Generalization of Plastic Hinge Method for Plate Problems†

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Abstract

Previously, one of the authors developed the new mechanisms of plastic hinge based on the plastic flow theory, derived the elastic-plastic and the plastic stiffness matrices of a beam element and establised the plastic hinge method for elastic-plastic analysis of space-framed structures including the case accompanied by large deflection. In this paper, the previous theory is further extended, and a general theory of the plastic hinge method is proposed based on the finite element method for the analysis of the plastic strength of plates. Applying this generalized plastic hinge method, several examples are analysed including the elastic-plastic large deflection problems, and the usefulness of this method is demonstrated.

KEY WORDS: (Plastic Hinge Method) (Plastic Collapse) (Ultimate Strength) (Plastic Interaction) (Finite Element Method)

1. Introduction

Generally, there exist two different methods to analyse elastic-plastic behavior of structural elements. One of these is the ordinary method, that the spread of the plastic zone is strictly considered introducing the plastic stress-strain relations based on the mathematical theory of plasticity to the regions where the yield condition is satisfied. The other one is the method of so-called plastic analysis, that the plastic deformation spread in the element is considered to be concentrated on the plastic hinges or plastic lines. The former method has been developed with the aid of the finite element method, which can evaluate accurately the plastic strength, and the usefulness of this method is well recognized. Before this method is established, the basic theory of the latter method was already founded as the analytical method for rigid plastic material[1], but problems which can be analyzed applying this method was limited. However, this method is very simple, and recently, the applicability of this method to plate problems has become to be paid attention in the combination with the finite element method [2], [3], [4].

More than ten years ago, one of the authors developed the new mechanism of plastic hinge based on the plastic flow theory, and derived the elastic-plastic and the plastic stiffness matrices for beam elements with plastic hinges.[5] Using these elements, the plastic hinge method is established for elastic-plastic analysis of space-frame structures including the case where large deflection is accompanied.[6] Furthemore, the idealized structural unit method is developed based on this plastic hinge method considering the buckling and the post-buckling behavior of the elements, which extremely shortens the computation time.[7], [8]

In this paper, the previous theory of plastic hinge method is further extended for the analysis of the plastic strength of plates in the combination with the finite element method, and the general theory of plastic hinge method is proposed. Applying this generalized plastic hinge method, several examples including elastic-plastic large deflection problems are analysed, and the usefulness of this method will be demonstrated.

2. Theory of Generalized Plastic hinge method

In this chapter, a generalized theory of the plastic hinge method will be derived in the combination with the finite element method. For the derivation of this theory, some assumptions are made, which are summarized as follows.

(1) The finite element method is applied based on the displacement method.

(2) A plastic hinge is formed at the nodal point of an element when the equivalent stress at this nodal point satisfies the yield condition. After the plastic hinge is formed, the plastic deformation is confined to the plastic hinge, and the inside of the element is still elastic.

(3) The behavior of the plastic hinge is the same as that of the elastic-perfectly plastic material, and is re-

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presented by the plastic flow theory.

2.1 Yield condition of the nodal point of an element

First, consider the finite element with m nodal points. The nodal forces, $\{X_n\}$, and the nodal displacements, $\{u_n\}$, of an element can be represented as follows.

$$\{X_n\}= \{X_1, X_2, \ldots, X_m\}^T \quad (1)$$

$$\{X_n\}= \{u_1, u_2, \ldots, u_m\}^T \quad (2)$$

If it is assumed that there exist s degrees of freedom at each nodal point, the nodal forces, $\{X_i\}$, and the nodal displacement, $\{u_i\}$, at the ith nodal point become

$$\{X_i\} = \{R_{1i}, R_{2i}, \ldots, R_{si}\}^T \quad (i = 1, 2, \ldots, m) \quad (3)$$

$$\{X_u\} = \{h_{1i}, h_{2i}, \ldots, h_{si}\}^T \quad (i = 1, 2, \ldots, m) \quad (4)$$

In this theory, the yield condition at the ith nodal point is described as follows.

The stresses at the ith nodal point are represented as the function of not only the nodal force of the ith nodal point, X_i, but also generally as the function of the This yield condition can be expressed in terms of the stress components, $\sigma_x, \ldots, \tau_{xy}, \ldots$, of the ith nodal point in the following form,

$$f_i(\sigma_{xi}, \ldots, \tau_{xyi}, \ldots) = 0 \quad (5)$$

The stresses at the ith nodal point are represented fundamentally as the function of not only the nodal force of the ith nodal point, X_i, but also generally as the function of the nodal forces of j nodal points, X_j, depending on the displacement functions assumed in the element. Therefore, the stresses, $\{\sigma_i\}$, at the ith nodal point can be expressed as follows.

$$\{\sigma_i\} = \{\sigma_i(X_1, X_2, \ldots, X_j)\} \; (j \leq m) \quad (6)$$

Substituting Eq. (6) into Eq. (5), the yield condition of the ith nodal point can be expressed by the function of j nodal forces in the following form.

$$F_i(X_1, X_2, \ldots, X_j) = 0 \; (j \leq m) \quad (7)$$

That is, the plastic hinge is formed at the ith nodal point, when j nodal forces satisfy Eq. (7).

2.2 Virtual work and plastic nodal displacement of the element

First, it is assumed that the deformation of the element is small, and the element is in the equilibrium condition under the nodal forces, $\{X_n + dX_n\}$, with the plastic hinges formed at s nodal points (s = 1, . . . , k). Under this condition, the internal and the external virtual works, δW_i and δW_e, are considered, which are done by the internal and the external forces, respectively, during the virtual displacement increments; $\{\delta du_n\}$.

Although the plastic hinge is formed at a nodal point where the extension of plastic region is confined, the plastic deformation may be produced. Then, the total nodal displacement increment, $\{du_n\}$ is represented as the sum of the elastic components, $\{du_n^e\}$, and the plastic components, $\{du_n^p\}$. That is

$$\{du_n\}= \{du_n^e\}+ \{du_n^p\} \quad (8)$$

For the entire region except at the plastic hinge the element is in the elastic state. The elastic strain components, $\{\epsilon^e\}$, in the element can be expressed with the aid of the strain-displacement matrix, [B], in the following form.

$$\{\epsilon^e\} = \{\epsilon_x^e, \epsilon_y^p, \ldots, \gamma_{xy}^e, \ldots\} = [B]\{u_n^e\} \quad (9)$$

For the elastic behavior of the element, the well-known relation between the nodal forces and nodal displacements is represented as follows.

$$\{X_n\} = [K^e]\{u_n^e\} \quad (10)$$

where $[K^e]$ is the elastic stiffness matrix of the element.

When the virtual displacement increments, $\{\delta du_n\}$, are applied to the element with the plastic hinges, the virtual work, δW_i, done by the internal forces can be represented as the sum of the elastic work, δW_i^e, and the plastic work, δW_i^p. That is

$$\delta W_i = \delta W_i^e + \delta W_i^p \quad (11)$$

In general, the elastic work increment per unit volume can be represented by the product of the stresses and the strain increments in the elastic range, and the plastic work increment per unit volume by that of the equivalent stress and the equivalent plastic strain increnent in the plastic region. Therefore, Eq. (11) can be expressed in the following form:

$$\delta W_i = \int_{V_{el}} \{\delta d\epsilon^e\}^T \{\sigma + d\sigma\} dV + \int_{V_{pl}} \{\delta \bar{d}\epsilon^p\}^T \{\bar{\sigma} + d\bar{\sigma}\} \bar{d}V \quad (12)$$

where V_{e1} and V_{p1} represent the volume of the elastic and the plastic regions of the element, respectively. In this theory, the inside of the element is assumed to be in the elastic state, and the first term of the right-hand side of Eq. (12) can be replaced by the product of the elastic virtual displacements, $\{\delta du_n^e\}$, the elastic stiffness matrix, $[K^e]$, and the elastic nodal displacement, $\{u_n + du_n\}$. On the other hand, the second term can be expressed by the product of the virtual plastic displacement increments, $\{\delta du_n^p\}$, and the nodal forces, $\{X_n + dX_n\}$, since the plastic deformation is confined only to the nodal point where the plastic hinge is formed. If it is assumed that the yield function, $F_i(X_1, X_2, \ldots, X_j)$, in Eq. (7) can be regarded as the plastic potential, the plastic nodal displacement increments at the ith nodal point of which nodal forces contribute to the yield condition at the ith nodal point are represented as

$$\{du_j^p\} = d\lambda_i \{\partial F_i/\partial X_j\} \tag{13}$$

where $d\lambda_i$ is a positive proportionality factor. Consequently, Eq. (12) can be expressed in the following form.

$$\delta W_i = \{\partial d\epsilon_n^e\}^T [K^e] \{u_n^e + du_n^e\} + \sum_{i=1}^{k} \delta d\lambda_i \{\partial F_i/\partial X_j\}^T \{X_j + dX_j\} \tag{14}$$

On the other hand, the virtual work done by the external forces is in the form,

$$\delta W_e = \{\delta du_n\}^T \{X_n + dX_n\} \tag{15}$$

Applying the principle of virtual work, $\delta W_e = \delta W_i$, the following equation is obtained.

$$\{\delta du_n\}^T \{X_n + dX_n\} = \{\delta du_n^e\}^T [K^e] \{u_n^e + du_n^e\} + \sum_{i=1}^{k} \delta d\lambda_i \{\partial F_i/\partial X_j\}^T \{X_j + dX_j\} \tag{16}$$

Considering the relation expressed by Eq. (10), the first term of the right-hand side of Eq. (16) becomes $\{\delta du_n^e\}^T \{X_n + dX_n\}$. Assuming that the nodal points are not relevant to the yield condition of the ith nodal point from the $j+1$ to the n th, the partial derivative of F_i with respect to these non-relevant nodal forces should be zero, that is $\{\partial F_i/\partial \Sigma_j\} = 0$. Taking this matter into account, the second term of the right hand side of Eq. (16) may be expressed in a more general form, $\sum_{i=1}^{k} \delta d\lambda_i \{\partial F_i/\partial X_n\}^T \{X_n + dX_n\}$. Consequently, Eq. (16) becomes as follows.

$$\{\delta du_n\}^T \{X_n + dX_n\} = \{\delta du_n^e\}^T \{X_n + dX_n\} + \sum_{i=1}^{k} \delta d\lambda_i \{\partial F_i/\partial X_n\}^T \{X_n + dX_n\} \tag{17}$$

From this equation, the nodal displacement increments, $\{du_n\}$, can be expressed in the following form:

$$\{du_n\} = \{du_n^e\} + \sum_{i=1}^{k} \delta d\lambda_i \{\partial F_i/\partial X_n\} \tag{18}$$

Eq. (18) indicates that the plastic displacement takes place not only at the ith nodal point but also at the other nodal points where plastic hinges are not formed, when the yield condition is expressed as the function of the nodal forces of these nodal points. However, the plastic hinges are not formed at these nodal points until the yield condition at the individual nodal point is satisfied.

2.3 Elastic-plastic and plastic stiffness matrices of the element with plastic hinges

In this section, the elastic-plastic or the plastic stiffness matrix of the element with plastic hinges will be derived. Here, it is assumed that the plastic hinges are formed at nodal points from the *1*st to the *k*th.

At the ith nodal point ($i = 1, 2, \ldots, k$) where the plastic hinge is formed, the following equation is satisfied.

$$F_i(X_n) = 0 \quad (i = 1, \ldots, k) \tag{19}$$

When unloading takes place at this nodal point, the increment of $F_i(X_n)$ becomes negative, that is

$$dF_i(X_n) < 0 \tag{20}$$

However, as far as loading condition continues, the following equation must be satisfied.

$$dF_i = \{\partial F_i/\partial X_n\}^T \{dX_n\} = 0 \; (i = 1, 2, \ldots, k) \tag{21}$$

Here, Eq. (10) can be expressed in the incremental form as

$$\{dX_n\} = [K^e] \{du_n^e\} \tag{22}$$

Substituting Eq. (18) into Eq. (22), the following equation is derived.

$$\{dX_n\} = [K^e] \left(\{du_n\} - \sum_{i=1}^{k} d\lambda_i \{\partial F_i/\partial X_n\} \right) \tag{23}$$

Substituting this equation into Eq. (21),

$$\{\partial F_i/\partial X_n\}^T [K^e] \left(\{du_n\} - \sum_{i=1}^{k} d\lambda_i \{\partial F_i/\partial X_n\} \right) = 0 \; (i = 1, 2, \ldots, k) \tag{24}$$

The above equation is a set of linear simultaneous equations about $d\lambda_i$. Defining the vector, $\{d\lambda\} = \{d\lambda_1, d\lambda_2, \ldots, d\lambda_k\}^T$, Eq. (24) becomes

$$[G] \{d\lambda\} = [H] \{du_n\} \tag{25}$$

where

$$G_{ij} = \{\partial F_i/\partial X_n\}^T [K^e] \{\partial F_j/\partial X_n\} \quad (i, j = 1, 2, \ldots, k) \tag{26}$$

$$H_{ir} = \{\partial F_i/\partial X_n\}^T [K^e] \quad (i = 1, 2, \ldots, k, \; r = 1, 2, \ldots, m \times s) \tag{27}$$

Solving Eq. (25) with respect to $\{d\lambda\}$, the following equation is obtained,

$$\{d\lambda\} = [G]^{-1} [H] \{du_n\} = [M] \{du_n\} \tag{28}$$

Substituting Eq. (28) into Eq. (23), the relation between the nodal force increments, $\{dX_n\}$, and the nodal displacement increments, $\{du_n\}$, as derived as follows.

$$\{dX_n\} = [K^p]\ \{du_n\} \tag{29}$$

where

$$[K^p] = [K^e] - \sum_{i=1}^{k} [K^e]\ \{\partial F_i/\partial X_n\}\ [M_{ir}] \tag{30}$$

$[K^p]$ in Eq. (29), which is represented by Eq. (30), is the elastic-plastic stiffness matrix of the element with plastic hinges. If $k=m$, this $[K^p]$ represents the plastic stiffness matrix of the element that the plastic hinges are formed at all nodal points.

For example, if the plastic hinge is formed at only the *1* st nodal point of the element, $[K^p]$ becomes as follwos.

$$[K^p] = [K^e] - [K^e]\ \{\partial F_1/\partial X_n\}\ \{\partial F_1/\partial X_n\}^T [K^e] / \{\partial F_1/\partial X_n\}^T [K^e]\ \{\partial F_1/\partial X_n\} \tag{31}$$

For the derivation of the above equations, the deformation of the element is assumed to be in the range of small displacement. In the case where the element is accompanied by large deformation, the elastic stiffness matrix varies with respect to the deformation. At the loading stage where the complete equilibrium of the entire structure is satisfied, the incremental form of the stiffness equation of the elastic element can be expressed in the following form, instead of Eq. (22).

$$\{dX_n\} = [K_l^e]\ \{du_n\} \tag{32}$$

where $[K_l^e]$ is the elastic stiffness matrix considering the large deformation, which varies with respect to the deformation. In contrast with this, the yield condition for plastic hinge is not influenced by large deformation. As far as the tangential stiffness of the element is concerned, $[K^e]$ changes to $[K_l^e]$. Therefore, the procedure used in the above case of small displacement is applied and the tangential elastic-plastic stiffness matrix, $[K_l^p]$, with large deformation can be expressed in the following form.

$$[K_l^p] = [K_l^e] - \sum_{i=1}^{k} [K_l^e]\ \{\partial F_i/\partial X_n\}\ [M_{ir}] \tag{33}$$

where $[M_{ir}]$ is a function of $\{\partial F_i/\partial X_n\}$ and $[K_l^e]$. This equation can be easily obtained from Eq. (30), replacing $[K^e]$ in Eq. (30) by $[K_l^e]$.

3. Analysis of plate problems by the generalized plastic hinge method

3.1 Procedure for analysis

In this chapter, the validity and usefulness of the generalized plastic hinge method will be demonstrated by analysing several plate problems. For the analysis, the triangular finite element is used. As for inplane deformation, the displacement function which guarantees constant strain in the element is assumed. As for out-of-plane deformation, two types of displacement functions are used, which are conforming and non-conforming with respect to the slope along the boundaries between the elements. The number of nodal points of this element is 3 ($m = 3$), and that of possible plastic hinges is also 3 ($k = 3$). In the range of small displacement, the inplane stresses, $\{\sigma_p\}$, and the bending stresses, $\{\sigma_b\}$, are represented respectively as follows.

$$\{\sigma_p\} = \{\sigma_{xp}, \sigma_{yp}, \tau_{xyp}\}^T = [A_p]\ \{X_{np}\} \tag{34}$$

$$\{\sigma_b\} = \{\sigma_{xb}, \sigma_{yb}, \tau_{xyb}\}^T = [A_b]\ \{X_{nb}\} \tag{35}$$

where

$$\{X_{nb}\} = \{X_1, Y_1, X_2, Y_2, X_3, Y_3\}^T \tag{36}$$

$$\{X_{nb}\} = \{Z_1, M_{x1}, M_{y1}, Z_2, M_{x2}, M_{y2}, Z_3, M_{x3}, M_{y3}\}^T \tag{37}$$

X, Y and Z ; nodal forces in x, y and z directions, respectively,

M_x and M_y : nodal bending moments about y and x axes, respectively.

The stresses, $\{\sigma_i\}$, at the *i*th nodal point can be represented as the sum of the inplane and the bending stresses, that is

$$\{\sigma_i\} = \{\sigma_{pi}\} + \{\sigma_{bi}\} = ([A_p] + [A_b])\ \{X_n\} \tag{38}$$

As is known from the above expression, the stresses at the *i*th nodal point of the element is the function of all nodal forces.

The yield condition at the *i*th nodal point of this element can be derived in terms of resultant stresses applying Mises's yield condition, that is

$$f_i = |m_i| + t_i^2 - 1 = 0 \tag{39}$$

where

$$m_i^2 = m_{xi}^2 - m_{xi}m_{yi} + m_{yi}^2 + 3m_{xyi}^2 \tag{40}$$

$$t_i^2 = t_{xi}^2 - t_{xi}t_{yi} + t_{yi}^2 + 3t_{xyi}^2 \tag{41}$$

$$m_{xi} = 2\sigma_{xbi}/3\sigma_Y,\ m_{yi} = 2\sigma_{ybi}/3\sigma_Y,\ m_{xyi} = 2\tau_{xybi}/3\sigma_Y \tag{42}$$

$$t_{xi} = \sigma_{xpi}/\sigma_Y,\ t_{yi} = \sigma_{ypi}/\sigma_Y,\ t_{xyi} = \tau_{xypi}/\sigma_Y \tag{43}$$

σ_Y ; yield stress of the material

The derivation of Eq. (39) is described in the appendix.

Substituting Eq. (38) into Eq. (39), the yield condition of the *i*th nodal point is expressed with respect to the nodal forces, that is

$$F_i(X_1, Y_1, Z_1, M_{x1}, M_{y1}, \ldots, M_{y3}) = 0 \qquad (44)$$

In the actual computation, the computation program is developed in such a way that a plastic hinge is formed at each nodal point strictly satisfying the yield condition, $F_i = 0$.

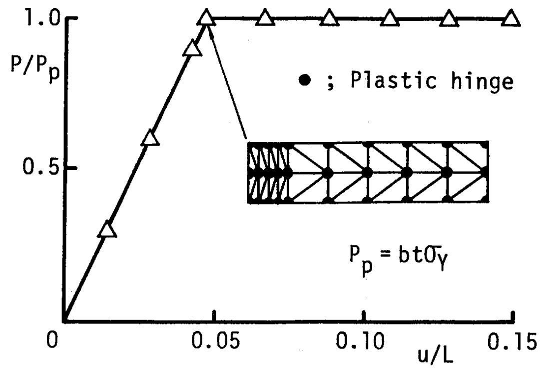
3.2 Analysis of small displacement problems

3.2.1 Analysis of inplane and out-of-plane behaviors of a strip

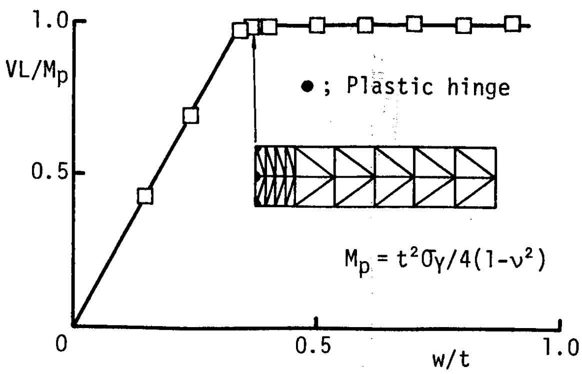
First, the fundamental characteristics of the yield condition of the plastic hinge are examined analysing the behavior of a cantilever using a non-conforming finite element. The loading condition is produced by forcing displacements at the free end of the cantilever in the following three manners.

CASE (A) ; Inplane displacements are applied.
CASE (B) ; Out-of-plane displacements are applied.
CASE (C) ; Inplane displacements are applied after the plastic hinges are formed at the fixed end by out-of-plane displacement.

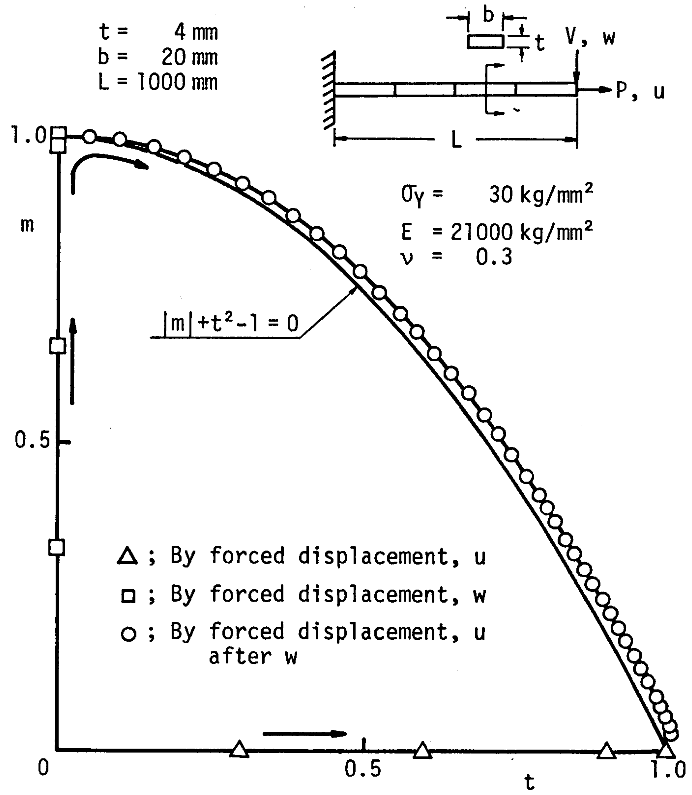
Figures 1 (a) and (b) show the relation between the load and displacement in CASE (A) and CASE (B), respectively. In CASE (A), the stresses are the same in the entire region of the element due to the nature of the assumed displacement function, and plastic hinges are formed at all nodal points of all elements at the same time. Consequently, the strip can carry no more load. In this case, when plastic hinges are produced at all nodal points, the stresses in the element satisfy the yield condition. From this fact, it is apparent that the plastic hinge method indicates exact the same collapsing load as that of the ordinary finite element method. On the other hand, in CASE (B), plastic hinges are formed at the nodal points of the fixed ends. At this instant, the collapsing mechanism is formed, and no more load is carried. The plastic strength of CASE (B) is somewhat higher than the exact value using the mesh division shown in Fig. 1. The accuracy of the collapsing load obtained by this method will be discussed in the following section.



(a) Load-displacement curve under forced displacement, u



(b) Load-deflection curve under forced displacement, w



(c) Interaction between t and m

Fig. 1 Plastic behaviors of plates under tension or/and bending

The result of the analysis that is performed for CASE (C) is shown in Fig. 1 (c), which represents the relation between the non-dimensionalized bending stress, m_i, and the inplane stress, t_i, of the nodal point at the fixed end. At the first stage of loading, only the bending stress, m_i, increases up to 1.0, and the plastic hinges are formed at the fixed end of the cantilever. Then, the inplane displacements are applied, and m_i decreases with the increase of t_i. Finally, m_i becomes zero when t_i becomes 1.0, as shown by ○ in Fig. 1 (c). The deviation from the exact interaction curve, $|m_i| + t_i^2 - 1 = 0$, is fundamentally due to the linearization of the non-linear analysis. However, if the magnitude of load increment at every step is taken as small as possible, the deviation may be negligible.

In the case of small displacement analysis, the interaction terms between the inplane and out-of-plane components do not appear in the stiffness matrix if no plastic hinge is formed. However, once the plastic hinge is formed, such interaction terms appear even in the small displacement analysis.

3.2.2 Analysis of collapsing strength of a plate under lateral load

First, using non-conforming elements, the collapsing load of a square plate is analyzed under uniformly distributed lateral load for different supporting conditions. Figure 2 (a) shows the load-deflection curve calculated for the square plate of which all edges are simply supported, and Fig. 2 (b) that of the square plate of which all edges are fixed. In each figure, • represents the location of the plastic hinge, and the first hinge is formed at point A on each load-deflection curve. As the load increases, plastic hinges are formed one by one and almost all nodal points become the plastic hinges. However, the load is still carried but very small increase. Then, it is difficult to define the exact collapsing load. Here, the approximate collapsing load is defined as that indicated by the intersecting point of the two tangential lines shown by the chain lines in Figs. (2) (a) and (b). The accuracy of the resulting collapsing load depends on the fineness of the mesh division.

To examine the effect of the fineness of the mesh division on the plastic collapsing load, a series of analyses is carried out using the conforming and the non-conforming elements. The collasping loads obtained are plotted in Figs. 3 (a) and (b) both for the simply supported edges and the fixed edges, respectively, together with the loads which gives the same deflection in the elastic range. In the case of the simply supported edges shown in Fig. 3 (a), the upper bound and the lower bound solutions of the collapsing load obtained by the plastic analysis[10] coincide with each other, and this gives the exact solution. As the fineness of the mesh increases, the collapsing strength by

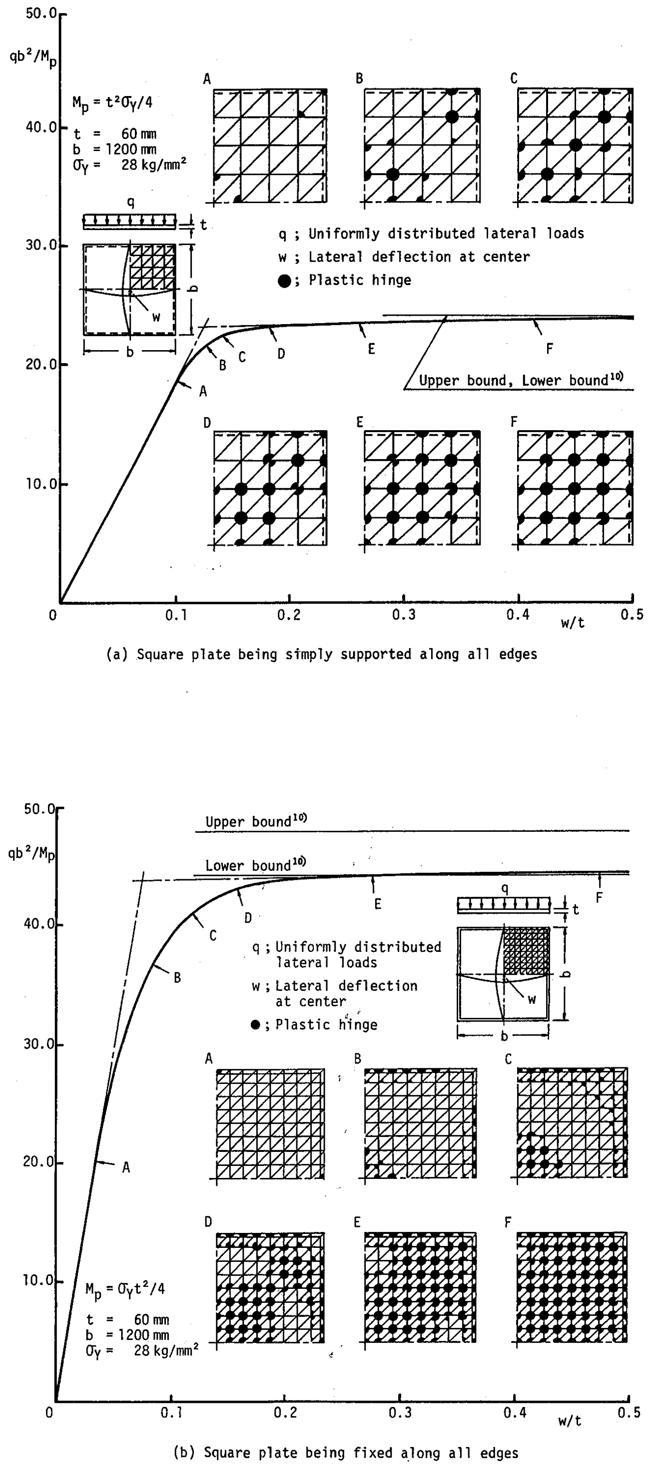


Fig. 2 Elastic-plastic analysis of square plates under uniformly distributed lateral loads

the conforming element approaches to this exact value from the upper side, and that by the non-conforming element from the lower side. On the other hand, in the case of the fixed edges shown in Fig. 3 (b), the upper bound and the lower bound solutions of the collapsing strength

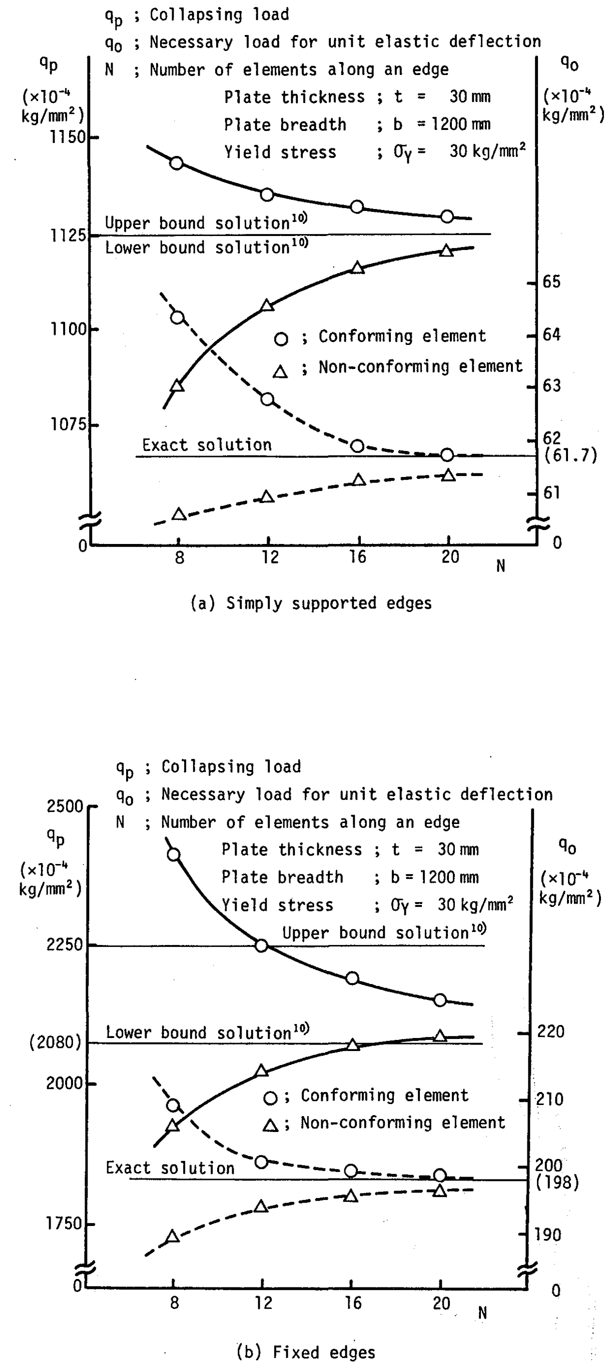


Fig. 3 Accuracy of the collapsing load and elastic deflection

do not coincide.[10)] In this case, the calculated collapsing load by the conforming element approaches from the upper side to a certain value between the upper and the lower bound solutions as the fineness of the mesh division increases, and that by non-conforming element from the lower side. Based on the fact in the above case, the exact collapsing load should be found between the two calculated loads. In both cases, the tendency of approaching the exact value is very similar to the case where the elastic deflection is computed.

For various boundary conditions, the collapsing loads of square plates are calculated under uniformly distributed lateral load. The collapsing loads and the location of the plastic hinges near the collapsing load are represented in Fig. 4.

3.3 Analysis of large deflection problems

In this section, the elastic-plastic behavior of a both end fixed strip under a centrally concentrated lateral load is analysed. In this case, the menbrane stresses due to large deflection can not be ignored. Figure 5 shows the relation between load and central deflection. As the load increases, the plastic hinges are formed at the center and both fixed ends. This load corresponds to that of point A in Fig. 5. Above this load, the membrane stress increases as the deflection increases, and further load is carried. Then, as the load increases, the number of the plastic hinges increases, and finally the plastic hinges are formed at all nodal points of all elements. Point D in Fig. 5 correspond to this condition. The result of the analysis represented by ○ and the solid line well agrees with that of experiment[6)] represented by the dashed line in Fig. 5.

Furthermore, the analysis is performed on the elastic-plastic large deflection behavior of a plate which is simply supported along all edges and is subjected to compression.

The results are shown in Ref. 9 . However, the compressive ultimate strength analysed by this method is somewhat lower than that obtained by the ordinal finite element analysis. One of the reason of this difference may be attributed to that the equilibrium condition for large deflection problem is very sensitive to the tangential stiffness matrix. In the actual behavior of the plate, the gradual expansion of the plastic zone in the element decreases gradually the stiffness of the element. In the conventional elastic-plastic analysis by the finite element method, this change of the stiffness is taken into account in evaluating the stiffness matrix. On the other hand, in this plastic hinge method, the stiffness changes suddenly when the plastic hinge is produced. For this reason, the difference in the evaluated magnitude of the stiffness matrix by both methods becomes large, when the effect of large deflection is coupled with plastification. Further investigation is necessary into the applicability of this method to such problems.

4. Conclusions

In this paper, the generalized theory of the plastic hinge method for plate problems is presented based on the plastic hinge method for one dimensional element in

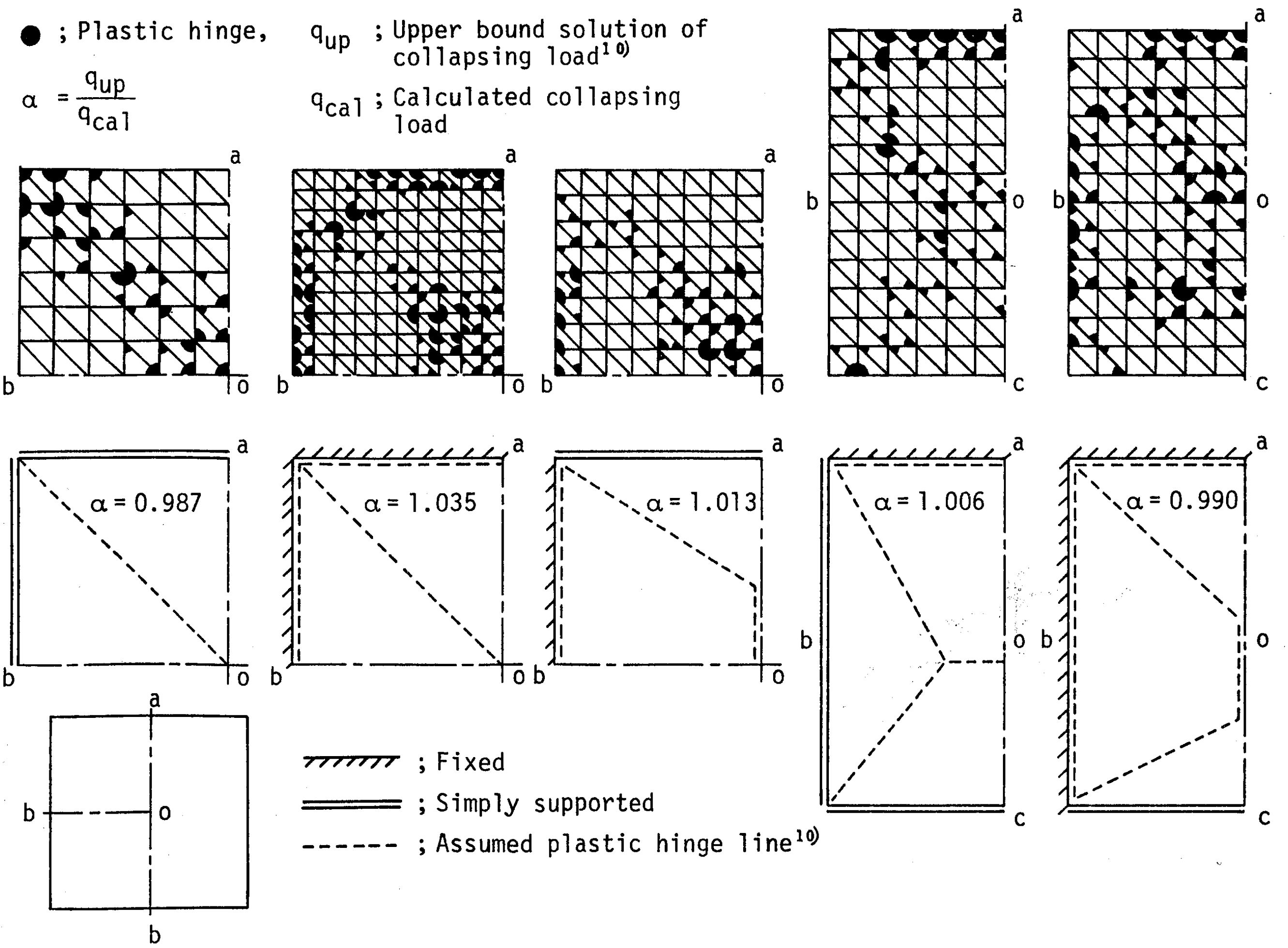


Fig. 4 Collapsing load and the distribution of plastic hinges near collapse of a square plate for various boundary conditions

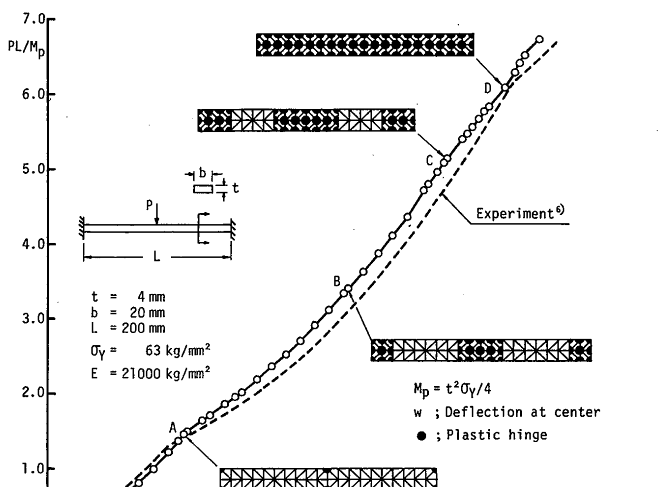


Fig. 5 Load-deflection curve of a strip clamped at both ends under centrally concentrated load

the combination with the finite element method. This theory has the following characteristics.

(1) In the elastic range, the results completely coinside with those by the ordinary finite element method.

(2) In the new theory, the plastic hinge is formed at a nodal point when the resultant equivalent stresses at the nodal point satisfy the yield condition. The yield condition which is derived from the above condition and represented by the function of the nodal forces is valid.

(3) When the plastic hinge is formed, the equivalent stress components at this nodal point varies satisfying the plasticity condition, and the elastic-plastic stiffness matrix is easily derived as the function of the elastic stiffness matrix and the derivatives of the yield function with respect to the nodal forces.

(4) In general, the plastic displacement takes place even at the nodal point where the plastic hinge is not formed, if the nodal force of this node is related to the plastification of the other nodal point. However, the plastic hinge is not formed at this nodal point until the yield condition of this nodal point is satisfied.

(5) As the fineness of the mesh division increases, the collapsing load calculated by this method approaches to the exact value.

Applying this generalized plastic hinge method, several problems including the elastic-plastic large deflection problem are analysed, and the usefulness and the validity of this method are demonstrated except for inelastic stability problems of plates.

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APPENDIX: YIELD CONDITION FOR THE PLASTIC HINGE AT A NODAL POINT

In this theory, the element is in the elastic state until the yield condtion at any nodal point is satisfied. When the yield condition is satisfied at a certain nodal point, the stress distributions at this nodal point are assumed as shown in Fig. A-1. In this state, the stress components, σ_{xb}, σ_{yb} and τ_{xyb}, which contribute to the bending and tortional moments can be expressed in terms of the yield stress, σ_Y, as follows.

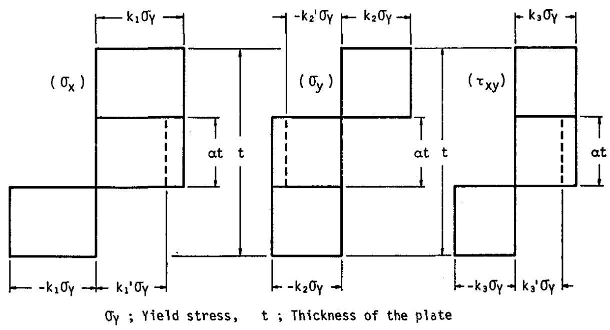

Fig. A-1 Distributions of stress componets at plastic hinge

$$\sigma_{xb} = k_1\sigma_Y,\quad \sigma_{yb} = k_2\sigma_Y,\quad \tau_{xyb} = k_3\sigma_Y \qquad \text{(A-1)}$$

For the stress distribution shown in Fig. A-1, the bending moment about y axis is calculated as

$$M_x = k_1\sigma_Y t^2/4 - k_1\sigma_Y(\alpha t)^2/4 = k_1 M_p(1-\alpha^2) \qquad \text{(A-2)}$$

In the above expression, $M_p = t^2\ \sigma_Y/4$ represents the fully plastic moment of the plate. Here, the following non-dimensionalized bending moment is defined, that is

$$m_x = M_x/M_p = k_1(1-\alpha^2) \qquad \text{(A-3)}$$

Similarly,

$$m_y = k_2(1-\alpha^2) \qquad \text{(A-4)}$$

$$m_{xy} = k_3(1-\alpha^2) \qquad \text{(A-5)}$$

The Mises yield condition in the plane stress state is represented as

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_Y^2 \qquad \text{(A-6)}$$

Substituting Eq. (A-1) into the above equation, the following expression is obtained.

$$k_1^2 - k_1k_2 + k_2^2 + 3k_3^2 = 1 \qquad \text{(A-7)}$$

Substituting Eqs. (A-3), (A-4) and (A-5) into Eq. (A-7),

$$m_x^2 - m_xm_y + m_y^2 + 3m_{xy}^2 = (1-\alpha^2)^2 \qquad \text{(A-8)}$$

On the other hand, the stress components, σ_{xp}, σ_{yp} and τ_{xyp}, which distribute in the middle portion of the thickness, αt, contribute to the inplane forces except the bending and the tortional moments. They are represented in the same manner as those in Eq. (A-1), that is

$$\sigma_{xp} = k'_1 \sigma_Y, \ \sigma_{yp} = k'_2 \sigma_Y, \ \tau_{xyp} = k'_3 \sigma_Y \quad \text{(A-9)}$$

The inplane forces, T_x, in the x direction due to the stress component, σ_{xp}, is

$$T_x = k'_1 \sigma_Y \alpha t = \alpha k'_1 T_Y \quad \text{(A-10)}$$

where $T_Y = t\sigma_Y$. Here, the following non-dimensionalized inplane forces are defined.

$$t_x = T_x/T_Y = \alpha k'_1 \quad \text{(A-11)}$$

$$t_y = \alpha k'_2 \quad \text{(A-12)}$$

$$t_{xy} = \alpha k'_3 \quad \text{(A-13)}$$

Substituting Eq. (A-9) into Eq. (A-6), and considering the relations expressed by Eqs. (A-11), (A-12) and (A-13), the following equation is obtained.

$$t_x^2 - t_x t_y + t_y^2 + 3t_{xy}^2 = \alpha^2 \quad \text{(A-14)}$$

From Eqs. (A-8) and (A-14), the yield condition is derived in the following form.

$$m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 = \{1 - (t_x^2 - t_x t_y + t_y^2 + 3t_{xy}^2)\}^2 \quad \text{(A-15)}$$

Here, the following quantities are defined.

$$m^2 = m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 \quad \text{(A-16)}$$

$$t^2 = t_x^2 - t_x t_y + t_y^2 + 3\,t_{xy}^2 \quad \text{(A-17)}$$

Substituting Eqs. (A-16) and Eq. (A-17) into Eq. (A-15), the following equation is obtained

$$|\,m\,| + t^2 - 1 = 0 \quad \text{(A-18)}$$

The above equation is the yield condition for the plastic hinge.