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Osaka J. Math.
16 (1979), 831-841

ON THE STRUCTURE OF THE CLASS GROUPS OF METACYCLIC GROUPS

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(Received September 18, 1978)
(Revised Jaunary 10, 1979)

Let Λ be a $\boldsymbol{Z}$-order in a semisimple $\boldsymbol{Q}$-algebra A. We mean by the class group of Λ the class group defined by using locally free left Λ-modules and denote it by $C(\Lambda)$. Define $D(\Lambda)$ to be the kernel of the natural surjection $C(\Lambda)\rightarrow C(\Omega)$ for a maximal $\boldsymbol{Z}$-order Ω in A containing Λ and $d(\Lambda)$ to be the order of $D(\Lambda)$.

Let $\boldsymbol{Z}G$ be the integral group ring of a finite group G. Then $\boldsymbol{Z}G$ can be regarded as a $\boldsymbol{Z}$-order in the semisimple $\boldsymbol{Q}$-algebra $\boldsymbol{Q}G$, and hence $C(\boldsymbol{Z}G)$ and $D(\boldsymbol{Z}G)$ can be defined.

In this paper we consider only finite groups. We will treat the semidirect product $G=N\cdot F$ of a group N by a group F. Define $D_0(\boldsymbol{Z}G)$ (resp. $C_0(\boldsymbol{Z}G)$) to be the kernel of the natural surjection $D(\boldsymbol{Z}G)\rightarrow D(\boldsymbol{Z}F)$ (resp. $C(\boldsymbol{Z}G)\rightarrow C(\boldsymbol{Z}F)$). First we will give

[I] *Let $N=N_1\times N_2$ be the direct product of groups N_1 and N_2 and $G=N\cdot F$ be the semidirect product of the group N by a group F. Assume that F acts on each N_i, $i=1, 2$. Denote by G_i the subgroup $N_i\cdot F$ of G, $i=1, 2$. Then $D(\boldsymbol{Z}F)\oplus D_0(\boldsymbol{Z}G_1)\oplus D_0(\boldsymbol{Z}G_2)$ (resp. $C(\boldsymbol{Z}F)\oplus C_0(\boldsymbol{Z}G_1)\oplus C_0(\boldsymbol{Z}G_2)$) is a direct summand of $D(\boldsymbol{Z}G)$ (resp. $C(\boldsymbol{Z}G)$).*

For an abelian group A and a positive integer q, $A^{(q)}$ denotes the q-part of A and $A^{(q')}$ denotes the maximal subgroup of A whose order is coprime to q. In particular, we write $O(A)=A^{(2')}$. For any module M over a group H we define $M^H=\{m\in M\mid \tau m=m \text{ for every } \tau\in H\}$.

We will apply [I] to some metacyclic groups. Denote by C_m the cyclic group of order m. Using induction technique we will give, as a refinement of a result in [1],

[II] *Let $G=C_n\cdot C_q$, and define e_p by $p^{e_p}\|n$ for each prime divisor p of n. Assume that C_q acts faithfully on each Sylow subgroup of C_n and that $(n, q)=1$. Then*

$$D(\boldsymbol{Z}G)\cong D(\boldsymbol{Z}C_q)\oplus \bigoplus_{p|n} D(\boldsymbol{Z}C_{p^{e_p}})^{C_q}\oplus\left(\boldsymbol{Z}\Big/\frac{q}{(2,q)}\boldsymbol{Z}\right)^{\sum_{F|n} e_p} \oplus \mathrm{Ind}_{C_n}^{G} D(\boldsymbol{Z}C_n)^{(q)}\oplus K\,,$$

where K is the complementary subgroup of $\bigoplus_{p|n} D(\boldsymbol{Z}C_{p^{e_p}})^{C_q}$ in $(D(\boldsymbol{Z}C_n)^{C_q})^{(q')}$ (cf. § 1).

Next we will study the class groups of generalized quaternion groups in connection with those of dihedral groups. Denote by H_n the generalized quaternion group of order $4n$; $H_n=\langle\sigma, \tau\,|\,\sigma^n=\tau^4=1,\ \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$ and by D_n the dihedral group of order $2n$; $D_n=\langle\sigma, \tau\,|\,\sigma^n=\tau^2=1,\ \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$. Fröhlich and Wilson have studied the 2-part of $D(\boldsymbol{Z}H_{p^t})$ for an odd prime p ([5], [11]), and Cassou-Noguès has given some information on $D(\boldsymbol{Z}H_n)$ for an odd integer n ([2]).

[III] *Let $n\geqq 3$ be an odd integer and define e_p by $p^{e_p}\|n$ for each prime divisor p of n. Then;*

i) $D(\boldsymbol{Z}H_n)\oplus D(\boldsymbol{Z}D_n)\cong D(\boldsymbol{Z}H_n/(\tau^2+1))\oplus D(\boldsymbol{Z}D_{2n})$

ii) $D(\boldsymbol{Z}H_n)\cong O(D(\boldsymbol{Z}D_{2n}))\oplus D(\boldsymbol{Z}D_n)^{(2)}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{\sum_{p|n} e_p}\oplus L,$

where L is an extension of $D(\boldsymbol{Z}D_n)^{(2)}$ by an elementary 2-group. In particular, if $n=p^t$ for an odd prime p,

$$D(\boldsymbol{Z}H_{p^t})\cong D(\boldsymbol{Z}D_{2p^t})\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^t\,.$$

1. Decomposition of class groups

The following theorem will play an essential part in this paper.

Theorem 1.1. *Let $N=N_1\times N_2$ be the direct product of groups N_1 and N_2 and $G=N\cdot F$ be the semidirect product of the group N by a group F. Assume that F acts on each N_i, $i=1, 2$. Denote by G_i the subgroup $N_i\cdot F$ of $G, i=1, 2$. Then $D(\boldsymbol{Z}F)\oplus D_0(\boldsymbol{Z}G_1)\oplus D_0(\boldsymbol{Z}G_2)$ (resp. $C(\boldsymbol{Z}F)\oplus C_0(\boldsymbol{Z}G_1)\oplus C_0(\boldsymbol{Z}G_2)$) is a direct summand of $D(\boldsymbol{Z}G)$ (resp. $C(\boldsymbol{Z}G)$). In particular, if $F=\{1\}$, $D(\boldsymbol{Z}G_1)\oplus D(\boldsymbol{Z}G_2)$ (resp. $C(\boldsymbol{Z}G_1)\oplus C(\boldsymbol{Z}G_2)$) is a direct summand of $D(\boldsymbol{Z}G)$ (resp. $C(\boldsymbol{Z}G)$).*

Proof. We denote the augmentation ideal of $\boldsymbol{Z}N$ (resp. $\boldsymbol{Z}N_i$) by I_N (resp. I_{N_i}). There is an exact sequence

$$0\to D_0(\boldsymbol{Z}G)\to D(\boldsymbol{Z}G)\overset{\alpha}{\to} D(\boldsymbol{Z}F)\to 0\,,$$

where α is induced by $M\to \boldsymbol{Z}G/(I_N)\otimes_{\boldsymbol{Z}G} M$. Let β: $D(\boldsymbol{Z}F)\to D(\boldsymbol{Z}G)$ be the induction map. Then it is easy to see that $\alpha\circ\beta=id_{D(\boldsymbol{Z}F)}$. So we have that $D(\boldsymbol{Z}G)\cong D(\boldsymbol{Z}F)\oplus D_0(\boldsymbol{Z}G)$ (cf. [10]).

Let $\mathfrak{a}$ be a projective left ideal of $\boldsymbol{Z}G_1$ such that the class $[\mathfrak{a}]$ is in $D_0(\boldsymbol{Z}G_1)$. Then $\boldsymbol{Z}G\otimes_{\boldsymbol{Z}G_1}\mathfrak{a}$ is isomorphic to $\boldsymbol{Z}N_2\otimes_{\boldsymbol{Z}}\mathfrak{a}$ as $\boldsymbol{Z}G$-modules. Since $[\boldsymbol{Z}G/(I_N)\otimes_{\boldsymbol{Z}G}(\boldsymbol{Z}N_2\otimes_{\boldsymbol{Z}}\mathfrak{a})]=[\boldsymbol{Z}G_1/(I_{N_1})\otimes_{\boldsymbol{Z}G}\mathfrak{a}]=0$ in $D(\boldsymbol{Z}F)$, $[\boldsymbol{Z}N_2\otimes_{\boldsymbol{Z}}\mathfrak{a}]$ is in $D_0(\boldsymbol{Z}G)$. Hence we have the map φ_1: $D_0(\boldsymbol{Z}G_1)\to D_0(\boldsymbol{Z}G)$ and similarly we get the map φ_2: $D_0(\boldsymbol{Z}G_2)\to$

$D_0(\boldsymbol{Z}G)$. Further, for a projective left ideal $\mathfrak{b}$ of $\boldsymbol{Z}G$ such that $[\mathfrak{b}]\in D_0(\boldsymbol{Z}G)$, $[\boldsymbol{Z}G_1/(I_{N_1})\otimes_{\boldsymbol{Z}G_1}(\boldsymbol{Z}G/(I_{N_2})\otimes_{\boldsymbol{Z}G}\mathfrak{b})]=0$ in $D(\boldsymbol{Z}F)$, so $[\boldsymbol{Z}G/(I_{N_2})\otimes_{\boldsymbol{Z}G}\mathfrak{b}]\in D_0(\boldsymbol{Z}G_1)$. Hence we have the map $\psi_1\colon D_0(\boldsymbol{Z}G)\to D_0(\boldsymbol{Z}G_1)$ and similarly we get the map $\psi_2\colon D_0(\boldsymbol{Z}G)\to D_0(\boldsymbol{Z}G_2)$. For every projective left ideal $\mathfrak{a}$ of $\boldsymbol{Z}G_1$ such that $[\mathfrak{a}]\in D_0(\boldsymbol{Z}G_1)$, $\psi_1\circ\varphi_1[\mathfrak{a}]=[\boldsymbol{Z}G/(I_{N_2})\otimes_{\boldsymbol{Z}G}(\boldsymbol{Z}N_2\otimes_{\boldsymbol{Z}}\mathfrak{a})]=[\boldsymbol{Z}G_1\otimes_{\boldsymbol{Z}G_1}\mathfrak{a}]=[\mathfrak{a}]$ in $D_0(\boldsymbol{Z}G_1)$. In $\psi_2\circ\varphi_1[\mathfrak{a}]=[\boldsymbol{Z}G/(I_{N_1})\otimes_{\boldsymbol{Z}G}(\boldsymbol{Z}N_2\otimes_{\boldsymbol{Z}}\mathfrak{a})]$, N_2 acts on $\boldsymbol{Z}G/(I_{N_1})$ and N_2 via group action and F acts on $\boldsymbol{Z}G/(I_{N_1})$ via group action, and we know that $\psi_2\circ\varphi_1[\mathfrak{a}]=[\boldsymbol{Z}G_2]=0$ in $D_0(\boldsymbol{Z}G_2)$. Consequently we see that $(\psi_1\oplus\psi_2)\circ(\varphi_1\oplus\varphi_2)=id_{D_0(\boldsymbol{Z}G_1)\oplus D_0(\boldsymbol{Z}G_2)}$. This implies that $D_0(\boldsymbol{Z}G_1)\oplus D_0(\boldsymbol{Z}G_2)$ is a direct summand of $D_0(\boldsymbol{Z}G)$.

If $F=\{1\}$, then $D_0(\boldsymbol{Z}G)=D(\boldsymbol{Z}G)$ and $D_0(\boldsymbol{Z}G_i)=D(\boldsymbol{Z}G_i)$, hence we see that $D(\boldsymbol{Z}G_1)\oplus D(\boldsymbol{Z}G_2)$ is a direct summand of $D(\boldsymbol{Z}G)$. The assertion for $C(\boldsymbol{Z}G)$ can be proved in the same way as for $D(\boldsymbol{Z}G)$.

Throughout this paper p stands for a rational prime. In case where G is metacyclic, (1.1) will become as follows.

Proposition 1.2. *Let $G=C_n\cdot C_q$ and define e_p by $p^{e_p}\|n$ for each $p|n$. Denote by G_p the subgroup $C_{p^{e_p}}\cdot C_q$ of G. Assume that $(n, q)=1$ and that* $\mathrm{Ker}(C_q\to\mathrm{Aut}\,C_{p^{e_p}})=C_r$ *for every $p|n$. Let d denote the order of C_q/C_r. Then*

$$D(\boldsymbol{Z}G)\cong D(\boldsymbol{Z}C_q)\oplus\bigoplus_{p|n}D_0(\boldsymbol{Z}G_p)\oplus M,$$

where M is an extension of an abelian group whose exponent divides d by the group $\mathrm{Ker}\,[\mathrm{Ind}_{C_n\times C_r}^{G}D(\boldsymbol{Z}C_n\times C_r)\to\bigoplus_{p|n}\mathrm{Ind}_{C_{p^{e_p}}\times C_r}^{G_p}D(\boldsymbol{Z}C_{p^{e_p}}\times C_r)]$.

Proof. It follows from (1.1) that $D(\boldsymbol{Z}C_q)\oplus\bigoplus_{p|n}D_0(\boldsymbol{Z}G_p)$ is a direct summand of $D(\boldsymbol{Z}G)$. Now we determine the remaining factor M. Define the subgroup $D_1(\boldsymbol{Z}C_n\times C_r)$ (resp. $D_1(\boldsymbol{Z}C_{p^{e_p}}\times C_r)$) of $D(\boldsymbol{Z}C_n\times C_r)$ (resp. $D(\boldsymbol{Z}C_{p^{e_p}}\times C_r)$) as the complementary subgroup of $D(\boldsymbol{Z}C_r)$. Then there is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\,\alpha & \longrightarrow & M & \longrightarrow & \mathrm{Ker}\,\gamma & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ind}_{C_n\times C_r}^{G}D_1(\boldsymbol{Z}C_n\times C_r) & \xrightarrow{\varphi} & D_0(\boldsymbol{Z}G) & \longrightarrow & \mathrm{Coker}\,\varphi & \longrightarrow & 0\\
 & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\
0 & \longrightarrow & \bigoplus_{p|n}\mathrm{Ind}_{C_{p^{e_p}}\times C_r}^{G_p}D_1(\boldsymbol{Z}C_{p^{e_p}}\times C_r) & \xrightarrow{\varphi'} & \bigoplus_{p|n}D_0(\boldsymbol{Z}G_p) & \longrightarrow & \mathrm{Coker}\,\varphi' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

where φ and φ' are the inclusion maps and α, β, and γ are the natural maps. By the induction theorem (cf. [3]) we know that the exponent of Coker φ divides

d, and hence the exponent of Ker γ also divides d. Next consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \operatorname{Ker}\alpha & \to & \operatorname{Ind}_{C_n\times C_r}^{G} D_1(ZC_n\times C_r) & \xrightarrow{\alpha} & \bigoplus_{p|n} \operatorname{Ind}_{C_{p^{e_p}}\times C_r}^{G_p} D_1(ZC_{p^{e_p}}\times C_r) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Ker}\tilde{\alpha} & \to & \operatorname{Ind}_{C_n\times C_r}^{G} D(ZC_n\times C_r) & \xrightarrow{\tilde{\alpha}} & \bigoplus_{p|n} \operatorname{Ind}_{C_{p^{e_p}}\times C_r}^{G_p} D(ZC_{p^{e_p}}\times C_r) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Ker}\delta & \to & \operatorname{Ind}_{C_n}^{G} D(ZC_r) & \xrightarrow{\delta} & \bigoplus_{p|n} \operatorname{Ind}_{C_r}^{G_p} D(ZC_r) & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

Since δ is injective, $\operatorname{Ker}\delta=0$ and so $\operatorname{Ker}\alpha\cong\operatorname{Ker}\tilde{\alpha}$. This completes the proof.

Let $N\cdot F$ be the semidirect product of a group N by a group F. For a ZN-module M and each $\tau\in F$, we define another ZN-module structure on M to be $\sigma\cdot m=\tau^{-1}\sigma\tau m$ where $\sigma\in N$ and $m\in M$, and denote it by M^{τ}. This yields the action of F on $D(ZN)$. Hence $D(ZN)$ can be regarded as a module over F.

Proposition 1.3. *Let $G=C_n\cdot C_q$ and define e_p by $p^{e_p}\|n$ for each $p|n$. Assume that C_q acts faithfylly on each Sylow subgroup of C_n and that $(n, q)=1$. Then*

$$D(ZG)\cong D(ZC_q)\oplus \bigoplus_{p|n} D(ZC_{p^{e_p}})^{C_q}\oplus\left(Z\Big/\frac{q}{(2,q)}Z\right)^{\sum_{p|n} e_p}\oplus\operatorname{Ind}_{C_n}^{G} D(ZC_n)^{(q)}\oplus K,$$

where K is the complementary subgroup of $\bigoplus_{p|n} D(ZC_{p^{e_p}})^{C_q}$ in $(D(ZC_n)^{C_q})^{(q')}$.

Proof. We have the induction map φ: $D(ZC_n)\to D_0(ZG)$ and the restriction map ψ: $D_0(ZG)\to D(ZC_n)$. It is known that $\operatorname{Coker}\varphi\cong\left(Z\Big/\frac{q}{(2,q)}Z\right)^{\sum_{p|n} e_p}$ ([1]). We see that $q\cdot\operatorname{Ker}\psi=0$. Then we have that φ: $D(ZC_n)^{(q')}\to D_0(ZG)^{(q')}$ is surjective and that ψ: $D_0(ZG)^{(q')}\to D(ZC_n)^{(q')}$ is injective. On the other hand for a ZC_n-module M, $ZG\underset{ZC_n}{\otimes} M\cong M\oplus M^{\tau}\oplus\cdots\oplus M^{\tau^{q-1}}$ as ZC_n-modules, where τ is a generator of C_q. So we see that $\psi\circ\varphi=\operatorname{trace}_{C_q}$. Since $q\cdot D(ZC_n)^{C_q}\subseteq \operatorname{trace}_{C_q}(D(ZC_n))\subseteq D(ZC_n)^{C_q}$, $\operatorname{trace}_{C_q}$: $(D(ZC_n)^{C_q})^{(q')}\to(D(ZC_n)^{C_q})^{(q')}$ is bijective. Hence ψ: $D_0(ZG)^{(q')}\to(D(ZC_n)^{C_q})^{(q')}$ is surjective and φ: $(D(ZC_n)^{C_q})^{(q')}\to D_0(ZG)^{(q')}$ is injective, and so both maps are bijective. Applying this argument to the subgroup $G_p=C_{p^{e_p}}\cdot C_p$ of G, we have the split exact sequence

$$0\to D(ZC_{p^{e_p}})^{C_q}\to D_0(ZG_p)\to\left(Z\Big/\frac{q}{(2,q)}Z\right)^{e_p}\to 0,$$

we note here that $D(ZC_{p^{e_p}})$ is a p-group and that p is coprime to q.

Now applying (1.2), we get that

$$D(ZG)\cong D(ZC_q)\oplus \bigoplus_{p|n} D(ZC_{p^{e_p}})^{C_q}\oplus\left(Z\Big/\frac{q}{(2,q)}Z\right)^{\sum_{p|n} e_p}\oplus$$
$$\mathrm{Ker}\,[\mathrm{Ind}_{C_n}^{G} D(ZC_n)\to \bigoplus_{p|n}\mathrm{Ind}_{C_{p^{e_p}}}^{C_p} D(ZC_{p^{e_p}})]\,.$$

Trivially the last factor is isomorphic to $\mathrm{Ind}_{C_n}^{G} D(ZC_n)^{(q)}\oplus \mathrm{Ker}\,[\mathrm{Ind}_{C_n}^{G} D(ZC_n)^{(q')}\to\bigoplus_{p|n}\mathrm{Ind}_{C_{p^{e_p}}}^{G_p} D(ZC_{p^{e_p}})]$, and further, from the above argument on the induction maps it follows that the second factor is isomorphic to the complementary subgroup of $\bigoplus_{p|n} D(ZC_{p^{e_p}})^{C_q}$ in $(D(ZC_n)^{C_q})^{(q')}$. This completes the proof.

2. Structure of $D(ZH_n)$

Throughout this section we assume that $n\geqq 3$ is an odd integer.

Lemma 2.1. *There are exact sequences*

$$0\to N\to D(ZH_n)\to D(ZD_n)\oplus D(ZH_n/(\tau^2+1))\to 0$$
$$0\to N'\to D(ZD_{2n})\to D(ZD_n)\oplus D(ZD_n)\to 0$$

where both N and N' are of odd order.

Proof. From the pullback diagrams

$$\begin{array}{ccc} ZH_n & \longrightarrow & ZH_n/(\tau^2+1)\\ \downarrow & & \downarrow\\ ZD_n\cong ZH_n/(\tau^2-1) & \longrightarrow & F_2D_n\end{array}$$

$$\begin{array}{ccc} ZD_{2n} & \longrightarrow & ZD_n\\ \downarrow & & \downarrow\\ ZD_n & \longrightarrow & F_2D_n\end{array}$$

we get the (Mayer-Vietoris) exact sequences (cf. [8])

$$K_1(ZD_n)\oplus K_1(ZH_n/(\tau^2+1))\to K_1(F_2D_n)\to D(ZH_n)\to D(ZD_n)\oplus D(ZH_n/(\tau^2+1))\to 0$$
$$K_1(ZD_n)\oplus K_1(ZD_n)\to K_1(F_2D_n)\to D(ZD_{2n})\to D(ZD_n)\oplus D(ZD_n)\to 0\,.$$

Hence it is sufficient to show that Coker $[K_1(ZD_n)\to K_1(F_2D_n)]$ is of odd order. Write $D_{2n}=\langle\rho,\sigma,\tau\,|\,\rho^2=\sigma^n=\tau^2=1,\ \rho\sigma=\sigma\rho,\ \rho\tau=\tau\rho,\ \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$ and $D_n=\langle\sigma,\tau\,|\,\sigma^n=\tau^2=1,\ \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$, and define $\Sigma_n\in ZD_{2n}$ (resp. $\Sigma_n\in ZD_n$) to be $\Sigma_n=\sum_{i=0}^{n-1}\sigma^i$. It has been shown [4] that $D(ZD_{2n})\cong D(ZD_{2n}/(\Sigma_n))$ and $D(ZD_n)\cong$

$D(\boldsymbol{Z}D_n/(\Sigma_n))$. Then we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\boldsymbol{Z}D_n) & \xrightarrow{\varphi} & K_1(\boldsymbol{F}_2D_n) & \to & D(\boldsymbol{Z}D_{2n}) & \to & D(\boldsymbol{Z}D_n)\oplus D(\boldsymbol{Z}D_n) & \to & 0 \\ & & & & \wr\downarrow & & \wr\downarrow & & \\ K_1(\boldsymbol{Z}D_n/(\Sigma_n)) & \xrightarrow{\varphi'} & K_1(\boldsymbol{F}_2D_{2n}/(\Sigma_n)) & \to & D(\boldsymbol{Z}D_{2n}/(\Sigma_n)) & \to & D(\boldsymbol{Z}D_n/(\Sigma_n))\oplus D(\boldsymbol{Z}D_n/(\Sigma_n)) & \to & 0. \end{array}$$

We see that $\operatorname{Coker}\varphi\cong\operatorname{Coker}\varphi'$ and that the latter is of odd order, since $K_1(\boldsymbol{F}_2D_n/(\Sigma_n))$ is so. This completes the proof.

Lemma 2.2. *There is a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc} 0 & \to & E & \to & D(\boldsymbol{Z}H_n) & \xrightarrow{\varphi} & D(\boldsymbol{Z}D_{2n}) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & E & \to & D(\boldsymbol{Z}H_n/(\tau^2+1)) & \xrightarrow{\varphi'} & D(\boldsymbol{Z}D_n) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & , \end{array}$$

where E is an elementary 2-group.

Proof. We will use the following notation;

R^d=the ring of integers of $\boldsymbol{Q}(\zeta_d+\zeta_d^{-1})$, where ζ_d is a primitive d-th root of unity,

$$R^d{}_p=\boldsymbol{Z}_p\underset{\boldsymbol{Z}}{\otimes}R^d,\quad \boldsymbol{Z}_pH_m=\boldsymbol{Z}_p\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Z}H_m,\quad \boldsymbol{Z}_pD_m=\boldsymbol{Z}_p\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Z}D_m.$$

Write $H_n=\langle\sigma,\tau\,|\,\sigma^n=\tau^4=1,\ \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$ and $\Sigma_n=\sum_{i=0}^{n-1}\sigma^i\in\boldsymbol{Z}H_n$. Then we see that $\operatorname{Nrd}((\boldsymbol{Z}_pD_{2n}/(\Sigma_n))^*)=(\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho]/(\Sigma_n))^*$ for every prime p, because $\boldsymbol{Z}_pD_{2n}/(\Sigma_n)$ is embedded into $M_2(\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho]/(\Sigma_n))$. Since we can prove by the same method as in [4, § 3] that $D(\boldsymbol{Z}[\sigma+\sigma^{-1},\rho])\cong D(\boldsymbol{Z}[\sigma+\sigma^{-1},\rho]/(\Sigma_n))$, we have that $D(\boldsymbol{Z}D_{2n})\cong D(\boldsymbol{Z}[\sigma+\sigma^{-1},\rho])$. Similarly we have that $D(\boldsymbol{Z}D_n)\cong D(\boldsymbol{Z}[\sigma+\sigma^{-1}])$. Now we express the class groups in idèlic form (cf. [6]). Then we have

$$D(\boldsymbol{Z}H_n)\cong\frac{\prod_{p|2n}\prod_{1\neq d|n}(R^d{}_p{}^*\times R^d{}_p{}^*)}{\prod_{1\neq d|n}(R^{d*}\times R^{d*}_+)\prod_{p|2n}n(\boldsymbol{Z}_pH_n{}^*)},$$

where $n(\boldsymbol{Z}_pH_n{}^*)=\{\operatorname{Nrd}(x)\,|\,(1,x)\in\boldsymbol{Z}_pH_n{}^*\hookrightarrow\boldsymbol{Z}_p\langle\tau\rangle^*\times\boldsymbol{Z}_pH_n/(\Sigma_n)^*\}$ and $R^{d*}_+=\{u\in R^{d*}\,|\,u$ is positive at all real places of $R^d\}$,

$$D(\boldsymbol{Z}D_{2n})\cong\frac{\prod_{p|2n}\prod_{1\neq d|n}(R^d{}_p{}^*\times R^d{}_p{}^*)}{\prod_{1\neq d|n}(R^{d*}\times R^{d*})\prod_{p|2n}u(\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho])},$$

where

$u(\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho])=\{y\,|\,(1,y)\in\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho]^*\hookrightarrow\boldsymbol{Z}_p\langle\rho\rangle^*\times\boldsymbol{Z}_p[\sigma+\sigma^{-1},\rho]/(\Sigma_n)^*\}$,

$$D(ZH_n/(\tau^2+1)) \cong \frac{\prod_{p|n} \prod_{1 \neq d|n} R^d{}_p{}^*}{\prod_{1 \neq d|n} R^{d*}_+ \prod_{p|n} n(Z_p H_n/(\tau^2+1)^*)},$$

where $n(Z_p H_n/(\tau^2+1)^*)=$
$\{\mathrm{Nrd}(x) \mid (1, x) \in Z_p H_n/(\tau^2+1)^* \hookrightarrow Z_p[\bar{\tau}]^* \times Z_p H_n/(\Sigma_n, \tau^2+1)^*\}$, and

$$D(ZD_n) \cong \frac{\prod_{p|n} \prod_{1 \neq d|n} R^d{}_p{}^*}{\prod_{1 \neq d|n} R^{d*} \prod_{p|n} u(Z_p[\sigma+\sigma^{-1}])},$$

where $u(Z_p[\sigma+\sigma^{-1}])=\{y \mid (1, y) \in Z_p[\sigma+\sigma^{-1}]^* \hookrightarrow Z_p{}^* \times Z_p[\sigma+\sigma^{-1}]/(\Sigma_n)^*\}$.

Hence there exist natural surjections φ: $D(ZH_n) \to D(ZD_{2n})$ and φ': $D(ZH_n/(\tau^2+1)) \to D(ZD_n)$. Then

$$\operatorname{Ker} \varphi \cong \frac{\prod_{1 \neq d|n} (R^{d*} \times R^{d*}) \prod_{p|2n} u(Z_p[\sigma+\sigma^{-1}, \rho])}{\prod_{1 \neq d|n} (R^{d*} \times R^{d*}_+) \prod_{p|2n} n(Z_p H_n{}^*)}.$$

Trivially $(R^{d*})^2 \subseteq R^{d*}_+$ for every $d|n$, $d \neq 1$. Since the degree of $Z_p H_n/(\Sigma_n)$ over its center is 4, $u(Z_p[\sigma+\sigma^{-1}, \rho])^2 \subseteq n(Z_p H_n{}^*)$ for every $p|n$. Hence $\operatorname{Ker} \varphi$ is an elementary 2-group. Similarly we can show that $\operatorname{Ker} \varphi'$ is an elementary 2-group.

Let ψ: $D(ZH_n) \to D(ZH_n/(\tau^2+1))$ and ψ': $D(ZD_{2n}) \to D(ZD_n)$ be the maps defined as follows; for $(x, y) \in (\prod_{p|2n} \prod_{1 \neq d|n} R^d{}_p{}^*) \times (\prod_{p|2n} \prod_{1 \neq d|n} R^d{}_p{}^*)$, ψ (the class of (x, y))=the class of y, and ψ' (the class of (x, y))=the class of y. In fact ψ (resp. ψ') is the map induced by the natural surjection $ZH_n \to ZH_n/(\tau^2+1)$ (resp. $ZD_{2n} \to ZD_{2n}/(\rho+1) \cong ZD_n$). It is clear that both ψ and ψ' are surjective. Further we have the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 \to N \to & D(ZH_n) & \xrightarrow{(\ ,\psi)} & D(ZD_n) \oplus D(ZH_n/(\tau^2+1)) & \to 0 \\
 & \downarrow \varphi & & \downarrow id \oplus \varphi' & \\
0 \to N' \to & D(ZD_{2n}) & \xrightarrow[(\ ,\psi')]{} & D(ZD_n) \oplus D(ZD_n) & \longrightarrow 0. \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

Since $\operatorname{Ker} \varphi$ and $\operatorname{Ker} \varphi'$ are 2-group, we get by (2.1) that $\operatorname{Ker} \varphi \cong \operatorname{Ker} \varphi'$. Thus we conclude the proof.

Theorem 2.3. *Let $n \geqq 3$ be an odd integer and define e_p by $p^{e_p} \| n$ for each $p|n$. Then:*

i) $D(ZH_n) \oplus D(ZD_n) \cong D(ZH_n/(\tau^2+1)) \oplus D(ZD_{2n})$

ii) $D(ZH_n) \cong O(D(ZD_{2n})) \oplus D(ZD_n)^{(2)} \oplus (Z/2Z)^{\sum_{p|n} e_p} \oplus L$,

where L is an extension of $D(ZD_n)^{(2)}$ by an elementary 2-group. In particular, if $n=p^t$ for an odd prime p,

$$D(ZH_{p^t})\cong D(ZD_{2p^t})\oplus(Z/2Z)^t.$$

Proof. By (2.2) we have the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & F & = & F & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & E & \to & D(ZH_n) & \longrightarrow & D(ZD_{2n}) & \to & 0 \\ & & \| & & \downarrow \psi & & \downarrow \psi' & & \\ 0 & \to & E & \to & D(ZH_n/(\tau^2+1)) & \to & D(ZD_n) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & . \end{array}$$

Since ψ' splits by (1.2), ψ splits also. Therefore

$$\begin{aligned} D(ZH_n)\oplus D(ZD_n) &\cong D(ZH_n/(\tau^2+1))\oplus F\oplus D(ZD_n) \\ &\cong D(ZH_n/(\tau^2+1))\oplus D(ZD_{2n}). \end{aligned}$$

For the proof of ii) we begin with the case $n=p^t$. It has been shown (e.g. [1], [4]) that $d(ZD_{p^t})$ and $d(ZD_{2p^t})$ are odd, and hence in this case the exact sequences in (2.2) split. On the other hand it is known that the 2-part of $D(ZH_{p^t}/(\tau^2+1))$ is an elementary 2-group of rank t ([11]). Therefore we see that

$$D(ZH_{p^t})\cong D(ZD_{2p^t})\oplus(Z/2Z)^t.$$

Next consider the general case. By (2.1) we see that

$$D(ZH_n)^{(2)}\cong D(ZD_n)^{(2)}\oplus D(ZH_n/(\tau^2+1))^{(2)}.$$

On the other hand, by (2.2), we have that $O(D(ZH_n))\cong O(D(ZD_{2n}))$. Thus we get

$$D(ZH_n)\cong O(D(ZD_{2n}))\oplus D(ZD_n)^{(2)}\oplus D(ZH_n/(\tau^2+1))^{(2)}.$$

There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & D(ZH_n/(\tau^2+1)) & \longrightarrow & D(ZD_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ 0 & \to & (Z/2Z)^{\sum_{p\mid n} e_p} & \to & \bigoplus_{p\mid n} D(ZH_{p^{e_p}}/(\tau^2+1)) & \to & \bigoplus_{p\mid n} D(ZD_{p^{e_p}}) & \to & 0. \end{array}$$

It can be shown along the same line as in (1.2) that α is surjective and splir, and by (2.2) E is an elementary 2-group. Therefore we see that

$$D(ZH_n/(\tau^2+1))^{(2)}\cong(Z/2Z)^{\sum_{p\mid n} e_p}\oplus L,$$

where L is an extension of $D(\boldsymbol{Z}D_n)^{(2)}$ by an elementary 2-group. We conclude the proof.

REMARK 2.4. When $n=p^t$, rank $E=t$. But it may be conjectured that rank $E-\sum_{p|n} e_p>0$ unless n is a power of an odd prime. In fact, when $n=15$, $E\cong C_2\times C_2\times C_2$ and in this case we get that $D(\boldsymbol{Z}H_{15})\cong C_2\times C_2\times C_2$. We note here the outline of the computation.

Since $D(\boldsymbol{Z}D_{30})=D(\boldsymbol{Z}D_{15})=\{1\}$ ([4]), the commutative diagram in the proof of (2.3) shows that $F=\{1\}$, and hence

$$E\cong D(\boldsymbol{Z}H_{15})\cong D(\boldsymbol{Z}H_{15}/(\tau^2+1))\,.$$

Along the same line as in the proof of [1, Théorème 3] we get that for an odd square-free integer n,

$$D(\boldsymbol{Z}H_n/(\tau^2+1))\cong \bigoplus_{p|n} D(\boldsymbol{Z}H_p/(\tau^2+1))\oplus \bigoplus_{\substack{1\neq d|n\\ d\neq \text{prime}}} (R^d/I^d)^*/\mathrm{Im}\, R^{d*}_+\,,$$

where $I^d=\prod_{p|d}(1-\zeta_p)(1-\zeta_p^{-1})R^d$. Further we see that there is a natural surjection $\bigoplus_{\substack{1\neq d|n\\ d\neq \text{prime}}} (R^d/I^d)^*/\mathrm{Im}\, R^{d*}_+\to D(\boldsymbol{Z}H_n/(\Sigma_n,\ \tau^2+1))$. On the other hand, we know that $\mathrm{Ker}[D(\boldsymbol{Z}H_n/(\tau^2+1))\to D(\boldsymbol{Z}H_n/(\Sigma_n,\ \tau^2+1))]$ is an elementary 2-group of rank $\sum_{p|n} 1$. Though this is true for every odd integer, here we give the proof for the square-free case. Expressing both groups in idèlic form (cf. the proof of (2.2)), we know that

$$\begin{aligned}
&\mathrm{Ker}[D(\boldsymbol{Z}H_n/(\tau^2+1))\to D(\boldsymbol{Z}H_n/(\Sigma_n,\ \tau^2+1))]\\
&\cong\frac{\prod_{1\neq d|n} R^{d*}_+ \prod_{p|n} \mathrm{Nrd}(\boldsymbol{Z}_pH_n/(\Sigma_n,\ \tau^2+1)^*)}{\prod_{1\neq d|n} R^{d*}_+ \prod_{p|n} n(\boldsymbol{Z}_pH_n/(\tau^2+1)^*)}\\
&\cong \prod_{p|n}\left(\frac{R^{p*}_+\mathrm{Nrd}(\boldsymbol{Z}_pH_p/(\Sigma_p,\ \tau^2+1)^*)}{R^{p*}_+ n(\boldsymbol{Z}_pH_p/(\tau^2+1)^*)}\right)\\
&\cong \bigoplus_{p|n} \mathrm{Ker}[D(\boldsymbol{Z}H_p/(\tau^2+1))\to D(\boldsymbol{Z}H_p/(\Sigma_p,\ \tau^2+1))]\\
&\cong(\boldsymbol{Z}/2\boldsymbol{Z})^{\sum_{p|n}1}\,.
\end{aligned}$$

Hence we have that for an odd square-free integer n

$$\begin{aligned}
D(\boldsymbol{Z}H_n/(\tau^2+1))&\cong \bigoplus_{p|n} D(\boldsymbol{Z}H_p/(\tau^2+1))\oplus D(\boldsymbol{Z}H_n/(\Sigma_n,\ \tau^2+1))\\
&\cong(\boldsymbol{Z}/2\boldsymbol{Z})^{\sum_{p|n}1}\oplus D(\boldsymbol{Z}H_n/(\Sigma_n,\ \tau^2+1)).
\end{aligned}$$

Now let us return to the case $n=15$. It is sufficient to show that $D=D(\boldsymbol{Z}H_{15}/(\Sigma_{15},\ \tau^2+1))\cong\boldsymbol{Z}/2\boldsymbol{Z}$. From the pullback diagram

$$\begin{array}{ccc} ZH_{15}/(\Sigma_{15}, \tau^2+1)) & \longrightarrow & Z[\zeta_{15}, \bar{\tau}] \\ \downarrow & & \downarrow \\ Z[\zeta_3, \bar{\tau}]\oplus Z[\zeta_5, \bar{\tau}] & \rightarrow & F_5[\zeta_3, \bar{\tau}]\oplus F_3[\zeta_5, \bar{\tau}] \end{array}$$

we get the exact sequence

$$K_1(Z[\zeta_3, \bar{\tau}])\oplus K_1(Z[\zeta_5, \bar{\tau}])\oplus K_1(Z[\zeta_{15}, \bar{\tau}]) \rightarrow K_1(F_5[\zeta_3, \bar{\tau}])\oplus K_1(F_3[\zeta_5, \bar{\tau}]) \rightarrow D \rightarrow 0 .$$

Taking the reduced norm, we have the exact sequence

$$Z_+^*\oplus Z[\zeta_5+\zeta_5^{-1}]_+^*\oplus Z[\zeta_{15}+\zeta_{15}^{-1}]_+^* \rightarrow F_5^*\oplus F_3[\zeta_5+\zeta_5^{-1}]^* \rightarrow D \rightarrow 0 .$$

On the other hand $Z[\zeta_{15}+\zeta_{15}^{-1}]_+^*=\{\varepsilon_1{}^a\varepsilon_2{}^b\varepsilon_3{}^c \mid a, b \text{ and } c \text{ are all odd or all even}\}$, where $\varepsilon_1=\zeta_{15}+\zeta_{15}^{-1}-1$, $\varepsilon_2=\zeta_{15}^2+\zeta_{15}^{-2}-1$ and $\varepsilon_3=\zeta_{15}^3+\zeta_{15}^{-3}+1$. A direct computation shows that $D\cong Z/2Z$.

REMARK 2.5. Let $\Lambda_{2n}=ZC_{2n}\cap \prod_{d|n} R^d\times R^d$. Cassou-Noguès has shown in [2] that there exists a surjection of $D(ZH_n)$ in $D(\Lambda_{2n})$ whose kernel is an elementary 2-group. It is seen in the proof of (2.2) that $D(\Lambda_{2n})\cong D(ZD_{2n})$. Hence a part of (2.2) and the final assertion of (2.3) are only restatements of the results of Cassou-Noguès.

REMARK 2.6. Recently, after this manuscript was written, T. Miyata has shown [9] that Res: $D(ZD_m)\rightarrow D(ZC_m)$ is injective for every integer $m>1$. Using this we know that the map φ in (2.2) has a close relation to the restriction $\mathrm{Res}_{C_{2n}}^{H_n}$: $D(ZH_n)\rightarrow D(ZC_{2n})$. Further we can extend the results to the case where n is even. Let $m>1$ be an integer and $H_m=\langle\sigma, \tau \mid \sigma^{2m}=1, \sigma^m=\tau^2, \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$. Then there is a natural surjection φ: $D(ZH_m)\rightarrow D(ZD_{2m})$ such that $\mathrm{Res}_{C_{2m}}^{D_{2m}}\circ\varphi=\mathrm{Res}_{C_{2m}}^{H_m}$. (When m is odd, φ is the map defined in (2.2).) From this we see that $\mathrm{Res}_{C_{2m}}^{H_m}(D(ZH_m))\cong D(ZD_{2m})$ and Ker φ=Ker $\mathrm{Res}_{C_{2m}}^{H_m}$ is an elementary 2-group.

We give here the outline of the proof. There are isomorphisms (for details see [6], [7])

$$\begin{aligned} C(ZG) &\cong J_{QG}/[J_{QG}, J_{QG}](QG)^*U(ZG) \\ &\cong \mathrm{Hom}_{\Omega_Q}(R_G, J_F)/\mathrm{Hom}_{\Omega_Q}(R_G, F^*)\mathrm{Det}(U(ZG)) , \end{aligned}$$

where R_G is the Grothendieck group of virtual characters of G. For each element of $D(ZG)$ we can choose representatives as follows;

a projective left ideal M

$\leftrightarrow\alpha=(\alpha_p)\in U(\mathfrak{M})\subseteq J_{QG}$, where $\mathfrak{M}$ is a maximal order of QG containing ZG, such that $M=\bigcap_p(Z_pG\alpha_p\cap QG)$

$\leftrightarrow\mathrm{Det}(\alpha)\in\mathrm{Hom}_{\Omega_Q}(R_G, J_F)$.

For a subgroup H of G, $\mathrm{Res}_H^G(M)$ has the representative $\rho_{G/H}(\mathrm{Det}(\alpha))$, where $\rho_{G/H}(\mathrm{Det}(\alpha))(\chi)=\mathrm{Det}_{\mathrm{Ind}_H^G\chi}(\alpha)$ for $\chi\in R_H$ (for details see Appendix in [7]).

Now we compute $\mathrm{Res}_{C_{2m}}^{H_m}$ and $\mathrm{Res}_{C_{2m}}^{D_{2m}}$ by using $\rho_{H_m/C_{2m}}$ and $\rho_{D_{2m}/C_{2m}}$. When m is odd, we have the commutative diagram with exact row and column

$$(*)\qquad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0\to \mathrm{Ker}\,\varphi \to & D(\boldsymbol{Z}H_m) & \xrightarrow{\quad\varphi\quad} & & D(\boldsymbol{Z}D_{2m}) & \to 0 & \\ & & \searrow^{\mathrm{Res}_{C_{2m}}^{H_m}} & & \downarrow{\scriptstyle \mathrm{Res}_{C_{2m}}^{D_{2m}}} & & \\ & & & & D(\boldsymbol{Z}C_{2m}) & & , \end{array}$$

where φ is the map defined in (2.2). Let m be even. Since $\mathrm{Res}_{C_{2m}}^{D_{2m}}$ is injective, we know that the natural map φ of $D(\boldsymbol{Z}H_m)\cong U(\mathcal{O})_+/\mathcal{O}_+^*\mathrm{Nrd}(U(\boldsymbol{Z}H_m))$ to $D(\boldsymbol{Z}D_{2m})\cong U(\mathcal{O})/\mathcal{O}^*\,\mathrm{Nrd}(U(\boldsymbol{Z}D_{2m}))$, where $\mathcal{O}=\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus \bigoplus_{\substack{d|2m\\ d\neq 1,2}} R^d$, is well defined. Hence we also have the diagram (*). Finally, $\mathrm{Ker}\,\varphi=\mathrm{Ker}\,\mathrm{Res}_{C_{2m}}^{H_m}$ is annihilated by 2 (the Artin exponent of H_m).

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