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A q-ANALOGUE OF YOUNG SYMMETRIZER*

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Let W be the symmetric group on the set of n letters $\{1, 2, \cdots, n\}$, s_i $(1 \leqq i \leqq n-1)$ the transposition $(i, i+1)$ in W, and $S=\{s_1, s_2, \cdots, s_{n-1}\}$. Then every element w of W can be expressed as $w=s_{i_1}s_{i_2}\cdots s_{i_l}$ $(1 \leqq i_\alpha \leqq n-1)$. We denote the minimal length of such an expression by $l(w)$, i.e., $l(w)=\min\{l\}$. Let $K=\boldsymbol{C}(q)$ be the field of rational functions in one variable q over the complex number field $\boldsymbol{C}$. The Hecke algebra $H=H(q)$ of W is defined as follows: H has a basis $\{h(w)\}_{w \in W}$ which is parametrized by the elements of W. The multiplication is characterized by the rules

$$\begin{aligned} &(h(s)+1)(h(s)-q) = 0\,, && \text{if } s \in S\,, \\ &h(w)h(w') = h(ww')\,, && \text{if } l(w)+l(w') = l(ww')\,. \end{aligned}$$

Notice that H is a q-analogue of the group algebra $\boldsymbol{C}W$ of W in the sense that when q is specialized to 1, H is specialized to $\boldsymbol{C}W$. It should also be mentioned that the Hecke algebra can be defined for a general Coxeter system (W, S) (see [2; Chap. 4, §2, Ex. 23]).

As is well-known, a complete set of mutually orthogonal primitive idempotents of $\boldsymbol{C}W$ is constructed by A. Young (see, for example, [6], [9]). Our main theorems are (3.10) and (4.5). In these theorems, we give a complete family of mutually orthogonal primitive idempotents of H, which is specialized to the one constructed by Young, when q is specialized to 1.

The present work was motivated by a question posed by Dr. M. Jimbo in connection with his investigation [7] of the Yang-Baxter equation in mathematical physics. The author would like to express his thanks to Dr. M. Jimbo.

1. Let (W, S) be a Coxeter system, w an element of W and $w=s_1s_2\cdots s_n$ $(s_i \in S)$ a reduced decomposition of w. See [2; Chap. IV] for the fundamental concepts concerning Coxeter systems. It is known and easily proved by using [2; Chap. IV, n° 1.5, Lemma 4] that the set

$$\{s_{i_1}s_{i_2}\cdots s_{i_p} \mid 1 \leqq i_1 < \cdots < i_p \leqq n,\ 0 \leqq p \leqq n\}$$

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is uniquely determined by w and does not depend on the choice of a reduced decomposition of w. If an element x of W is contained in this set, we write $x\leqq w$. The partial order defined in this way is called the Bruhat order.

Assume, from now on, that W is finite. It is known that every representation of the Hecke algebra $H=H(q)$ can be afforded by a W-graph [5]. The precise definition of a W-graph is irrelevant here. What we need is that, for every finite dimensional representation ρ_q of H, by an appropriate choice of a basis of the representation space, the elements $h(w)(w\in W)$ are represented by matrices over $\boldsymbol{C}[q]$. Hence we can obtain a representation ρ_1 of W by the specialization $q\rightarrow 1$. This fact is used, for example, in the following way.

Let $\chi_q=\text{trace }\rho_q$, $\chi_1=\text{trace }\rho_1$ and $\chi_q=\sum_i m_i\chi_{i,q}$ the irreducible decomposition of χ_q. By [3], we have

$$\sum_{w\in W}\chi_q(h(w))\chi_q(q^{-l(w)}h(w^{-1}))/\sum_{w\in W}q^{l(w)} = \sum_i m_i^2(d_{i,1}/d_{i,q}),$$

where $d_{i,q}$ is the generic degree of $\chi_{i,q}$ [1; Definition (2.4)] which is known to be a polynomial in q, and $d_{i,1}=(d_{i,q})_{q\rightarrow 1}$, which is equal to the degree (i.e., the dimension of the representation space) of the representation affording $\chi_{i,q}$. By the specialization $q\rightarrow 1$, we get

$$\sum_{w\in W}\chi_1(w)\chi_1(w^{-1})/\text{card } W = \sum_i m_i^2 .$$

Hence ρ_q is irreducible if and only if ρ_1 is irreducible.

We will use this kind of "specialization argument" very often without mentioning the details.

From now on, we assume that W is the n-th symmetric group acting on $\{1, 2, \cdots, n\}$ and $S=\{s_1, s_2, \cdots, s_{n-1}\}$, where $s_i=(i, i+1)$. See [6] for the fundamental concepts concerning symmetric groups.

For each partition λ of n, we can define two standard tableaux $T_+=T_+(\lambda)$ and $T_-=T_-(\lambda)$, e.g., if $\lambda=(5\ 4^2\ 1)$,

$$T_+(\lambda) = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & \\ 14 & & & & \end{matrix}$$

$$T_-(\lambda) = \begin{matrix} 1 & 5 & 8 & 11 & 14 \\ 2 & 6 & 9 & 12 & \\ 3 & 7 & 10 & 13 & \\ 4 & & & & \end{matrix}$$

We omit the exact definition of $T_\pm(\lambda)$. Let $I_+=I_+(\lambda)$ (resp. $I_-=I_-(\lambda)$) be the set of s_i's which preserve each row (resp. column) of $T_+(\lambda)$ (resp. $T_-(\lambda)$) as a set.

For example, if $\lambda=(5\ 4^2\ 1)$, then

$$I_+ = \{s_1, s_2, s_3, s_4, s_6, s_7, s_8, s_{10}, s_{11}, s_{12}\}$$

and

$$I_- = \{s_1, s_2, s_3, s_5, s_6, s_8, s_9, s_{11}, s_{12}\} .$$

Let $W_\pm = W_\pm(\lambda)$ be the parabolic subgroups of W generated by $I_\pm$, and $H_\pm = \sum_{w\in W_\pm} Kh(w)$. Then H are subalgebras of $H_\pm$. Let

$$e_+ = e_+(\lambda) = \sum_{w\in W_+} h(w) \tag{1.1}$$

and

$$e_- = e_-(\lambda) = \sum_{w\in W_-} (-q)^{-l(w)} h(w) . \tag{1.2}$$

Since, for each $s\in I_+$,

$$e_+ = \sum_{\substack{w\in W_+ \\ sw>w}} (1+h(s))h(w) ,$$

we have

$$h(s)e_+ = qe_+ .$$

Hence

$$h(w)e_+ = q^{l(w)}e_+ \qquad (w\in W_+) .$$

In the same way, we can show that

$$h(w)e_+ = e_+h(w) = q^{l(w)}e_+ \qquad (w\in W_+) ,$$

and

$$h(w)e_- = e_-h(w) = (-1)^{l(w)}e_- \qquad (w\in W_-) .$$

From these equalities, we get

$$e_\pm^2 = P_\pm e_\pm ,$$

where

$$P_\pm = P_\pm(\lambda) = \sum_{w\in W^\pm} q^{\pm l(w)} .$$

The left H-modules $He_\pm$ are isomorphic to the induced representations $H \underset{H_\pm}{\otimes} \mathcal{E}_\pm$, where $\mathcal{E}_\pm$ are the one-dimensional $H_\pm$-modules denfied by

$$h(w)v = q^{l(w)}v \qquad (v\in \mathcal{E}_+)$$

and

$$h(w)v = (-1)^{l(w)}v \qquad (v\in \mathcal{E}_-)$$

By the classical result of A. Young and by the specialization argument, we have

$$\dim_K \mathrm{Hom}_H(He_\pm, He_\mp) = 1 .$$

Take (non-zero) intertwining operators

$$f_\pm \in \mathrm{Hom}_H(He_\mp, He_\pm) .$$

The images of $f_\pm$ do not depend on the choice of $f_\pm$. Thus we have the following result.

Proposition 1.3. *Let $V_\pm = V_\pm(\lambda)$ be the images of $f_\pm$. Then $V_\pm$ are irreducible H-modules and*

$$f_\pm\colon V_\mp \xrightarrow{\sim} V_\pm .$$

Every irruducible representation of H can be realized uniquely as V_+ (or as V_-).

REMARK. It is known that every irreducible representation of H is absolutely irreducible [1].

2. The purpose of this section is to construct a q-analogue of the Young symmetrizer. The main result of this section is (2.2.1).

2.1. First, let us determine f_+ explicitly. For this purpose, it suffices to determine $f_+(e_-)$. Since

$$f_+(e_-) = e_-(P_+^{-1}P_-^{-1}f_+(e_-))e_+$$

and

$$e_-h(x)h(w)h(y)e_+ = (-1)^{l(x)}q^{l(y)}e_-h(w)e_+ \qquad (x \in W_-,\ y \in W_+) ,$$

$f_+(e)$ is of the form

(2.1.1) $$\textstyle\sum_{w\in X} a_w e_-h(w)e_+ \qquad (a_w \in K) ,$$

where

$$\begin{aligned} X = \{w \in W \mid sw > w \quad &\text{for every} \quad s \in I_-(\lambda), \text{ and} \\ wt > w \quad &\text{for every} \quad t \in I_+(\lambda)\} . \end{aligned}$$

Let T_1 and T_2 be standard tableaux which belong to the partition λ, and $[T_2, T_1]$ the permutation which transforms T_1 to T_2. We write $[T\pm]$ (resp. $[\pm T]$, $[\pm\mp]$) for $[T, T_\pm]$ (resp. $[T_\pm, T][T_\pm, T_\mp]$), e.g., if $\lambda = (5\ 4^2\ 1)$ and

$$T = \begin{matrix} 1 & 2 & 4 & 7 & 14 \\ 3 & 5 & 6 & 8 & \\ 9 & 10 & 11 & 13 & \\ 12 & & & & \end{matrix}$$

then

$$[T+]=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14\\ 1 & 2 & 4 & 7 & 14 & 3 & 5 & 6 & 8 & 9 & 10 & 11 & 13 & 12\end{pmatrix}$$

and

$$[T-]=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14\\ 1 & 3 & 9 & 12 & 2 & 5 & 10 & 4 & 6 & 11 & 7 & 8 & 13 & 14\end{pmatrix}$$

If i and $i+1$ are in the same row of T_+, then $[T+](i)<[T+](i+1)$. Hence

(2.1.2) $$[T+]s>[T+] \qquad (s\in I_+)\,.$$

In the same way, we can show that

(2.1.3) $$[T-]s>[T-] \qquad (s\in I_-)\,.$$

Note that $[T_1, T_2][T_2, T_3]=[T_1, T_3]$ and $[+-]W_-[-+]$ consists of permutations which preserve each column of T_+. Hence we can restate [9; Lemma (4.2.A)] as follows.

Lemma 2.1.4. *For $z\in W$, the following two conditions are equivalent*:

(i) $$zW_+z^{-1}\cap[+-]W_-[-+]=\{1\}\,.$$

(ii) $$z\in([+-]W_-[-+])W_+\,.$$

In fact (ii)⇒(i) is trivial. Consversely, assume (i). Let T be the transform of T_+ by z, i.e., $z=[T+]$. If there are two numbers a, b which appear in the same row of T and the same column of T_+, then the transposition (a, b) belongs to $zW_+z^{-1}\cap[+-]W_-[-+]$. This contradicts (i). Hence we get (ii) by [9; Lemma (4.2.A)].

Let $[-+]z$ $(\neq[-+])$ be an element of X. By (2.1.2) and (2.1.3), $[-+]$ is also an element of X. Hence

$$[-+]z\notin W_-[-+]W_+$$

by [2; Chap 4, §1, Ex. 3]. By (2.1.4),

$$zW_+z^{-1}\cap[+-]W_-[-+]\neq\{1\}\,,$$

i.e., we can find elements $x_\pm\in W_\pm$ such that

$$([-+]z)x_+=x_-([-+]z)\,, \qquad x_\pm\neq 1\,.$$

By the equality

$$\begin{aligned} e_-h([-+]zx_+)e_+ &= q^{l(x_+)}e_-h([-+]z)e_+ \\ &= e_-h(x_-[-+]z)e_+=(-1)^{l(x_-)}e_-h([-+]z)e_+\,, \end{aligned}$$

we conclude that

$$e_-h([-+]z)e_+ = 0 . \tag{2.1.5}$$

Hence (2.1.1) is of the form

$$a\cdot e_-h([-+])e_+ \qquad (a\in K) .$$

Since $f_+\neq 0$, $a\neq 0$. Note that the above argument shows also that

$$e_-h([T-])^{-1}h([T+])e_+ = b\cdot e_-h([-+])e_+$$

with some $b\in K$. By the specialization $q\to 1$, b specializes to 1. Hence $b\neq 0$. Thus we may assume that

$$f_+(e_-) = e_-h([T-])^{-1}h([T+])e_+ .$$

By the same argument as above, we can also show that

$$f_-(e_+) = e_+h([T+])^{-1}h([T-])e_-$$

(up to scalar multiple).

2.2 Now let us construct a q-analogue of the Young symmetrizer. Since $f_+(e_-)\in V_+$,

$$f_+f_-f_+(e_-) = cf_+(e_-) \qquad (c = c(q)\in K) ,$$

i.e.,

$$e_-h_-^{-1}h_+e_+h_+^{-1}h_-e_-h_-^{-1}h_+e_+ = ce_-h_-^{-1}h_+e_+ ,$$

where $h_\pm=h([T+])$. Hence

$$(h_-e_-h_-^{-1}\cdot h_+e_+h_+^{-1})^2 = c(h_-e_-h_-^{-1}\cdot h_+e_+h_+^{-1}) . \tag{2.2.1}$$

By the specialization $q\to 1$, $(h_-e_-h_-^{-1})(h_+e_+h_+^{-1})$ specializes to the Young symmetrizer (corresponding to the standard tableau T). Hence $c=c(T)\neq 0$.

2.3. For a standard tableau T which belongs to a partition λ, let

$$E(T) = c(T)^{-1}(h([T-])e_-(\lambda)h([T-])^{-1})(h([T+])e_+(\lambda)h([T+])^{-1}) .$$

Let us consider when

$$E(T_1)E(T_2) = 0$$

for two different standard tableaux.

If T_1 and T_2 belong to different partitions, $E(T_1)E(T_2)=0$. In fact, if χ_q is an irreducible character of H such that $\chi_q(E(T_1)) = m$ ($\neq 0$, $\in \boldsymbol{Z}$), then $\chi_1(E(T_1))_{q\to 1}=m$. By (3.9) below, which will be proved without using the results of (2.3), the specialization $E(T_1)_{q\to 1}$ is well defined and equal to the Young symmetrizer. Hence $m=1$. In the same way we can show that $\chi_q(E(T_2))=0$. Hence $E(T_1)$ and $E(T_2)$ are (primitive) idempotents which belong to different irreducible representation of H. Hence $E(T_1)E(T_2)=0$.

Assume that T_1 and T_2 belong to the same partition λ.

Lemma 2.3.1. *If $T_1 \neq T_2$ and $l([T_1-])\geqq l([T_2-])$, then $E(T_1)E(T_2)=0$.*

Proof. It suffices to prove

(2.3.2) $$e_+(\lambda)h([T_1-])^{-1}h([T_2-])e_-(\lambda) = 0\,.$$

By using the fact

$$l(w) = \operatorname{card}\{(i, j)\,|\,1\leqq i<j\leqq n,\ w(i)>w(j)\} \qquad (w\in W)\,,$$

it is easy to see that

(2.3.3) $$l([T+])+l([T-]) = l([+-])$$

for any standard tableau T. By our assumption,

(2.3.4) $$l([+-])\geqq l([T_1+])+l([T_2-])\,.$$

Let

$$Y = \{x_1x_2\,|\,x_1\leqq [T_1+]^{-1},\ x_2\leqq [T_2-]\}\,.$$

Then $Y\cap W_+[+-]W_-=\phi$ by (2.3.4). Since we can express $h([T_1+])^{-1}h([T_2-])$ as a linear combination

$$\textstyle\sum_{y\in Y} a_y h(y) \qquad (a_y\in K)\,,$$

the argument of 2.1 shows (2.3.2).

3. The purpose of this section is to determine the scalar $c=c(q)$ which appeared in (2.2.1). Our main result of this section is (3.8).

Let us define a linear functional tr on H by

$$tr\, h(w) = \begin{cases} g & (w=1) \\ 0 & (w\neq 1)\,, \end{cases}$$

where

(3.1) $$g = (q-1)(q^2-1)\cdots(q^n-1)/(q-1)^n = \textstyle\sum_{w\in W} q^{l(w)}\,.$$

It is known [4] that

(3.2) $$tr(h(x)h(y)) = \begin{cases} gq^{l(x)} & (xy=1) \\ 0 & (xy\neq 1) \end{cases}$$

and

$$(3.3)\qquad tr(h_1h_2)=tr(h_2h_1)\qquad (h_1, h_2\in H)\,.$$

By specializing q to a prime power r, $H(q)$ specializes to a $\boldsymbol{C}$-algebra $H(r)$ which can be identified with a subalgebra of the group ring $\boldsymbol{C}GL_n(r)$ (see [3]). It is easy to see that the restriction of the character of the regular representation of $\boldsymbol{C}\,GL_n(r)$ to $H(r)$ equals the specialization $tr_{q\to r}$. It is known [3] that every irreducible character of $H(r)$ can be uniquely obtained by restricting an irreducible character of $GL_n(r)$ (which is extended to a linear functional on $\boldsymbol{C}\,GL_n(r)$). Let $\chi(\lambda)$ be the character of $V_\pm(\lambda)$ (see (1.3)) and $\tilde{\chi}(\lambda)$ the irreducible character of $GL_n(r)$ corresponding to $\chi(\lambda)_{q\to r}$ in the above sense. Let $\tilde{d}(\lambda, r)$ be the multiplicity of $\tilde{\chi}(\lambda)$ in the regular representation of $GL_n(r)$, which is also the degree of $\tilde{\chi}(\lambda)$. Then

$$(3.4)\quad \tilde{d}(\lambda, r)=\frac{\prod_{i>j}(r^{\lambda_i+(m-i)}-r^{\lambda_j+(m-j)})}{\prod_i (r-1)(r-1)^2\cdots(r^{\lambda_i+(m-i)}-1)}\times\frac{(r-1)(r^2-1)\cdots(r^n-1)}{r^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}}\,,$$

where $\lambda=\{\lambda_1\geqq\lambda_2\geqq\cdots\lambda_m\geqq 0\}$. (See [8].) Let $d(\lambda, q)$ be the polynomial such that $d(\lambda, r)=\tilde{d}(\lambda, r)$ for any prime power r. The above argument shows that

$$(3.5)\qquad tr=\textstyle\sum_\lambda d(\lambda, q)\chi(\lambda)\,,$$

where λ runs over the set of partitions of n. We have

$$\begin{aligned}
(3.6)\qquad & tr(h_-e_-h_-^{-1}\cdot h_+e_+h_+^{-1}) && \\
&= q^{-l([T-])}\,tr(h_-e_-h([-+])e_+h_+^{-1}) && \text{(by (2.1.5))}\\
&= q^{-l(T-])}\,tr(h([-+])e_+h_+^{-1}h_-e_-) && \text{(by (3.3))}\\
&= q^{-l([T-])-l([T+])}\,tr(h([-+])e_+h([+-])e_-) && \text{(by (2.1.5))}\\
&= q^{-l([-+]))}\textstyle\sum_{x_\pm\in W_\pm}(-q)^{-l(x_-)}\,tr(h([-+])h(x_+)h([+-])h(x_-)) && \\
&= q^{-l([-+]))}\textstyle\sum_{x_\pm\in W_\pm}(-q)^{-l(x_-)}\,tr(h([-+]x_+)h([+-]x_-)) && \\
& && \text{(by (2.1.2) and (2.1.3))}\\
&= q^{-l([-+]))}(q^{l([-+])}g) && \text{(by (3.2) and (2.1.4))}\\
&= g\,. &&
\end{aligned}$$

On the other hand, (2.2.1) implies that $E=c^{-1}h_-e_-h_-^{-1}\cdot h_+e_+h_+^{-1}$ is an idempotent of $V_+(\lambda)h_+^{-1}$. By the specialization $q\to 1$, E specializes to a primitive idempotent. Hence the value of the character $\chi(\lambda)$ at E specializes to 1. But a character value at an idempotent must be an integre. Hence E is primitive. Hence

$$(3.7)\qquad tr(c^{-1}h_-e_-h_-^{-1}h_+e_+h_+^{-1})=d(\lambda, q)\,.$$

By (3.6) and (3.7),

$$c=\frac{g}{d(\lambda, q)} \tag{3.8}$$

By (3.4), c can be also expressed as follows

$$c=\frac{\prod_i (q-1)(q^2-1)\cdots(q^{\lambda_i+(m-i)}-1)}{\prod_{i>j}(q^{\lambda_i+(m-i)}-q^{\lambda_j+(m-j)})}\, q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\cdot(q-1)^{-n}\,. \tag{3.9}$$

Let us restate our results as a theorem.

Theorem 3.10. *Let λ be a partition of n and $\{T_1, \cdots, T_f\}$ the standard tableaux which belong to λ. Assume that*

$$l([T_i-])\geqq l([T_j-])\,, \qquad \text{if} \quad i<j\,.$$

For each standard tableau T, let

$$E(T)=c^{-1}(h([T-])e_-(\lambda)h([T-])^{-1}(h([T+])e_+(\lambda)h([T+])^{-1})\,,$$

where

$$c=\frac{g}{d(\lambda, q)}\,.$$

Then $E(T_1), \cdots, E(T_f)$ are primitive idempotents which belong to $\chi(\lambda)$, and

$$E(T_i)E(T_j)=0\,, \qquad \text{if} \quad i<j\,.$$

(See (1.1) and (1.2) for $e_\pm$, section 2.1 for $[T\pm]$, (3.1) for g, (3.4) and the subsequent lines for $d(\lambda, q)$.)

4. Orthogonalization of idempotents

The purpose of this section is to give a procedure to construct an orthogonal family of idempotents from a given family of idempotents. By applying this procedure to the famliy of idempotents $\{E(T)\}$ which was obtained in the preceding section, we get a complete family of mutually orthogonal, primitive idempotents of H.

4.1. Let X be a partially ordered set of cardinality n. Let $I=\{1, 2, \cdots, n\}$ and A be the set of bijections $a: I\to X$ such that a^{-1} is order preserving. If a is an element of A and if $a(i)$ and $a(i+1)$ are not comparable, we define a new element of A by

$$a'(j)=\begin{cases} a(j) & (j\neq i, i+1) \\ a(i+1) & (j=i) \\ a(i) & (j=i+1)\,. \end{cases}$$

If $b(\in A)$ can be obtained from a by applying this operation several times, we say that b is equivalent to a and write $a\sim b$. This relation is an equivalence relation.

Lemma 4.2. *Any two elements of A are equivalent to each other.*

Proof. Let $a, b\in A$ such that

$$\begin{aligned} a(k) &= b(k) \qquad (k<i) \\ a(i) &\neq b(i) . \end{aligned}$$

Let $a(i)=a_0$ and $b^{-1}(a_0)=j$. Then $j>i$ and $a_0=b(j)$ is not comparable with any one of $\{b(i), b(i+1), \cdots, b(j-1)\}$. In fact, if $b(j)$ is comparable with $b(k)$ $(i\leqq k<j)$, then $a_0=b(j)>b(k)$. But $a^{-1}(b(j))=i$ and $b(k)\notin \{b(1), \cdots, b(i-1), a_0\} = \{a(1), \cdots, a(i)\}$, hence $a^{-1}(b(k))>i$. Since $k<j$, this is a contradiction.

Now we can define an element c of A by

$$c(k) = \begin{cases} b(k) & (1\leqq k<i) \\ b(j) & (k=i) \\ b(k-1) & (i<k\leqq j) \\ b(k) & (j<k\leqq n) . \end{cases}$$

Then $b\sim c$ and $a(k)=c(k)$ $(k<i+1)$. Thus, by an induction on i, we can show that $a\sim b$.

4.3. Let X be a set of idempotents in a ring with 1. Let us define a relation $\leqq$ in X by

$$(\sharp) \qquad \begin{aligned} &e\leqq e' \text{ if there exists a sequence} \\ &e = e_0, e_1, \cdots, e_n = e' \text{ of elements of } X \\ &\text{such that } e_ie_{i+1}\neq 0 \qquad (0\leqq i<n) . \end{aligned}$$

Assume that

$$\text{(4.3.1)} \qquad \text{the relation} \leqslant \text{defined by } (\sharp) \text{ is a partial order} .$$

We can define A for this partially ordered set.

REMARK. If from the beginning, X is totally ordered and satisfies

$$\text{(4.3.2)} \qquad ee' = 0 \quad \text{if} \quad e>e' ,$$

then (4.3.1) is automatically satisfied. For example the set $\{E(T_1), \cdots, E(T_f)\}$ satisfies (4.3.2) with any total order such that $l([T-])\geqslant l([T'-])$ whenever $E(T)\geqslant E(T')$.

Lemma 4.4. *Let X be a set of idempotents. Let $x \in X$, $a \in A$, $i = a^{-1}(x)$ and $E(a, x) = (1-a(1)) \cdots (1-a(i-1))a(i)$. Then $\{E(a, x)\}_{x \in X}$ are mutually orthogonal idempotents, and each element $E(a, x)$ is independent of $a \in A$.*

Proof. If $i > j$, then $a(i)a(j) = 0$. Hence

$$\begin{aligned} &a(i)(1-a(1)) \cdots (1-a(i-1))a(i) = a(i)\,, \\ &a(i)(1-a(1)) \cdots (1-a(i)) = 0\,, \\ &a(i)(1-a(1)) \cdots (1-a(j-1))a(j) = 0 \qquad (i > j)\,. \end{aligned}$$

From these equalities, we can conclude that $E(a, a(i))$ are mutually orthogonal idempotents.

To show that every $E(a, a(i))$ is independent of a, it is enough to prove that

$$E(a, a(j)) = E(a', a(j)) \tag{4.4.1}$$

if a' is obtained from a by the transposition $(i, i+1)$. There is nothing to prove for $j < i$. For $j = i$,

$$E(a, a(i)) = (1-a(1)) \cdots (1-a(i-1))a(i)$$

and

$$E(a', a(i)) = (1-a'(1)) \cdots (1-a'(i))a'(i+1)\,,$$

since $a'(i+1) = a(i)$. Since

$$a'(i)a'(i+1) = a(i+1)a(i) = 0\,,$$

we have $E(a', a(i)) = E(a, a(i))$. For $j = i+1$,

$$E(a, a(i+1)) = (1-a(1)) \cdots (1-a(i))a(i+1)$$

and

$$E(a', a(i+1)) = (1-a'(1)) \cdots (1-a'(i-1))a'(i)\,,$$

since $a'(i) = a(i+1)$. Since

$$a(i)a(i+1) = a'(i+1)a'(i) = 0\,,$$

we have $E(a', a(i)) = E(a, a(i))$. Since

$$\begin{aligned} (1-a'(i))(1-a'(i+1)) &= (1-a(i+1))(1-a(i)) \\ &= (1-a(i))(1-a(i+1))\,, \end{aligned}$$

(4.4.1) holds for $j > i+1$.

By the above lemma, we can define a set of mutually orthogonal idempotents

$$X^0 = \{x^0 \mid x \in X\}\,,$$

where, $x^0=E(a,x)$ for some $a\in A$.

Theorem 4.5. *The set*

$$\{E(T)^0 \mid T \text{ standard tableau}\}$$

is a complete family of mutually orthogonal primitive idempotents in H.

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