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ON NON-SINGULAR FPF-RINGS I

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A ring R is right finitely pseudo Frobenius (FPF) if every finitely generated faithful right R-module generates the category of right R-modules. In [2], C. Faith has shown that a commutative ring R is FPF if and only if (1) The total quotient ring K of R is injective, and (2) Every finitely generated faithful ideal is projective. In particular, as in case that R is a commutative semiprime ring, he has also shown that R is FPF if and only if the total quotient ring K of R is injective and R is semihereditaty.

On the other hand, S. Page [8] has proved that a (Von Neumann) regular ring R is (right) FPF if and only if R is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings. Therefore we shall require a characterization of arbitrary FPF-rings, which involves above results.

In this paper, we shall concerned with non-singular rings. In section 1, we shall give a characterization of non-singular (resp. semihereditary) FPF-rings, which involves the theorems of C. Faith and S. Page. Further we shall give another characterization of commutative semiprime FPF-rings. In section 2, we shall present some examples.

0. Preliminaries

Throughout this paper, we assume that a ring R has identity and all modules are unitary.

Let R be a ring and M (resp. N) be a right (resp. left) R-module. Then we use $r_R(M)$ (resp. $l_R(N)$) to denote the right (resp. left) annihilator ideal of M (resp. N), and we use $Tr_R(M)$ to denote the trace ideal of M, i.e. $Tr_R(M)=\sum_{f\in M^*} f(M)$, where M^* means that the dual module of M. Further we use $Z_r(M)$ to denote the singular submodule of M, and $L_r(M)$ (resp. $L_l(N)$) to denote the lattice of right (resp. left) R-submodules of M (resp. N).

For any right R-module M, M is said to have the extending property of modules for $L_r(M)$ if for any A in $L_r(M)$, there exists a direct summand A^* of M such that $A\subseteq_e A^*$, where the notation $A\subseteq_e A^*$ means that A is an essential submodule of A^*.

For any ring R, we use $B(R)$ to denote the set of all central idempotents in R, and we use $BS(R)$ to denote the collection of all maximal ideal of $B(R)$.

A ring R is said to be right bounded if every essential right ideal contains a nonzero two-sided ideal which is essential as a right ideal. In section 1, if R is a non-singular ring, we shall show an elementary property of right bounded ring.

1. A characterization of non-singular FPF-rings

The purpose of this section is to give a characterization of non-singular FPF-rings. First we prepare some lemmas.

We recall that a ring R is right bounded if every essential right ideal contains a nonzero two-sided ideal of R which is essential as a right ideal.

Lemma 1. *For a non-singular ring R, the following conditions are equivalent.*

(1) *R is right bounded.*

(2) *For any finitely generated right R-module M, $r_R(Z_r(M))\subseteq_e R_R$.*

Proof. (1)$\Rightarrow$(2). Let B be a complement submodule of $Z_r(M)$ in M. Then since $M/(Z_r(M)\oplus B)$ and $(Z_r(M)\oplus B)/B$ are singular right R-modules, so that M/B is also singular. Let $\bar{m}_1, \bar{m}_2, \cdots, \bar{m}_n$ be a set of generators of M/B. Then $r_R(M/B)=\bigcap_i r_i(\bar{m}_i R)$ is an essential right ideal of R, because R is right bounded and $r_R(\bar{m}_i)$ is an essential right ideal. On the other hand, since $Z_r(M) \cong(Z_r(M)\oplus B)/B\lesssim M/B$, we conclude that $r_R(Z_r/M))$ is an essential ideal of R. (2)$\Rightarrow$(1). Let I be an essential right ideal of R. Then since R/I is a cyclic singula r right R-module, (2) implies that $r_R(R/I)\subseteq_e R_R$. Thus R is right bounded.

Lemma 2. *Let R be a right non-singular right bounded ring. Then for any finitely generated right R-module M, M is a faithful right R-module if and only if $M/Z_r(M)$ is a faithful right R-module.*

Proof. First we assume that M is a faithful right R-module and set $I= r_R(Z_r(M))\cap r_R(M/Z_r(M))$. Choose an element a of I, then $M\cdot a\cdot r_R(Z_r(M))=0$. Thus $a\cdot r_R(Z_r(M))=0$ since M is faithful. While by Lemma 1, $r_R(Z_r(M))$ is an essential right ideal of R, so a must be zero since R is right non-singular. Hence $I=0$. Moreover since $r_R(Z_r(M))$ is an essential right ideal of R, we conculde that $M/Z_r(M)$ is a faithful right R-module. Conversely, if $M/Z_r(M)$ is faithful, then evidently M is faithful.

Lemma 3 ([5, Proposition 1]). *Let R be a non-singular right FPF-ring. Then R is right bounded.*

Proof. See [5].

Lemma 4 ([8, Corollary]). *Let R be a non-singular right FPF-ring and let Q be the maximal right quotient ring of R. Then the multiplication map $Q\otimes_R Q \cong Q$ is an isomorphism and Q is flat as a right R-module.*

Proof. See [8].

Now we can give a characterization of non-singular FPF-rings.

Theorem 1. *Let R be a ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.*

(1) *R is a non-singular right FPF-ring.*

(2) (i) *R is right bounded.*

(ii) *The multiplication map $Q\otimes_R Q\cong Q$ is an isomorphism and Q is flat as a right R-module.*

(iii) *For any finitely generated right ideal I of R, $Tr_R(I)\oplus r_R(I)=R$ (as ideals).*

Proof. (1)$\Rightarrow$(2). (i) and (ii) are evident by Lemmas 3 and 4. In order to prove (iii), let I be a finitely generated right ideal of R. First we claim that $r_R(I)=eR$ for some central idempotemt e of R. It is easy to see that $r_R(I)=r_Q(I)\cap R$ and $r_Q(I)=eQ$ for some central idempotent e of Q since Q is a regular right self-injective ring. While [9, proposition 3] shows that $B(R)=B(Q)$. Hence $r_R(I)=eR$. Now I is a finitely generated faithful right ideal of $(1-e)R$. Since $(1-e)R$ is also a non-singular right FPF-ring, we see that $Tr_{(1-e)R}(I)=(1-e)R$. Note that $Tr_R(I)=Tr_{(1-e)R}(I)=(1-e)R$. Therefore $Tr_R(I)\oplus r_R(I)=eR\oplus(1-e)R=R$.

(2)$\Rightarrow$(1). First we shall show that R is a right non-singular ring. Let x be an element of $Z_r(R)$. By (iii), $Tr_R(xR)=eR$ for some central idempoetnt e of R. It can be easily seen that $Tr_R(xR)\subseteq Z_r(R)$, hence e is in $Z_r(R)$. This implies $e=0$, so $Z_r(R)=0$. Now let M be a finitely generated faithful right R-module. Since R is a right bounded ring, by Lemma 2, $M/Z_r(M)$ is also faithful. If $M/Z_r(M)$ generates the category of right R-modules, then clearly M generates the category of right R-modules. Therefore we may assume that M is non-singular. The non-singularity of M miplies that $\mathrm{Hom}_R(M,Q)\neq 0$. While it is well known that $\mathrm{Hom}_R(M,Q)$ is isomorphic to $\mathrm{Hom}_Q(M\otimes_R Q,Q)$ as abelian groups. Hence $\mathrm{Hom}_Q(M\otimes_R Q,Q)\neq 0$. Then [6, Proposition 1] say that $\mathrm{Hom}_Q(M\otimes_R Q,Q)$ is a nonzero finitely generated left Q-module. Let $f_1, f_2, \cdots, f_n$ be a set of generators of $\mathrm{Hom}_Q(M\otimes_R Q,Q)$ and set $I=\sum_{i=1}^{n} f_i(M)$. We can write $I=\sum_{i,j=1}^{n,m} a_{ij}R$ for some $a_{ij}\in Q$. Further we set $J=\{r\in R\,|\,ra_{ij}\in R\}$. Then we define an R-homomorphism φ: $R\to(Q/R)^{nm}$ by $\varphi(r)=((\overline{ra_{ij}}))_{i,j=1}^{n,m}$. Since $\mathrm{Ker}(\varphi)=J$, we obtain an exact sequence $0\to R/J\to(Q/R)^{nm}$. Therefore the condition (ii) implies that $Q=QJ$. We claim that J is an essential left ideal of R. Choose f_i such that $f_i(M)\neq 0$. Then since $Q=QJ$, there exists an element r in J such that $ra_{i1}\neq 0$. Hence $rf_i(M)=r(\sum_{i=1}^{m} a_{ij}R)\neq 0$ and it is contained in R, so $\mathrm{Hom}_R(M,R)\neq 0$. This proof is valid for all finitely generated non-singular right R-modules. Consequently, we see that the dual module of every non-zero finitely generated non-singular right R-module is not zero. Therefore it is easily seen that for any in-essential right ideal L, $l_R(L)$ is not zero. In this case [3, Theorem 3.15] shows that R is a left non-singular. Let K be a complement right ideal of J in R. Then an exact sequence $0\to J\to J\oplus K\to K\to 0$ implies the exact sequence

$$\begin{array}{ccccccc} 0\to & Q\otimes_R J & \to & Q\otimes_R (J\oplus K) & \to & Q\otimes_R K & \to 0. \\ & \wr\| & & \wr\| & & \wr\| & \\ & QJ & & QJ\oplus QK & & QK & \\ & \| & & \| & & & \\ & Q & & Q & & & \end{array}$$

Thus $QK=0$, so J is an essential left ideal of R. Furthermore since $Q=QJ$, we can write that $1=\sum_{i=1}^{t} q_ib_i$ for some $q_i\in Q$ and $b_i\in J$. Set $J'=\sum_{i=1}^{t} Rb_i$. Clearly $J'\subseteq J$ and $QJ'=Q$. Hence J' is also an essential left ideal of R. Next we set $H=\sum_{i=1}^{t} b_iI$. H is a finitely generated right ideal of R. We claim that $r_R(H)=0$. If $r_R(H)$ is not zero, then there exists a central idempotent e of R such that $r_R(H)=eR$ by the condition (iii). Note that $J'\cdot Ie=0$. Hence $Ie=0$ since R is left non-singular. This shows that $f_i(Me)=0$ for all $i=1,2,\cdots,n$. We shall show that $Me=0$. We assume not, then since Me is non-singular, $\mathrm{Hom}_R(Me,R)\cong\mathrm{Hom}_Q(Me\otimes_R Q,Q)\cong\mathrm{Hom}_Q(M\otimes_R Qe,Q)$. Thus $\mathrm{Hom}_Q(Me\otimes_R Q,Q)$ is a nonzero direct summand of $\mathrm{Hom}_Q(M\otimes_R Q,Q)$. Therefore there exists a nonzero f_j such that $f_j(Me)\neq 0$. But this is impossible, so $Me=0$. While since M is faithful, $e=0$, hence $r_R(H)=0$, as claimed. Thus H is a generator in the category of right R-modules by the condition (iii). It follows that M is also a generator in the category of right R-modules. Now the proof is complete.

REMARK. If R is a commutative semiprime ring, then the condition (iii) of (2) of Theorem 1 shows that R is a semihereditary ring and the condition (ii) implies that the total quotient ring of R coincides the maximal quoteint ring of R. Hence the theorem of C. Faith follows from Theorem 1. Further, later, we shall give another characterization of commutative semiprime FPF-rings.

If R is a regular ring, the condition (ii) implies that R is a right self-injective. Furthermore, the conditions (i) and (iii) implies that R is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings by [5, Corollary of Theorem 2]. Therefore the theorem of S. Page follows.

Next we consider semihereditary FPF-rings. If R is a commutative FPF-ring, then by Theorem 1, R is semihereditary. However, for arbitrary non-singular FPF-ring R, it is not known whether R is semihereditary. In this paper, we shall give a characterization of semihereditary FPF-rings, and by this characterization, we shall give a necessary and sufficient condition for non-singular FPF-rings to be semihereditary.

Theorem 2. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is right semihereditary and right FPF.*

(2) (i) *R is right bounded and right non-singular.*

(ii) *For any positive integer n, $(nR)_R$ has the extending property of modules for $L_r(nR)$.*

(iii) *For any finitely generated idempotent right ideal I of R, there exists a central idempotent e of R such that $RI=eR$.*

Proof. (1)$\Rightarrow$(2). (i) is clear by Lemma 3 and semihereditarity of R. Next we show (ii). Since R is right semihredirary right FPF, Theorem 1 and [4, Theorem 5.18] show that all finitely generated non-singular right R-modules are projective. Given a positive integer n and any right submodule K of $(nR)_R$, then let K^* be the closure of K in $(nR)_R$. Now nR/K^* is a finitely generated non-singular right R-module, so K^* is a direct summand of $(nR)_R$. Hence $(nR)_R$ has the extending property of modules for $L_r(nR)$. In order to prove (iii), let I be a finitely generated idempotent right ideal of R. Then we show that $Tr_R(I)=RI$. Evidently, $RI\subseteq Tr_R(I)$. Let f be any element of the dual module I^* of I, and a be any element of I. Then since I is an idempotent right ideal of R, $a=\sum_{i=1}^{n} b_i c_i$ for some elements b_i, $c_i\in I$. Thus $f(a)=\sum_{i=1}^{n} f(b_i)c_i\in RI$, so $Tr_R(I)=RI$. While Theorem 1 shows that there exists a central idempotent e of R such that $Tr_R(I)=eR$. Therefore (iii) follows.

(2)$\Rightarrow$(1). First we show that any finitely generated non-singular right R-modules are projective. Let M be a finitely generated non-singular right R-module. Then we have an exact sequence $0\rightarrow K\rightarrow R^n\rightarrow M\rightarrow 0$ for some positive integer n. This implies that K is a closed submodule of $(nR)_R$, so K is a direct summand of $(nR)_R$. Hence M is projective. To prove that R is right FPF, it suffices to show that every finitely generated faithful non-singular right R-module is a generator in the category of right R-modules since R is right bounded. Let M be a finitely generated faithful non-singular right R-module. Then since M is projective, M^*, the dual module of M, is finte finitely generated. Let $m_1, m_2, \cdots, m_n$ be a set of generators of M and $f_1, f_2, \cdots, f_n$ be a set of generators of M^*. We set $I=\sum_{i=1}^{n} f_i(M)$. Then I is projective, so we can write that $Tr_R(I)=\sum_{i,j=1}^{m} Rg_i(a_j)R$ for some $g_i\in I^*$ and $a_j\in I$. Moreover by the Dual basis lemma, we see that $a_j=$

$\sum_{t=1}^{k} b_t g_t(a_j)$ for some $b_t \in I$. Set $J = \sum_{i,j=1}^{m} g_i(a_j)R$. Then $g_i(a_j) = g_i(\sum_{t=1}^{k} b_t g_t(a_j)) = \sum_{t=1}^{k} g_i(b_t) g_t(a_t) \in J^2$. Thus $J = J^2$. Therefore by the condition (iii), there exists a central idempotent e of R such that $RJ = eR$. Note that $Tr_R(J) = RJ$. Thus $RJ = Tr_R(J) = Tr_R(I)$. Next we show that $Tr_R(M) = Tr_R(I)$. Since $RI \subseteq Tr_R(I)$, it is clear that $Tr_R(M) \subseteq Tr_R(I)$. Let f be any nonzero element of I^* and a be any nonzero element of I. Then $a = \sum_{i,j=1}^{n} f_i(m_j) r_{ij}$ for some $r_{ij} \in R$. Hence $f(a) = f(\sum_{i,j=1}^{n} f_i(m_j) r_{ij}) = \sum_{i,j=1}^{n} f(f_i(m_j)) r_{ij}$. Observing that $ff_i \in \mathrm{Hom}_R(M, R)$ for all i, we conclude that $f(a) \in Tr_R(M)$. Hence $Tr_R(M) = Tr_R(I)$. Therefore $Tr_R(M) = eR$. On the other hand, it is easily seen that $e = 1$ since M is faithful and projective. Thus M is a generator in the category of right R-modules.

Corollary 1. *Let R be a right semihereditary and right FPF-ring. Then R is left FPF if and only if R is left bounded.*

Proof. If R is left FPF, then clearly R is left bounded. Conversely, we assume that R is left bounded. Then by Lemma 2, it suffices to show that every finitely generated faithful non-singular left R-module is a generator in the category of left R-modules. Since by Theorem 2, all finitely generated non-singular right R-modules are projective, Theorem 1 and [4, Theorem 5.18] show that all finitely generated non-singular left R-modules are projective. Let M be a finitely generated faithful non-singular left R-module. Then M is projective. Further since M^*, the dual module of M, is also projective, we set $I = r_R(M^*)$ and choose any $r \in r_R(M^*)$. Then for any $f \in M^*$ and $m \in M$, $(fr)(m) = f(m)r = 0$. Hence $f(M) \cdot r_R(M^*) = 0$. Furthermore, $(r_R(M^*) \cdot f(M))^2 = 0$. Now since R is semiprime, $r_R(M^*) \cdot f(M) = 0$. Hence $f(r_R(M^*) \cdot M) = 0$. While since M is projective, so $r_R(M^*) \cdot M = 0$. Therefore $r_R(M^*)$ is zero since M is faithful. Hence M^* is a generator in the categoty of right R-modules since R is right FPF. In this case we have also that M is a generator in the category of left R-modules. Therefore R is left FPF.

Corollary 2. *Let R be a non-singular right FPF-ring. Then R is semihereditary if and only if for any positive integer n, nR has the extending property of modules for $L_r(nR)$.*

G. Bergman [1, Theorem 4.1] has proved that a commutative ring R is semihereditary if and only if

(1) R is a P·P-ring, and

(2) For any $M \in BS(R)$, R/MR is a *Prüfer* domain.

Therefore combining Theorem 2 with the theorem of G. Bergman, we have another characterization of commutative semiprime FPF-rings.

Corollary 3. *Let R be a commutative ring. Then the following conditions are equivalent.*

(1) *R is semiprime FPF-ring.*

(2) *$R\oplus R$ has the extending property of modules for $L(R\oplus R)$ and for any $M\in BS(R)$, R/MR is a Prüfer domain.*

Proof. (1)$\Rightarrow$(2). It is clear by Theorem 2 and the theorem of G. Bergman. (2)$\Rightarrow$(1). Let x be any element of Q, the maximal qutoient ring of R, and set $M=xR+R$. Then M is faithful and projective since $R\oplus R$ has the extending property. While since there is an exact sequence $0\rightarrow J\rightarrow R\oplus R\rightarrow M\rightarrow 0$, where $J=\{r\in R\,|\,xr\in R\}$. Hence J is a direct summand of $R\oplus R$, so projective. Therefore clearly $JQ=Q$. In this case, [4, Theorem 5.18] shows that $Q\underset{R}{\otimes}Q\cong Q$, and Q is flat as a R-module. On the other hand, evidently, R is a P.P-ring by the extending property of $R\oplus R$. Thus the theorem of G. Bergman and Theorem 1 show that R is a semiprime FPF-ring.

2. Examples

In this section, we present some examples to illistrate the idea of this paper.

EXAMPLE 1. There exists a non-singular ring such that right bounded, but not right FPF.

Proof. Let F be a field and let $F_n=F$ for all $n=1, 2, \cdots$. We set $T=\prod_n F_n$ and $K=\sum_n \oplus F_n+F\cdot 1_T$. It is easily seen that T is a commutative regular self-injective ring. Since $S=\oplus F_n$ is an ideal of T, S is a regular ideal of K, and since $K/S\cong F$, K is a regular ring. Note that T is a maximal quotient ring of K. We set $R=\begin{pmatrix} K & S \\ K & K \end{pmatrix}$. It is clear that $Q=\begin{pmatrix} T & T \\ T & T \end{pmatrix}$ is a maximal right and left quotient ring of R. Hence R is a right and left non-singular ring. We show that R is right bounded. Let I be a right ideal of R. Then I is of the form, $I=\begin{pmatrix} A & AS \\ C & D \end{pmatrix}$, where A, C, D are ideals of K such that $D\subseteq C$ and $CS=DS$. Thus I is an essential right ideal of R if and only if $A, D\subseteq_e R_R$. Now, if I is an essential right ideal of R, $J=\begin{pmatrix} (A\cap D) & (A\cap D)S \\ (A\cap D) & (A\cap D) \end{pmatrix}$ is clearly a two-sided ideal, and essential as a right ideal of R. Therefore R is right bounded. Next set $e=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $eR=\begin{pmatrix} K & S \\ 0 & 0 \end{pmatrix}$ is a finitely generated faithful right ideal of

R. While $Tr_R(eR)=ReR=\begin{pmatrix} K & S \\ K & S \end{pmatrix}\neq R$, so eR is not a generator in the category of right R-modules. Therefore R is not right FPF.

EXAMPLE 2. There exists a non-singular ring R suhc that $Tr_R(M)\oplus r_R(M)=R$ (as ideals) for any finitely generated non-singular right R-module M, but not right FPF.

Proof. Choose fields $F_1, F_2, \cdots$, set $R_n=M_n(F_n)$ for all $n=1,2,\cdots$, and set $T=\prod_n R_n$. Let M be a maximal two-sided ideal of T which contains $\sum_n \oplus R_n$. Then T/M be a simple right and left self-injective regular ring. Hence all finitely generated non-singular right T/M-modules are projective, so by [7, Lemma 1], $Tr_R(M)\oplus r_R(M)=R$ (as ideals) for any finitely generated non-singular right $R(=T/M)$-module M. On the other hand, [5, Proposition 2] states that R is not right bounded. Thus by Theorem 1, R is not right FPF.

EXAMPLE 3. There exists a semihereditary ring such that the condition (i) and (iii) of (3) of Theorem 2 are satisfied, but not satisfy the condition (ii). (This example is due to H. Kambara).

Proof. Let F be a field and let $F_n=F$ for all $n=1, 2, \cdots$. We set $T=\prod_n M_{2^n}(F_n)$ and set $(*)=\{x=(x_n)\in T\mid$ there exists a positive integer n, and for all $m\geqq n$, $x_m=\begin{pmatrix} x_{11} & \cdots & x_{12^n} \\ \vdots & & \vdots \\ x_{2^n1} & \cdots & x_{2^n2^n} \end{pmatrix}$, where each $x_{ij}=\begin{pmatrix} a_{ij} & & 0 \\ & a_{ij} & \\ 0 & & \ddots \\ & & & a_{ij} \end{pmatrix}$ $(i, j=1, 2, \cdots, 2^n)$ and $x_n=(a_{ij})_{i,j=1}^{2^n}\}$. Let R be a F-sub-algebra of T generated by $\oplus M_{2^n}(F_n)$ and $(*)$. Note that R is a regular ring and T is a maximal right quotient ring of R. Then [5, Theorem 2] states that R is right bounded. Let x be an element of R. We may assume that $x\notin \oplus M_{2^n}(F_n)$. Then $x=(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots)$, and $x_n=(a_{ij})_{i,j=1}^{2^n}$ and $(0\neq)$ $x_{n+m}=(x_{ij})_{i,j=1}^{2^n}$, and $x_{ij}=\begin{pmatrix} a_{ij} & & 0 \\ & a_{ij} & \\ 0 & & \ddots \\ & & & a_{ij} \end{pmatrix}$ $(i, j=1, 2, \cdots, 2^n)$.

Since $M_{2^n}(F_n)$ is a simple ring and $x_n\neq 0$, $M_{2^n}(F_n)x_nM_{2^n}(F_n)=M_{2^n}(F_n)$, so $M_{2^{n+m}}(F_{n+m})x_{n+m}M_{2^{n+m}}(F_{n+m})=M_{2^{n+m}}(F_{n+m})$ for all $m=1, 2, \cdots$. Thus $RxR=(M_2(F_1)x_1M_2(F_1), \cdots, M_{2^n}(F_n), \cdots)$. Set $e=(e_1, \cdots, e_{n-1}, 1, 1, 1, \cdots)$, where $e_i=1$ if $x_i\neq 0$, and $e_i=0$ if $x_i=0$. Clearly, e is a central idempotent of R, so $RxR=eR$. Therefore the condition (iii) is satisfied. While since R is not self-injective, the condition (ii) does not satisfy.

EXAMPLE 4. There exists a semihereditary ring R such that the condition (i) and (ii) of (3) of Thoerem 2 are satisfied, but not satisfy the condition (iii).

Proof. Let F be a field and V be a countable, infinite dimensional vector space over F, and set $R=\mathrm{End}_R(V)$, i.e. R is a right full linear ring. Hence R is a prime rgeular and right self-injective ring. By [5, Theorem 1], R is right bounded, but does not satisfy the condition (iii) by the proof of Corollary to [5, Theorem 2].

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