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Osaka University
Let $W$ be the symmetric group on the set of $n$ letters $\{1, 2, \ldots, n\}$, $s_i \ (1 \leq i \leq n-1)$ the transposition $(i, i+1)$ in $W$, and $S = \{s_1, s_2, \ldots, s_{n-1}\}$. Then every element $w$ of $W$ can be expressed as $w=s_1s_2\cdots s_{i_1}(1 \leq i_1 \leq n-1)$. We denote the minimal length of such an expression by $l(w)$, i.e., $l(w) = \min \{l\}$. Let $K = \mathbb{C}(q)$ be the field of rational functions in one variable $q$ over the complex number field $\mathbb{C}$. The Hecke algebra $H = H(q)$ of $W$ is defined as follows: $H$ has a basis $\{h(w)\}_w \subseteq W$ which is parametrized by the elements of $W$. The multiplication is characterized by the rules

$$(h(s)+1)(h(s)-q) = 0, \quad \text{if } s \in S,$$

$$h(w)h(w') = h(ww'), \quad \text{if } l(w)+l(w') = l(ww').$$

Notice that $H$ is a $q$-analogue of the group algebra $\mathbb{C}W$ of $W$ in the sense that when $q$ is specialized to 1, $H$ is specialized to $\mathbb{C}W$. It should also be mentioned that the Hecke algebra can be defined for a general Coxeter system $(W, S)$ (see [2, Chap. 4, §2, Ex. 23]).

As is well-known, a complete set of mutually orthogonal primitive idempotents of $\mathbb{C}W$ is constructed by A. Young (see, for example, [6], [9]). Our main theorems are (3.10) and (4.5). In these theorems, we give a complete family of mutually orthogonal primitive idempotents of $H$, which is specialized to the one constructed by Young, when $q$ is specialized to 1.

The present work was motivated by a question posed by Dr. M. Jimbo in connection with his investigation [7] of the Yang-Baxter equation in mathematical physics. The author would like to express his thanks to Dr. M. Jimbo.

1. Let $(W, S)$ be a Coxeter system, $w$ an element of $W$ and $w=s_1s_2\cdots s_n \ (s_i \in S)$ a reduced decomposition of $w$. See [2; Chap. IV] for the fundamental concepts concerning Coxeter systems. It is known and easily proved by using [2; Chap. IV, n° 1.5, Lemma 4] that the set

$$\{s_{i_1}s_{i_2}\cdots s_{i_p} | 1 \leq i_1 < \cdots < i_p \leq n, 0 \leq p \leq n\}$$

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is uniquely determined by \( w \) and does not depend on the choice of a reduced decomposition of \( w \). If an element \( x \) of \( W \) is contained in this set, we write \( x \leq w \). The partial order defined in this way is called the Bruhat order.

Assume, from now on, that \( W \) is finite. It is known that every representation of the Hecke algebra \( H=H(q) \) can be afforded by a \( W \)-graph [5]. The precise definition of a \( W \)-graph is irrelevant here. What we need is that, for every finite dimensional representation \( \rho_q \) of \( H \), by an appropriate choice of a basis of the representation space, the elements \( h(w)(w \in W) \) are represented by matrices over \( C[q] \). Hence we can obtain a representation \( \rho_1 \) of \( W \) by the specialization \( q \to 1 \). This fact is used, for example, in the following way.

Let \( \chi_q = \text{trace } \rho_q \), \( \chi_1 = \text{trace } \rho_1 \) and \( \chi_q = \sum_i m_i \chi_i,q \) the irreducible decomposition of \( \chi_q \). By [3], we have

\[
\sum_{w \in W} \chi_q(h(w))\chi_q(q^{-1}(w)h(w^{-1}))/\sum_{w \in W} q^{\ell(w)} = \sum_i m_i(d_{i,q}/d_{i,1}) ,
\]

where \( d_{i,q} \) is the generic degree of \( \chi_i,q \) [1; Definition (2.4)] which is known to be a polynomial in \( q \), and \( d_{i,1} = (d_{i,q})_{q \to 1} \), which is equal to the degree (i.e., the dimension of the representation space) of the representation affording \( \chi_i,q \). By the specialization \( q \to 1 \), we get

\[
\sum_{w \in W} \chi_i(w)\chi_i(w^{-1})/\text{card } W = \sum_i m_i^2 .
\]

Hence \( \rho_q \) is irreducible if and only if \( \rho_1 \) is irreducible.

We will use this kind of "specialization argument" very often without mentioning the details.

From now on, we assume that \( W \) is the \( n \)-th symmetric group acting on \( \{1, 2, \cdots, n\} \) and \( S = \{s_1, s_2, \cdots, s_{n-1}\} \), where \( s_i = (i, i+1) \). See [6] for the fundamental concepts concerning symmetric groups.

For each partition \( \lambda \) of \( n \), we can define two standard tableaux \( T_+ = T_+(\lambda) \) and \( T_- = T_-(\lambda) \), e.g., if \( \lambda = (5 \ 4 \ 2 \ 1) \),

\[
T_+(\lambda) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 \\
14 &
\end{array}
\]

\[
T_-(\lambda) = \begin{array}{cccc}
1 & 5 & 8 & 11 & 14 \\
2 & 6 & 9 & 12 \\
3 & 7 & 10 & 13 \\
4 &
\end{array}
\]

We omit the exact definition of \( T_\pm(\lambda) \). Let \( I_+ = I_+(\lambda) \) (resp. \( I_- = I_-(\lambda) \)) be the set of \( s_i \)'s which preserve each row (resp. column) of \( T_+(\lambda) \) (resp. \( T_-(\lambda) \)) as a set.
For example, if \( \lambda = (5 \ 4 \ 2 \ 1) \), then
\[
I_+ = \{ s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_{10}, s_{11}, s_{12} \}
\]
and
\[
I_- = \{ s_1, s_2, s_3, s_5, s_6, s_8, s_9, s_{11}, s_{12} \}.
\]

Let \( W_\pm = W_\pm(\lambda) \) be the parabolic subgroups of \( W \) generated by \( I_\pm \), and
\[
H_\pm = \sum_{w \in W_\pm} K h(w).
\]
Then \( H \) are subalgebras of \( H_\pm \). Let
\[
(1.1) \quad e_+ = e_+(\lambda) = \sum_{w \in W_+} h(w)
\]
and
\[
(1.2) \quad e_- = e_-(\lambda) = \sum_{w \in W_-} (-q)^{-t(w)} h(w).
\]

Since, for each \( s \in I_+ \),
\[
e_+ = \sum_{w \in W_+} (1 + h(s)) h(w),
\]
we have
\[
h(s) e_+ = q e_+.
\]

Hence
\[
h(w) e_+ = q^{t(w)} e_+ \quad (w \in W_+).
\]

In the same way, we can show that
\[
h(w) e_+ = e_+ h(w) = q^{t(w)} e_+ \quad (w \in W_+),
\]
and
\[
h(w) e_- = e_- h(w) = (-1)^{t(w)} e_- \quad (w \in W_-).
\]

From these equalities, we get
\[
e^2_\pm = P_\pm e_\pm,
\]
where
\[
P_\pm = P_\pm(\lambda) = \sum_{w \in W_\pm} q^{t(w)}.
\]

The left \( H \)-modules \( H e_\pm \) are isomorphic to the induced representations \( H \otimes \mathcal{E}_\pm \), where \( \mathcal{E}_\pm \) are the one-dimensional \( H_\pm \)-modules denoted by
\[
h(w) v = q^{t(w)} v \quad (v \in \mathcal{E}_+)
\]
and
\[
h(w) v = (-1)^{t(w)} v \quad (v \in \mathcal{E}_-)
\]
By the classical result of A. Young and by the specialization argument, we have
\[ \dim_k \text{Hom}_H(He_\pm, He_\pm) = 1. \]
Take (non-zero) intertwining operators
\[ f_\pm \in \text{Hom}_H(He_\pm, He_\pm). \]
The images of \( f_\pm \) do not depend on the choice of \( f_\pm \). Thus we have the following result.

**Proposition 1.3.** Let \( V_\pm = V_\pm(\lambda) \) be the images of \( f_\pm \). Then \( V_\pm \) are irreducible \( H \)-modules and
\[ f_\pm : V_\mp \longrightarrow V_\pm. \]
Every irreducible representation of \( H \) can be realized uniquely as \( V_+ \) (or as \( V_- \)).

REMARK. It is known that every irreducible representation of \( H \) is absolutely irreducible [1].

2. The purpose of this section is to construct a \( q \)-analogue of the Young symmetrizer. The main result of this section is (2.2.1).

2.1. First, let us determine \( f_+ \) explicitly. For this purpose, it suffices to determine \( f_+(e_-) \). Since
\[ f_+(e_-) = e_-(P_+^{-1}P_-^{-1}f_+(e_-))e_+ \]
and
\[ e_-h(x)h(y)e_+ = (-1)^{i(x)}q^{i(\sigma)}e_-h(w)e_+, \quad (x \in W_-, y \in W_+), \]
f_+(e) is of the form
\[ \sum_{w \in X} a_w e_-h(w)e_+ \quad (a_w \in K), \]
where
\[ X = \{ w \in W | sw > w \} \quad \text{for every} \quad s \in I_-(\lambda), \quad \text{and} \]
\[ wt > w \quad \text{for every} \quad t \in I_+(\lambda) \}.

Let \( T_1 \) and \( T_2 \) be standard tableaux which belong to the partition \( \lambda \), and \([T_2, T_1]\) the permutation which transforms \( T_1 \) to \( T_2 \). We write \([T \pm]\) (resp. \([\pm T], [\pm \mp] \)) for \([T, T\pm]\) (resp. \([T_\pm, T][T_\pm, T_\mp] \)), e.g., if \( \lambda = (5, 4^2, 1) \)
\[
\begin{align*}
T & \quad = \\
1 & \quad 2 \quad 4 \quad 7 \quad 14 \\
3 & \quad 5 \quad 6 \quad 8 \\
9 & \quad 10 \quad 11 \quad 13 \\
12 & \quad
\end{align*}
\]
then

\[[T^+] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 2 & 4 & 7 & 14 & 3 & 5 & 6 & 8 & 9 & 10 & 11 & 13 & 12 \end{pmatrix}\]

and

\[[T^-] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 3 & 9 & 12 & 2 & 5 & 10 & 4 & 6 & 11 & 7 & 8 & 13 & 14 \end{pmatrix}\]

If \(i\) and \(i+1\) are in the same row of \(T_\ast\), then \([T^+](i) < [T^+](i+1)\). Hence

\[(2.1.2)\quad [T^+]s > [T^+]\quad (s \in L_\ast)\,.

In the same way, we can show that

\[(2.1.3)\quad [T^-]s > [T^-]\quad (s \in L_\ast)\,.

Note that \([T_1, T_2][T_3, T_4] = [T_1, T_3]\) and \([++][-][-++]\) consists of permutations which preserve each column of \(T_\ast\). Hence we can restate [9; Lemma (4.2.A)] as follows.

**Lemma 2.1.4.** For \(z \in W\), the following two conditions are equivalent:

(i) \quad \(zW_+z^{-1} \cap [++]W_-[++] = \{1\}\).

(ii) \quad \(z \in ([++]W_-[++)W_+\).

In fact (ii) \(\Rightarrow\) (i) is trivial. Conversely, assume (i). Let \(T\) be the transform of \(T_\ast\) by \(z\), i.e., \(z = [T^+]\). If there are two numbers \(a, b\) which appear in the same row of \(T\) and the same column of \(T_\ast\), then the transposition \((a, b)\) belongs to \(zW_+z^{-1} \cap [++]W_-[++)\). This contradicts (i). Hence we get (ii) by [9; Lemma (4.2.A)].

Let \([++]z\) \((= [++]\)) be an element of \(X\). By (2.1.2) and (2.1.3), \([++]\) is also an element of \(X\). Hence

\([++]z \in W_-[++)W_+\)

by [2; Chap 4, §1, Ex. 3]. By (2.1.4),

\[zW_+z^{-1} \cap [++]W_-[++) \neq \{1\},\]

i.e., we can find elements \(x_\pm \in W_\pm\) such that

\([++]z)x_+ = x_-(z)z, \quad x_\pm \neq 1\).

By the equality

\[e_-h([++]z)x_+ e_+ = q^{(s_\pm)}e_-h([++]z)e_+\]

\[= e_-h(x_-[++]z)e_+ = (-1)^{t(s_\pm)}e_-h([++]z)e_+\],
we conclude that

\[(2.1.5) \quad e_- h([-+]) e_+ = 0.\]

Hence (2.1.1) is of the form

\[a \cdot e_- h([-+]) e_+ \quad (a \in K).\]

Since \(f_+ \neq 0, a \neq 0\). Note that the above argument shows also that

\[e_- h([T-])^{-1} h([T+]) e_+ = b \cdot e_- h([-+]) e_+\]

with some \(b \in K\). By the specialization \(q \to 1\), \(b\) specializes to 1. Hence \(b \neq 0\). Thus we may assume that

\[f_+(e_-) = e_- h([T-])^{-1} h([T+]) e_+.\]

By the same argument as above, we can also show that

\[f_-(e_+) = e_+ h([T+])^{-1} h([T-]) e_-\]

(up to scalar multiple).

2.2. Now let us construct a \(q\)-analogue of the Young symmetrizer. Since \(f_+(e_-) \in V_+\),

\[f_+ f_-(e_-) = c f_+(e_-) \quad (c = c(q) \in K),\]

i.e.,

\[e_- h^{-1} h_+ e_+ h^{-1} h_+ e_+ e_+ = c e_- h^{-1} h_+ e_+ ,\]

where \(h_\pm = h([T+])\). Hence

\[(2.2.1) \quad (h_- e_- h^{-1} \cdot h_+ e_+ h^{-1}) = c (h_- e_- h^{-1} \cdot h_+ e_+ h^{-1}).\]

By the specialization \(q \to 1\), \((h_- e_- h^{-1}) (h_+ e_+ h^{-1})\) specializes to the Young symmetrizer (corresponding to the standard tableau \(T\)). Hence \(c = c(T) \neq 0\).

2.3. For a standard tableau \(T\) which belongs to a partition \(\lambda\), let

\[E(T) = c(T)^{-1} (h([T-]) e_-(\lambda) h([T-])^{-1}) (h([T+]) e_+(\lambda) h([T+])^{-1}).\]

Let us consider when

\[E(T_1) E(T_2) = 0\]

for two different standard tableaux.

If \(T_1\) and \(T_2\) belong to different partitions, \(E(T_1) E(T_2) = 0\). In fact, if \(\chi_q\) is an irreducible character of \(H\) such that \(\chi_q(E(T_1)) = m \neq 0, \in \mathbb{Z}\), then \(\chi_q(E(T_1))_{q \to 1} = m\). By (3.9) below, which will be proved without using the results
of (2.3), the specialization $E(T_1)_{q^+1}$ is well defined and equal to the Young symmetrizer. Hence $m=1$. In the same way we can show that $\chi_q(E(T_2))=0$. Hence $E(T_1)$ and $E(T_2)$ are (primitive) idempotents which belong to different irreducible representation of $H$. Hence $E(T_1)E(T_2)=0$.

Assume that $T_1$ and $T_2$ belong to the same partition $\lambda$.

**Lemma 2.3.1.** If $T_1 \neq T_2$ and $l([T_1+-]) \geq l([T_2+-])$, then $E(T_1)E(T_2)=0$.

**Proof.** It suffices to prove

$$e_+(\lambda) h([T_1+-])^{-1} h([T_2+-]) e_-(\lambda) = 0. \tag{2.3.2}$$

By using the fact

$$l(w) = \text{card} \{ (i, j) | 1 \leq i < j \leq n, \, w(i) > w(j) \} \quad (w \in W),$$

it is easy to see that

$$l([T+]) + l([T-]) = l([+-]) \tag{2.3.3}$$

for any standard tableau $T$. By our assumption,

$$l([+-]) \geq l([T_1+])+l([T_2-]). \tag{2.3.4}$$

Let

$$Y = \{ x_1, x_2 | x_1 \leq [T_1+]-1, \, x_2 \leq [T_2-] \} .$$

Then $Y \cap W_+[+-]W_-=\emptyset$ by (2.3.4). Since we can express $h([T_1+])^{-1} h([T_2-])$ as a linear combination

$$\sum_{y \in Y} a_y h(y) \quad (a_y \in K),$$

the argument of 2.1 shows (2.3.2).

3. The purpose of this section is to determine the scalar $c=c(q)$ which appeared in (2.2.1). Our main result of this section is (3.8).

Let us define a linear functional $tr$ on $H$ by

$$tr h(w) = \begin{cases} g & (w=1) \\ 0 & (w \neq 1) \end{cases},$$

where

$$g = (q-1)(q^2-1) \cdots (q^n-1)/(q-1)^n = \sum_{w \in W} q^{l(w)}. \tag{3.1}$$

It is known [4] that

$$tr(h(x)h(y)) = \begin{cases} gq^{l(x)} & (xy=1) \\ 0 & (xy \neq 1) \end{cases} \tag{3.2}$$
and

\[(3.3) \quad \text{tr}(h_1 h_2) = \text{tr}(h_2 h_1) \quad (h_1, h_2 \in H).\]

By specializing \(q\) to a prime power \(r\), \(H(q)\) specializes to a \(C\)-algebra \(H(r)\) which can be identified with a subalgebra of the group ring \(C\text{GL}_n(r)\) (see \[3\]). It is easy to see that the restriction of the character of the regular representation of \(C\text{GL}_n(r)\) to \(H(r)\) equals the specialization \(\text{tr}_{e \cdot r}\). It is known \[3\] that every irreducible character of \(H(r)\) can be uniquely obtained by restricting an irreducible character of \(C\text{GL}_n(r)\) (which is extended to a linear functional on \(C\text{GL}_n(r)\)).

Let \(\chi(\lambda)\) be the character of \(V_\pm(\lambda)\) (see \(1.3)\) and \(\tilde{\chi}(\lambda)\) the irreducible character of \(C\text{GL}_n(r)\) corresponding to \(\chi(\lambda)_{e \cdot r}\) in the above sense. Let \(d(\lambda, r)\) be the multiplicity of \(\tilde{\chi}(\lambda)\) in the regular representation of \(C\text{GL}_n(r)\), which is also the degree of \(\tilde{\chi}(\lambda)\).

Then

\[(3.4) \quad d(\lambda, r) = \prod_{i \geq 1} \frac{(r \lambda_i + (m-i)) - r \lambda_{i+1}}{(r-1)(r-1)^2 \cdots (r \lambda_i + (m-i) - 1)} \times \frac{(r-1)(r^2-1) \cdots (r^n-1)}{r^{(m-1)+(m-2)+ \cdots}},\]

where \(\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \lambda_m \geq 0\}\) (See \[8\].) Let \(d(\lambda, q)\) be the polynomial such that \(d(\lambda, r) = d(\lambda, r)\) for any prime power \(r\).

The above argument shows that

\[(3.5) \quad \text{tr} = \sum_{\lambda} d(\lambda, q) \chi(\lambda),\]

where \(\lambda\) runs over the set of partitions of \(n\). We have

\[(3.6) \quad \text{tr}(h \cdot e \cdot h^{-1} \cdot h_+ e_+ h_+^{-1})
\begin{align*}
&= q^{-l(\lambda-\emptyset)} \text{tr}(h \cdot e \cdot h([+\cdot]) e_+ h_+^{-1}) \\&= q^{-l(\lambda-\emptyset)} \text{tr}(h([+\cdot]) e_+ h_+ h^{-1}) \\&= q^{-l(\lambda-\emptyset)} \text{tr}(h([+\cdot]) e_+ h([+\cdot]) e_+) \\&= q^{-l(\lambda-\emptyset)} \sum_{x \in W_+} (q^{-l(\pm)} e_+ h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot])) \\&= q^{-l(\lambda-\emptyset)} \sum_{x \in W_+} (q^{-l(\pm)} e_+ h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot]) h([+\cdot])) \\&= q^{-l(\lambda-\emptyset)} (q^{-l(\lambda-\emptyset)} g) \\&= g.
\end{align*}\]

On the other hand, \((2.2.1)\) implies that \(E = e^{-1} h \cdot e \cdot h^{-1} \cdot h_+ e_+ h_+^{-1}\) is an idempotent of \(V_+(\lambda) h_+^{-1}\). By the specialization \(q \rightarrow 1\), \(E\) specializes to a primitive idempotent. Hence the value of the character \(\chi(\lambda)\) at \(E\) specializes to 1. But a character value at an idempotent must be an integer. Hence \(E\) is primitive. Hence

\[(3.7) \quad \text{tr}(e^{-1} h \cdot e \cdot h^{-1} \cdot h_+ e_+ h_+^{-1}) = d(\lambda, q).\]

By \((3.6)\) and \((3.7)\),
\[ c = \frac{g}{d(\lambda, q)} \]

By (3.4), \( c \) can be also expressed as follows

\[ c = \frac{\prod (q-1)(q^2-1) \cdots (q^{\lambda_i}(m-i)-1)}{\prod (q^{\lambda_j}(m-j)-q^{\lambda_j}(m-j))} q^{(m_2-1) + (m_2-2) + \cdots + (m_2-n)}. \]

Let us restate our results as a theorem.

**Theorem 3.10.** Let \( \lambda \) be a partition of \( n \) and \( \{T_1, \ldots, T_f\} \) the standard tableaux which belong to \( \lambda \). Assume that

\[ l([T_i-]) \geq l([T_j-]), \quad \text{if} \quad i < j. \]

For each standard tableau \( T \), let

\[ E(T) = c^{-1}h([T-])e_-(\lambda)h([T-])^{-1}h([T+])e_+(\lambda)h([T+])^{-1}, \]

where

\[ c = \frac{g}{d(\lambda, q)}. \]

Then \( E(T_1), \ldots, E(T_f) \) are primitive idempotents which belong to \( \chi(\lambda) \), and

\[ E(T_i)E(T_j) = 0, \quad \text{if} \quad i < j. \]

(See (1.1) and (1.2) for \( e_\pm \), section 2.1 for \( [T \pm] \), (3.1) for \( g \), (3.4) and the subsequent lines for \( d(\lambda, q) \).)

4. Orthogonalization of idempotents

The purpose of this section is to give a procedure to construct an orthogonal family of idempotents from a given family of idempotents. By applying this procedure to the family of idempotents \( \{E(T)\} \) which was obtained in the preceding section, we get a complete family of mutually orthogonal, primitive idempotents of \( H \).

4.1. Let \( X \) be a partially ordered set of cardinality \( n \). Let \( I = \{1, 2, \ldots, n\} \) and \( A \) be the set of bijections \( a: I \rightarrow X \) such that \( a^{-1} \) is order preserving. If \( a \) is an element of \( A \) and if \( a(i) \) and \( a(i+1) \) are not comparable, we define a new element of \( A \) by

\[ a'(j) = \begin{cases} a(j) & \text{if } j \neq i, i+1 \\ a(i+1) & \text{if } j = i \\ a(i) & \text{if } j = i+1. \end{cases} \]
If \( b(\in A) \) can be obtained from \( a \) by applying this operation several times, we say that \( b \) is equivalent to \( a \) and write \( a \sim b \). This relation is an equivalence relation.

**Lemma 4.2.** *Any two elements of \( A \) are equivalent to each other.*

Proof. Let \( a, b \in A \) such that
\[
\begin{align*}
a(k) &= b(k) \quad (k < i) \\
a(i) &= b(i) .
\end{align*}
\]

Let \( a(i) = a_0 \) and \( b^{-1}(a_0) = j \). Then \( j > i \) and \( a_0 = b(j) \) is not comparable with any one of \( \{b(i), b(i+1), \ldots, b(j-1)\} \). In fact, if \( b(j) \) is comparable with \( b(k) \) \((i \leq k < j)\), then \( a_0 = b(j) > b(k) \). But \( a^{-1}(b(j)) = i \) and \( b(k) \in \{b(1), \ldots, b(i-1), a_0 \} = \{a(1), \ldots, a(i)\} \), hence \( a^{-1}(b(k)) > i \). Since \( k < j \), this is a contradiction.

Now we can define an element \( c \) of \( A \) by
\[
c(k) = \begin{cases} 
b(k) & (1 \leq k < i) \\
b(j) & (k = i) \\
b(k-1) & (i < k \leq j) \\
b(k) & (j < k \leq n) .
\end{cases}
\]

Then \( b \sim c \) and \( a(k) = c(k) \ (k < i + 1) \). Thus, by an induction on \( i \), we can show that \( a \sim b \).

4.3. Let \( X \) be a set of idempotents in a ring with 1. Let us define a relation \( \leq \) in \( X \) by
\[
e \leq e' \text{ if there exists a sequence } \\
(\#) \quad e = e_0, e_1, \ldots, e_n = e' \text{ of elements of } X \\
such that \quad e_ie_{i+1} \neq 0 \quad (0 \leq i < n) .
\]

Assume that
\[\text{(4.3.1)} \quad \text{the relation } \leq \text{ defined by (\#)} \text{ is a partial order} .\]

We can define \( A \) for this partially ordered set.

**Remark.** If from the beginning, \( X \) is totally ordered and satisfies
\[\text{(4.3.2)} \quad ee' = 0 \quad \text{if } \ e \geq e' ,\]
then (4.3.1) is automatically satisfied. For example the set \{\( E(T_1), \ldots, E(T_p) \)\} satisfies (4.3.2) with any total order such that \( l([T]) \geq l([T']) \) whenever \( E(T) \geq E(T') \).
Lemma 4.4. Let $X$ be a set of idempotents. Let $x \in X$, $a \in A$, $i = a^{-1}(x)$ and $E(a, x) = (1 - a(1)) \cdots (1 - a(i - 1))a(i)$. Then $\{E(a, x)\}_{x \in X}$ are mutually orthogonal idempotents, and each element $E(a, x)$ is independent of $a \in A$.

Proof. If $i > j$, then $a(i)a(j) = 0$. Hence
\[
a(i)(1 - a(1)) \cdots (1 - a(i - 1))a(i) = a(i),
\]
\[
a(i)(1 - a(1)) \cdots (1 - a(i)) = 0,
\]
\[
a(i)(1 - a(1)) \cdots (1 - a(j - 1))a(j) = 0 \quad (i > j).
\]

From these equalities, we can conclude that $E(a, a(i))$ are mutually orthogonal idempotents.

To show that every $E(a, a(i))$ is independent of $a$, it is enough to prove that
\[
E(a, a(j)) = E(a', a(j))
\]
if $a'$ is obtained from $a$ by the transposition $(i, i + 1)$. There is nothing to prove for $j < i$. For $j = i$,
\[
E(a, a(i)) = (1 - a(1)) \cdots (1 - a(i - 1))a(i)
\]
and
\[
E(a', a(i)) = (1 - a'(1)) \cdots (1 - a'(i))a'(i + 1),
\]
since $a'(i + 1) = a(i)$. Since
\[
a'(i)a'(i + 1) = a(i + 1)a(i) = 0,
\]
we have $E(a', a(i)) = E(a, a(i))$. For $j = i + 1$,
\[
E(a, a(i + 1)) = (1 - a(1)) \cdots (1 - a(i))a(i + 1)
\]
and
\[
E(a', a(i + 1)) = (1 - a'(1)) \cdots (1 - a'(i - 1))a'(i),
\]
since $a'(i) = a(i + 1)$. Since
\[
a(i)a(i + 1) = a'(i + 1)a'(i) = 0,
\]
we have $E(a', a(i)) = E(a, a(i))$. Since
\[
(1 - a'(i))(1 - a'(i + 1)) = (1 - a(i + 1))(1 - a(i))
\]
\[
= (1 - a(i))(1 - a(i + 1)),
\]
(4.4.1) holds for $j > i + 1$.

By the above lemma, we can define a set of mutually orthogonal idempotents
\[
X^o = \{x^o | x \in X\},
\]
where, $x^a = E(a, x)$ for some $a \in A$.

**Theorem 4.5.** The set

$$\{E(T)^a | T \text{ standard tableau}\}$$

is a complete family of mutually orthogonal primitive idempotents in $H$.

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**References**


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