(\beta_1, \cdots, \beta_n)$ be a homogeneous polynomial of degree 2 on $\mathbb{C}^{n+1}$ defined by (1.2). It is obvious that $Q(\beta_1, \cdots, \beta_n)$ has following properties:

1) $Q(\beta_1, \cdots, \beta_n)$ is alternating with respect to $\beta_1, \cdots, \beta_n$.
2) $Q(\beta_1, \cdots, \beta_n)$ is alternating with respect to $\beta_1, \cdots, \beta_n$.
3) if $\{\beta_1, \cdots, \beta_n\} \subset \{\beta_1, \cdots, \beta_n\}$, $Q(\beta_1, \cdots, \beta_n) \equiv 0$.

Furthermore we have a following lemma which gives the relations between these polynomials.

**Lemma 2.1.** On $\pi^{-1}(U_{i_1, \cdots, i_n})$,

(a) $Q(\beta_1, \cdots, \beta_n, k \beta_{n+1}) = -Q(\beta_1, \cdots, \beta_n, k \beta_{n+1})$

$$= \sum_{i=1}^{n} (-1)^{i+n+1} \frac{p_{1, \cdots, i, \cdots, n}}{p_{1, \cdots, } Q(\beta_1, \cdots, \beta_n, k \beta_{n+1})}$$

(b) $Q(\beta_1, \cdots, \beta_n, i_1, \cdots, i_{n+k})$

$$= \frac{p_{1, \cdots, i_1, \cdots, i_k}}{p_{1, \cdots, } Q(\beta_1, \cdots, \beta_n, i_1, \cdots, i_k)}$$

Proof. Straightforward computation.

Let $(i_1, \cdots, i_n)$ be an $n$-tuples such that

$$1 \leq i_1 < i_2 < \cdots < i_n \leq m+n$$

and let $(i_1, \cdots, i_n, s_1, \cdots, s_m)$ be the permutation of $(1, \cdots, m+n)$ such that

$$1 \leq s_1 < \cdots < s_m \leq m+n$$.

For a permutation $(i_1, \cdots, i_m)$ of $(1, \cdots, m)$, we introduce a linear order $\preceq$ on

$$\{1, \cdots, m+n\}$$

by $i_{j_1} \preceq \cdots \preceq i_{j_k} \preceq \cdots \preceq s_{i_m}$. We denote $\beta = (\beta_1, \cdots, \beta_n) \mid \beta_1 \preceq \cdots \preceq \beta_n$. 


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[...]

The associated lexicographical order on $C(i_1 \cdots i_n, -\mathcal{O})$ is called an admissible order with respect to $(i_1, \ldots, i_n)$. If the linear order $-\mathcal{O}$ on \{1, \ldots, $m+n$\} is given by $i_1 \mathcal{O} \cdots \mathcal{O} i_n \mathcal{O} \cdots \mathcal{O} s_m$, the admissible order is called principal with respect to \{1, \ldots, $i_n$\}. For an admissible order with respect to $(i_1, \ldots, i_n)$, we define a subset $I(i_1 \cdots i_n, -\mathcal{O})$ of $C(i_1 \cdots i_n, -\mathcal{O})$ by

\[
\left\{ \beta = (\beta_1, \ldots, \beta_n) \mid \beta = (i_1, \ldots, i_t, \ldots, i_n, s_l), \quad l = 1, \ldots, n; \quad t = 1, \ldots, m, \quad \text{or} \quad \beta = (i_1, \ldots, i_n) \right\}.
\]

Note that $I(i_1 \cdots i_n, -\mathcal{O}) = I(i_1, \ldots, i_n, -\mathcal{O})$ for $-\mathcal{O}$, $-\mathcal{O}'$ admissible orders, with respect to $(i_1, \ldots, i_n)$ and the number of elements in $I(i_1 \cdots i_n, -\mathcal{O})$ is $mn + 1$. Moreover $Q(\beta_1 \cdots i_n) = 0$ for $\beta \in I(i_1 \cdots i_n, -\mathcal{O})$ by (2.2) 3).

For an admissible order $-\mathcal{O}$ with respect to $(i_1, \ldots, i_n)$, we define a holomorphic $r$-form $q_{i_1 \cdots i_n}^\mathcal{O} \bigotimes^r C^{\mathcal{O}+1}$ by

\[
q_{i_1 \cdots i_n}^\mathcal{O} = \bigwedge_{\beta \in C(i_1 \cdots i_n, -\mathcal{O})} dQ(\beta_1 \cdots i_n)
\]

where we take the exterior product of $dQ(\beta_1 \cdots i_n)$ according to the admissible order $-\mathcal{O}$ on $C(i_1 \cdots i_n, -\mathcal{O})$. If the admissible order $-\mathcal{O}$ is principal, we denote $q_{i_1 \cdots i_n}^\mathcal{O}$ by $q_{i_1 \cdots i_n}^\mathcal{O}$.

**Lemma 2.2.** Let $-\mathcal{O}$, $-\mathcal{O}'$ be admissible orders with respect to $(i_1, \ldots, i_n)$. Then we have

\[
q_{i_1 \cdots i_n}^{\mathcal{O}}(z) = \varepsilon(-\mathcal{O}, -\mathcal{O}')q_{i_1 \cdots i_n}^{\mathcal{O}'}(z)
\]

for $z \in \pi^{-1}(V_{i_1 \cdots i_n})$, where $\varepsilon(-\mathcal{O}, -\mathcal{O}') \in \{\pm 1\}$.

**Proof.** Let $-\mathcal{O}$ be a linear order on \{1, \ldots, $m+n$\} given by $i_1 \mathcal{O} \cdots \mathcal{O} i_n \mathcal{O} \cdots \mathcal{O} s_m \mathcal{O} \cdots \mathcal{O} s_m$. Since the symmetric group of $m$ elements is generated by transpositions \{(k, k+1) \mid k = 1, \ldots, m-1\}, we may assume that the admissible order $-\mathcal{O}'$ is given by a linear order

\[
i_1 \mathcal{O}' \cdots \mathcal{O}' i_n \mathcal{O}' \cdots \mathcal{O}' s_{i_1} \mathcal{O}' \cdots \mathcal{O}' s_{i_{k+1}} \mathcal{O}' \cdots \mathcal{O}' s_{i_k} \mathcal{O}' \cdots \mathcal{O}' s_{i_{k+2}} \mathcal{O}' \cdots \mathcal{O}' s_{i_m}.
\]

Let $\beta$ be an element of $C(i_1 \cdots i_n, -\mathcal{O}) - I(i_1 \cdots i_n, -\mathcal{O})$. Then $\beta$ is of the form either

1) $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_t \equiv s_{i_t} s_{i_{t+1}}$ for any $t = 1, \ldots, n$,
2) $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_t = s_{i_t}$ for some $t$ and $\beta_a \equiv s_{i_a}$ for $a \neq t$,
3) $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_t = s_{i_{t+1}}$ for some $t$ and $\beta_a \equiv s_{i_a}$ for $a \neq t$,
4) $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_t = s_{i_{t+1}}$, $\beta_{t+1} = s_{i_t}$ for some $t+1 < n$,
5) $\beta = (\beta_1, \ldots, \beta_{n-2}, s_{i_{k+1}}, s_{i_k})$.

In the cases of 1), 2) and 3), $\beta \in C(i_1 \cdots i_n, -\mathcal{O}) - I(i_1 \cdots i_n, -\mathcal{O})$. In the case of 4),

\[
Q(\beta_1 \cdots i_n) = Q(\beta_1 \cdots s_{i_{k+1}} s_{i_k} \beta_{t+2} \cdots \beta_{n-1} i_n) = -Q(\beta_1 \cdots s_{i_{k+1}} s_{i_k} \beta_{t+2} \cdots \beta_{n-1} i_n).
\]
by (2.2) 1). Note that $(\beta_1 \cdots \beta_{t-1}s_i s_{i+1} \beta_{t+1} \cdots \beta_n) \in C(i_1 \cdots i_n, \emptyset) - I(i_1 \cdots i_n, \emptyset)$. In the case of 5), we have

$$Q(\beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n) = -Q(\beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$$

$$+ \sum_{\nu=1}^{n} (-1)^{\nu+1} \frac{P_{\nu}(i_{\nu}(k+1))}{\nu!} Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$$

$$+ \sum_{\nu=1}^{n} (-1)^{\nu+1} \frac{P_{\nu}(i_{\nu}(k+1))}{\nu!} Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$$

by Lemma 2.1 (a). Note that $(\beta_1, \cdots, \beta_{n-2}, s_{i+1}, s_{i+1}) \in C(i_1, \cdots, i_n, \cdots) - I(i_1, \cdots i_n, \emptyset)$, $i_{\nu} \in s_{i+1}$ and $i_{\nu} \in s_{i+1}$. By (2.2) 1), $Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$ where $\beta_1, \cdots, \beta_n$ is a permutation of $i_1, \beta_1, \cdots, i_n$ such that $s_{i+1} \hat{i}_1 \cdots \hat{i}_n \emptyset \emptyset \emptyset$. If $Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$, then $(\beta_1, \cdots, \beta_n)$ is a permutation of $i_1, \beta_1, \cdots, i_n$ such that $s_{i+1} \hat{i}_1 \cdots \hat{i}_n \emptyset \emptyset \emptyset$. If $Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$, then $(\beta_1, \cdots, \beta_n)$ is of the form of the case 2). Similarly, $Q(i_1 \beta_1 \cdots \beta_n \cdot s_{i+1} s_{i} \hat{i}_1 \cdots \hat{i}_n)$ is of the form of the case 3). Now we get our claim by taking differential.

q.e.d.

Let $(i_1 \cdots i_f \cdots i_{i_f} \cdots i_{i_m})$ be a permutation of $(1 \cdots m+n)$. We define a linear order $<$ on $(1, \cdots, m+n)$ by $i_1 < i_2 < \cdots < i_f < i_{i_f} < \cdots < i_m$. We define a set $C(i_1 \cdots i_f \cdots i_{i_f})$ by $\beta = (\beta_1, \cdots, \beta_n) | i_1 < \cdots < i_{i_f}$ and a subset $I(i_1 \cdots i_f \cdots i_{i_f}, \emptyset)$ of $C(i_1 \cdots i_f \cdots i_{i_f}, \emptyset)$ by

$$\left\{ \begin{array}{c}
\beta \in C(i_1 \cdots i_f \cdots i_{i_f}, \emptyset) \\
\beta = (i_1 \cdots i_{i_1} \cdots i_{i_f} \cdots i_{i_m}) \\
\text{or} \\
\beta = (i_1 \cdots i_{i_f} \cdots i_{i_m}) \\
\text{or} \\
\beta = (i_1 \cdots i_f \cdots i_{i_f})
\end{array} \right\}$$

Lemma 2.3.

$$dQ(\beta_1 \cdots \beta_n) = \varepsilon(i_1 \cdots i_n, i_1 \cdots i_f \cdots i_{i_f} \cdots i_{i_m})$$

where $\varepsilon(i_1 \cdots i_n, i_1 \cdots i_f \cdots i_{i_f} \cdots i_{i_m}) \in \{\pm 1\}$ and the exterior product is taken according to the lexicographical order induced from the linear order $<$.q.e.d.

Proof. Note that there is a natural bijection between $C(i_1 \cdots i_n, \emptyset) - I(i_1 \cdots i_n, \emptyset)$ and $C(i_1 \cdots i_f \cdots i_{i_f}, \emptyset) - I(i_1 \cdots i_f \cdots i_{i_f}, \emptyset)$. We denote this map by

$$f: C(i_1 \cdots i_n, \emptyset) - I(i_1 \cdots i_n, \emptyset) \rightarrow C(i_1 \cdots i_f \cdots i_{i_f}, \emptyset) - I(i_1 \cdots i_f \cdots i_{i_f}, \emptyset).$$

Then, for $\beta \in C(i_1 \cdots i_n, \emptyset) - I(i_1 \cdots i_n, \emptyset)$, $Q(\beta i_1 \cdots i_n)$ and $Q(f(\beta) i_1 \cdots i_f \cdots i_{i_f})$ coincide up to sign by (2.2) 1) and 2).q.e.d.
Let \((i_1 \cdots i_j \cdots i_n s_i \cdots s_m)\) be a permutation of \((1, \cdots, m+n)\). We define a linear order \(<\) on \(\{1, \cdots, m+n\}\) by

\[
i_1 < \cdots < i_j < \cdots < i_n < s_k < i_{j+1} < \cdots < s_k < \cdots < s_m.
\]

We define a set \(C(i_1 \cdots i_j \cdots i_n s_k, <)\) by \(\{\beta = (\beta_1 \cdots \beta_n) | \beta_1 < \cdots < \beta_k\}\) and a subset \(I(i_1 \cdots i_j \cdots i_n s_k, <)\) by

\[
\beta \in C(i_1 \cdots i_j \cdots i_n s_k, <) = \begin{cases} 
\beta = (i_1 \cdots i_j \cdots i_n s_k), \\
\beta = (i_1 \cdots i_j \cdots i_n s_k), \\
\beta = (i_1 \cdots i_j \cdots i_n s_k), \\
\beta = (i_1 \cdots i_j \cdots i_n s_k) \text{ or } \\
\beta = (i_1 \cdots i_j \cdots i_n s_k)
\end{cases}
\]

\(t = 1, \cdots, k, \cdots m,\)

\(l = 1, \cdots, j, \cdots, m,\)

**Lemma 2.4.** For \(l = 1, \cdots, j, \cdots, n, t = 1, \cdots, k, \cdots, m, Q(i_1 \cdots i_j \cdots i_n s_k) = (-1)^{t+j}Q(i_1 \cdots i_j \cdots i_n s_k)\).

Proof. The first part is nothing but Lemmas 2.1 (b). Noting that only three terms of \(Q\) are non trivial in our case, we get the second part by the definition. q.e.d.

Now we define a linear order \(<'\) on \(\{1, \cdots, m+n\}\) by \(i_1 <' \cdots <' i_j <' \cdots <' i_n <' s_k <' s_l <' \cdots <' s_k <' \cdots <' s_m,\)

We define a set \(C(i_1 \cdots i_j \cdots i_n s_k, <')\) by \(\{\beta = (\beta_1 \cdots \beta_n) | \beta_1 <' \cdots <' \beta_k\}\) and a subset \(I(i_1 \cdots i_j \cdots i_n s_k, <')\) by \(I(i_1 \cdots i_j \cdots i_n s_k, <')\) by \(I(i_1 \cdots i_j \cdots i_n s_k, <')\). We put

\[
\begin{align*}
V(i_1 \cdots i_j \cdots i_n s_k, <') &= C(i_1 \cdots i_j \cdots i_n s_k, <') - I(i_1 \cdots i_j \cdots i_n s_k, <') \\
V(i_1 \cdots i_j \cdots i_n s_k, <') &= C(i_1 \cdots i_j \cdots i_n s_k, <') - I(i_1 \cdots i_j \cdots i_n s_k, <')
\end{align*}
\]

Let \(\tilde{h} = \{(1, \cdots, m+n), <) \rightarrow \{(1, \cdots, m+n), <'\}\) be an order preserving bijection defined by

\[
\tilde{h}(i) = i \text{ for } i \neq i_j, s_k \\
\tilde{h}(i_j) = s_k \\
\tilde{h}(s_k) = i_j,
\]

Then \(\tilde{h}\) induces order preserving bijections

\[
h : C(i_1 \cdots i_j \cdots i_n s_k, <) \rightarrow C(i_1 \cdots i_j \cdots i_n s_k, <')
\]

and
h: \( I(i_1 \cdots i_j \cdots i_n s_k, <) \rightarrow I(i_1 \cdots i_j \cdots i_n i_j, <') \).

Hence, we have an order preserving bijection

\[
h: V(i_1 \cdots i_j \cdots i_n s_k, <) \rightarrow V(i_1 \cdots i_j \cdots i_n i_j, <') .
\]

**Proposition 2.5.** On \( \pi^{-1}(V_{i_1 \cdots i_n}) \),

\[
\bigwedge_{\gamma \in V(i_1 \cdots i_j \cdots i_n s_k, <)} dQ(\beta i_1 \cdots i_j \cdots i_n s_k)
\]

\[
= \varepsilon(i_1 \cdots i_j \cdots i_n s_k, i_1 \cdots i_j \cdots i_n i_j) \left( \frac{p_{i_1 \cdots i_j \cdots i_n s_k}}{p_{i_1 \cdots i_n}} \right) \bigwedge_{\gamma \in V(i_1 \cdots i_j \cdots i_n i_j, <')} dQ(\gamma i_1 \cdots i_j \cdots i_n i_j)
\]

where \( \varepsilon(i_1 \cdots i_j \cdots i_n s_k, i_1 \cdots i_j \cdots i_n i_j) \) is constant and valued in \( \{\pm 1\} \), and \( t = r - (n-1)(m-1) \).

**Proof.** By Lemma 2.4, we have

\[
Q(i_1 \cdots i_j \cdots i_n i_j, s_k) = \pm Q(i_1 \cdots i_j \cdots i_n i_j, s_k)
\]

for \( l = 1, \ldots, j, \ldots, n, t = 1, \ldots, k, \ldots, m \). In other words, for

\[
\beta = (i_1 \cdots i_j \cdots i_n s_k, i_1 \cdots i_j \cdots i_n i_j) (l = 1, \ldots, j, \ldots, n; \ t = 1, \ldots, k, \ldots, m)
\]

\[
Q(\beta i_1 \cdots i_j \cdots i_n s_k) = \pm Q(h(\beta)i_1 \cdots i_j \cdots i_n i_j).
\]

We put

\[
S(i_1 \cdots i_j \cdots i_n s_k)
\]

\[
= \left\{ \beta \in V(i_1 \cdots i_j \cdots i_n s_k, <) \left| \beta = (i_1 \cdots i_j \cdots i_n i_j, l = 1, \ldots, j, \ldots, n; \ t = 1, \ldots, k, \ldots, m) \right. \right\}
\]

and

\[
S(i_1 \cdots i_j \cdots i_n i_j)
\]

\[
= \left\{ \beta \in V(i_1 \cdots i_j \cdots i_n i_j, <) \left| \beta = (i_1 \cdots i_j \cdots i_n i_j, l = 1, \ldots, j, \ldots, n; \ t = 1, \ldots, k, \ldots, m) \right. \right\}.
\]

Obviously \( h(S(i_1 \cdots i_j \cdots i_n s_k)) = S(i_1 \cdots i_j \cdots i_n i_j) \). Now we claim that on \( \pi^{-1}(U_{i_1 \cdots i_n}) \)

\[
Q(\beta i_1 \cdots i_j \cdots i_n s_k) = \pm \frac{p_{i_1 \cdots i_j \cdots i_n s_k}}{p_{i_1 \cdots i_n}} Q(h(\beta)i_1 \cdots i_j \cdots i_n i_j)
\]

\[
+ \sum_{\gamma \in S(\beta)} P_\gamma \left( \frac{p_{\beta \gamma \lambda \cdots}}{p_{i_1 \cdots i_n}} \right) Q(\gamma i_1 \cdots i_j \cdots i_n i_j),
\]

where \( P_\gamma \left( \frac{p_{\beta \gamma \lambda \cdots}}{p_{i_1 \cdots i_n}} \right) \) denotes a polynomial of \( \frac{p_{\beta \gamma \lambda \cdots}}{p_{i_1 \cdots i_n}} \), for each.
\( \beta \in V(i_1 \cdots i_l \cdots i_n s_k) \).

Case 1. \( \beta = (i_1 \cdots i_j \cdots i_{l-1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q}) \) where \( l = 1, \ldots, j, \ldots, n-1, \)

\( l < \alpha_k (a=1, \ldots, t), \mu_b \neq k (b=1, \ldots, q). \)

By Lemma 2.1 (b) and (2.2) 1) 2),

\[
Q(i_1 \cdots i_j \cdots i_{l-1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q} i_1 \cdots i_n^s_k) = 0
\]

\[
\sum_{s_{l+1}} (-1)^{l^1 + 1} \frac{Q(i_1 \cdots i_j \cdots i_{l-1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q} i_1 \cdots i_n^s_k)}{p_{i_1 \cdots i_n}}
\]

Then

\[
\gamma = (i_1 \cdots i_j \cdots i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q})
\]

for \( a \geq l ).

By Lemma 2.1 (a) and (2.2) 2),

\[
(2.7) \quad Q(i_1 \cdots i_j \cdots i_{l+1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q} s_k) = -Q(i_1 \cdots i_j \cdots i_{l+1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q} s_k) + \sum_{s_{l+1}} (-1)^{l^1 + 1} \frac{Q(i_1 \cdots i_j \cdots i_{l+1} i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots s_{p_q} s_k)}{p_{i_1 \cdots i_n}}
\]

Note that

\[
Q(i_1 \cdots i_j \cdots i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots i_n^s_k) \equiv 0 \quad (t + l = n-1),
\]

\[
Q(i_1 \cdots i_j \cdots i_{a_1} \cdots i_{a_l} s_j s_{p_1} \cdots i_n^s_k) \equiv 0 \quad (t + l = n-1)
\]

and
\[ Q(i_1 \cdots \hat{i}_j \cdots i_{a_1} \cdots i_{a_r} s_{j \cdot i_{a_r}} \cdots s_{j \cdot i_{a_r+1}} \cdots i_j \cdots i_{a_r}) = 0 \]

if \( q \geq 2 \). Thus the first term in the right hand side of (2.7) is identically zero. Obviously

\[ Q(i_1 \cdots \hat{i}_j \cdots i_{a_1} \cdots i_{a_r} s_{j \cdot i_{a_r}} \cdots s_{j \cdot i_{a_r+1}} \cdots i_j \cdots i_{a_r}) = -Q(i_{j_1} \cdots \hat{i}_j \cdots i_{a_1} \cdots i_{a_r} s_{j_1} \cdots s_{j_r} \cdots i_j \cdots i_{a_r}) \]

by (2.2) 1). Inductively we get

(2.8) \[ Q(i_1 \cdots \hat{i}_j \cdots i_{a_1} \cdots i_{a_r} s_{j \cdot i_{a_r}} \cdots s_{j \cdot i_{a_r+1}} \cdots i_j \cdots i_{a_r}) = \sum \limits_{\gamma \in A(\beta)} P_{\gamma}(\cdots, \frac{p_{\lambda_1 \cdots \lambda_s}}{p_{i_1 \cdots i_n}}, \cdots) Q(\gamma_{i_1} \cdots \hat{i}_j \cdots i_{a_r}) \]

for some polynomial functions \( P_{\gamma} \). Hence we get our claim (2.5) in this case. By the same way, we can show our claim in the following cases:

Case 2. \( \beta = (i_1, \cdots, i_{l_1}, \cdots, i_{l_t}, a_{r+1}, \cdots, a_{t_k}) \)
\( l = 1, \cdots, n, t \geq 0, q \geq 2, l < a_r \neq j \)
\( (a = 1, \cdots, t) \mu_b \neq k (b = 1, \cdots, q) \).

Case 3. \( \beta = (i_1, \cdots, i_{l_1}, \cdots, i_{l_t}, a_{r+1}, \cdots, a_{t_k}) \)
\( l = 1, \cdots, n, t \geq 0, q \geq 2, l < a_r \neq j \)
\( (a = 1, \cdots, t) \mu_b \neq k (b = 1, \cdots, q) \).

Case 4. \( \beta = (i_1, \cdots, i_{l_1}, \cdots, i_{l_t}, a_{r+1}, \cdots, a_{t_k}) \)
\( l = 1, \cdots, n, t \geq 0, q \geq 2, l < a_r \neq j \)
\( (a = 1, \cdots, t) \mu_b \neq k (b = 1, \cdots, q) \).

Hence, on \( \pi^{-1}(V_{i_1, \cdots, i_n}) \), we have
\[ dQ(\beta_{i_1} \cdots \hat{i}_j \cdots i_{a_r}) = \pm dQ(h(\beta)i_1 \cdots \hat{i}_j \cdots i_{a_r}) \]

for \( \beta \in S(i_1, \cdots, i_{a_r}) \) and
\[ dQ(\beta_{i_1} \cdots \hat{i}_j \cdots i_{a_r}) = \pm \frac{p_{i_1 \cdots i_{a_r}}}{p_{i_1 \cdots i_n}} dQ(h(\beta)i_1 \cdots \hat{i}_j \cdots i_{a_r}) \]
\[ + \sum \limits_{\gamma \in A(\beta)} P_{\gamma}(\cdots, \frac{p_{\lambda_1 \cdots \lambda_s}}{p_{i_1 \cdots i_n}}, \cdots) dQ(\gamma_{i_1} \cdots \hat{i}_j \cdots i_{a_r}) \]

for \( \beta \in V(i_1, \cdots, i_{a_r}, \prec) - S(i_1, \cdots, i_{a_r}) \).

Since \( h \) is order preserving and the number of elements in \( S(i_1, \cdots, i_{a_r}) \) is \( (n-1)(m-1) \), we get Proposition 2.5.
\[ \text{q.e.d.} \]

**Proposition 2.6.** For \( n \)-tuples \( (i_1, \cdots, i_n), (j_1, \cdots, j_n) \) \( (1 \leq i_1 < \cdots < i_n \leq m+n, \)
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on \( \pi^{-1}(V_{i_1 \cdots i_n}) \), where \( \varepsilon(j_1 \cdots j_n, i_1 \cdots i_n) \) is constant and valued in \( \{ \pm 1 \} \).

Proof. It is enough to see that for \( n \)-tuples \( (i_1, \ldots, i_n) \) and \( (i_1', \ldots, i_n') \) \((1 \leq i_1 < \cdots < i_j < i_{j+1} < \cdots < i_n \leq m+n)\)

\[
q_{i_1' \cdots i_n'} = \varepsilon(i_1' \cdots i_n'; j_1 \cdots j_n') \left( \frac{p_{j_1 \cdots j_n}}{p_{i_1' \cdots i_n'}} \right)^t q_{i_1' \cdots i_n'}
\]

on \( \pi^{-1}(V_{i_1' \cdots i_n'}) \).

By Lemma 2.2, 2.3 and Proposition 2.5, the equality (2.9) holds on

\[
\pi^{-1}(V_{i_1 \cdots i_n}) \cap \pi^{-1}(V_{i_1' \cdots i_n'}).
\]

Since \( \varphi_{i_1' \cdots i_n'} \) and \( \varphi_{i_1 \cdots i_n} \) are holomorphic forms on \( \mathbb{C}^{m+1} \), the equality (2.9) holds on \( \pi^{-1}(V_{i_1 \cdots i_n}) \).

q.e.d.

Lemma 2.7. For \( n \)-tuples \( (i_1, \ldots, i_n) \), \( (j_1, \ldots, j_n) \), \( (k_1, \ldots, k_n) \),

\( \varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) \varepsilon(j_1 \cdots j_n, k_1 \cdots k_n) \varepsilon(k_1 \cdots k_n, i_1 \cdots i_n) = 1 \) on \( V_{i_1 \cdots i_n} \cap V_{j_1 \cdots j_n} \cap V_{k_1 \cdots k_n} \).

Proof. Since

\[
\varphi_{i_1 \cdots i_n}(z) = (p_{i_1 \cdots i_n}(z))^t \left( \bigwedge_{\beta \in G(i_1', \ldots, i_n')} (dp_{\beta}) + \text{other terms} \right) \varphi_{i_1 \cdots i_n}(z) \equiv 0 \quad \text{for} \quad z \in \pi^{-1}(V_{i_1 \cdots i_n}).
\]

By Proposition 2.6, we get

\[
\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) \varepsilon(j_1 \cdots j_n, k_1 \cdots k_n) \varepsilon(k_1 \cdots k_n, i_1 \cdots i_n) = 1
\]

on \( \pi^{-1}(V_{i_1 \cdots i_n}) \cap \pi^{-1}(V_{j_1 \cdots j_n}) \cap \pi^{-1}(V_{k_1 \cdots k_n}) \). Since \( \varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) \) is constant, we get our claim.

q.e.d.

Lemma 2.8 (Principle of monodromy). Let \( G \) be an abelian group and \( M \) a simply connected manifold. Let \( \mathcal{U} = \{ U_a \}_a \) be an open covering of \( M \) such that each \( U_a \) is connected. Then \( H^1(\mathcal{U}, G) = (0) \).


Applying Lemma 2.8, for the complex Grassmann manifold \( G_{m+n, n}(\mathbb{C}) \) and the system of transition functions \( \{ \varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) \} \), we get a system of constant functions \( \{ \delta(i_1 \cdots i_n) \} (\delta(i_1 \cdots i_n): V_{i_1 \cdots i_n} \rightarrow \{ \pm 1 \}) \) such that \( \varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) = \delta(j_1 \cdots j_n)^{-1} \delta(i_1 \cdots i_n) \). We put \( q_{i_1 \cdots i_n} = \delta(i_1 \cdots i_n) q_{i_1 \cdots i_n} \). Then, by Proposition 2.6, we have
By Proposition 1.2, a compact complex hypersurface $X$ of $G_{m+n,n}(C)$ is the complete intersection of $G_{m+n,n}(C)$ and an irreducible subvariety $Y$ of codimension 1 in $P^n(C)$. Let $(F)$ denote the homogeneous ideal associated to $Y$. Note that the degree of homogeneous polynomial $F$ on $C^{m+1}$ is the degree of $X$ and $W_{1 \cdots n} = \{ (x) \in V_{1 \cdots n} | F(x) = 0 \}$.

**Lemma 2.9.** On $\pi^{-1}(W_{1 \cdots n}), \alpha_1 \cdots \alpha_n \wedge dF \neq 0$.

**Proof.** Suppose that there is a point $x_0 \in \pi^{-1}(W_{1 \cdots n})$ such that $(\alpha_1 \cdots \alpha_n \wedge dF)_{x_0} = 0$. Since $\pi^{-1}(X)$ is a complex submanifold of $C^{m+1}$, there are an open neighborhood $U$ of $x_0$ in $C^{m+1}$ and holomorphic functions $f_j (j=1, \cdots, r+1)$ such that $U \cap \pi^{-1}(X) = \{ z \in U | f_j(z) = 0, j=1, \cdots, r+1 \}$ and $(df_j)_{x_0} = 0$ are linearly independent for $z \in U \cap \pi^{-1}(X)$. By the Nullstellensatz for prime ideals ([4] chap. 2A Theorem 7),

$$f_j = \sum_{\alpha} q_{j\alpha} Q_\alpha + h_j F$$

where $q_{j\alpha}, h_j$ are holomorphic functions on $U$ and $Q_\alpha$ are generators of the ideal $I(G_{m+n,n}(C))$. Thus we have

$$(df_j)_{x_0} = \sum_{\alpha} q_{j\alpha}(x_0)(dQ_\alpha)_{x_0} + h_j(x_0)(dF)_{x_0}.$$  

By Lemma 2.1 a) and b) and (2.2), we see that for each $Q_\alpha$

$$(dQ_\alpha)_{x_0} = \sum_{\gamma \in \pi^{-1}(G_{1 \cdots n,0})} C_\alpha(\gamma)(dQ(\gamma_1 \cdots \gamma_n))_{x_0}$$

for some $C_\alpha(\gamma) \in C$. Hence, $\bigwedge_{j=1}^{r+1} (df_j)_{x_0} = c(q_{1 \cdots n} \wedge dF)_{x_0}$ for some $c \in C$ and hence $\bigwedge_{j=1}^{r+1} (df_j)_{x_0} = 0$. This is a contradiction.  

We define a local holomorphic section $t_1 \cdots n$ of the line bundle $N$ on $W_{1 \cdots n}$ by

$$t_{1 \cdots n}(x) = (s_{1 \cdots n} (q_{1 \cdots n} \wedge dF))_x$$

for $x \in W_{1 \cdots n}$.

**Lemma 2.10.** The system of transition functions associated to the local trivialization $(W_{1 \cdots n}, t_1 \cdots n)$ of the line bundle $N$ is $(\epsilon^* g_1 \cdots n, j_1 \cdots j_n 2^{r+\alpha-i})$, where $a$ is the degree of $X$. In particular, $N = \epsilon^* E^{2r+\alpha-i}$.

**Proof.** By Lemma 2.9, we have $t_{1 \cdots n}(x) \neq 0$ for any $x \in W_{1 \cdots n}$. Since $Q(\beta_1 \cdots \beta_n)$ are of degree 2 and $F$ is of degree $a$, 

\[ (2.10) \]

$$q_{j_1 \cdots j_n} = \left( \frac{p_{j_1 \cdots j_n}}{p_{j_1 \cdots j_n}} \right)^i_{q_{j_1 \cdots j_n}} \text{ on } \pi^{-1}(V_{1 \cdots n}).$$
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\[ t_{j_1 \ldots j_n}(x) = \left( \frac{p_{j_1 \ldots j_n}(x)}{p_{j_1 \ldots j_n}} \right)^{-1 + 2r + a} t_{j_1 \ldots j_n}(x) \]

\[ = (t^* g_{i_1 \ldots i_n, j_1 \ldots j_n}(x))^{2r + a - t} t_{j_1 \ldots j_n}(x) \]
on \( W_{j_1 \ldots j_n} \cap W_{i_1 \ldots i_n} \), by (2.10).

q.e.d.

The canonical line bundle \( K \) of \( P^\mu(C) \), the holomorphic line bundle of co-vectors of bi-degree \((\mu, 0)\) on \( P^\mu(C) \), is isomorphic to \( E^{\mu+1} \). By (2.1) and Lemma 2.10,

(2.12)

\[ K(X) = t^* E^{\mu+n-a}, \]
since \( t = r - (n-1)(m-1) \).

**Remark.** Let \( j : G_{m+n}(C) \to P^\mu(C) \) be the inclusion. Then \( K(G_{m+n}(C)) = j^* E^{m+n} \) ([1] §16). Let \( X \) be a compact complex submanifold of codimension 1 in \( G_{m+n}(C) \) and \( t_0 : X \to G_{m+n}(C) \) the inclusion. Then \( K(X) = (j \circ t_0)^* E^{m+n-a} \), by considering the normal bundle \( N(X, G_{m+n}(C)) \) of \( X \) in \( G_{m+n}(C) \) and by Proposition 1.2.

The first Chern class of \( X \), which is the Chern class of the dual bundle \( K(X)^* \) of \( K(X) \), is the cohomology class containing the form \((m+n-a) \omega \), where \( \omega = t^* \Omega \) is the Kähler form on \( X \) associated to the induced Kähler metric on \( X \).

We shall determine a local section \( k_{i_1 \ldots i_n} \) of \( K(X)^* \) on each \( W_{i_1 \ldots i_n} \) so that the system of transition functions associated to the local trivialization \( (W_{i_1 \ldots i_n}, k_{i_1 \ldots i_n}) \) is \((t^* g_{i_1 \ldots i_n, j_1 \ldots j_n})^{a-(m+n)} \). We put

(2.13)

\[ l_{i_1 \ldots i_n} = (-1)^{\chi (i_1 \ldots i_n)} \left( \prod_{\langle a_1, \ldots, a_n \rangle \subset \{ i_1, \ldots, i_n \}} \frac{\partial}{\partial u_{i_1 \ldots i_n, a_1 \ldots a_n}} \right) \]
on \( U_{i_1 \ldots i_n} \), where we take the exterior product of \( \frac{\partial}{\partial u_{i_1 \ldots i_n, a_1 \ldots a_n}} \) according to the natural lexicographical order. Then \( (U_{i_1 \ldots i_n}, l_{i_1 \ldots i_n}) \) is the local trivialization of the holomorphic line bundle \( K \) on \( P^\mu(C) \) and the system of transition functions is \((g_{i_1 \ldots i_n, j_1 \ldots j_n})^{\mu+1}) \).

**Lemma 2.11.** Let \( k_{i_1 \ldots i_n} \) be a local holomorphic section of \( K(X)^* \) on \( W_{i_1 \ldots i_n} \) defined by

(2.14)

\[ k_{i_1 \ldots i_n}(x) = l_{i_1 \ldots i_n}(x) \sqcup t_{i_1 \ldots i_n}(x) \]

for \( x \in W_{i_1 \ldots i_n} \), where \( \sqcup \) denotes the right interior multiplication. Then the system of transition functions associated to the local trivialization \( (W_{i_1 \ldots i_n}, k_{i_1 \ldots i_n}) \) of \( K(X)^* \) is \((t^* g_{i_1 \ldots i_n, j_1 \ldots j_n})^{\mu+1}) \).

Proof. By (2.1) and Lemma 2.10, \((k_{i_1 \ldots i_n}, W_{i_1 \ldots i_n}) \) is a local trivialization of \( K(X)^* \) and the system of transition functions is \((t^* g_{i_1 \ldots i_n, j_1 \ldots j_n})^{\mu+1} + 2r + a - t \). Since \(-(\mu+1) + 2r + a - t = a - (m+n)\), we get our claim.

q.e.d.
3. The relation between volumes

Let \( C_n \) denote the set \( \{(i_1, \ldots, i_n) | 1 \leq i_1 < \cdots < i_n \leq m+n\} \). For an element \( i=(i_1, \ldots, i_n) \in C_n \), we put

\[
q_i = \sum H_{\lambda_1: \lambda_r}^i dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r}
\]

where the summation runs over all \( (\lambda_1, \cdots, \lambda_r) \in C_n^{ \times r} \) such that \( \lambda_1 < \cdots < \lambda_r \) with respect to the lexicographical order \( < \) on \( C_n \). Note that \( H_{\lambda_1: \lambda_r}^i \) are homogeneous polynomials of degree \( r \).

**Proposition 3.1.** There exist homogeneous polynomials \( H_{\lambda_1: \lambda_r} \) of degree \((n-1)(m-1)\) on \( C^{n+1} \) such that

\[
H_{\lambda_1: \lambda_r}^i = p_i^j H_{\lambda_1: \lambda_r} \text{ on } \pi^{-1}(V_i) \text{ for each } i \in C_n.
\]

**Proof.** By (2.10), we have

\[
H_{\lambda_1: \lambda_r}^i = \left(\frac{p_i}{p_j}\right)^{(n-1)(m-1)} H_{\lambda_1: \lambda_r} \text{ on } \pi^{-1}(V_i \cap V_j) \text{ for each } (\lambda_1, \cdots, \lambda_r). \text{ Thus we get}
\]

\[
\frac{H_{\lambda_1: \lambda_r}}{p_i^*} = \left(\frac{p_j}{p_i}\right)^{(n-1)(m-1)} \frac{H_{\lambda_1: \lambda_r}}{p_j^*}
\]

On \( V_i \cap V_j \). Hence, \( \{H_{\lambda_1: \lambda_r}^i/p_i\}_{i \in C_n} \) define a holomorphic section of the line bundle \( j^*H^{(n-1)(m-1)} \). Note that a holomorphic section of line bundle \( H^{(n-1)(m-1)} \) on \( P^n(C) \) is nothing but a homogeneous polynomial of degree \((n-1)(m-1)\) on \( C^{n+1} \). By Proposition 1.1, there is a homogeneous polynomial \( H_{\lambda_1: \lambda_r} \) of degree \((n-1)(m-1)\) on \( C^{n+1} \) such that

\[
\frac{H_{\lambda_1: \lambda_r}}{p_i^{(n-1)(m-1)}} = \frac{H_{\lambda_1: \lambda_r}}{p_i^*} \text{ on } V_i.
\]

Thus we get (3.2). q.e.d.

Now we have

\[
q_i = p_i^* \sum H_{\lambda_1: \lambda_r} dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r}
\]

on \( \pi^{-1}(V_i) \) for each \( i \in C_n \), and hence

\[
q_i \wedge dF = p_i^* \sum G_{\lambda_1: \lambda_{r+1}} dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_{r+1}}
\]

on \( \pi^{-1}(W_i) \), where \( G_{\lambda_1: \lambda_{r+1}}(\lambda_1 < \cdots < \lambda_{r+1}) \) are homogeneous polynomials of degree \((n-1)(m-1)+(a-1)\).

For homogeneous polynomials \( P_1, \ldots, P_s \) on \( C^{n+1} \), we put
\[ dP_1 \wedge \cdots \wedge dP_s = \sum P_{\lambda_1 \cdots \lambda_s} \lambda_1 d\lambda_1 \wedge \cdots \wedge d\lambda_s \]

where the summation runs over all \((\lambda_1, \cdots, \lambda_s) \in C_\mu \times \cdots \times C_\mu\) such that \(\lambda_1 < \cdots < \lambda_s\) with respect to the lexicographical order \(<\) on \(C_\mu\), and we define

\[ ||dP_1 \wedge \cdots \wedge dP_s||^2(z) = \sum |P_{\lambda_1 \cdots \lambda_s}(z)|^2 \]

for \(z \in C_{\mu+1}\). Then we have

\[ ||g_i \wedge dF||^2(z) = |p_i(z)|^2 \sum |G_{\lambda_1 \cdots \lambda_r}(z)|^2 \]

for \(z \in \pi^{-1}(W_i)\).

Now we can define a \(C^\infty\)-function \(\varphi: X \to \mathbb{R}\) by

\[ \varphi(x) = \frac{||g_i \wedge dF||^2(z)}{|p_i(z)|^2 ||z||^{2[(\mu-1)(\mu-1)+1]}}, \]

where \(z \in \pi^{-1}(x)\).

Note that \(\varphi(x) = (\sum |G_{\lambda_1 \cdots \lambda_r}(z)|^2 ||z||^{2[(\mu-1)(\mu-1)+1]} / ||z||^{2(\mu-1)(\mu-1)+1})\) for \(z \in \pi^{-1}(x), x \in X\).

Since the dual bundle \(K(X)^*\) of the canonical line bundle \(K(X)\) is the line bundle of \((mn-1)\) vectors of bi-degree \((mn-1, 0)\), the set of hermitian fiber metrics on \(K(X)^*\) and the set of positive volume elements on \(X\) are canonically in one to one correspondence. Let \(\vartheta\) denote the volume element on \(X\) corresponding to the fiber metric \(\vartheta^* ||z||^{2(a-n-m)}\) on \(K(X)^*\). Then the curvature form of the connection determined by the fiber metric \(\vartheta^* ||z||^{2(a-n-m)}\) is \((m+n-a)\omega\), where \(\omega = \vartheta^* \Omega\) is the Kähler form of the induced metric on \(X\).

Now the relation between two volume elements \(\omega_{mn+1}\) and \(\vartheta\) is given by the following Proposition.

**Proposition 3.2.** Let \(\varphi\) be a \(C^\infty\)-function on \(X\) defined by (3.9). Then

\[ \omega_{mn+1} = \frac{(mn-1)!}{(2\pi)^{mn-1}} \varphi \vartheta \quad \text{on} \quad X. \]

We need several lemmas to prove Proposition 3.2. Note that the norm defined by (3.7) does not depend on the choice of unitary cartesian coordinates on \(C_{\mu+1}\). That is, for a unitary matrix \(A \in U(\mu+1)\) and homogeneous polynomials \(P_j\), we put \(P_j(w) = P_j(A^{-1}w)\) for \(w \in C_{\mu+1}\). Then

\[ ||dp_1 \wedge \cdots \wedge dp_s||^2(z) = ||dp_1' \wedge \cdots \wedge dp_s'||^2(w) \]

for \(w = Az, z \in C_{\mu+1}\).

In order to prove Proposition 3.2, it suffices to verify (3.10) at an arbitrary point \(x_0 \in X\). Fix a point \(x_0 \in X\) and let \(z_0\) denote an element of \(C_{\mu+1}\) such that
For an element $A \in U(\mu + 1)$, let $p_I'$ denote $p_I' = \sum_{I} A_I p_I$, where $A_I = (A_I)$, and put $w = (\cdots, p'_I, \cdots)$. For a homogeneous polynomial $P$ of degree $k$ on $\mathbb{C}^{n+1}$, put $P'(w) = P(A^{-1}w)$, $P'_I(w) = P'(w)(p'_I)^k$, where $i_0 = (1, \ldots, n) \in C_n$ and put $u'_{i,0}(x) = p'_I(x)/p'_I(z)(z \in \pi^{-1}(x))$. \( \lambda \in C_n, (\lambda \neq i_0) \).

**Lemma 3.3.** If $x_0 \in W, (i \in C_n)$, there is an element $A \in U(\mu + 1)$ such that $p'_I(x_0) = 1$, $p'_J(x_0) = 0$ for $j \in C_n, j \neq i_0$ and $(dQ'(\beta, i)_0)_{x_0} (\beta \in C(i, -1) = I(i, -1))$ (where the order is principal with respect to $i$), $(dF'_i)_{x_0}$ are linear combination of $\lambda \in C(i, <) - I(i, <)$, $(du'_{i,0,1}\cdots [n-1+1])_{x_0}$.

**Proof.** By a routine computation of linear algebra.

Now we put $p_I = \sum_{I} B_I p'_I$ and $C_{\lambda} = (\partial u_{i, \lambda}/\partial u'_{i,0, \gamma})(x_0)$.

**Lemma 3.4.**

(3.12) $C_{\lambda} = (B_{\lambda})^{-2}(B_{\lambda} B_{i,0} - B_{i,0} B_{\lambda})$

for $\lambda \neq i, \nu \neq i_0, \lambda, \nu \in C_n$.

**Proof.** Straightforward computation.

Let $J(i_0, <)$ denote $I(i_0, <) - \{i_0, (12\cdots n-1n+1)\}$. We put $J(i_0, <) = \{\nu_1, \cdots, \nu_m, (k = 1, \cdots, mn-2), C(i, -1) - I(i, -1) = \{\beta_1, \cdots, \beta_r\}$ with $\beta_{1\cdots r+1}(l = 1, \cdots, r-1)$ and $C(i_0, <) - I(i_0, <) = \{\lambda_1, \cdots, \lambda_r\}$ with $\lambda_s < \lambda_{s+1} (s = 1, \cdots, r-1)$.

**Lemma 3.5.** Let $k_1$ be the holomorphic section of $K(X)^{\star}$ on $W_i$ defined in Lemma 2.12. Then, at $x_0 \in W_i$,

(3.13) $k_1(x_0) = (-1)^{\tau(i)^{-1} \cdot \delta(i) \cdot \det (C_{\lambda})^{-1}}$

$\times \left(\frac{p'_i(x_0)}{p_i} \right)_{x_0} \partial Q'(\beta_1, i)_{x_0} \cdots \partial Q'(\beta_r, i)_{x_0} F'_{x_0} (x_0)$

$\times (\partial u'_{i,0,12\cdots [n-1n+1]} \cdots (\partial u'_{i,0,\Lambda,s})_{x_0}.$

**Proof.** For a homogeneous polynomial $P$ of degree $k$ on $\mathbb{C}^{n+1}$, put $P_i = P(p'_i)$ on $U_i$. By the definition,

$t_i(x_0) = \delta(i) \partial Q'(\beta_1, i) \cdots \partial Q'(\beta_r, i) \partial F'_{x_0} (x_0)$.

Thus

$t_i(x_0) = \delta(i) (\partial Q'(\beta_1, i) \cdots \partial Q'(\beta_r, i) \partial F'_{x_0})_{x_0}$

$= \delta(i) (p'_i p_i(x_0) \partial Q'(\beta_1, i) \cdots \partial Q'(\beta_r, i) \partial F'_{x_0})_{x_0}$

On the other hand, we have
\[ \det (C^i_v)(\alpha \in U_{\beta_i}, \gamma < \delta - i) \frac{\partial}{\partial u_{i, \mu}} x_0 = \left( \frac{\partial}{\partial u'_{i, \mu}} x_0 \right) \]

By the definition of \( k_i \),
\[
k_i(x_0) = (-1)^{\gamma(i)} \delta(i) \cdot [\det (C^i_v)]^{-1} (p_i, p_i)(x_0)^{2r+s} \]
\[
\times \left( \frac{\partial}{\partial u'_{i, \mu}} x_0 \right) \cap (dQ'_{i, i_{10}}, \ldots, \cdots, dF'_{i_0})(x_0) \]

By Lemma 3.3, we get (3.13). q.e.d.

Now the local expression of the volume element \( \nu \) at \( x_0 \) is given by the following Lemma.

**Lemma 3.6.**

(3.14) \[ \nu(x_0) = (\sqrt{-1})^{mn-1} |\det (C^i_v)|^2 \cdot |(p_i, p_i)(x_0)|^{-2(m+n+2r)} \]
\[
\times \frac{\partial Q'_{i, i_{10}}, \ldots, \cdots, dF'_{i_0}}{\partial (u'_{i, \mu_{12} - n - 1x + 3}, u'_{i, \lambda_1}, \ldots, u'_{i, \lambda_r})} (x_0) \]

where \((d\eta')(x_0) = (du'_{i, \mu_1}, \ldots, \cdots, \cdots, du'_{i, \gamma_{mn-1}})\)

**Proof.** By the definition, \( \nu \) is the volume element on \( X \) corresponding to the fiber metric \( \epsilon^*|z|^{2(m+n+2r)} \) on \( K(X)^* \). Note that
\[ 1 + \sum_{i \in U} |(p_i, p_i)(x_0)|^2 = |(p_i, p_i)(x_0)|^2. \]

Put
\[ T_i(x_0) = (-1)^{\gamma(i)} \delta(i) \cdot [\det (C^i_v)]^{-1} \]
\[
\times (p_i^*, p_i^*)(x_0)^{2r+s} \frac{\partial Q'_{i, i_{10}}, \ldots, \cdots, dF'_{i_0}}{\partial (u'_{i_1, \mu_{12} - n - 1x + 3}, u'_{i, \lambda_1}, \ldots, u'_{i, \lambda_r})} (x_0) \]

Then \( \nu(x_0) \) is given by
\[ \frac{1}{|T_i(x_0)|^2 |(p_i^*, p_i^*)(x_0)|^{2(m+n+2r)} (d\eta' \wedge d\eta')(x_0). \]

Hence
\[ \nu(x_0) = (\sqrt{-1})^{mn-1} |\det (C^i_v)|^2 \cdot |(p_i^*, p_i^*)(x_0)|^{2(m+n+2r)} \]
\[
\times \frac{\partial Q'_{i, i_{10}}, \ldots, \cdots, dF'_{i_0}}{\partial (u'_{i_1, \mu_{12} - n - 1x + 3}, u'_{i, \lambda_1}, \ldots, u'_{i, \lambda_r})} (x_0) \]

q.e.d.

**Lemma 3.7.** At \( x_0 \in W_i \),

(3.15) \[ \varphi(x_0) = |(p_i^*, p_i^*)(x_0)|^{2r} \frac{\partial Q'_{i, i_{10}}, \ldots, \cdots, dF'_{i_0}}{\partial (u'_{i_1, \mu_{12} - n - 1x + 3}, u'_{i, \lambda_1}, \ldots, u'_{i, \lambda_r})} (x_0)^2. \]
Proof. Fix $c \in \mathbb{C}^*$ so that $|c_0(x_0)|^2 = 1$. Then $|c|^2 \cdot \left( \sum_{a \in \mathbb{C}_x} |(p_{a}\nu\nu)(x_0)|^2 \right) = 1$ and $|c|^2 = |(p_{\nu\nu}\nu\nu)(x_0)|^{-2}$. Note that

$$\varphi(x_0) = \frac{||q_{i} \wedge dF||^2 c_0(x_0)}}{|c|^{2}} = ||q_{i} \wedge dF'||^2 (1, 0, ..., 0) \quad \text{by (3.11).}$$

Since

$$\frac{\partial O'(\beta_{a}, i)}{\partial p_{i}}(1, 0, ..., 0) = 0 \quad \text{for} \quad k = 1, ..., \nu, \frac{\partial F'}{\partial p_{i}}(1, 0, ..., 0) = 0$$

and

$$\frac{\partial O'(\beta_{a}, i)}{\partial p_{i}}(1, 0, ..., 0) = \frac{\partial F'}{\partial u_{i}}(x_0) \quad \text{for} \quad j \in \mathbb{C}_x, j \neq i_0,$$

we have

$$||q_{i} \wedge dF||^2 (1, 0, ..., 0) = ||dO'(\beta_{a}, i) \wedge \cdots \wedge dO'(\beta_{a}, i) \wedge dF'||^2 (1, 0, ..., 0)$$

and

$$= \left| \frac{\partial O'(\beta_{a}, i)}{\partial u_{i}}(x_0) \right|^2$$

by Lemma 3.3. \quad \text{q.e.d.}

By Lemma 3.3, the Kähler form $\omega$ of the induced metric on $X$ is given by

$$\omega_{x_0} = \frac{\sqrt{-1}}{2\pi} \left( \sum_{\nu \not= \nu_0} du_{i_0, \nu} \wedge d\bar{u}_{i_0, \nu} \right) \quad \text{at} \quad x_0 \in X.$$

Hence,

$$\omega^m_{x_0} = \frac{\sqrt{-1}}{2\pi} \frac{(m-1)!}{(2\pi)^{m-1}} \left( d\varphi' \wedge \bar{d}\varphi' \right)_{x_0}.$$

Lemma 3.8.

$$|\det (C_i)|^2 = |(p_{\nu\nu}\nu\nu)(x_0)|^{2(\nu+1)}$$

Proof. Put $D_i^\nu = B_i^\nu B_i^\nu - B_i^\nu B_i^\nu$ for $\lambda \not= i, \nu \not= i_0, \lambda, \nu \in \mathbb{C}_x$. Note that

$$|\det (D_i)|^2 = \det (D_i) \cdot \det (\bar{D}_i) = \det \left( \sum_{\nu \not= \nu_0} D_i^\nu D_i^\nu \right)$$

and that

$$\sum_{\nu \not= \nu_0} D_i^\nu D_i^\nu = \sum_{\nu \not= \nu_0} \left( B_i^\nu B_i^\nu - B_i^\nu B_i^\nu \right) \left( B_i^\nu B_i^\nu - B_i^\nu B_i^\nu \right)$$

$$= \sum_{\nu \not= \nu_0} \left( B_i^\nu B_i^\nu - B_i^\nu B_i^\nu \right) \left( B_i^\nu B_i^\nu - B_i^\nu B_i^\nu \right)$$

$$= \delta_{\lambda\nu} |B_i^\nu|^2 + B_i^\nu B_i^\nu.$$
since $\sum_{\alpha \in \mathcal{O}, \beta} B_{\alpha}^* \overline{B}_\beta^\nu = \delta_{\lambda^\nu}$.

Thus
\[
|\det(D_{\beta^\nu})|^2 = |\det(\delta_{\lambda^\nu} | B_{\lambda^\nu} |^2 + B_{\lambda^\nu}^\nu \overline{B}_{\nu}^\lambda)|^2
= |B_{\lambda^\nu}^\nu |^2 \det(\delta_{\lambda^\nu} + (B_{\lambda^\nu} \overline{B}_{\nu}^\lambda | B_{\nu}^\lambda |^2))
= |B_{\lambda^\nu}^\nu |^2 (1 + \sum_{\lambda < \nu} |B_{\lambda^\nu}^\nu | B_{\nu}^\lambda |^2)
= |B_{\lambda^\nu}^\nu |^{2(m-1)}.
\]

Now
\[
|\det(C_{\beta^\nu})|^2 = |B_{\lambda^\nu}^\nu |^{-2} |\det(D_{\beta^\nu})|^2
= |B_{\lambda^\nu}^\nu |^{-2} \times |B_{\lambda^\nu}^\nu |^{2(m-1)} = |B_{\lambda^\nu}^\nu |^{-2(m-1)}.
\]

Since $B_{\lambda^\nu}^\nu = (p_{\lambda^\nu} / p_{\lambda^\nu}^\nu)(x_0)$, we get our claim. q.e.d.

Proof of Proposition 3.2.

By Lemma 3.6, Lemma 3.7 and Lemma 3.8, we have
\[
\varphi(x_0)b_{x_0} = (\sqrt{-1})^{(m-n-1)^2} |(p_{l^\nu} / p_{l^\nu}^l)(x_0)|^{2(m-n-2r+\alpha+1+1)}(d\eta' \wedge d\bar{\eta}')_{x_0}.
\]

Since
\[
\begin{align*}
    r-t & = (m-1)(n-1) = mn-(m+n)+1, \\
    \mu+1+t-2r-m-n & = \mu+1-r-(m+n)-mn+m+n-1 \\
    & = \mu+1-r-mn-1 = 0.
\end{align*}
\]

Hence
\[
\varphi(x_0)b_{x_0} = (\sqrt{-1})^{(m-n-1)^2}(d\eta' \wedge d\bar{\eta}')_{x_0}.
\]

Now our claim follows from (3.16).


Let $g_0$ denote the Kähler metric on $X$ induced from the Fubini-Study metric on $P^n(C)$. Then $(X, g_0)$ is an Einstein manifold if and only if $\varphi$ is a constant function on $X$.

Proof. The Ricci form of the Kähler metric $g_0$ on $X$ is the curvature form of the connection of type (1.0) on the holomorphic line bundle $K(X)^*$ determined by the volume element $\omega^{mn-1}$. Suppose that $g_0$ is Einstein, that is, the Ricci form is a constant multiple of the Kähler form $\omega$. Then the Ricci form is harmonic. On the other hand, the volume element $\nu$ determines the curvature form $(m+n-a)\omega$, which is also harmonic. Since the Ricci form and $(m+n-a)\omega$ are both curvature forms of the bundle $K(X)^*$, they are cohomologous. Thus the Ricci form must be $(m+n-a)\omega$. Since $\omega^{mn-1}$ and $\nu$ define the same curvature form, $d'd'' \log \varphi=0$, and hence $\log \varphi$ is a harmonic function on $X$. This implies that $\varphi$ is a constant function. Conversely, if $\varphi$ is a constant function, then the
metric $g_0$ is Einstein. q.e.d.

4. The dual map and Veronese map

In this section we recall the dual map and Veronese map due to Hano [5]. Let $\wedge^{r+1}(C^{m+1})^*$ denote the $(r+1)$-th exterior product of the dual space of the vector space $C^{m+1}$. We identify the tangent space of $C^{m+1}$ at a point with $C^{m+1}$ itself. We regard $(q_i \wedge dF)_z$ as an element in $\wedge^{r+1}(C^{m+1})^*$. Let $(z_{\lambda_1, ..., \lambda_{r+1}})$ be the standard base of $\wedge^{r+1}(C^{m+1})^*$. Then

\[(q_i \wedge dF)_z = (p_i(z))^t \sum G_{\lambda_1, ..., \lambda_{r+1}}(z)z_{\lambda_1, ..., \lambda_{r+1}}\text{ for } z \in \pi^{-1}(W_i).\]

Now we define a map $G: C^{m+1} \rightarrow \wedge^{r+1}(C^{m+1})^*$ by

\[(4.1) \quad G(z) = \sum G_{\lambda_1, ..., \lambda_{r+1}}(z)z_{\lambda_1, ..., \lambda_{r+1}}.\]

We denote by $P^e(C)$ the complex projective space associated to the complex vector space $\wedge^{r+1}(C^{m+1})^*$, where $e+1 = \left(\frac{\mu+1}{r+1}\right)$. Since the map $G: C^{m+1} \rightarrow \wedge^{r+1}(C^{m+1})^*$ is a polynomial map of degree $(n-1)(m-1)+(a-1)$ and $G(z) \neq 0$ for $z \in \pi^{-1}(X)$, it induces a holomorphic map $g: X \rightarrow P^e(C)$. We call $g$ the dual map of $X$ in $P^e(C)$. Let $|w|^2$ be the norm of an element $w$ in $\wedge^{r+1}(C^{m+1})^*$. We regard $|w|^2$ as the fiber metric on $E$ over $P^e(C)$. Its reciprocal image under $g$ is the restriction of $\sum |G_{\lambda_1, ..., \lambda_{r+1}}(z)|^2$ to $\pi^{-1}(X)$ and is a fiber metric on $\wedge^{r+1}(C^{m+1})^*$. Then

\[
\pi^*g^*\Omega' = \frac{-1}{2\pi} d'd'' \log (\sum |G_{\lambda_1, ..., \lambda_{r+1}}(z)|^2).
\]

Now our claim follows from Corollary of Proposition 3.2. q.e.d.

Let $S_k$ be the vector space of homogeneous polynomials on $C^{m+1}$ of degree $k$ and $S_k^*$ the dual space of $S_k$. We denote by $P^d(C)$ the complex projective space associated to $S_k^*$, where $d+1 = \dim S_k$. Each point $z \in C^{m+1}$ defines a linear function $\Psi(z)$ on $S_k$ given by $\Psi(z)(P) = P(z)$ for $P \in S_k$. We denote by $\psi$ the map $z \mapsto \Psi(z)$. The polynomial map $\Psi$ induces an injective holomorphic map.
(4.2) \( \psi: P^m(C) \rightarrow P^d(C) \)

if \( k \geq 1 \). The map \( \psi \) is called the Veronese map of degree \( k \).

For simplicity we denote the Plücker coordinate \((\cdots, p, \cdots)\) by \((\omega, \cdots, \omega_\mu)\).

With respect to the hermitian inner product on \( S_k \) induced from the one on \( C^{k+1} \), the set of all monomials

\[ (4.3) \ x_\omega \cdots x_\omega_\mu/(\nu_0! \cdots \nu_\mu!)^{1/2}, \nu_0 + \cdots + \nu_\mu = k \]

is a unitary base of \( S_k \). Moreover

\[ (4.4) \ |x_\omega \cdots x_\omega_\mu/(\nu_0! \cdots \nu_\mu!)^{1/2}|^2 = ||z||^2/k! . \]

Obviously the reciprocal image of the standard line bundle over \( P^d(C) \) under the map \( \psi \) is \( E^k \). By (4.4), if \( \Omega'' \) denotes the Fubini-Study form on \( P^d(C) \), then \( \psi^* \Omega'' = k \Omega \). That is, the Veronese map \( \psi \) is homothetic and the ratio of the metrics is the degree \( k \) of the map \( \psi \).

Now we specify \( k \) to be \((n-1)(m-1)+(a-1)\), and define a linear map \( L: S_{(n-1)(m-1)+a-1} \rightarrow \wedge^{r+1}(C^{n+1})^* \) so that \( L \circ \psi = G \) on the cone \( \pi^{-1}(X) \). Let \((\xi_{\omega_0}, \cdots, \omega_\mu)\) be the dual base of the unitary base of \( S_{(n-1)(m-1)+a-1} \) chosen above. Since \( G_{\lambda_1 \cdots \lambda_{r+1}} \) is of degree \((n-1)(m-1)+(a-1)\),

\[ (4.5) \ a_{\lambda_1 \cdots \lambda_{r+1}} = \sum_{\omega_0 \cdots \omega_\mu} a(\lambda_0 \cdots \lambda_{r-1}; \nu_0 \cdots \nu_\mu)(x_\omega \cdots x_\omega_\mu)/(\nu_0! \cdots \nu_\mu!)^{1/2}. \]

Using these coefficients, a linear map \( L \) is defined by

\[ (4.6) \ L(\xi_{\omega_0} \cdots \omega_\mu) = \sum_{\lambda_0 \cdots \lambda_{r+1}} a(\lambda_0 \cdots \lambda_{r+1}; \nu_0 \cdots \nu_\mu)\xi_{\lambda_0 \cdots \lambda_{r+1}}. \]

By the way \( L \) is defined, it is clear that

\[ (L \circ \psi)(z) = G(z) \quad \text{for} \quad z \in \pi^{-1}(X). \]

Consider the rational map \( l: P^d(C) \rightarrow P^r(C) \) induced from the linear map \( L: S_{(n-1)(m-1)+a-1} \rightarrow \wedge^{r+1}(C^{n+1})^* \). The map \( l \) is holomorphic at a point \( x \in P^d(C) \) if the image under \( L \) at a point of \( S_{(n-1)(m-1)+a-1} \) lying over \( x \) is not zero. Since \( ||g \wedge dF||^2 \) vanishes nowhere on \( \pi^{-1}(W) \), \( L \) does not vanishes at each point on the image of \( \pi^{-1}(X) \) under \( \psi \). Therefore \( l \) is holomorphic on \( \psi(X) \).

**Proposition 4.2.** Let be \( \psi \) the Veronese map of degree \((n-1)(m-1)+(a-1)\) of \( P^m(C) \) into \( P^d(C) \) and let \( g \) be the dual map of \( X \) into \( P^r(C) \). Then there is a projective transformation \( l \) of \( P^d(C) \) into \( P^r(C) \) which is holomorphic on \( \psi(X) \) and satisfies the equality \((l \circ \psi)(x) = g(x)\) for \( x \in X \). Moreover the induced metric on \( X \) is Einstein if and only if the restriction of \( l \) to \( \psi(X) \) is everywhere locally isometric with respect to the induced metric on \( \psi(X) \) and the Fubini-Study metric on \( P^r(C) \).

Now we have the following Lemma due to Hano ([5] Lemma 7).

**Lemma 4.3.** Let \( \Phi \) be a linear map of \( C^{n+1} \) into \( C^{n+1} \) and \( \phi \) the induced projective transformation of \( P'(C) \) into \( P'(C) \). Let \( U \) be a connected algebraic submanifold in \( P'(C) \) which is not contained in any hyperplane in \( P'(C) \). We equip on \( U \) the metric induced from a Fubini-Study metric on \( P'(C) \), and on \( P'(\mathbb{C}) \) a Fubini-Study metric. Suppose that the restriction of \( \phi \) to \( U \) is holomorphic and locally isometric everywhere, then \( \Phi \) is a constant multiple of an isometry, and particularly \( \Phi \) is injective.

Now we have the following necessary condition from Lemma 4.3.

**Proposition 4.4** (cf. [5] Hano §8). Let \( X \) be a hypersurface of \( G_{m+n,n}(\mathbb{C}) \) of degree \( a \). If the induced metric on \( X \) Einstein, then

\[
(4.8) \quad \dim (S(I_{(m-1)+(a-1)}(\mathbb{C})) - I(\mathbb{C})) = \binom{\mu+1}{r+1},
\]

where \( I_{(m-1)+(a-1)} = S_{(m-1)+(a-1)} \cap I(X) \).

**Proof.** For \( P \in S(I_{(m-1)+(a-1)} \cap \mathbb{C}) \), the equation \( \langle x, P \rangle = 0, x \in S^n \cap I_{(m-1)+(a-1)} \), defines a hyperplane in \( P^d(C) \). By the definition of the Veronese map \( \psi \), a homogeneous polynomial \( P \) in \( S(I_{(m-1)+(a-1)}) \) defines a hyperplane containing \( \psi(X) \) if and only if \( P \) belongs to \( I_{(m-1)+(a-1)} \). Thus, the minimal linear variety \( P(I^d(C)) \) containing \( \psi(X) \) is the intersection of these hyperplanes each of which is associated to a polynomial in \( I_{(m-1)+(a-1)} \). Its dimension \( d' \) is given by \( \dim (S(I_{(m-1)+(a-1)}) - I_{(m-1)+(a-1)}) - 1 \). Let \( C^{d'+1} \) be the subspace in \( S(I_{(m-1)+(a-1)}) \) perpendicular to the subspace \( I_{(m-1)+(a-1)} \). Let \( L' \) be the restriction to \( C^{d'+1} \) of the linear map \( L : S(I_{(m-1)+(a-1)}) \to r+1(C^{a+1})^* \), and let \( l' \) be the restriction to \( P^d(C) \) of projective transformation \( l \). Now the connected algebraic submanifold \( \psi(X) \) in \( P^d(C) \) is not contained in any hyperplane of \( P^d(C) \). By Proposition 4.2, the restriction to \( \psi(X) \) of \( l' \) is everywhere locally isometric. Applying Lemma 4.3, to \( \psi(X) \) in \( P^d(C) \), we see that the linear map

\[
L' : C^{d'+1} \to r+1(C^{a+1})^*
\]

is injective, and hence we get (4.8).

**5. Proof of Theorem**

Let \( J \) denote the ideal \( I(G_{m+n,n}(\mathbb{C})) \) of homogeneous polynomials \( S \) on \( C^{n+1} \).

**Lemma 5.1.** Let \( J_k \) denote \( J \cap S_k \). Then

\[
\dim (S_k/J_k) = \prod_{i=1}^n \prod_{j=1}^{n+i} \frac{k+i-j}{j-i}
\]
Proof. By Proposition 1.1, the inclusion \( j: G_{m+n}(C) \to P^n(C) \) induces a surjective linear map
\[
j^*: H^0(P^n(C), H^k) \to H^0(G_{m+n}(C), j^*H^k).
\]
Noting that \( H^0(P^n(C), H^k) \) is the space of homogeneous polynomials \( S_k \) of degree \( k \),
\[
\text{Ker} j^* = \{ P \in S_k | P(z) = 0 \text{ for any } z \in \pi^{-1}(G_{m+n}(C)) \}
\]
\[
= J \cap S_k.
\]
Hence, \( \dim (S_k/J_k) = \dim H^0(G_{m+n}(C), j^*H^k) \).

On the other hand, by a Theorem of Borel-Weil [2] and the dimension formula of Weyl [10], we have
\[
\dim H^0(G_{m+n}(C), j^*H^k) = \prod_{i=1}^{m+n} \frac{k+j-i}{j-i}.
\]
q.e.d.

**Lemma 5.2.** Let \( I_k \) denote \( I(X) \cap S_k \). Then
\[
\dim (S_k/I_k) = \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a})
\]
if \( k \geq a \), where \( a \) is the degree of \( X \).

Proof. Let \([X]\) denote the non-singular divisor defined by \( X \) and \( \{X\} \) the holomorphic line bundle on \( G_{m+n}(C) \) defined by \([X]\). Then there is an exact sequence
\[
0 \to j^*H^{k-a} \to j^*H^k \to \hat{\ell}^*H^k \to 0
\]
of holomorphic sheaves on \( G_{m+n}(C) \). (cf. [6])

Then (5.1) induces the following exact sequence of cohomologies
\[
0 \to H^0(G_{m+n}(C), j^*H^{k-a}) \to H^0(G_{m+n}(C), j^*H^k) \to H^0(X, \ell^*H^k) \to H^0(G_{m+n}(C), j^*H^{k-a}) \to \ldots.
\]

Since \( H^0(G_{m+n}(C), j^*H^{k-a}) = 0 \) if \( k \geq a \), by a theorem of Bott [2],
\[
\dim H^0(X, \ell^*H^k) = \dim H^0(G_{m+n}(C), j^*H^k) - \dim H^0(G_{m+n}(C), j^*H^{k-a}).
\]
On the other hand, \( j^*: H^0(P^n(C), H^k) \to H^0(G_{m+n}(C), j^*H^k) \) is surjective, and hence \( \ell^*: H^0(P^n(C), H^k) \to H^0(X, \ell^*H^k) \) is surjective if \( k \geq a \). Noting that \( \text{Ker} \ell^* = I(X) \cap S_k \), we have
\[
\dim (S_k/I_k) = \dim H^0(X, \ell^*H^k)
\]
\[
= \dim H^0(G_{m+n}(C), j^*H^k) - \dim H^0(G_{m+n}(C), j^*H^{k-a})
\]
\[
= \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a}).
\]
q.e.d.
Proof of Theorem. Put \( k = (n - 1)(m - 1) + (a - 1) \). If \( n \geq 2 \) and \( m \geq n \), then \( k \geq a \). Thus, by Lemma 5.2,

\[
\dim (S_{(n-1)(m-1) + (a-1)}/(n-1)(m-1) + (a-1)) = \dim (S_{(n-1)(m-1) + (a-1)}/(n-1)(m-1) + (a-1)) - \dim (S_{(n-1)(m-1) - 1}/(n-1)(m-1) - 1).
\]

By Lemma 5.1, we see that \( \dim (S_k/I_k) \) is increasing in \( k \). Hence, it is enough to prove the following inequality (5.3) by Proposition 4.4;

\[
(5.3) \quad \dim (S_{\mu-(m+n)+2}/I_{\mu-(m+n)+2}) > \left( \frac{\mu+1}{mn} \right).
\]

By Lemma 5.1, we have

\[
\dim (S_k/I_k) = \frac{(k+1)(k+2)^n \cdots (k+m)^n(k+m-1)^{n-1} \cdots (k+m+n-1)}{1 \cdot 2 \cdot n \cdots m \cdot (m+1)^{n-1} \cdots (m+n-1)}.
\]

Thus

\[
\dim (S_{\mu-(m+n)+2}/I_{\mu-(m+n)+2}) = \left( \frac{\mu+1}{mn} \right)
\]

\[
= (\mu+1)\mu^2(\mu-1)^2 \cdots (\mu-m-n+2)^n(\mu-m-1)^{n-1} \cdots (\mu-m-n+3) \frac{1 \cdot 2 \cdot 3 \cdots n \cdots m \cdots (m+1)^{n-1} \cdots (m+n-1)}{1 \cdot 2 \cdot n \cdots m \cdots (m+1)^{n-1} \cdots (m+n-1)}
\]

\[
\times (mn-m-n+1)(mn-m-n+2)^2 \cdots (mn-m-1)^{n-1}(mn-n)^n \frac{1 \cdot 2 \cdot n \cdots m \cdots (m+1)^{n-1} \cdots (m+n-1)}{1 \cdot 2 \cdot n \cdots m \cdots (m+1)^{n-1} \cdots (m+n-1)}
\]

\[
> \frac{1}{(mn)!} (\mu+1)\mu^3(\mu-1)^3 \cdots (\mu-m-n+2)^n(\mu-m-1)^{n-1} \cdots (\mu-m-n+3) \cdot (\mu+1)\mu(\mu-1) \cdots (\mu+2-mn)
\]

\[
- (mn-m-n+1)(mn-m-n)^2 \cdots (mn-n)^n \cdot (mn-n+1)^{n-1} \cdots (mn-1)
\]

Now we have

\[
(\mu+1)\mu^3(\mu-1)^3 \cdots (\mu-m-n+2)^n(\mu-m-1)^{n-1} \cdots (\mu-m-n+3) \cdot (\mu+1)\mu(\mu-1) \cdots (\mu+2-mn)
\]

\[
- (\mu+1)(\mu-1) \cdots (\mu-m-n+3) \{ (\mu-1)^2 \cdots (\mu-m+3) \cdot (\mu+4) \cdots (\mu-m+n+2) \}
\]

\[
> (\mu+1)\mu(\mu-1) \cdots (\mu-m+n+3) (mn-m-n+2).
\]

On the other hand,

\[
(m-n+3) - (mn-n-m+2) = \left( \frac{m+n}{n} \right) - 2mn + m + n > 0.
\]
Thus we have

\[
(\mu+1)\mu(\mu-1)\cdots(\mu-mn+3)(mn-n+2)
- (mn-1)\cdots(mn-m-n+1)^{n-1}(mn-n)\cdots(mn-m)^{n-1} \cdots
\times (mn-m-n+1) > (\mu+1)\mu\cdots(\mu-mn+3)(mn-m-n+2)- (2mn-m-m) \cdots
\times (mn-m-n+2)(mn-m-n+1) > 0.
\]

Hence, we get (5.3).

q.e.d.

\textbf{Remark.} In the case of }G_{5,2}(C)\text{, we can see that if the degree }a(X)\text{ of }X\text{ satisfies }a(X) \geq 3\text{ a hypersurface }X\text{ is not an Einstein manifold with respect to the induced metric by the same way. But we do not know the cases when }a(X)=1,2.

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\textbf{References}


