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ON CHARACTERISTIC CLASSES OF KÄHLER FOLIATIONS

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0. Introduction

The purpose of this note is to study Kahler foliations, which are defined by requiring the transition functions to be holomorphic isometries of a Kahler manifold (see Definition 1.1), by adopting the method of [6] [7] [10]. In some sense Kahler foliations are the holomorphic analogue of Riemannian foliations and characteristic classes of the latter have been profoundly investigated by Lazarov and Pasternack (cf.[9], also see [6]). However from the view point of characteristic classes, the situations are completely different. Namely the vanishing phenomenon of the Pontrjagin classes of the normal bundles in the Riemannian case is much stronger than that in the smooth case (cf. strong vanishing theorem of Pasternack [12] and the Bott’s vanishing theorem [1]). By contrast, we do not have any strong vanishing phenomenon in the Kahler foliations. This fact reflects in the secondary characteristic classes. For example, all the secondary classes of smooth foliations are zero on Riemannian foliations, but some of the secondary classes of holomorphic foliations may be non-zero on Kahler foliations. A new ingredient of our context is the Kahler form which is a closed 2-form defined for any Kahler foliation.

In §1 we define Kahler foliations and construct characteristic classes of them and in §2 we compute the cohomology of certain truncated Weil algebra. In §§3 and 4, we study the relationships of our characteristic classes with those of Riemannian and holomorphic foliations. Finally in §5 we consider deformations of Kahler foliations.

1 Construction of the characteristic classes

In this section we define the notion of Kahler foliations and construct characteristic classes of them.

DEFINITION 1.1. A codimension $n$ Kahler foliation $F$ on a smooth manifold $M$ is a maximal family of submersions $f_{a}: U_{a} \rightarrow (C^{n}, g_{a})$, where $U_{a}$ is an

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open set in $M$ and $g_\alpha$ is a Kahler metric on $\mathbb{C}^n$, satisfying the condition: for every $x \in U_\alpha \cap U_\beta$ there exists a local holomorphic isometry $\gamma_{\alpha\beta}$ such that $f_\alpha = \gamma_{\alpha\beta} \circ f_\beta$ near $x$.

This is a holomorphic version of the notion of Riemannian foliations. Since $\gamma_{\alpha\beta}$ is a holomorphic isometry, patching together the pull backs of unitary frame bundle of $(\mathbb{C}^n, g_\alpha)$ by the map $f_\alpha$, we obtain a principal $U(n)$-bundle $\pi: U(F) \to M$. We call it the unitary frame bundle of the foliation $F$. Let $\theta_\alpha^\beta$ and $\vartheta_\alpha^\beta$ be the canonical form and the unique torsionfree Hermitian connection form of $(\mathbb{C}^n, g_\alpha)$. Since holomorphic isometrics preserve these forms, we can define global 1-forms $\theta_0$ and $\theta_1$ on $U(F)$ such that $\theta_0|_{\pi^{-1}(U_\alpha)}$, $\theta_1|_{\pi^{-1}(U_\alpha)}$ are the pull backs of $\theta_\alpha^\beta$, $\vartheta_\alpha^\beta$. Let $E_C(n)$ be the group generated by parallel transformations and unitary transformations on $\mathbb{C}^n$ (which is a semi-direct product of $\mathbb{C}^n$ and $U(n)$). The pair $(\theta_0, \theta_1)$ is an $e_c(n)$-valued 1-form, where $e_c(n)$ is the Lie algebra of $E_C(n)$. $(\theta_0, \theta_1)$ defines a d.g.a. map

$$\phi: W(e_c(n)) \to \Omega^*(U(F))$$

where $W(e_c(n))$ is the Weil algebra of $e_c(n)$ and $\Omega^*(U(F))$ is the de Rham complex of $U(F)$. Let $\omega^i, \omega_j^i, \Omega^i, \Omega_j^i \in W(e_c(n))$ be the universal connection and curvature forms corresponding to the usual basis (over $\mathfrak{R}$) of $e_c(n) = \mathbb{C}^n + \mathfrak{u}(n) \subset \mathbb{R}^{2n} + \mathfrak{so}(2n)$. If we denote $\theta^i, \theta_j^i, \Theta^i, \Theta_j^i$ for the $\phi$-images of $\omega^i, \omega_j^i, \Omega^i, \Omega_j^i$ respectively, then they satisfy the following equations (cf. [8])

$$(i) \quad \Theta^i = d\theta_i + \theta^i_{\jmath} \wedge \theta_j^i = 0 \quad \text{(torsionfree-ness)}$$
$$(ii) \quad d\theta_j^i = -\theta_{ij} \wedge \Theta_j^i + \Theta_j^i$$
$$(iii) \quad \Theta_j^i \wedge \theta^j = 0 \quad \text{(the first Bianchi's identity)}$$

Therefore Ker $\phi$ contains an ideal $I$ of $W(e_c(n))$ generated by the following elements.

$$(i) \quad \Omega^i$$
$$(ii) \quad \text{elements whose "length" } l \text{ is greater than } n,$$

where $l$ is defined by the conditions:

$$l(\omega_j^i) = l(\Omega^i) = 0, \ l(\omega^i) = 1 \text{ and } l(\Omega_j^i) = 2.$$  

$$(iii) \quad \Omega_j^i \wedge \omega^j.$$  

If we denote $W(e_c(n)) = W(e_c(n))/I$, then $\phi$ induces a d.g.a. map

$$\phi: W(e_c(n)) \to \Omega^*(U(F)).$$

Now suppose that the normal bundle of $F$ is trivialized by a cross section $s: M \to U(F)$, then we obtain

$$H^*(\tilde{W}(e_c(n))) \to H^*_B(U(F)) \xrightarrow{s^*} H^*_B(M).$$
We denote $BKT_n$ for the classifying space for codimension $n$ Kahlerian Haeffliger structures with trivial normal bundles which is defined similarly as the ordinary Haeffliger structures. Since the above construction is functorial, we obtain a homomorphism

$$\phi: H^*(W(e_c(n))) \rightarrow H^*(BKT_n; R).$$

Considering $U(n)$-basic elements of $W(e_c(n))$ we can also define a d.g.a. map $\phi: W(e_c(n))_{U(n)} \rightarrow \Omega^*(M)$ and this yields a homomorphism

$$\phi: H^*(W(e_c(n))_{U(n)}) \rightarrow H^*(BKT_n; R),$$

where $BKT_n$ denotes the classifying space for codimension $n$ Kahlerian Haeffliger structures.

The above is our construction of characteristic classes of Kahler foliations.

2. Cohomology of $W(e_c(n))$

Here we compute the cohomology of $W(e_c(n))$. $W(e_c(n))$ has a decreasing filtration $F^p$ defined by $F^p = \{x \in W(e_c(n)); l(x) \geq p\}$ where $l$ is the function on $W(e_c(n))$ induced by the length on $W(e_c(n))$. Let $\{E^p_q, d_r\}$ be the spectral sequence associated with this filtration. If we define $M_p = \{x \in W(e_c(n)); l(x) = p$ and $x$ contains no $\omega\}$, then $u(n)$ acts on $M_p$ by the Lie derivative. Thus $M_p$ is a $u(n)$-module. Let $C^q(u(n); M_p)$ be the set of $q$-cochains on $u(n)$ with coefficients in $M_p$. Then

$$E^p_q = \Lambda^q(u(n)) \otimes M_p \simeq C^q(u(n); M_p)$$

and this identification is compatible with the differentials. Thus

$$E^p_q \simeq H^q(u(n); M_p) .$$

Let $M^u_p$ be the $u(n)$-invariant subspace of $M_p$.

**Lemma 2.1.** $H^*(u(n); M_p) \simeq H^*(u(n)) \otimes M^u_p$.

*Proof.* Let $Z$ be the center of $u(n)$. Then we have $u(n) = \mathfrak{u}(n) \oplus Z$. $Z$-action on $M_p$ is obtained by differentiating the action of $S^1 \subset u(n)$ on $M_p$. Since $S^1$ is compact, $M_p$ breaks up as a sum of $S^1$-invariant (= $L_z$-invariant, $I$ is the generator of $Z$) subspaces $M_p = M^Z_p \oplus W$. Then the assertion follows from an argument in Corollary IV 2.2 [11]. q.e.d.

Now let $N_p = \{x \in W(e_c(n)); l(x) = p, x$ contains no $\omega\}$. Then we show

**Lemma 2.2.** $N^u_p / I \cap N^u_p \simeq M^u_p$.

*Proof.* We have a short exact sequence of $u(n)$-modules:
0 \to I \to W(e_c(n)) \to W(e_c(n)) \to 0.

This induces a long exact sequence

\[ 0 \to H^0(u(n); I) \to H^0(u(n); W(e_c(n))) \to H^0(u(n); W(e_c(n))) \]
\[ \to H^1(u(n); I) \to H^1(u(n); W(e_c(n))) \to \cdots, \]

where we have \( H^0(u(n); I) = I^{\uparrow(n)} \), \( H^0(u(n); W(e_c(n))) = N^{\uparrow(n)}_p (= \bigoplus N^{\uparrow(n)}_p) \) and \( H^0(u(n); W(e_c(n))) = M^{\uparrow(n)}_p (= \bigoplus M^{\uparrow(n)}_p) \). Now by the argument of Lemma 2.1,

\[ H^1(u(n); I) \approx H^1(u(n); \bigoplus I^{\uparrow(n)}) \]
\[ \text{and} \]
\[ H^1(u(n); W(e_c(n))) \approx H^1(u(n)) \otimes N^{\uparrow(n)}. \]

Therefore the map \( H^1(u(n); I) \to H^1(u(n); W(e_c(n))) \) is injective, which implies that the homomorphism \( N^{\uparrow(n)}_p \to M^{\uparrow(n)}_p \) is surjective. This completes the proof.

By virtue of Lemma 2.2, it is enough to determine \( N^{\uparrow(n)}_p \cap N^{\uparrow(n)}_p \) instead of \( M^{\uparrow(n)}_p \). To simplify the computation, we consider the complexification of \( N_p \). If we put \( \psi = \omega + \sqrt{-1} \omega \sigma^i, \psi_j = \Omega_j + \sqrt{-1} \Omega_j^i \) \((i, j = 1, \ldots, n)\), then \( \bigoplus (N \otimes \mathbb{C}) \) is multiplicatively generated by \( \psi, \overline{\psi}, \psi_j \) where \( \overline{\psi} = \omega - \sqrt{-1} \omega \sigma^i \) \((\overline{\psi}_j = \text{not necessary because } \overline{\psi}_j = -\psi_j)\). The action of an element \( A = (a^i_j) \in U(n) \) is given as follows;

\[ A \cdot \psi = \sum a^i_j \psi, \quad A \cdot \overline{\psi} = \sum a^i_j \overline{\psi}, \quad \text{and} \]
\[ A \cdot \psi_j = \sum a^i_j \psi_j. \]

Let \( \langle \psi \rangle, \langle \overline{\psi} \rangle, \langle \psi_j \rangle \) be the complex vector spaces with bases \( \{\psi^i, \ldots, \psi^n\} \), \( \{\overline{\psi}^i, \ldots, \overline{\psi}^n\} \), \( \{\psi_j^i, \ldots, \psi_j^n\} \) respectively and let \( \langle \psi \rangle^m, \langle \overline{\psi} \rangle^m, \langle \psi_j \rangle^m \) be the tensor products of \( m \)-copies of them. \( U(n) \) acts on these spaces by the diagonal action. Now we define an action of \( S_p \times S_q \times S_r \) \((p, q, r) \) on \( \langle \psi \rangle^p \otimes \langle \overline{\psi} \rangle^q \otimes \langle \psi_j \rangle^r \) by

\[ (\sigma_1, \sigma_2, \sigma_3) \cdot \psi \overline{\psi} \psi_j^r = \psi^{\sigma_1(i)} \overline{\psi}^{\sigma_2(j)} \psi_j^{\sigma_3(k)} \]

where \( (\sigma_1, \sigma_2, \sigma_3) \in S_p \times S_q \times S_r \) and \( \psi \overline{\psi} \psi_j^r \) denote \( \psi^{i_1} \cdots \psi^{i_p}, \overline{\psi}^{j_1} \cdots \overline{\psi}^{j_q}, \psi_j^{r_1} \cdots \psi_j^{r_r} \) respectively, etc.. If we use the same letter \( S_{p,q,r} \) for a linear endomorphism on \( \langle \psi \rangle^p \otimes \langle \overline{\psi} \rangle^q \otimes \langle \psi_j \rangle^r \) given by

\[ S_{p,q,r} \langle \psi \overline{\psi} \psi_j^r \rangle = \sum sgn(\sigma_p) sgn(\sigma_q) sgn(\sigma_r) \cdot \psi^{\sigma_1(i)} \overline{\psi}^{\sigma_2(j)} \psi_j^{\sigma_3(k)} \]

where \( (\sigma_p, \sigma_q, \sigma_r) \) ranges over all elements of \( S_{p,q,r} \), then the \( S_{p,q,r} \)-invariant subspaces \((= \text{Im } S_{p,q,r})\) is equal to \( \Lambda^p \langle \psi \rangle \otimes \Lambda^q \langle \overline{\psi} \rangle \otimes S^r \langle \psi_j \rangle \subset N_{p+q+2 \cdot} \otimes \mathbb{C} \). We denote this subspace by \( N_{p,q,r} \). Since \( S_{p,q,r} \) is \( U(n) \)-equivariant,

\[ N_{p,q,r}^{\uparrow(n)} \subset \langle \psi \rangle^p \otimes \langle \overline{\psi} \rangle^q \otimes \langle \psi_j \rangle^r \]
We can consider $\langle \phi^r \otimes \phi^r \otimes \phi^r \rangle \approx \langle \phi^{p+r} \otimes \phi^{q+r} \rangle$ by the map $\phi^r \phi^r \phi^r \rightarrow \phi^{p+r} \phi^{q+r}$. Since this map is $U(n)$-equivariant,

$$
\langle \phi^p \otimes \phi^q \otimes \phi^r \rangle^{U(n)} = \langle \phi^{p+r} \otimes \phi^{q+r} \rangle^{U(n)}.
$$

**Lemma 2.3.** The vector space $\langle \psi^\tau \otimes \psi^\tau \rangle^{U(m)}$ is non trivial only for $l=m$ and in that case it has a basis of the following tensors: $\sum_i \phi^{\tau(i)}$ where $\sigma \in S_l$.

**Proof.** Let $f = \sum_{i,j} \alpha^{i,j} \phi^{i,j} \in \langle \psi^\tau \otimes \psi^\tau \rangle^{U(m)}$ where $\alpha^{i,j}$ are coefficients in $C$. $f$ is $u(n)$-invariant if and only if $\alpha^{i,j} = a_k a_L \alpha^{K,L}$, for any $(a_i) \in U(n)$. Taking $(a_i)$ to be the diagonal matrix with entries $a_i = 1$ for $i \neq k$, $a_k = \lambda (|\lambda| = 1)$, $a_i = 0$ for $i \neq j$, we can show that the entries of $I$ and $J$ coincide. Letting $I_0 = (1, \cdots, l)$ we prove the following

$$ (2.4) \quad \alpha^{i,j} = \sum_{s,t} \alpha^{t,s(l_0)} $$

by the decreasing induction on $r =$ the number of distinct elements in $I$. For $r = m$ (2.4) clearly holds. Suppose (2.4) is true for $r = s + 1$. We consider the special case when $I = J = (1, \cdots, 1, 2, \cdots, 2, \cdots, s, \cdots, s)$. The other cases can be treated similarly. We can regard $f$ as an $R$-multilinear function: $(C^n)^{2l} \rightarrow C$ by considering $\phi^l$ (resp. $\bar{\phi}^l$) to be the mapping: $C^n \rightarrow C$ given by

$$ \phi^l(z_1, \cdots, z_n) = z_1 (\text{resp.} \bar{\phi}^l(z_1, \cdots, z_n) = \bar{z}_1). $$

We denote $e_i = (0, \cdots, 1, \cdots, 0)$. Since $\alpha^{K,L} = 0$ unless $K$ coincides with $L$ as sets,

$$ f(e_{s+1}^{a_1}, e_s^{a_2}, \cdots, e_{s+1}^{a_{s-1}}, (e_s + e_{s+1})^{a_s-1}, e_s, e_s^{a_1}, \cdots, e_{s-1}^{a_{s-1}}, e_{s+1}^{a_s}) = 0 $$

where $e_i^{a_i}$ denotes $e_{a_1} \cdots e_{a_{s-1}} e_i$ and the same rule for other letters. Since the transformation $A$ of $C^n$ given by

$$ A \cdot e_s = 1 + (e_s - e_{s+1}), \quad A \cdot e_{s+1} = 1 - (e_s + e_{s+1}) $$

$$ A \cdot e_i = e_i \quad (t \neq s) $$

is a unitary transformation, we have

$$ 1/2 f(e_{s+1}^{a_1}, \cdots, e_{s+1}^{a_{s-1}}, e_s - e_{s-1}, e_s^{a_1}, \cdots, e_{s-1}^{a_{s-1}}, (e_s + e_{s+1})^{a_s}) = 0. $$

Calculation shows

$$ f(e_{s+1}^{a_1}, \cdots, e_{s+1}^{a_{s-1}}, e_s - e_{s-1}, e_s^{a_1}, \cdots, e_{s-1}^{a_{s-1}}, (e_s + e_{s+1})^{a_s}) $$

$$ = \alpha^{i,j} \sum_{\lambda} \alpha^{\lambda^{s+1}, \cdots, s_{s+1}} \lambda^{s+1, \cdots, s_{s+1}} \cdots \lambda^{s+1, \cdots, s_{s+1}} $$

where $1^{a_1}$ denotes $1, \cdots, 1$ etc.. Therefore by the induction assumption (2.4) holds for $r = s$. This completes the proof. q.e.d.
Now we define elements $s_k$ and $\Phi$ of $M^{\mu(s)}$ by

$$s_k = \text{Trace} (\psi_j^k) \quad \text{for } k \text{ even},$$

$$= \sqrt{-1} \text{Trace} (\psi_j^k) \quad \text{for } k \text{ odd},$$

$$\Phi = \sqrt{-1} \sum \psi_j^1 \psi^1 = \sum \omega^i \omega^{*i}.$$

These forms are real because $\psi_j^1$ is skew-Hermitian.

**Proposition 2.4.** $M^{\mu(s)} = \mathbb{R}[s_1, \ldots, s_n, \Phi]/\{\text{degree } > 2n\}.$

**Proof.** By Lemma 2.3 a basis for $(\langle \psi \rangle^s \otimes \langle \psi \rangle^s \otimes \langle \psi \rangle^s)^{\mu(s)}$ is given by the tensors

$$\omega(\sigma) = \sum_{j} \psi_j^{s(\sigma(I,J))} \psi_j^{(1)} \psi_j^{(2)}$$

where $(\sigma(I,J), \sigma(I,J) \otimes \psi_j^1 \otimes \psi_j^s \otimes \psi_j^s)^{\mu(s)}$ is spanned by the tensors $\omega(\sigma): \sigma \in S_{p+r}$. If we denote $\omega(\sigma)$ for the image of the projection $N^{\mu(s)}_{p,s,r} \rightarrow N^{\mu(s)}_{p,s,r} \cap (I \otimes \mathbb{C})$, then $\omega(\sigma) = 0$ for $\sigma$ such that $\sigma(1, \ldots, p) \neq (1, \ldots, p)$ by the Bianchi's identity. Therefore

$$(N^{\mu(s)} / N^{\mu(s)} \cap I) \otimes \mathbb{C} = \bigoplus_{p,s,r} N^{\mu(s)}_{p,s,r} / N^{\mu(s)}_{p,s,r} \cap (I \otimes \mathbb{C})$$

Taking the real part of this space we obtain our proposition. q.e.d.

Now we define a d.g.a. $KW_n$ as follows. Let $T_s \in W(e_c(n))$ be the Chern-Simons' transgression form of $s_k$ (cf. [3]) and $u_s = \pi(T_s)$, where $\pi: W(e_c(n)) \rightarrow W(e_c(n))$ is the projection. Then clearly $du_s = s_k$. We define $KW_n$ to be the subalgebra generated by $u_s, s_k, \Phi$. It is easy to see that $KW_n$ is isomorphic to

$$E(u_s, \ldots, u_t) \otimes \hat{R}[s_1, \ldots, s_n, \Phi]$$

where $E$ denotes the real exterior algebra and $\hat{R}[\ ]$ is the real polynomial algebra truncated by the elements of degree $\geq 2n$. Now recall that, in our spectral sequence computing the cohomology of $W(e_c(n))$, $E^{p,q}_r = H^q(u(n)) \otimes M^{\mu(s)}_p$. The above results show that the inclusion of the subalgebra $KW_n$ in $W(e_c(n))$ induces an isomorphism on the $E_1$-term. Therefore, by the spectral sequence comparison theorem, we obtain

**Theorem 2.5.** $H^*(W(e_c(n))) \approx H^*(KW_n).$

Let $I = (i_1, \ldots, i_s)$ and $J = (j_1, \ldots, j_t)$ be $s$ and $t$-tuples of positive integers with $i_1 < \cdots < i_s$ and $j_1 \leq \cdots \leq j_t$. We denote $u_{i_1} \Phi_k$ for $u_{i_1} \Phi_k \in KW_n$. Note that if $|J| + k > n$, then $u_{i_1} \Phi_k = 0$ where $|J| = j_1 + \cdots + j_t$. Now the technique of Vey in [5] shows the following.
Proposition 2.6. A basis for $H^*(KW_n)$ is given by the classes of the elements $u_i s_f \Phi^k$ with

(i) $i_1 + |J| + k > n$
(ii) $i_1 \leq j_1$, where we understand $i_1 = \infty$ if $I = \phi$ and similarly for $j_1$.

Since $W(e_c(n))^{\mu(n)} = M^{\mu(n)}$, we can determine $H^*(W(e_c(n))_{U(n)})$ as follows.

Proposition 2.7. The classes of the elements $s_f \Phi^k$ with $|J| + k \leq n$ form a basis of $H^*(W(e_c(n))_{U(n)})$.

Now the class $\Phi$ has the following geometric meaning. On each Kahler manifold there is defined a 2-form called the Kahlerian form and holomorphic isometries between Kahler manifolds preserve these forms. Therefore if $F$ is a codimension $n$ Kahler foliation on a smooth manifold $M$ defined by submersions $f_a: U_a \rightarrow (C^*, g_a)$ (see Definition 1.1), then the local forms $f_a^* (\text{Kahler form of } g_a)$ on $U_a$ define a global 2-form $\Phi(F)$ on $M$ which is closed. We call $\Phi(F)$ the Kahler form of the foliation $F$. On the other hand, from the definition of our characteristic classes, we have a closed 2-form $\phi(\Phi)$ on $M$. We have

Proposition 2.8. $\phi(\Phi) = 1/2 \Phi(F)$.

Proof. Since $\Phi = \sum_0^\infty \omega^* \omega^*$, we have $\phi(\Phi) = \sum_0^\infty \theta^* \theta^*$. But it is easy to see that this form is the lift of $1/2 \Phi(F)$ to $U(F)$. q.e.d.

3. Relation with Riemannian case

Let $F$ be a codimension $n$ Riemannian foliation on a smooth manifold $M$ and let $O(F)$ be the orthonormal frame bundle of $F$. Let $E(n)$ be the group of Euclidean motions on $R^*$, $e(n)$ the Lie algebra of $E(n)$ and $\overline{W}(e(n))$ the Weil algebra of $e(n)$. Then in [6], [7], [10] a characteristic homomorphism

$$\phi: H^*(\overline{W}(e(n))) \rightarrow H^*(BRT_n; \mathcal{R})$$

was constructed, where $BRT_n$ is the classifying space for codimension $n$ Riemannian Haefliger structures with trivial normal bundles and $\overline{W}(e(n))$ is the quotient algebra of $W(e(n))$ by some ideal. Let $f_{2n} \in I(\tilde{\omega}(n))$ be defined by $f_{2n}(X) = \text{Trace}(X^{2n})$ for $X \in \tilde{\omega}(n)$ and for even $n$ let $\chi \in I(\tilde{\omega}(n))$ be the Euler form. We can consider $f_{2n}, \chi$ to be elements of $\overline{W}(e(n))$. Let $T_{f_{2n}}, TX$ be the transgression forms of $f_{2n}, \chi$ respectively. If we set $c_{2n} = \pi(f_{2n}), h_{2n} = \pi(T_{f_{2n}}), \chi = \pi(\chi)$ and $h_2 = \pi(TX)$ where $\pi: W(e(n)) \rightarrow \overline{W}(e(n))$ is the projection, then the subcomplex $RW_n$ of $\overline{W}(e(n))$ generated by $h_{2n}, c_{2n}$ and if $n$ is even also by $h_2, \chi$ is a finite complex expressed as

$$RW_n = E(h_2, h_4, \ldots, h_{n-2}) \otimes \mathcal{R}[c_2, c_4, \ldots, c_{n-2}] \quad n \text{ odd},$$
$$= E(h_2, h_4, \ldots, h_{n-2}, h_2) \oplus \mathcal{R}[c_2, c_4, \ldots, c_{n-2}, \chi] \quad n \text{ even}.$$
Furthermore let $r_p \in W(e(n))$ be the "$p$-th scalar curvature" defined in [10]. If we denote

$$E_n = E(h_2, \ldots, h_{n-1}) \quad n \text{ odd},$$

$$= E(h_2, \ldots, h_{n-2}, h_n) \quad n \text{ even},$$

then all the forms of $r_p E_n$ are closed and therefore $RW_n \oplus \sum_{p \text{ even}} r_p E_n$ is a subcomplex of $W(e(n))$. The inclusion induces an isomorphism on cohomology. Namely

$$H^*(W(e(n))) \cong H^*(RW_n) \oplus \sum_{p \text{ even}} r_p E_n$$

(see Theorem 3.1 of [10]).

Now we have the forgetful map

$$BK^r_n \to BR^r_{2n} \text{ and } BK^s_n \to BR^s_{2n}.$$ 

Let $i: e_c(n) \to e(2n)$ be the natural inclusion.

**Proposition 3.1.** The following diagrams are commutative.

\[
\begin{array}{ccc}
H^*(W(e(2n))) & \longrightarrow & H^*(BR^r_{2n}; R) \\
\downarrow & & \downarrow \\
H^*(W(e_c(n))) & \longrightarrow & H^*(BK^r_s; R) \\
\downarrow & & \downarrow \\
H^*(W(e(2n))_{0(2n)}) & \longrightarrow & H^*(BR^r_{2n}; R) \\
\downarrow & & \downarrow \\
H^*(W(e_c(n))_{U(n)}) & \longrightarrow & H^*(BK^r_s; R).
\end{array}
\]

The homomorphisms $i^*: H^*(W(e(2n))) \to H^*(W(e_c(n)))$, $i^*: H^*(W(e(2n))_{0(2n)}) \to H^*(W(e_c(n))_{U(n)})$ in terms of $h_i$, $c_j$, $x$, $r_p$, $u_i$, $s_j$, etc. can be completely determined. For example, as is well known, the image under $i^*$ of monomials on $c_j$ (=the Pontrjagin classes) can be uniquely described as polynomials on $s_j$ (=the Chern classes) and of course $i^*(\chi) =$ the $n$-th Chern class. The formula for the image of $h_i$ can be easily deduced from the definitions. We omit the detailed description of these formulas. Here we only mention the formula for the class $r_p$.

**Proposition 3.2.**

$$i^*[r_p] = (-1)^{(n(n-1)+p)/2}2^{n^2}(2n-p)!/[n-p]/2)\left[\Phi^{n-p/2} \sum a_i s_i\right]$$

where $a_i = (-1)^{\frac{i}{2}}(i+1)/2$ $n!/(n-i_1)\cdots(n-i_1-\cdots-i_{-1})$ for $I = (i_1, \ldots, i_l)$.

The proof of this proposition can be given by calculations using the Bianchi's identity and is left to the readers.
This proposition shows that the $p$-th scalar curvature $r_p$ for a Kähler foliation can be expressed in terms of the Kähler form and the Chern forms.

4. Relation with holomorphic case

A Kähler foliation can be regarded as a holomorphic foliation by forgetting the Kähler structures. We recall the construction of the characteristic classes of holomorphic foliations given by Bott in [2].

Let $F$ be a holomorphic foliation on a smooth manifold $M$ and let $J^2_c(F)$ be the bundle of holomorphic 2-jets of $F$. We have a d.g.a. map

$$\phi: W(gl(n; C)) \to \Omega^*(J^2_c(F)) \otimes C$$

defined by $\phi(\omega^j) = \theta^j, \phi(\Omega^j) = d\theta^j + \theta^k \wedge \theta^j$ where $\omega^j, \Omega^j$ are the universal connection and curvature forms of $gl(n; C)$ in terms of the natural basis and $\theta^j$ is the second order canonical forms on $J^2_c(F)$. $\phi$ has a kernel $I$ generated by monomials on $\Omega^j$ with degree $>2n$. Therefore if we set $\tilde{W}(gl(n; C)) = W(gl(n; C))/I$ and assume that the normal bundle of $F$ is trivialized by a cross section $s: M \to J^2_c(F)$, then we obtain a homomorphism

$$\phi^*: H^*(\tilde{W}(gl(n; C))) \to H^*(J^2_c(F); C) \to H^*(M; C).$$

Since this construction is functorial, we have

$$\phi: H^*(W(gl(n; C))) \to H^*(BT_n C; C).$$

Let $s_i \in I(gl(n; C))$ be given by $s_i(X) = \text{Trace}(\sqrt{-1} X)'$ for $X \in gl(n; C)$ and let $u_i = Ts_i$; the transgression form of $s_i$. $u_i$ and $s_i$ can be considered as elements of $W(gl(n; C))$ and we use the the same letters for their images in $\tilde{W}(gl(n; C))$. Now let $W^c_s$ be the subalgebra of $W(gl(n; C))$ generated by the elements $s_i, u_i$. Then we may write $W^c_s = E(u_0, \cdots, u_n) \otimes C[s_0, \cdots, s_n]$ as usual (see [2]) and the inclusion $i: W^c_s \to W(gl(n; C))$ induces an isomorphism on cohomology.

**Theorem 4.1.** Let $F$ be a codimension $n$ Kähler foliation on a smooth manifold $M$ with a trivialized normal bundle. Then the class $\phi(u_is_i)$ is a real class and coincides with $\phi(u_is_i)$.

Proof. This follows from the definitions of the characteristic classes of Kähler and holomorphic foliations. The point here is the fact that the $s_i$-form of a complex vector bundle with a Hermitian connection is a real form.

q.e.d.

**Remark 4.2.** Bott[2] has also defined characteristic classes of holomorphic foliations whose normal bundles are not necessarily trivial by comparing Bott and Hermitian connections. For a Kähler foliation these classes are all
zero because the unique torsionfree Hermitian connection is also a Bott connection.

5. Continuous variation

In this section we study continuous variations of our characteristic classes.

Definition 5.1 An element $\alpha$ of $H^*(KW_n)$ is called rigid if for any one parameter family $F_t$ of codimension $n$ Kahler foliations on a smooth manifold $M$, the classes $\phi(\alpha)(F_t)$ is constant with respect to $t$, namely $\frac{d}{dt}(\phi(\alpha)(F_t))=0$ holds.

By the same argument as in Heitsch [5] we obtain

Proposition 5.2. The class $[u_j,s]_i$ is rigid if $i_1+|J|>n+1$.

We conjecture that these classes are the only rigid classes. Thus the classes of $u_js_j\Phi^k$ would be non rigid if $k>0$ or $k=0$, $i_1+|J|=n+1$. We cannot prove this conjecture at the moment. In the following we prove partial solution to it.

Let $(M,g)$ be a Kahler manifold of dimension $n$ and $\pi: U(M) \to M$ the unitary frame bundle of $M$. We define a smooth family of codimension $n$ Kahler foliations $F(M, t)$ on $U(M)$ as follows. Let $(M, t^2g)$ be the Kahler manifold obtained from $M$ by the scale change $g \to t^2g$ ($t>0$). Then $F(M, t)$ is a foliation on $U(M)$ defined by pulling back the Kahler structure of $(M, t^2g)$ by the projection $\pi$.

The unitary frame bundle of this foliation $U(F(M, t))$ has a cross section

$$s_t: U(M) \to U(F(M, t)) \subset U(M) \times U(M, t^2g)$$

$$x \mapsto (x, x/t)$$

where $U(M, t^2g)$ is the unitary frame bundle of the Kahler manifold $(M, t^2g)$. From the definition of characteristic classes, we obtain

Proposition 5.3.

$$[u_j,s_j\Phi^k](F(M, t)) = t^2(((Ts)_j(M)\pi^*(s_j(M))\pi^*(\Phi(M, g)))$$

where $s_j(M)$ is the characteristic form of $M$ corresponding to $s_j$, $(Ts)_j(M)$ is the Chern-Simons' transgression form of $s_j$ and $\Phi(M, g)$ is the Kahler form of $(M, g)$.

Now we show the following result.

Proposition 5.4. Let $N(=2^{n-1}n)$ be the number of the bases in Proposition 2.6 with $i_1=1$, $J=(n-k)$, $k \geq 1$. Then there is a surjective homomorphism
We prepare several lemmas.

**Lemma 5.5.** Let $P^k(C)$ be the complex projective space with the standard Kähler metric. Then the classes of $T_{s_1}(M)(T_{s_1}(M))_\pi(s_1(M))$ are linearly independent in $H^*(U(P^k(C)); R)$.

Proof. By a well known theorem (see [4] for example), we have an isomorphism

$$\Delta: H^*(U(P^k(C)); R) \rightarrow H^*[\alpha]/(\alpha^{k+1}) \otimes E(u_1, \ldots, u_k)$$

where the right hand side is the cohomology of a differential complex $R[\alpha]/(\alpha^{k+1}) \otimes E(u_1, \ldots, u_k)$ with a differential $d$ defined by $d\alpha=0$, $du_i=\alpha^i$. The isomorphism $\Delta$ satisfies

$$\Delta([T_{s_1}(M)s_1(M)]) = [\alpha^{l_1}u_{i_1}].$$

Therefore for the proof of the lemma, it is enough to show that $[\alpha^{l_1}u_{i_1}]$ are linearly independent in $H^*[\alpha]/(\alpha^{k+1}) \otimes E(u_1, \ldots, u_k)$. But this can be easily checked by a spectral sequence argument. q.e.d.

**Lemma 5.6.** Let $T$ be the complex 1-dimensional torus with the standard Kähler metric and $M$ be a disjoint union of product of $P^n\sim_k(C)$ and $T^k$. Then $[\Phi^k(F(M, 1))]$ are linearly independent in $H^*(U(M); R)$.

Proof. Similarly as in Lemma 5.5, we have isomorphisms

$$\Delta_k: H^*(U(P^n\sim_k(C) \times T^k); R) \rightarrow H^*[\alpha]/(\alpha^{k+1}) \otimes E(u_1, \ldots, u_k)$$

for $0 \leq k \leq n$.

Let $\pi_k: H^*[T^k; R] \rightarrow H^{2k}[T^k; R]$ be the projection onto the part with degree $2k$ and $i_k: P^n\sim_k(C) \times T^k \rightarrow \bigsqcup_i P^n\sim_i(C) \times T^i$ be the inclusion. Then one can easily show

$$(1 \otimes \pi_k \otimes 1) \dot{\circ} \Delta_k \circ i_k^*: [u_1, \ldots, u_n, \Phi^k](F(M, 1))$$

$$= 0$$

$$= [T_{s_1}(M)s_1(M)] \otimes [\Phi(T^k)] \otimes u_{l_2}$$

where $I_1 = I \cap \{1, \ldots, n-k\}$, $I_2 = I \cap \{n-k+1, \ldots, n\}$.

Hence the image of $u_1, \ldots, u_n, \Phi^k$ by the map

$$(1 \otimes \pi_k \otimes 1) \dot{\circ} \Delta_k \circ i_k^*: H^*(U(M); R) \rightarrow H^*[\alpha]/(\alpha^{k+1}) \otimes E(u_1, \ldots, u_n)$$

is

$$[T_{s_1}(P^n\sim_k(C))(T_{s_1}(P^n\sim_k(C)))_\pi(s_1(P^n\sim_k(C)))].$$
Therefore in view of Lemma 5.5 we complete the proof. q.e.d.

Proof of Proposition 5.2 Choose homology classes \( x(I, k) \in H_*(U(M); \mathbb{Z}) \) so that the matrix \( \langle x(I, k), [u_i] \rangle (F(M, 1)) \) is non-singular. We introduce an \( N \)-vector valued parameter \( t = (t(I, k)) \in \mathbb{R}^N \) and put
\[
x(t) = \sum f_{t(I, k)} x(I, k) \in H_*(B\Gamma^*; \mathbb{Z})
\]
where \( f_{t(I, k)}: U(M) \rightarrow B\Gamma^* \) is the classifying map of the foliation \( F(M, t(I, k)) \).

Then
\[
\langle x(t), [u_i]s_{n-k} \Phi^k \rangle = \sum_{j,k} t(J, k) \langle x(J, l), [u_i]s_{n-k} \Phi^k \rangle (F(M, 1)).
\]
Therefore if we define a map \( \lambda: \mathbb{R}^N \rightarrow \mathbb{R}^N \) by \( \lambda(t) = \langle x(t), [u_i]s_{n-k} \Phi^k \rangle \), then the Jacobian matrix of \( \lambda \) at \( t(I, k) = k - 1 \) is \( \langle x(I, k), [u_i]s_{n-k} \Phi^k \rangle (F(M, 1)) \) which is non-singular. This completes the proof. q.e.d.

References