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Introduction. In [16], R. Kirby and P. Melvin study invariants of 3-manifolds $\tau_r$ ($r \geq 3$) introduced by E. Witten [38], N. Reshetikhin and V.G. Turaev [31], and W.B.R. Lickorish [25, 26, 27] (see also [18]). In particular, Kirby and Melvin calculated $\tau_3$ and $\tau_4$ explicitly. Let $M$ be a closed, oriented 3-manifold obtained from an (integral) framed link $L$. Then $\tau_3(M)$ can be written as follows [16, §6].

$$\tau_3(M) = c^{-\sigma} \sqrt{2}^{-n} \sum_{S \leq L} \sqrt{-1}^{S \cdot S}.$$  

Here $n$ is the number of components of $L$, $\sigma$ is the signature of its linking matrix, $c=\exp(\pi\sqrt{-1}/4)$, the sum is taken over all sublinks of $L$ including the empty sublink, and $S \cdot S$ is the sum of all the entries in the linking matrix of $S$.

In this paper, we generalize $\tau_3$ and define another series of invariants of 3-manifolds. Let $q$ be a primitive $N$-th $(2N$-th, resp.) root of unity for an odd (even, resp.) positive integer $N$. Put

$$Z_N(M; q) = \left( \frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{i^\top A i},$$

where $G_N(q) = \sum_{h \in \mathbb{Z}/N\mathbb{Z}} q^{h^2}$ (a Gaussian sum), $A$ is the linking matrix of $L$, $l$ is regarded as a column vector, and $l^\top$ is its transposed row vector. One can easily see that $Z_2(M; \sqrt{-1}) = \tau_3(M)$. We will show that these are all invariants for $M$ (Theorem 1.3). As Kirby and Melvin proved for $\tau_3(M)$, $Z_N(M; q)$ is also invariant under homotopy equivalence. More precisely, it is determined by the first Betti number of $M$ and the linking pairing on $\text{Tor} H_1(M; \mathbb{Z})$ for any $N$ and $q$ (Proposition 2.5, Corollary 2.6).

We will express the absolute value of $Z_N(M; q)$ in terms of the cohomology ring of $M$ with $\mathbb{Z}/N\mathbb{Z}$-coefficients (Theorem 3.2). When $|Z_N(M; q)| \neq 0$, we can also determine its phase (Theorem 4.5). It is a generalization of the Brown invariant $\beta(M)$ [16, §6] defined by the linking matrix using the signature and Brown's invariant [2] for $\mathbb{Z}/4\mathbb{Z}$-valued quadratic forms on a $\mathbb{Z}/2\mathbb{Z}$-vector space.

We can also calculate $Z_N(M; q)$ explicitly for 3-manifolds with linking pair-
ings which are members of generator system of linking pairings on finite abelian groups (Theorem 5.1). We also show that when \( M \) is a cyclic covering space of an oriented link, \( Z_N(M; q) \) is essentially equivalent to the link invariant introduced by E. Date, M. Jimbo, K. Miki, and T. Miwa [4] using chiral Potts models (Proposition 6.3).

Other purpose of this paper is to describe various relationship of our invariants with quantum field theory, quantum groups, and \( U(1) \) gauge theory. It is known that \( Z_N(M; q) \) can be obtained from solutions to the polynomial equations associated with \( Z/NZ \)-fusion rules [20, 21, 30]. It is also defined using a quasitriangular Hopf algebra as \( \tau_r(M) \) [6, 31, 16] (§7). If \( N \) is even, the absolute values of our invariants coincide with the invariants of T. Gocho [8], which is defined via \( U(1) \) gauge theory with charge \( N \) (§8). We can also prove that invariants of R. Dijkgraaf and E. Witten [5] can be described using our invariants if \( G=Z/NZ \) (§9).

For basic concepts concerning 3-manifolds and links we refer the reader to [3, 11, 32].

We thank T. Gocho, M. Jimbo, T. Kohno, and J. Murakami for their useful conversations.

1. Definition of invariants. An oriented link in the 3-sphere \( S^3 \) is a finite collection of disjoint, smoothly embedded, oriented circles \( L_1, L_2, \ldots, \) and \( L_n \) in \( S^3 \). An (oriented, integral) framed link is an oriented link, each component \( L_i \) being provided with a framing \( f_i \) which is an isotopy class of a section of the projection \( \partial N(L_i) \rightarrow L_i \). We can obtain a connected, closed, oriented 3-manifold \( M_L \) by surgery on \( S^3 \) along a framed link \( L \). \( M_L \) is the result of gluing \( n \) copies of \( D^2 \times S^1 \) to \( S^3 - \cup_{i=1}^n \text{int} N(L_i) \) so that the \( i \)-th \( \partial D^2 \times \{*\} \) is identified with \( f_i \). It is well known [24, 37] that each connected, closed, oriented 3-manifold can be obtained by surgery on \( S^3 \) along a certain framed link.

Let \( A=(\lambda_{ij}) \) (\( 1 \leq i, j \leq n \)) be the linking matrix of \( L \), that is, \( \lambda_{ij}=\text{lk}(L_i, L_j) \) and \( \lambda_{ii}=\text{lk}(L_i, f_i) \). Here \( \text{lk}(\cdot, \cdot) \) denotes the linking number in \( S^3 \). Denote by \( \sigma(A) \) the signature of \( A \) (the number of positive eigenvalues — the number of negative eigenvalues). Let \( N \) and \( d \) be coprime integers (\( N \geq 2, d \geq 1 \)) with \( N+d \) odd and put \( q=\exp(d\pi \sqrt{-1})/N \). Note that \( q \) is a primitive \( N \)-th root of unity if \( N \) is odd and a primitive \( 2N \)-th root of unity if \( N \) is even. Now we consider the following formula:

\[
Z_N(M, L; q) = \left( \frac{G_N(q)}{|G_N(q)|} \right)^{\sigma(A)} |G_N(q)|^{-n} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{i\lambda_{ii}},
\]

where \( M \) is obtained by surgery on \( S^3 \) along \( L \) and \( G_N(q)=\sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{i\lambda_{ii}} \). \( G_N(q) \) is called a Gaussian sum and its properties are well-known (see Lemma 4.4).

Remark 1.2. For \( N \) odd, \( q^{i\lambda_{ii}} \) is well-defined since \( q \) is an \( N \)-th root of
unity. For $N$ even, we can also easily see that it is well-defined though $q$ is a $2N$-th root of unity. In both case, we can regard $l \mapsto \mu l A l$ as a quadratic form in the following sense. If $N$ is odd, a quadratic form on $(\mathbb{Z}/N\mathbb{Z})^n$ is a function $Q: (\mathbb{Z}/N\mathbb{Z})^n \to \mathbb{Z}/N\mathbb{Z}$ satisfying $Q(ax) = a^2 Q(x)$ as usual. If $N$ is even, a $\mathbb{Z}/2N\mathbb{Z}$-valued quadratic form on $(\mathbb{Z}/N\mathbb{Z})^n$ associated to $(\cdot, \cdot)$ is a function $Q: (\mathbb{Z}/N\mathbb{Z})^n \to \mathbb{Z}/2N\mathbb{Z}$ satisfying $Q(ax) = a^2 Q(x) \in \mathbb{Z}/2N\mathbb{Z}$ and $Q(x+y) = Q(x) + Q(y) + 2 \cdot (x, y) \in \mathbb{Z}/N\mathbb{Z}$. Here $(\cdot, \cdot): (\mathbb{Z}/N\mathbb{Z})^n \times (\mathbb{Z}/N\mathbb{Z})^n \to \mathbb{Z}/N\mathbb{Z}$ is a symmetric bilinear (not assumed to be non-singular) form, and $2: \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/2N\mathbb{Z}$ is a homomorphism sending $1$ to $2$. In this case, $Q(l) = \mu l A l \mod 2N$ with a lift $l \in (\mathbb{Z})^n$ of $l$ and $(l, l')$ is `$\mu l A l$' mod $N$. (This definition coincides with that in [2, 10, 28] for the case that $N=2$ and $(\cdot, \cdot)$ is non-singular.) Then the sum $\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{\mu l A l}$ is written as $\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{Q(l)}$ and is an invariant of quadratic forms.

**Theorem 1.3.** $Z_N(M, L; q)$ is a topological invariant of $M$ and does not depend on any choice of $L$.

Proof. Two unoriented framed links $L$ and $L'$ determine the same closed 3-manifold if and only if $L'$ may be obtained from $L$ by Kirby moves; “stabilization” and “handle sliding” (see [15]). Two framed links $L$ and $L'$ are related by a stabilization if they are identical except for elimination or insertion of a splitted, unknotted component $L_i$ with framing $f_i$ such that $\text{lk}(L_i, f_i) = \pm 1$. $L$ and $L'$ are related by a handle sliding if they are identical except for changing a component $L_i$ by $L_i = L_j \# b f_i$ with framing $f'_i$ such that $\text{lk}(L_i, f'_i) = \text{lk}(L_j, f_j) + \text{lk}(L_i, f_i) \pm 2 \text{lk}(L_i, f_j)$. Here $\# b$ means the band connected sum with $b$ a band connecting $f_i$ and $L_j$. The sign is + if the orientations of $f_i$ and $L_j$ are coherent and — otherwise.

Now from a theorem of R. Kirby [15], it suffices to verify that a stabilization, a handle sliding, and reversing of an orientation do not change $Z_N(M, L; q)$. Assume first that two framed links $L$ and $L'$ are related by a stabilization. We assume that $L'$ is obtained from $L$ by inserting a splitted, unknotted component. Then denoting by $A$ the linking matrix of $L$ with $n$ components, that of $L'$ is given by

$$A' = \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}.$$ 

Since the size and the signature of $A'$ is $n+1$ and $\sigma(A) \pm 1$ respectively, we have

$$Z_N(M, L; q) = \left( \frac{G_N(q)}{G_N(q)} \right)^{-\sigma(A) \pm 1} G_N(q) \left| \frac{G_N(q)}{G_N(q)} \right|^{-n-1} \sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{\mu l A l} \sum_{h \in (\mathbb{Z}/N\mathbb{Z})^n} q^{h h \cdot k}.$$ 

Since $\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^n} q^{h h \cdot k} = G_N(q)$ or $G_N(q)$ (the complex conjugate), we obtain $Z_N(M, L'; q) = Z_N(M, L; q)$. 

Let \( L \) and \( L' \) be two framed links related by a handle sliding such that \( L'=L \# f_i \). Then the linking matrix \( A'=(\lambda'_{ij}) \) of \( L' \) satisfies

\[
\begin{align*}
\lambda'_{ss} &= \lambda_{ss} + \lambda_{tt} + 2\lambda_{tt}, \\
\lambda'_{is} &= \lambda_{is} \pm \lambda_{it} & (i \neq s), \\
\lambda'_{ij} &= \lambda_{ij} \pm \lambda_{tt} & (j \neq s), \\
\lambda'_{ij} &= \lambda_{ij} & (i \neq s, j \neq s).
\end{align*}
\]

Hence \( A'=TA'T \) holds with \( T_{ii}=1, T_{tt}=\pm 1 \) and \( T_{ij}=0 \) otherwise, where \( T=(T_{ij}) \). Putting \( n'=T^{-1}n \), we have

\[
\sum_{i' \in \mathbb{Z}/N\mathbb{Z}^n} q^{n'A'n'} = \sum_{i' \in \mathbb{Z}/N\mathbb{Z}^n} q^{n'M}.
\]

Since \( n \) and \( \sigma(A) \) remain unchanged under this transformation, we have \( Z_N(M, L'; q)=Z_N(M, L; q) \).

If \( L' \) is a framed link which is obtained from \( L \) by reversing orientation of a component \( L_k \), then the linking matrix of \( L' \) is \( tSAS \), where \( S=(S_{ij}) \) with \( S_{ij}=0 \) (\( i \neq j \)), \( S_{ii}=1 \) (\( i \neq k \)), and \( S_{kk}=-1 \). So \( Z_N(M, L'; q)=Z_N(M, L; q) \) by a similar way as above.

This completes the proof. \( \blacksquare \)

By Theorem 1.3 we have topological invariants of \( M \).

**Definition 1.4.** Let \( M \) be a connected, closed, compact 3-manifold obtained by surgery on \( S^3 \) along a framed link \( L \). Then we put \( Z_N(M, q)=Z_N(M, L; q) \).

**2. Fundamental properties.** In this section we study fundamental properties of the invariant \( Z_N(M; q) \).

First of all, we note that \( Z_N(S^3; q)=1 \) for any \( N \) and \( q \). If \( M \) is obtained from a framed link \( L \), the mirror image of \( L \) gives \( -M \), \( M \) with the opposite orientation. Since the linking matrix of the mirror image of \( L \) is \( -A \) with \( A \) the linking matrix of \( L \), we have

**Proposition 2.1.** For a closed, oriented 3-manifold \( M \),

\[
Z_N(-M; q) = Z_N(M; q).
\]

The split union of two framed links gives the connected sum of the corresponding 3-manifolds. So we have

**Proposition 2.2.** If \( M_1 \) and \( M_2 \) are closed, oriented 3-manifolds, then

\[
Z_N(M_1 \# M_2; q) = Z_N(M_1; q)Z_N(M_2; q).
\]

\( Z_N(M; q) \) also factors associated with a factorization of \( N \).
Proposition 2.3. If $N = N_1 N_2$ with coprime integers $N_1$ and $N_2$, then

$$Z_N(M; q) = Z_{N_1}(M; q^{N_2}) Z_{N_2}(M; q^{N_1}).$$

Proof. Let $l \in (\mathbb{Z}/N\mathbb{Z})^*$ is uniquely expressed as $l = N_1 l_1 + N_2 l_2$ for $l_1 \in (\mathbb{Z}/N_1\mathbb{Z})^*$ and $l_2 \in (\mathbb{Z}/N_2\mathbb{Z})^*$. Hence we have

$$\sum_{l \in (\mathbb{Z}/N\mathbb{Z})^*} q^{i_{l_1} N_1 l_1 + i_{l_2} N_2 l_2 + 2N_1 N_2 A_1 A_2} = \sum_{l_1 \in (\mathbb{Z}/N_1\mathbb{Z})^*} q^{i_{l_1} N_1 l_1} \sum_{l_2 \in (\mathbb{Z}/N_2\mathbb{Z})^*} q^{i_{l_2} N_2 l_2},$$

where the second equality holds since $q^{2N_1 N_2} = 1$. In a similar way, we obtain $G_N(q) = G_{N_1}(q^{N_2}) G_{N_2}(q^{N_1})$. Therefore $Z_N(M; q)$ factors as above. \[\square\]

As R. Kirby and P. Melvin state for $\tau_3(M)$ [16, 6.2 Remark], $Z_N(M; q)$ is also a homotopy invariant (see Corollary 2.6 below) for every $N$ and $q$. To prove this, we review results of M. Kneser and P. Puppe [17], A.H. Durfee [7], and R.H. Kyle [22].

Let $B$ and $B'$ be symmetric integral matrices. $B$ and $B'$ are said to be stably equivalent (or closely related in [22]) if they are equivalent under the equivalence relation generated by the following $Q_1$ and $Q_2$:

$$Q_1: B \leftrightarrow SBS \text{ with } S \text{ integral and unimodular},$$

$$Q_2: B \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\]$$

As in the previous section, let $M$ be a 3-manifold obtained by surgery on $S^3$ along a framed link $L$ and $A$ its linking matrix. Summarizing results in [17, 7, 22], we can conclude that stable equivalence class is determined by the first Betti number of $M$ and the linking pairing on $(\text{Tor} \, H^1(M; \mathbb{Z}), \lambda)$. More precisely, the following proposition holds.

**Proposition 2.4.** Stable equivalence class of linking matrices of framed links is determined by the first Betti number of the 3-manifold $M$ obtained from it and $(\text{Tor} \, H^1(M; \mathbb{Z}), \lambda)$, that is, two linking matrices $A$ and $A'$ are stably equivalent if and only if $M$ and $M'$ satisfy (1) and (2) below, where $M$ (resp.) is obtained from framed link $L$ (resp.) with linking matrix $A$ (resp.)

1. The first Betti numbers of $M$ and $M'$ are equal.
2. There exists an isomorphism between $\text{Tor} \, H^1(M; \mathbb{Z})$ and $\text{Tor} \, H^1(M'; \mathbb{Z})$ which induces an isomorphism between the linking pairings $\lambda$ and $\lambda'$.}

Here the linking pairing on $(\text{Tor} \, H^1(M; \mathbb{Z})$ is defined as follows. An exact sequence of coefficient groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$
gives rise to a long exact sequence of homology groups of $M$:
\[
\rightarrow H_d(M; \mathbb{Q}) \xrightarrow{\eta_*} H_1(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta_*} H_1(M; \mathbb{Z}) \xrightarrow{\iota_*} H_*(M; \mathbb{Q}) \rightarrow ,
\]
where $\delta_*$ is the connecting homomorphism. The linking pairing
\[
\lambda: \text{Tor} H_1(M; \mathbb{Z}) \times \text{Tor} H_1(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}
\]
is defined by $\lambda(\alpha, \beta) = \alpha \cdot \beta$ where $\delta_* \hat{\beta} = \beta$ and a dot means the intersection product
\[
H_1(M; \mathbb{Z}) \times H_2(M; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.
\]
One can easily check that $\lambda$ is well-defined.

By the above proposition, we immediately have the following proposition.

**Proposition 2.5.** If $M$ and $M'$ satisfy the conditions (1) and (2) in Proposition 2.4, then $Z_N(M; q) = Z_N(M'; q)$.

**Proof.** Since the corresponding linking matrices $A$ and $A'$ are stably equivalent, we have $Z_N(M; q) = Z_N(M'; q)$ as in the proof of Theorem 1.4. □

Clearly two 3-manifolds which are homotopy equivalent satisfy the conditions (1) and (2). So we have

**Corollary 2.6.** If $M$ and $M'$ are homotopy equivalent, then $Z_N(M; q) = Z_N(M'; q)$.

### 3. Absolute value.
In this section we calculate the absolute value of $Z_N(M; q)$ and give its topological meaning.

First of all we prepare a lemma which will be used frequently in this paper. A proof is an easy exercise.

**Lemma 3.1.** Let $z$ be a primitive $N$-th root of unity. Then
\[
\sum_{\substack{z \in \mathbb{Z}/N\mathbb{Z}^*}} z^{y} = \begin{cases} N^* & \text{if } y = 0 \in (\mathbb{Z}/N\mathbb{Z})^*, \\ 0 & \text{if } y \neq 0 \in (\mathbb{Z}/N\mathbb{Z})^*, \end{cases}
\]
where we regard $x$ and $y$ as column vectors.

Now $|Z_N(M; q)|$ is given as follows. This generalizes [16, Theorem 6.3].

**Theorem 3.2.** If there exists $\alpha$ in $H^1(M; \mathbb{Z}/N\mathbb{Z})$ with $\alpha \cup \alpha \cup \alpha = 0$, then $Z_N(M; q) = 0$. Otherwise $|Z_N(M; q)| = |H^1(M; \mathbb{Z}/N\mathbb{Z})|^{12}$ where $| \cdot |$ in the right hand side is the order of the set.

**Proof.** Let $M$ be a 3-manifold obtained by surgery on $S^3$ along an $n$-
component framed link $L$. From (1.1), we have

$$|Z_N(M; q)| = |G_N(q)|^{-n} \cdot \sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{1Al}.$$

We first calculate $|G_N(q)|^2 = N$.

$$|G_N(q)|^2 = \sum_{h,h' \in \mathbb{Z}/N\mathbb{Z}} q^{h^2-h'^2}$$

$$= \sum_h \sum_{h''} q^{h'^{h''}h} \quad (h' = h'' + h)$$

$$= N.$$

The last equality follows from Lemma 3.1 putting $n=1, x=h'', y=h, z=q^2$ since $q^2$ is a primitive $N$-th root of unity.

Next we calculate the absolute value of $\sum q^{1Al}$. In a similar way as above, we have

$$|\sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{1Al}|^2 = \sum_{l',l} q^{1Al'-1Al}$$

$$= \sum_{l',l''} \sum_l q^{1Al''} \quad (l' = l'' + l)$$

$$= N^m \sum_{l'' \in \ker \Lambda_A} q^{l''1Al''},$$

where $L_A$ is a linear map $L_A: (\mathbb{Z}/N\mathbb{Z})^n \rightarrow (\mathbb{Z}/N\mathbb{Z})^n, l \mapsto Al$. The last equality follows from Lemma 3.1 putting $x=l$ and $y=Al''$. Therefore we have

$$|Z_N(M; q)|^2 = |\sum_{i \in \ker \Lambda_A} q^{1Al}|.$$

Now there are two cases to consider.

Case 1: $N$ is odd. Recall that $q$ is an $N$-th root of unity. For $l \in \ker L_A$, we have $1Al=0$ in $\mathbb{Z}/N\mathbb{Z}$ and $q^{1Al}=1$. Hence $|Z_N(M; q)|^2$ is equal to the order of $\ker L_A$. By Lemma 3.3 below, we have

$$|Z_N(M; q)| = |H^1(M; \mathbb{Z}/N\mathbb{Z})|^1/2.$$

In this case $\alpha \cup \alpha \subset \alpha = 0$ holds for any $\alpha$ in $H^1(M; \mathbb{Z}/N\mathbb{Z})$, because the cup product is skew-symmetric and the order of $H^1(M; \mathbb{Z}/N\mathbb{Z})$ is odd. Hence we obtain Theorem 4.1 for $N$ odd.

Case 2: $N$ is even. In this case $q$ is a $2N$-th root of unity. As in Remark 1.3 we regard $l \mapsto 1Al$ as a map $(\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/2N\mathbb{Z}$. We denote the restriction of this map to $\ker L_A$ by $\phi: \ker L_A \rightarrow \{0, N\} \subset \mathbb{Z}/2N\mathbb{Z}$. Then $\phi$ is a homomorphism because

$$(1 + 1') A(1 + 1') = 1Al + 1'Al' + 2 \cdot 1Al'$$

and $2 \cdot 1Al'$ can be divided by $2N$. Therefore we have

...
By Lemmas 3.3 and 3.4 below, we obtain

\[ |Z_N(M; q)| = \begin{cases} \frac{1}{2} |\ker L_A| & \varphi \equiv 0, \\ 0 & \text{otherwise.} \end{cases} \]

This completes the proof. □

**Lemma 3.3.** \( \ker L_A \) is isomorphic to \( H^1(M; Z/NZ) \).

**Proof.** Since \( M \) is a union of \( S^3 \)-int \( N(L) \) and \( n \) copies of \( D^2 \times S^1 \), we have the Mayer-Vietoris exact sequence below.

\[
\cdots \rightarrow H^1(M; Z/NZ) \rightarrow H^1(S^3\text{-int } N(L); Z/NZ) \oplus \oplus H^1(D^2 \times S^1; Z/NZ) \rightarrow \cdots
\]

Hence \( H^1(M; Z/NZ) \) is isomorphic to \( \ker f \). Moreover \( f \) corresponds to a matrix \( \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \) as a map \( f: (Z/NZ)^n \oplus (Z/NZ)^n \rightarrow (Z/NZ)^{2n} \), where \( 1_n \) is the \( n \times n \) identity matrix. Since \( \ker f \) is isomorphic to \( \ker L_A \), we obtain Lemma 3.3. □

**Lemma 3.4.** Let \( N \) be even. With the isomorphism \( \iota \) in Lemma 3.3, the next diagram commutes:

\[
\begin{array}{ccc}
\ker L_A & \xrightarrow{\varphi} & \{0, N\} \subset \mathbb{Z}/2NZ \\
\iota \downarrow & & \downarrow \times \frac{1}{2} \\
H^1(M; Z/NZ) & \xrightarrow{\psi} & \left\{0, \frac{N}{2}\right\} \subset H^3(M; Z/NZ) = Z/NZ,
\end{array}
\]

where \( \psi \) is defined by \( \psi(\alpha) = \alpha \cup \alpha \cup \alpha \).

**Proof.** Let \( l \) be an element in \( \ker L_A \), and put \( \alpha = \iota(l) \). We calculate \( \alpha \cup \alpha \cup \alpha \) in the Poincaré dual and we will show that \( \alpha \cup \alpha \cup \alpha \) is equal to \( \varphi(l)/2 \).

Let \( S \) be a branched surface representing the Poincaré dual modulo \( Z/NZ \) of \( \alpha \) in \( M=\left(\mathbb{R}^3 - \text{int } N(L)\right) \cup \cup_{i=1}^m D^2 \times S^1 \) such that branch locus of \( S \) is a union of disjoint circles in \( S^3 - N(L) \) and the number of sheets meeting along each circle is a multiple of \( N \). Since \( [S] \) is the Poincaré dual of \( \iota(l) \), \( S \cap \partial N(L_i) \) is a union of \( \tilde{l}_i \) circles in \( \partial N(L_i) \), each of which is parallel to the framing \( f_i \), where \( \tilde{l}_i \in Z \) is a lift of \( l_i \in Z/NZ \) with \( l_i = (l_1, \ldots, l_n) \). Let \( m_i \) be a meridian of \( L_i \) in \( S^3 - N(L) \). Since \( [m_i] \)s generate \( H_1(M; Z) \), we may assume that branch locus
of $S$ is a union of $m_i$'s. Let $a_i N$ be the number of sheets of $S$ meeting along $m_i$.

Since the boundary of $S \cap (S^3 - \text{int} N(L))$ consists of $\bar{I}_i$ copies of $f_i$ in $\partial N(L_i)$, we have $\sum a_i N [m_i] = \sum \bar{I}_i [f_i]$ in $H_i(S^3 - \text{int} N(L); \mathbb{Z})$. Moreover the classes $[f_i]$'s are determined by

$$
\begin{pmatrix}
[f_1] \\
\vdots \\
[f_n]
\end{pmatrix}
= A
\begin{pmatrix}
[m_1] \\
\vdots \\
[m_n]
\end{pmatrix}
$$

Hence we obtain a relation between $a_i$'s and $\bar{I}_i$'s:

$$
\begin{pmatrix}
a_1 N \\
\vdots \\
a_n N
\end{pmatrix}
= A
\begin{pmatrix}
\bar{I}_1 \\
\vdots \\
\bar{I}_n
\end{pmatrix}
$$

Now we calculate the self-intersection of $S$. Since $S - \cup m_i$ is orientable, we can push $S$ in a normal direction. There are self-intersections near $m_i$ as in Figure 3.1. Hence we have

\[
[S][S] = \sum (1 + 2 + \cdots + (a_i N - 1)) [m_i] = \sum \frac{a_i N}{2} [m_i] \in H_i(M; \mathbb{Z} / N \mathbb{Z}).
\]

Since $[S][m_i] = \bar{I}_i$, we obtain

$$
[S][S][S] = \sum \frac{a_i N}{2} \bar{I}_i \\
= \frac{1}{2} \bar{I} A \bar{I} = \frac{1}{2} \varphi(I).
$$

This is the required formula.  

\section*{Remark 3.4.} The above lemma also follows algebraically from [35, Theorem I], which states that

$$
\alpha \cup \alpha \cup \beta = \frac{N^2}{2} \lambda(\alpha, \beta) \in \mathbb{Z} / N \mathbb{Z}
$$
for $\alpha, \beta \in H^1(M; \mathbb{Z}/N\mathbb{Z})$. Here $\alpha, \beta \in \text{Tor} \ H_1(M; \mathbb{Z})$ satisfy $\lambda(\alpha, x) = \alpha(x)/n \in \mathbb{Q}/\mathbb{Z}$ and $\lambda(\beta, x) = \alpha(x)/n \in \mathbb{Q}/\mathbb{Z}$ for any $x \in \text{Tor} \ H_1(M; \mathbb{Z})$.

4. Phase. Now we study the phase of $Z_N(M; q)$.

We use the following notations for an odd integer $x$: (cf. [33])

$$
\varepsilon(x) = (-1)^{(x-1)/2} = \begin{cases} 1 & x \equiv 1 \mod 4, \\ -1 & x \equiv 3 \mod 4, \end{cases}
$$

$$
\omega(x) = (-1)^{(x^2-1)/8} = \begin{cases} 1 & x \equiv \pm 1 \mod 8, \\ -1 & x \equiv \pm 3 \mod 8. \end{cases}
$$

Note that $\varepsilon: (\mathbb{Z}/4\mathbb{Z})^* \to \{1, -1\}$ and $\omega: (\mathbb{Z}/8\mathbb{Z})^* \to \{1, -1\}$ are homomorphisms.

By Theorem 3.1, $Z_N(M; q) \neq 0$ if $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \mathbb{Z}/N\mathbb{Z})$. So we assume this in the following of this section.

We put

$$
Z_N(A; q) = \left( \frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} \sqrt{N^{-n}} \sum_{i \in \mathbb{Z}/N\mathbb{Z}} q^{iA},
$$

for an integral symmetric $n \times n$ matrix $A$. $Z_N(M; q) = Z_N(A; q)$ if $M$ is obtained from a framed link $L$ with linking matrix $A$. We note that if $N$ is odd, $A$ may be regarded as a matrix in $\mathbb{Z}/N\mathbb{Z}$ and if $N$ is even, the diagonal entries in $A$ may be regarded as integers modulo $2N$ and the off-diagonal entries modulo $N$. We will try to diagonalize $A$ to calculate the phase.

From Proposition 2.3 we will restrict ourselves to the case $N=p^m$ with $p$ prime for a while.

If $p$ is odd, we can diagonalize $A$ as a matrix in $\mathbb{Z}/N\mathbb{Z}$, that is, there exists a matrix $S \in \text{SL}(n, \mathbb{Z})$ such that

$$
^tSAS \equiv \bigoplus_{j=1}^n (a_j) \mod p^m.
$$

If $p=2$, we cannot diagonalize $A$ itself in general, but it is proved that one can diagonalize the block sum of $A$ and $(1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \cdots \oplus (2^{m-1}) \oplus (-2^{m-1})$, that is, there exists a matrix $S \in \text{SL}(n+2m, \mathbb{Z})$ such that

$$
^tS(A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2) \oplus \cdots \oplus (2^{m-1}) \oplus (-2^{m-1})) \equiv \bigoplus_{j=1}^{n+2m} (a_j),
$$

where the diagonal entries are considered modulo $2^{m+1}$ and the off-diagonal entries modulo $2^m$. (In fact, it can be proved that $A \oplus (1) \oplus (2) \oplus \cdots \oplus (2^{m-1})$ is diagonalizable, using the technique to diagonalize $\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \oplus (2i)$. Note that the phase of $Z_N(A; q)$ remains unchanged by replacing $A$ with $A \oplus (1) \oplus (-1) \oplus (2) \oplus (-2)$.
Now the phase of $Z_p^m(A, q)$ is equal to that of $(G_p^m(q))^{-\sigma(A)} \sum_{k \in \mathbb{Z}/p^m \mathbb{Z}} q^{s/k^2}$. Thus we only need to calculate the sum

$$G_N(a; q) = \sum_{k \in \mathbb{Z}/N \mathbb{Z}} q^{s/k^2}$$

for an integer $a$ and a prime-power $N$. Note that a Gaussian sum $G_N(q)$ is equal to $G_N(l; q)$. Let $q = \exp(d\pi \sqrt{-1/p^n})$ with $(d, p) = 1$ and $d + p$ odd, and $a = p^e c$ with $(p, c) = 1$. If $p$ is odd, we also write $q$ as $\exp(2b\pi \sqrt{-1/p^n})$ putting $d = 2b$. We can describe the above sum as follows.

**Lemma 4.4.**

1. $p$ is odd. Let $(\frac{x}{p})$ be Legendre's symbol, that is, $(\frac{x}{p}) = 1$ if there exists an integer $l$ such that $l^2 \equiv x \mod p$, and $(\frac{x}{p}) = -1$ otherwise. Then

$$G_p^m(a; q) = \begin{cases} p^m & \text{if } k-m \geq 0, \\ \sqrt{p}^{-m+k} & \text{if } k-m < 0 \text{ and even,} \\ (\frac{c}{p}) (\frac{b}{p}) \sqrt{p}^{-m+k} & \text{if } k-m < 0 \text{ and odd, and } p \equiv 1 \mod 4, \\ (\frac{c}{p}) (\frac{b}{p}) \sqrt{-1} \sqrt{p}^{-m+k} & \text{if } k-m < 0 \text{ and odd, and } p \equiv 3 \mod 4. \end{cases}$$

2. $p=2$. Put $\xi = \exp(\pi \sqrt{-1/4})$. Then

$$G_2^m(a; q) = \begin{cases} 2^m & \text{if } k-m \geq 0, \\ 0 & \text{if } k-m = 0, \\ \xi^{cd} \sqrt{2^{m+k}} & \text{if } k-m < 0 \text{ and even,} \\ \xi^{(c)(m)} \sqrt{2^{m+k}} & \text{if } k-m < 0 \text{ and odd.} \end{cases}$$

Proof. For the case that $k-m \geq 0$ or the case that $k-m = 0$ and $p$ is odd, the formulas follow since $q^s = 1$. If $k-m = 0$ and $p$ is even, $G_p^m(a, q) = 0$ since $q^s = -1$. The case that $p=2$ and $k-m < 0$, the formula follows from $G_2^m(a, q) = 2G_2^{m-1}(a, q) \ (m \geq k+3)$ and direct computations for $m= k+1$ and $k+2$. The case that $p$ is odd and $k-m < 0$ is well-known. For a proof, see for example [23, Chapter IV, §3]. (There are some errors in [23], which one can easily fix.) The proof is complete. □

From this lemma, we know that the phase of $Z_N(A; q)$ takes only eight values. So we define $\phi_N(A; q) \in \mathbb{Z}/8\mathbb{Z}$ as follows.

We first consider the case that $N = p^m$ for an odd prime $p$. Let $a_j$'s be diagonal entries when $A$ is diagonalized as in (4.2). Let $a_j = p^{i_j} e_j$ with $(p, c_j) = 1$.
for $a_j \neq 0$. Note that we can assume $k_j - m$ is always negative. We put $n_+$ and $n_-$ as follows.

\[
\begin{align*}
n_+ &= \# \{ a_j \mid k_j - m \text{ is odd and } \left( \frac{c_j}{p} \right) = 1 \} \\
n_- &= \# \{ a_j \mid k_j - m \text{ is odd and } \left( \frac{c_j}{p} \right) = -1 \}
\end{align*}
\]

Here $\# \{ \cdot \}$ means the number of elements in $\{ \cdot \}$. Then $\phi_p(A; q) \in \mathbb{Z}/8\mathbb{Z}$ is defined as follows.

\[
\phi_p(A; q) =
\begin{cases}
2 \left( \left( \frac{b}{p} \right) - 1 \right) n_+ - 2 \left( \left( \frac{b}{p} \right) + 1 \right) n_- & \text{if } p \equiv 1 \mod 4 \text{ and } m \text{ is even,} \\
2 \left( \left( \frac{b}{p} \right) - 1 \right) n_+ - 2 \left( \left( \frac{b}{p} \right) + 1 \right) n_- - 2 \left( \left( \frac{b}{p} \right) - 1 \right) \sigma(A) & \text{if } p \equiv 1 \mod 4 \\
2 \left( \left( \frac{b}{p} \right) - 1 \right) n_+ - 2 \left( \left( \frac{b}{p} \right) + 1 \right) n_- & \text{if } p \equiv 3 \mod 4 \text{ and } m \text{ is even,} \\
2 \left( \left( \frac{b}{p} \right) - 1 \right) n_+ - 2 \left( \left( \frac{b}{p} \right) + 1 \right) n_- - 2 \left( \left( \frac{b}{p} \right) \right) \sigma(A) & \text{if } p \equiv 3 \mod 4 \text{ and } m \text{ is odd.}
\end{cases}
\]

Then from Lemma 4.4 and $\left( \frac{x}{p} \right) = \pm 2^{(\ell - 1)}$, it follows that $\phi_p(A; q) \pi \sqrt{-1}/4$ is the phase of $Z_p(A; q)$.

Next we consider the case $N = 2^m$. Let $a_j$'s be diagonal entries when $A$ is diagonalized as in (4.3). Let $a_j = 2^k c_j$ with $c_j$ odd for $a_j \neq 0$. Here we assume $k_j - m < 0$ as before. Then $\phi_2(A; q)$ is defined by

\[
\phi_2(A; q) = \begin{cases}
d \sum_{k_j - m : \text{even}} c_j + \varepsilon(d) \sum_{k_j - m : \text{odd}} \varepsilon(c_j) - d \sigma(A) & \text{if } m \text{ is even,} \\
d \sum_{k_j - m : \text{even}} c_j + \varepsilon(d) \sum_{k_j - m : \text{odd}} \varepsilon(c_j) - \varepsilon(d) \sigma(A) & \text{if } m \text{ is odd.}
\end{cases}
\]

From Lemma 4.4, the phase of $Z_2(A; q)$ is $\phi_2(A; q) \pi \sqrt{-1}/4$.

According to Proposition 2.3, we define $\phi_N(A; q)$ for an arbitrary $N$ by using

\[
\phi_N(A; q) = \phi_N(A; q^{N_2}) + \phi_N(A; q^{N_2}) \in \mathbb{Z}/8\mathbb{Z},
\]

where $N = N_1 N_2$ with coprime integers $N_1$ and $N_2$.

For a closed, oriented 3-manifold $M$, we define $\phi_N(M; q) = \phi_N(A; q)$ for the linking matrix $A$ of a framed link which gives $M$. Summarizing the above argument we have the next proposition.

**Theorem 4.5.** If $\alpha \cup \alpha \cup \alpha = 0$ for any $\alpha \in H^1(M; \mathbb{Z}/N \mathbb{Z})$, then

\[
Z_N(M, q) = \exp \left( \frac{\pi \sqrt{-1}}{4} \phi_N(M; q) \right)|H^1(M; \mathbb{Z}/N \mathbb{Z})|^{|/2},
\]

where $\phi_N(M; q) \in \mathbb{Z}/8\mathbb{Z}$ is defined above. In particular $\phi_N(M; q)$ is a topological
invariant of $M$.

**Remark 4.6.** By definition, $\beta(M) = -\phi_4(M; \sqrt{-1})$ is the Brown invariant [16, §6]. See [2, 10, 28] for Brown's invariant of $\mathbb{Z}/4\mathbb{Z}$-valued quadratic forms on a $\mathbb{Z}/2\mathbb{Z}$-vector space.

As applications of Theorem 4.5, we calculate $Z_N(M; q)$ for $\mathbb{Z}/p\mathbb{Z}$-homology spheres. (A closed, oriented 3-manifold $M$ is called a $\mathbb{Z}/p\mathbb{Z}$-homology sphere if $H_i(M; \mathbb{Z}/p\mathbb{Z}) = H_i(S^3; \mathbb{Z}/p\mathbb{Z})$ for all $i$.)

**Corollary 4.7.** Let $N = 2^m$ and $q = \exp(d\pi\sqrt{-1}/N)$. If $M$ is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, then the value of $Z_N(M; q)$ is as follows.

$$Z_N(M; q) = \begin{cases} \xi^{-d\mu(M)} & \text{if } m \text{ is even,} \\ \omega(|H_1(M; \mathbb{Z})|) \xi^{-\varepsilon(d)\mu(M)} & \text{if } m \text{ is odd.} \end{cases}$$

where $\xi = \exp(\pi\sqrt{-1}/4)$ and $\mu(M)$ is the $\mu$-(or Rochlin) invariant of $M$ (the signature modulo 16 of a spin 4-manifold with boundary $M$).

Proof. Since $H^1(M; \mathbb{Z}/N\mathbb{Z}) = 0$, we calculate the phase. After a change of basis we may assume that $A$ is diagonal (mod $2N$) with diagonal entries $a_j$. Since $M$ is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, $a_j$ is always odd. We also assume that $a_j = 1, 3, 5,$ or $7$ because there exists an odd integer $l$ such that $cl^2 = 1, 3, 5,$ or $7$ mod $2N$ for any odd integer $c$. Let $n_c$ be the number of $c$'s in these diagonal entries ($c = 1, 3, 5,$ or $7$).

For $m$ even, by the definition of $\phi_N(M; q)$, we have

$$\phi_N(M; q) \equiv d(n_1 + 3n_3 + 5n_5 + 7n_7 - \sigma(A)) \mod 8.$$ 

Since $\mu(M) \equiv \sigma(A) - (n_1 + 3n_3 + 5n_5 + 7n_7) \mod 8$ (see [16, Appendix C]), we obtain the required formula. For $m$ odd, we have

$$\phi_N(M; q) = \varepsilon(d)(n_3 - n_3 - 5n_5 - 7n_7 - \sigma(A))$$

Thus $\phi_N(M; q) + \varepsilon(d) \mu(M) \equiv -4\varepsilon(d)(n_3 + n_5) \mod 8$. Since $\varepsilon(d) = \pm 1$, we have

$$\phi_N(M; q) \equiv -\varepsilon(d) \mu(M) + 4(n_3 + n_5) \mod 8.$$ 

Moreover since

$$|H_1(M; \mathbb{Z})| = \pm \det A \equiv \pm 3^{n_3} 5^{n_5} 7^n = \pm 3^{n_3}(-3)^{n_3} (-1)^n \mod 8,$$

we obtain $\omega(|H_1(M; \mathbb{Z})|) = (-1)^{n_3 + n_5}$. Therefore we obtain the required formula.

**Corollary 4.8.** Let $N = p^m$ with odd prime $p$ and $q$ an $N$-th root of unity.
If $M$ is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere, then

$$Z_N(M; q) = \left(\frac{r}{p}\right)^m$$

where $r = |H_1(M; \mathbb{Z})|$ and $\left(\frac{r}{p}\right)$ is Legendre's symbol.

Proof. Adding a splitted, unknotted component if necessary, we assume that $\det A$ is positive so that $r = \det A$. Let $b, a_j's, n_+, \text{ and } n_-$ be as in the notation of the definition of $\phi_p(A; q)$. Since $M$ is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere, $(p, a_j) = 1$ and so $k_j = 0$ for any $j$. We also note that $r = \det A = \prod a_j \mod p$.

Thus we have

$$\left(\frac{r}{p}\right) = \left(\prod a_j \right) = \prod \left(\frac{a_j}{p}\right) = \begin{cases} 1 & \text{if } n_- \text{ is even,} \\ -1 & \text{if } n_- \text{ is odd.} \end{cases}$$

For $m$ even, we have $n_+ = n_- = 0$. Hence $Z_N(M; q) = 1$.

Next we consider the case that $m$ is odd. In this case, $n_+ + n_- = n_-$, the size of $A$. So $n_+ = n - n_-$. We also have $\sigma(A) \equiv 0 \mod 4$ since $\det A > 0$.

If $p \equiv 1 \mod 4$, then by definition, we have

$$\phi_N(M; q) = 2 \left(\left(\frac{b}{p}\right) - 1\right) n_+ - 2 \left(\left(\frac{b}{p}\right) + 1\right) n_- - \left(\left(\frac{b}{p}\right) - 1\right) \sigma(A)$$

$$= 2 \left(\left(\frac{b}{p}\right) - 1\right) (n - \sigma(A)) - 4n_-$$

$$\equiv 4n_- \mod 8.$$ 

If $p \equiv 3 \mod 4$, then we also have

$$\phi_N(M; q) = 2 \left(\frac{b}{p}\right) n_+ - 2 \left(\frac{b}{p}\right) n_- - 2 \left(\frac{b}{p}\right) \sigma(A)$$

$$= 2 \left(\frac{b}{p}\right) (n - \sigma(A)) - 4 \left(\frac{b}{p}\right) n_-$$

$$\equiv 4n_- \mod 8.$$

Therefore we obtain the value of $Z_N(M; q)$ as above, completing the proof. ■

5. Calculation for generators of linking pairings

Any linking pairing is a direct sum of the following linking pairings [36, 14]:

$$(p^{-k} r) (k \geq 1), \ A_k(n) (k \geq 1), \ E_b^k(k \geq 1), \text{ and } E_l^k(k \geq 2),$$

where $p$ is odd, prime integer, $r$ is 1 or a fixed quadratic non-residue modulo $p$, and $n = 1(k = 1), \pm 1(k = 2), \pm 1 \text{ or } \pm 3(k \geq 3)$. Here we use the notation of
Since \(Z_N(M; q)\) is an invariant of first Betti numbers and linking pairings (Proposition 2.5), and linking pairings split as above, we can calculate \(Z_N(M; q)\) if we know them for 3-manifolds with the linking pairings above from Proposition 2.3. Note that the free part of the first homology affects \(Z_N(M; q)\) only by absolute values (Theorem 3.2).

In the following, we denote \(Z_N(M; q)\) by \(Z_N(\Lambda; q)\) if the linking pairing on \(H_1(M; \mathbb{Z})\) is isomorphic to \(\Lambda\) in the above.

**Theorem 5.1.** Let \(p\) and \(p'\) be odd, prime integers \((p' = \ell p')\), and \(b, b'\) integers with \((p, b) = 1\) and \((p', b') = 1\), and \(d\) an odd integer. Put \(q = \exp(2b\pi\sqrt{-1}/p^m)\), \(q' = \exp(2b'\pi\sqrt{-1}/p'^m)\), \(q'' = \exp(d\pi\sqrt{-1}/2^m)\), and \(\zeta = \exp(\pi\sqrt{-1}/4)\). We also use the notations (4.1).

1. The case \(\Lambda = (p^{-k} r)\).

\[
Z_{p^m}((p^{-k} r); q''') = \begin{cases} 
1 & \text{for } (*, 0, *, *), \\
-\omega(p) \left(\frac{r}{p}\right) & \text{for } (*, 1, 0, 1), \\
-\varepsilon(d) \omega(p) \left(\frac{r}{p}\right) \sqrt{-1} & \text{for } (*, 1, 0, 3), \\
\left(\frac{r}{p}\right) & \text{for } (*, 1, 1, 1), \\
\left(\frac{r}{p}\right) \sqrt{-1} & \text{for } (*, 1, 1, 3).
\end{cases}
\]

\[
Z_{r^m}((p^{-k} r); q) = \begin{cases} 
\sqrt{p^m} & \text{for } (+or 0, *, 0, *), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{p^m} & \text{for } (+or 0, *, 1, 1), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^m} & \text{for } (+or 0, *, 1, 3), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{p^k} & \text{for } (-, 0, *, *), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^k} & \text{for } (-, 1, *, 1), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^k} & \text{for } (-, 1, 0, 3), \\
\left(\frac{r}{p}\right) \left(\frac{b}{p}\right) \sqrt{-1} \sqrt{p^k} & \text{for } (-, 1, 1, 3).
\end{cases}
\]

Here \((\cdot, \cdot, \cdot, \cdot)\) is (sign of \(k - m, k\) mod 2, \(m\) mod 2, \(p\) mod 4).

2. The case \(\Lambda = A^k(n), E^k_6,\) or \(E^k_1\).
\[ Z_{2m}(A^4(1); q') = \begin{cases} \xi^{-d} \sqrt{2^m} & \text{for } (+, *, 0), \\ \xi^{-r(d)} \sqrt{2^m} & \text{for } (+, *, 1), \\ 0 & \text{for } (0, *, *) \end{cases} \]
\[ \omega(d) \sqrt{2^k} & \text{for } (-, 0, *), \\ \omega(d) \sqrt{2^k} & \text{for } (-, 1, *). \]
\[ Z_{2m}(A^4(3); q') = \begin{cases} \xi^{-3r(d)} \sqrt{2^m} & \text{for } (+, 1, 1), \\ 0 & \text{for } (0, *, *), \\ \omega(d) \sqrt{2^k} & \text{for } (-, 0, *), \\ \omega(d) \sqrt{2^k} & \text{for } (-, 1, *). \end{cases} \]

Here \((*, *, *)\) is (sign of \(k - m, k \mod 2, m \mod 2\)).

\[ Z_{2m}(A^4(-1); q') = Z_{2m}(A^4(1); q') \] (complex conjugate).
\[ Z_{2m}(A^4(-3); q') = Z_{2m}(A^4(3); q') \]

\[ Z_{2m}(E^4_0; q') = \begin{cases} 2^m & \text{if } k \geq m, \\ 2^k & \text{if } k < m. \end{cases} \]
\[ Z_{2m}(E^4_1; q') = \begin{cases} (-1)^{m+k} 2^m & \text{if } k \geq m, \\ 2^k & \text{if } k < m. \end{cases} \]

\[ Z_{2m}(A^4(n); q) = \begin{cases} -1 & \text{if } m \text{ and } k \text{ are odd, and } p \equiv \pm 3 \mod 8, \\ 1 & \text{otherwise}. \end{cases} \]

\[ Z_{2m}(E^4_0; q) = Z_{2m}(E^4_1; q) = 1 \]

Proof. For \((p^{*-k} r)\), we consider the lens space \(L(p^k, r)\). It can be obtained from a framed link with linking matrix of the form
\[
\begin{pmatrix}
a_1 & 1 & 0 & \cdots & 0 \\
1 & a_2 & 1 & \cdots & 0 \\
0 & 1 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 \\
0 & 0 & \cdots & 1 & a_n
\end{pmatrix}.
\]

Here the continued fraction
\[
a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n}}} \]
is equal to $p^k/r$. See for example [32]. So we can calculate $Z_{p^m}(p^{-k} r); q)$ using Theorem 4.5. The value $Z_{q^m}((p^{-k} r); q')$ can be calculated using Corollary 4.8. $Z_{p^m}(p^{-k} r); q'')$ can be obtained from Corollary 4.7 and the fact

$$2\mu(L(\alpha, \beta)) \equiv 2(\alpha+1) - 4(\beta | \alpha) \mod 16,$$

where $(\beta | \alpha)$ is the Jacobi symbol [12, Theorem 8.14]. Note that our definition of the $\mu$-invariant differs from that in [12].

For $A^k(1)$ and $A^k(3)$, we choose linking matrices of the form

$$(2^k) \quad \text{and} \quad ( -1)^{k+1} \begin{pmatrix} 4^{m+1} - (-2)^k/3 \ 2^{m+1} \\ 3 \end{pmatrix},$$

respectively. (Note that they are diagonal in $Z/2^{m+1} Z$.) Then we can calculate $Z_{p^m}(A^k(n); q') (n=1, 3)$ using Theorem 4.5. Since if the linking pairing for a 3-manifold $M$ is $A^k(n)$, then that for $-M$ is $A^k(-n)$, the values $Z_{p^m}(A^k(n); q') (n=-1, -3)$ are obtained from Proposition 2.1.

To calculate $Z_{q^m}(E_0^k; q') (m \neq k)$, we use the relation (see [14])

$$A^k(1) \oplus 2A^k(-1) = A^k(-1) \oplus E_0^k.$$

Since $Z_{q^m}(A^k(-1); q') \neq 0$ for $m \neq k$, we obtain $Z_{q^m}(E_0^k; q')$ from Proposition 2.2. For $m=k$, we can directly calculate it choosing $(0 2^k \pmatrix{0 \ 2^k \ 0})$ as a linking matrix for $E_0^k$.

Using the relations (see [14] again)

$$3A^k(1) = A^k(3) \oplus E_0^k \quad \text{and} \quad E_0^k \oplus A^k(1) = E_0^k \oplus A^k(-3),$$

we can obtain $Z_{q^m}(E_0^k; q')$ for any $m$.

The values $Z_{p^m}(A^k(n); q), Z_{q^m}(E_0^k; q)$ and $Z_{p^m}(E_0^k; q)$ are easily obtained from Corollary 4.7.

The proof is complete. \[\square\]

**Remark 5.2.** The series $\{Z_N(\cdots); q\}$ is not a complete invariant of linking pairings. For example $Z_N(32A^1(1) \oplus 16A^3(1); q) = Z_N(16A^1(1) \oplus 24A^3(1); q)$ for any $N$ and $q$ but $32A^1(1) \oplus 16A^3(1)$ is not equivalent to $16A^1(1) \oplus 24A^3(1)$.

From Theorem 5.1, we have another condition for $Z_N(M; q)$ to be zero.

**Corollary 5.3.** $Z_N(M; q)=0$ if and only if there exists $x \in H_1(M; Z)$ of order $2^m$ with $\lambda(x, x)=c|2^m$, where $N=2^m b$ with $b$ odd, $c$ is an odd integer, and $\lambda$ is the linking pairing on $\text{Tor} \, H_1(M; Z)$.

Proof. From the above theorem and Proposition 2.2, $Z_N(M; q)=0$ if and only if the linking pairing has a direct summand of the form $A^k(n)$. If $Z_N(M; q)=0$ then the existence of an element $x$ as in the statement of the corollary
follows easily. Conversely, suppose that there exists \( x \) as above. Then since the linking pairing restricted to the cyclic group generated by \( x \) is non-singular, it has \( A^k(n) \) as a direct summand with \( n \equiv c \mod 8 \) (see [36, Lemma (1)]). The proof is complete.

6. Invariants for links. For an oriented link \( L \) in \( S^3 \) (without framing) and an integer \( s(\geq 2) \), one can construct the \( s \)-fold cyclic branched covering space branched along \( L \) associated with the kernel of a map \( H_1(S^3-L; \mathbb{Z}) \to \mathbb{Z}/s\mathbb{Z} \) sending each meridian to 1. Since it is a closed, oriented 3-manifold, we can define \( Z_N(L; q, s) \) to be \( Z_N(M(L, s); q) \), where \( M(L, s) \) is the \( s \)-fold cyclic branched covering space as above. \( Z_N(L; q, s) \) is an invariant of \( L \) for every \( s \) since \( M(L, s) \) is uniquely determined by \( L \) and \( s \).

A framed link description for \( M(L, s) \) is given by S. Akubult and R. Kirby [1]. Denoting a Seifert matrix for \( L \) constructed from a connected Seifert surface by \( V \), its linking matrix is given by \( V \otimes B + t V \otimes t B \), where \( B = (B_{ij}) \) with \( B_{ij} = 1 \) for \( 1 \leq i, j \leq s - 1 \) and \( B_{ij} = 0 \) otherwise. So we have

**Lemma 6.1.**

\[
Z_N(L; q, s) = \left( \frac{G_N(q)}{|G_N(q)|} \right)^{-\sigma(A)} \sqrt{N^{-(s-1)}} \sum_{t \in \mathbb{Z}/N\mathbb{Z}^{s-1}} q^{t|A|},
\]

where \( A = V \otimes B + t V \otimes t B \) and \( g \) is the size of \( V \).

Note that if \( s = 2 \), \( \sigma(A) \) is just \( \sigma(L) \), the signature of \( L \) [29, 34].

In [4], E. Date, M. Jimbo, K. Miki, and T. Miwa define link invariants using generalized chiral Potts models. They are given as follows.

**Definition 6.2.** [4]. Let \( N \) be a positive odd integer, \( q \) a primitive \( N \)-th root of unity, and \( C \) an \((s-1) \times (s-1)\) integral matrix \((s > 1)\). For an oriented link \( L \) with Seifert matrix \( V \) of size \( g \), we put

\[
\tau(L; N, q, s, C) = \sqrt{N^{-(s-1)}} \sum_{t \in \mathbb{Z}/N\mathbb{Z}^{s-1}} q^{t(V \otimes C)}.
\]

Since \( t(V \otimes C + t V \otimes t C) l = 2(\tau(V \otimes C)) l \), we have

**Proposition 6.3.** Let \( q = \exp \left( 2b\pi \sqrt{-1}/N \right) \) and \( q' = \exp \left( (N+1) b\pi \sqrt{-1}/N \right) \) with \( (b, N) = 1 \). Then

\[
Z_N(L; q', s) = \left( \frac{G_N(q')}{|G_N(q')|} \right)^{-\sigma(A)} \tau(L; N, q, s, B),
\]

where \( A = V \otimes B + t V \otimes t B \) and \( B \) is as above. Note that \( q' \) is also a primitive \( N \)-th root of unity because \( N \) is odd.

**Remark 6.4.** For a positive even integer \( N \) and a primitive \( N \)-th root of unity \( q \),
\[ \tau(L; N, q, s, C) = \sqrt[N]{q^{s(s-1)}} \sum_{i \in \{X, X^\prime\}} q^{i+v \otimes c} \]

is also an invariant of a link \( L \). This follows from the fact that the above formula is invariant of \( S \)-equivalence class [3, 29, 34] of Seifert matrices for links. Proposition 6.3 also holds in this case. \((q') \) is now a primitive \( 2N \)-th root of unity.)

The cyclotomic invariant \( T_N(L) \) [19] is given by \( \tau(L; N, \exp(2\pi \sqrt{-1}/N), 2, (1)) \) for an integer greater than 1. (See also [9, 13].) So we have

**Proposition 6.5.** Put \( q = \exp((N+1)\pi \sqrt{-1}/N) \). Then

\[ T_N(L) = \left( \frac{G_N(q)}{|G_N(q)|} \right)^{\sigma(L)} Z_N(L; q, 2). \]

For relations of the cyclotomic invariants to the polynomial invariants for links, see [9, 19].

7. A family of quasitriangular Hopf algebras. We will give another description for \( Z_N(M; q) \) using representations of some algebras. A Hopf algebra \( \mathcal{A} \) is an algebra over a field \( k \) with comultiplication \( \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \), counit \( \varepsilon : \mathcal{A} \rightarrow k \) and antipode \( \gamma : \mathcal{A} \rightarrow \mathcal{A} \). Let \( R \) be an element in \( \mathcal{A} \otimes \mathcal{A} \). The pair \( (A, R) \) is called a quasitriangular Hopf algebra [6] if \( R \) is invertible in \( \mathcal{A} \otimes \mathcal{A} \), \( P \circ \Delta(a) = R \Delta(a) R^{-1} \) for any \( a \in \mathcal{A} \), where \( P \) is the permutation operator \( (P(x \otimes y) = y \otimes x) \), and

\[ (\Delta \otimes \text{id}) (R) = R_{12} R_{23} \]
\[ (\text{id} \otimes \Delta) (R) = R_{12} R_{13} \]
where \( R_{12} = R \otimes 1 \), \( R_{23} = 1 \otimes R \), and \( R_{13} = \sum \alpha_i \otimes 1 \otimes \beta_i \) for \( R = \sum \alpha_i \otimes \beta_i \).

Let \( r \) be a positive integer and \( q \) a primitive \( r \)-th root of unity. We define a quasitriangular Hopf algebra \( \mathcal{A}_q \) over the field \( \mathbb{Q}(q) \). The algebra \( \mathcal{A}_q \) is generated by \( 1, K, \) and \( K^{-1} \) with relation \( K' = 1 \). A comultiplication, counit and antipode are defined by \( \Delta(K) = K \otimes K \), \( \varepsilon(K) = 1 \) and \( \gamma(K) = K^{-1} \), respectively. Let \( R \) be \( r^{-1} \sum_{i,j=0}^{r-1} q^{-ij} K^i \otimes K^j \). Then we have

**Lemma 7.1.** \( (A_q, R) \) is a quasitriangular Hopf algebra.

**Proof.** The inverse element of \( R \) is given by \( r^{-1} \sum_{i,j=0}^{r-1} q^{ij} K^i \otimes K^j \) because

\[ R \cdot r^{-1} \sum_{i',j'} q^{i'j'} K^{i'} \otimes K^{j'} \]
\[ = r^{-2} \sum_{i',j'-ii} q^{i'j'-ij} K^{i+i'} \otimes K^{j+j'} \]
\[ = r^{-2} \sum_{i',j',k} q^{ik}(\sum_j q^{-(i+j)k}) K^{i+i'} \otimes K^k \]
\[ = r^{-1} \sum_k (\sum_i q^{ik}) \cdot 1 \otimes K^k \]
\[ = 1. \]
Here the third and fourth equalities follow from Lemma 3.1.

Since \( A_q \) is commutative, we have \( R \Delta(a) R^{-1} = \Delta(a) = P \circ \Delta(a) \). Moreover

\[
R_{13} R_{23} = r^{-2} \sum q^{-i-j'j'} K^iK^{i'}K^{j+j'}
\]

\[
= r^{-2} \sum q^{-i-k}(\sum q^{(i-j')}) K^iK^{i'}K^k, \quad (k = j + j')
\]

\[
= r^{-1} \sum q^{-ik} K^iK^{i'}K^k
\]

\[
= (\Delta \otimes \text{id})(R).
\]

A similar calculation shows \( (\text{id} \otimes \Delta)(R) = R_{13} R_{12} \).

Since \( A_q \) is commutative, all irreducible representation spaces are one-dimensional. We denote these representations by \( \{V_j\}_{j=0,1,\ldots,r-1} \), with the action \( \rho_j(K) \) given by the multiplication by \( q^j \). For representations \( \rho_i: A_q \to \text{End} (V_i) \) and \( \rho_j: A_q \to \text{End} (V_j) \), a tensor product representation is defined by \( (\rho_i \otimes \rho_j) \circ \Delta: A_q \to \text{End} (V_i \otimes V_j) \). The action \( \rho_j^* \) on the dual space \( V_j^* \) induced from the antipode \( \gamma \) is given by the multiplication by \( q^{-j} \). We can easily see that \( (A_q, R, v, \{V_j\}) \) is a modular Hopf algebra [31] putting \( v = r^{-1} \sum_{i,j=0}^{r-1} q^{i(i-j)} K^i \). With this algebra \( (A_q, R, v, \{V_j\}) \), we can construct invariants of 3-manifolds according to [31]. We survey an outline of the procedure for constructing them.

Let \( L \) be a framed link and consider its diagram. We assume that its framing \( f_i \) of a component \( L_i \) is parallel to \( L_i \) in the plane. A coloring of \( L \) is an assignment of \( V_j \) to each component of \( L \). Now we associate an operator \( \Omega \) with each crossing of a colored framed link as follows.

\[
\begin{align*}
V_i & \quad V_j \\
q^{ij} & \quad \text{(a)}
\end{align*}
\]

\[
\begin{align*}
V_i & \quad V_j^* \\
q^{-ij} & \quad \text{(b)}
\end{align*}
\]

Figure 7.1.

If the crossing is as in Figure 7.1(a), then \( \Omega \) is a homomorphism from \( V_i \otimes V_j \) to \( V_j \otimes V_i \) given by \( x \otimes y \mapsto (P \circ ((\rho_i \otimes \rho_j) R))(x \otimes y) \). It follows that \( \Omega(x \otimes y) = q^{ij}(y \otimes x) \) because

\[
((\rho_i \otimes \rho_j) R)(x \otimes y) = r^{-1} \sum q^{-i'j'}(\rho_i(K^{i'}))x \otimes \rho_j(K^{j'})y
\]

\[
= r^{-1} \sum q^{i'i'} \sum q^{(i-j')j'}(x \otimes y)
\]

\[
= q^{ij}(x \otimes y),
\]
where the last equality follows from Lemma 3.1 again. If the crossing is as in Figure 7.1(b), then $\Omega$ is a homomorphism from $V_t \otimes V^*_t$ to $V^*_t \otimes V_t$ given by $P \circ ((\rho \otimes \rho^*))R$ and we see that $\Omega(x \otimes y^*) = q^{ij}(y^* \otimes x)$. Similar calculations show that if the crossing is positive, then $\Omega$ is the multiplication by $q^{ij}$ (and the interchanging of the coordinate) and if the crossing is negative, then $\Omega$ is the multiplication by $q^{-ij}$.

Then we can obtain an invariant of a 3-manifold as the sum of the products $\prod_{\text{positive crossings}} q^{ij} \prod_{\text{negative crossings}} q^{-i'j'}$ for all colorings after some normalization.

$Z_N(M; q)$ corresponds to this invariant putting $r=2N$ for $N$ even and $r=N$ for $N$ odd.

8. Operator invariants for 3-dimensional cobordism and invariants of Gocho

As in [31] we can extend the invariants $Z_N(M; q)$ to operator invariants of 3-dimensional cobordisms with non-empty parametrized boundaries, using the modular Hopf algebra structure in $A_q$ described in §7. In this section, we define them by using linking matrices, and prove that invariants of T. Gocho [8] are essentially the absolute values of our invariants. See [31, §4] for the precise definition of 3-dimensional cobordisms with parametrized boundaries.

We denote by $G^T_G (G^B_G, \text{resp.})$ a horizontal line segment with $g$ arcs glued to the top (bottom, resp.), which is embedded in $S^3$ as described in Figures 8.1 and 8.2. Each arc has a framing (or parametrization) indicated by a thin line parallel to it in the plane.

Let $G^T_G (G^B_G, \text{resp.})$ be a farmed link obtained by eliminating short segments between arcs from $G^T_G (G^B_G, \text{resp.})$ as in Figures 8.3 and 8.4.
Let \((M, F', F)\) be a 3-dimensional cobordism with connected \(M\) whose parametrized boundaries are \(F'\) and \(F\). For simplicity we assume that \(F'\) and \(F\) are connected surfaces of genus \(g'\) and \(g\) respectively. We can represent \(M\) by Dehn surgery on \(S^3\) as follows. We consider graphs \(G_{g'}^{g}^T\) and \(G_{g}^{g}^T\), and a framed link \(L\) in \(S^3\), where \(L\) is located between \(G_{g'}^{g}\) and \(G_{g}^{g}\) as shown in Figure 8.5. With suitably chosen \(L\), we can put \(M = M_L - (\text{int } N(G_{g'}^{g}) \cup \text{int } N(G_{g}^{g}))\), where \(M_L\) is a 3-manifold obtained by Dehn surgery in \(S^3\) along \(L\), and \(N(G_{g'}^{g})\) and \(N(G_{g}^{g})\) are tubular neighborhoods of \(G_{g'}^{g}\) and \(G_{g}^{g}\) respectively.

Let \(V_{g}\) be an \(N\)-dimensional complex vector space with basis \(\{e_h\}\), where \(N\) is an integer greater than 1 and \(h \in (\mathbb{Z}/N\mathbb{Z})^g\). \(V_{g}^{\ast}\) is its dual with dual basis \(\{e_h^{\ast}\}\). We define an operator invariant of \(M\) in \(V_{g}^{\ast} \otimes V_{g} \cong \text{Hom}(V_{g'}, V_{g})\) by

\[
Z_{N}(M; q) = \left( \frac{G_{N}(q)}{|G_{N}(q)|} \right)^{-\sigma(A)} \left[ G_{N}(q) \right]^{-\langle q' \rangle_{\mathbb{Z}} - \langle q \rangle_{\mathbb{Z}}} \sum_{h' \in (\mathbb{Z}/N\mathbb{Z})^g} \sum_{i' \in (\mathbb{Z}/N\mathbb{Z})^g} \sum_{j' \in (\mathbb{Z}/N\mathbb{Z})^g} \left( \frac{A_{i' j'}}{h'} \right) e_{h'}^{\ast} \otimes e_{h},
\]

where \(q\) and \(G_{N}(q)\) are as in §1, \(A\) is the linking matrix of \(G_{g'}^{g} \cup L \cup G_{g}^{g}\), and \(n\)
is the number of components of $L$. In a similar way as the proof of Theorem 1.3, we can show that this is a topological invariant of $M$ as a 3-dimensional cobordism with parametrized boundary.

The following proposition is a corollary to [31, Theorem 4.5]. We give a direct proof using the formula above.

**Proposition 8.1.** If a 3-dimensional cobordism $(M, F_1, F_3)$ is a composition of two cobordisms $(M_1, F_1, F_2)$ and $(M_2, F_2, F_3)$, then for some integer $c$

$$Z_N(M; q) = \zeta^c Z_N(M_2; q) \circ Z_N(M_1; q),$$

where $Z_N(M_1; q) \in V_{g_1} \otimes V_{g_2} = \text{Hom}(V_{g_1}, V_{g_2})$, $Z_N(M_2; q) \in \text{Hom}(V_{g_2}, V_{g_3})$, $g_i$ is the genus of $F_i$, and $\zeta = \exp(\pi\sqrt{-1}/4)$.

**Proof.** For simplicity, we assume that $F_1 = F_3 = 0$. We present $M_1$ and $M_2$ by $L_1 \cup G^T_{\mathbb{R}_1}$ and $G^B_{\mathbb{R}_2} \cup L_2$ respectively, where $M_1 = M_{1 \cup \text{int} N(G^T_{\mathbb{R}_1})}$ and $M_2 = M_{2 \cup \text{int} N(G^B_{\mathbb{R}_2})}$. Then $M$ is presented by a framed link $L_1 \cup L_0 \cup L_2$, where $L_0$ is a framed link obtained from $G^T_{\mathbb{R}_1}$ and $G^B_{\mathbb{R}_2}$ by gluing arcs as shown in Figure 8.6.

![Figure 8.6](image)

Let $A, A_1,$ and $A_2$ be the linking matrices of $L_1 \cup L_0 \cup L_2, L_1 \cup G^T_{\mathbb{R}_1},$ and $G^B_{\mathbb{R}_2} \cup L_2$ respectively. We have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix},$$

where 0's are zero matrices with suitable sizes. Hence we have

$$\begin{pmatrix} l_1 \\ h \end{pmatrix} A \begin{pmatrix} l_1 \\ h \end{pmatrix} = \begin{pmatrix} l_1 \\ h \end{pmatrix} A_1 \begin{pmatrix} l_1 \\ h \end{pmatrix} + \begin{pmatrix} l_2 \\ h \end{pmatrix} A_2 \begin{pmatrix} l_2 \\ h \end{pmatrix}.$$
(G_N(q)| G_N(q)|)^{\sigma(A_1) - \sigma(A_2)}$. Since the phase of a Gaussian sum has a value of eighth root of unity, we obtain the required formula.

Let $M_g$ be the mapping class group of a closed surface of genus $g$. With this proposition we obtain a representation of $M_g$ to $PU(V_g) = U(V_g)/U(1)$ as follows. Let $F$ be a closed surface with parametrization of genus $g$ and $f: F \to F$ a homeomorphism. We denote by $C_f$ the mapping cylinder of genus $g$ that is, $F \times [0, 1]$ with parametrization in $F \times \{1\}$ induced by $f$. For fixed $N$ and $q$, we have a map $M_g \to PU(V_g), f \mapsto Z_N(C_f; q)$. By Proposition 8.1 this map becomes a representation.

In the case that $N$ is even and $q = \exp(\pi\sqrt{-1}/N)$, this representation coincides with a representation constructed by T. Gocho [8]. Let $N$ and $q$ as above in the following of this section. By a geometric method based on $U(1)$ gauge theory, Gocho constructed a representation $\rho_g$ of $M_g$ to $PU(V_g)$ which factors $Sp(2g; \mathbb{Z}) \to f^* H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$. The representation $\rho_g: Sp(2g; \mathbb{Z}) \to PU(V_g)$ is given by the next formulas.

\[
\begin{align*}
\rho_g \begin{pmatrix} 0 & -e^g \\ e^g & 0 \end{pmatrix} e_h &= \sqrt{N}^{-1/2} \sum_{h'} q^{g^{h'h}} e_{h'}, \\
\rho_g \begin{pmatrix} X & 0 \\ 0 & tX^{-1} \end{pmatrix} e_h &= e^{X^{-1} h}, \\
\rho_g \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} e_h &= q^{-1/2} Y^{h'} e_h.
\end{align*}
\]

Here $X \in GL(g; \mathbb{Z})$ and $Y$ is a $g \times g$ symmetric integral matrix. Note that 
\[
\begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}, \begin{pmatrix} X & 0 \\ 0 & tX^{-1} \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}
\]
generate $Sp(2g; \mathbb{Z})$. We can check that this representation coincides with our representation by calculating about generators of $M_g$. In [8], Gocho also defines a topological invariant of $M$ by

\[
W_N(M) = \sqrt{N}^{-1} \langle \rho_g(f_0) e_0, e_0^* \rangle \in C/U(1),
\]

where $M$ is presented by a Heegaard splitting $M = H_g \cup (-H_g)$ with $H_g$ a handlebody of genus $g$. Noting that $W_N(S^3) = \sqrt{N}^{-1}, Z_N(H_g; q) = \sqrt{N^{1/2}} e_0,$ and 
\[
Z_N(-H_g; q) = \sqrt{N^{1/2}} e_0^*,
\]
we immediately have the next proposition.

**Proposition 8.2.** Let $N$ be even. Then we have

\[
\frac{W_N(M)}{W_N(S^3)} = |Z_N(M; \exp \frac{\pi \sqrt{-1}}{N})|,
\]

where $W_N(M)$ is Gocho's invariant defined above.
9. Invariants of Dijkgraaf and Witten for $G = \mathbb{Z}/N\mathbb{Z}$.

In this section we will show relations between our invariants and invariants of R. Dijkgraaf and E. Witten.

Let $G$ be $\mathbb{Z}/N\mathbb{Z}$. We choose a class $q \in H^1(BG, U(1))$. Since $H^1(BG, U(1)) \cong H^1(\mathbb{Z}/N\mathbb{Z})$ (see for example [11, Lemma 9.2]) for a classifying space $BG$ for $G$, we regard $q$ as a (not necessarily primitive) $N$-th root of unity with an inclusion $\mathbb{Z}/N\mathbb{Z} \to U(1)$. Let $M$ be a closed orientable 3-manifold. In [5], Dijkgraaf and Witten defined invariants as the sum over all possible $G$ bundles over $M$:

$$D_N(M; q) = \sum_{\gamma \in \text{Hom}(\mathbb{Z}/N\mathbb{Z}, G)} \langle f^*_\gamma, [M] \rangle \in \mathbb{C},$$

where $f_\gamma : M \to BG$ is a classifying map corresponding to $\gamma$ and $\langle f^*_\gamma, [M] \rangle \in U(1)$. We regard $U(1)$ as the set of units in $\mathbb{C}$ and the sum is taken in $\mathbb{C}$.

**Proposition 9.1.** Let $N$ be a positive integer, $K$ a divisor of $N$, and $q$ an $N^2$-th (primitive) root of unity. Then the following formulas hold.

For $N$ odd $D_N(M; q^NK) = Z_N^h(M; q^N)$.

For $N$ even $D_N(M; q^NK) = Z_N^h(M; q^K)$.

Before we prove this proposition, we show some lemmas. Since $H^1(M; \mathbb{Z}/N\mathbb{Z}) \cong H^1(M; \mathbb{Z})$, we denote by $\tilde{\gamma}$ the corresponding element to $\gamma$ in $H^1(M; \mathbb{Z}/N\mathbb{Z})$.

**Lemma 9.2.**

$$\langle f^*\tilde{\gamma}, [M] \rangle = q^{\delta \ast \gamma \ast (\delta \ast \gamma)^{-1}(M)}$$

where $\delta : H^1(M; \mathbb{Z}/N\mathbb{Z}) \to H^1(M; \mathbb{Z})$ is the connecting homomorphism with respect to an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \to 0$ and $\cup : H^1(M; \mathbb{Z}/N\mathbb{Z}) \times H^1(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/N\mathbb{Z})$.

**Proof.** Let $\gamma' \in \text{Hom} (\mathbb{Z}/N\mathbb{Z}, G) = \text{Hom} (G, G)$ be the identity map which is the monodromy representation of a classifying space $EG \to BG$. We denote by $\tilde{\gamma}'$ a corresponding element to $\gamma'$ in $H^1(BG, G)$. Some calculations show that $\gamma' \cup \delta \ast (\gamma')$ is a generator of $H^1(BG, G) \cong \mathbb{Z}/N\mathbb{Z} \cong H^1(BG, U(1))$, where $\delta : H^1(BG, \mathbb{Z}/N\mathbb{Z}) \to H^1(BG, \mathbb{Z})$ is the connecting homomorphism. Let $q$ be $\exp(m \cdot 2\pi \sqrt{-1}/N) \in H^1(BG, U(1)) \subset U(1)$. Then $f^*_\gamma q = \exp(m(\tilde{\gamma} \cup \delta \ast (\tilde{\gamma}))) \cdot 2\pi \sqrt{-1}/N \in H^2(M, U(1)) = U(1)$ because $\tilde{\gamma} = f^*_\gamma \tilde{\gamma}'$. Hence we have the required formula. $\blacksquare$

The following lemma is obtained in a similar way as a proof of Lemma 3.4.

**Lemma 9.3.** Let $l \in \ker L_A \subset (\mathbb{Z}/N\mathbb{Z})^*$ be the corresponding element to $\tilde{\gamma}$ under the isomorphism $\iota$ in Lemma 3.3. Then we have
where $\bar{l} \in \mathbb{Z}^n$ is a lift of $l$ and $A$ is the linking matrix of the framed link.

Proof of Proposition 9.1. By Lemmas 3.3, 9.2, and 9.3, we have

$$D_N(M; q^{NK}) = \sum_{\bar{l} \in \ker L_A} q^{\bar{l}^tA\bar{l}}$$

with $L_A: (\mathbb{Z}/N\mathbb{Z})^n \to (\mathbb{Z}/N\mathbb{Z})^n$, $l \mapsto Al$.

For $N$ odd, we have

$$Z_{N^{2/4}}(M; q^K)Z_{K}(M; q^{-N^2/K})$$

$$= (\frac{\Gamma}{|\Gamma|})^{-\sigma} |\Gamma|^{-\sigma} \sum_{l_1 \in (\mathbb{Z}/N\mathbb{Z})^n} q^{l_1^tA_{l_1}l_1} \sum_{l_2 \in (\mathbb{Z}/N\mathbb{Z})^n} q^{-N^2K^{-1}l_2l_2}$$

$$= (\frac{\Gamma}{|\Gamma|})^{-\sigma} |\Gamma|^{-\sigma} \sum_{l_1} q^{l_1^tA_{l_1}l_1+2N^2l_1^tA_{l_1}} (l_1 = l_1 + NK^{-1}l_2)$$

$$= (\frac{\Gamma}{|\Gamma|})^{-\sigma} |\Gamma|^{-\sigma} \sum_{l_1} q^{l_1^tA_{l_1}l_1} \sum_{l_2} q^{2N^2(l_1^t+l_2^t)A_{l_1}} (l_1 = h + Nl_2)$$

$$= (\frac{\Gamma}{|\Gamma|})^{-\sigma} |\Gamma|^{-\sigma} N^2 \sum_{\bar{l} \in \ker L_A} q^{\bar{l}^tA_{\bar{l}}\bar{l}}$$

where $\Gamma = G_{N^{2/4}}(q^K)G_K(q^{-N^2/K})$. Similar calculations show $\Gamma = N$. Hence we obtain the required formula.

For $N$ even the required formula is obtained in a similar way. ■

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