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FILTRATIONS ON THE LOG DE RHAM COMPLEX

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Abstract

Given a log smooth log scheme X over $\operatorname{Spec}\mathbb{C}$, in this article we analyze and compare different filtrations defined on the log de Rham complex $\omega_X^\bullet$ associated to X. We mainly refer to the articles of Ogus ([23]), Danilov ([1]), Ishida ([16]). In this context, we analyze two filtrations on $\omega_X^\bullet$: the decreasing Ogus filtration $\tilde{L}^\bullet$, which is a sort of extension of the Deligne weight filtration $W_\bullet$ to log smooth log schemes over $\operatorname{Spec}\mathbb{C}$, and an increasing filtration, which we call the Ishida filtration and denote by $I_\bullet$, defined by using the Ishida complex $\tilde{\Omega}_X^\bullet$ of X. Moreover, we have the Danilov de Rham complex $\Omega_X^\bullet(\log D)$ with logarithmic poles along $D = X - X_{\mathrm{triv}}$ (X_{triv} being the trivial locus for the log structure on X), endowed with an increasing weight filtration (the Danilov weight filtration $\mathcal{W}_\bullet$). Then we prove that the Danilov de Rham complex $\Omega_X^\bullet(\log D)$ coincides with the log de Rham complex $\omega_X^\bullet$ and the Ishida filtration $I_\bullet$ (which is a globalization of the Danilov weight filtration $\mathcal{W}_\bullet$) coincides with the opposite Ogus filtration $\tilde{L}^{-\bullet}$.

Introduction and motivations

Given an algebraic variety X, even singular over the complex field $\mathbb{C}$, Deligne described the mixed Hodge structure of it by using a smooth hyper-covering $\pi_\bullet\colon X_\bullet \to X$ of X ([3, §8.2]), and by applying the theory he had developed in [2], at each term X_i of the hyper-resolution. By descent theory, he showed that the mixed Hodge structure on X^{an} comes from the Hodge structure of each X_i^{an}. Indeed, for every i, he considered an open immersion of X_i into a proper smooth scheme $\overline{X}_i$ over $\operatorname{Spec}\mathbb{C}$, whose complement $\overline{X}_i - X_i$ is a normal crossing divisor D_i (by [9]), and analyzed the Hodge and weight filtrations on each de Rham complex with logarithmic poles $\Omega_{\overline{X}_i}^\bullet(\log D_i)$. Then, the mixed Hodge structure on $H^\bullet(X^{\mathrm{an}},\mathbb{C})$ is related to the Hodge structures on $H^\bullet(X_i^{\mathrm{an}},\mathbb{C}) \cong \mathbb{H}^\bullet\bigl(\overline{X}_i, \Omega_{\overline{X}_i}^\bullet(\log D_i)\bigr)$, for each i.

A similar approach was used by Du Bois ([4]). He introduced a category $\mathrm{C}_{\mathrm{diff}}(X)$, which can be seen as a "filtered version" of the Herrera-Lieberman category ([8]): the objects of $\mathrm{C}_{\mathrm{diff}}(X)$ are filtered complexes, and the morphisms are $\mathcal{O}_X$-linear maps of complexes, which are compatible with the filtration. Working in this filtered category, Du Bois proved that a part of the mixed Hodge structure (namely, the Hodge filtra-

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tion in cohomology) of a singular variety X over $\operatorname{Spec}\mathbb{C}$ can be described by using a complex, $\underline{\underline{\Omega}}^\bullet_X$, which belongs to the derived category $\mathrm{D}_{\mathrm{diff}}(X)$. This complex is also constructed through the use of a proper smooth hyper-covering $\pi_\bullet\colon X_\bullet \to X$, by taking the classical de Rham complex $\Omega^\bullet_{X_i}$ of each X_i, and defining $\underline{\underline{\Omega}}^\bullet_X$ as the direct image $\mathbb{R}\pi_{\bullet *}\Omega^\bullet_{X_\bullet}$. The filtration on $\underline{\underline{\Omega}}^\bullet_X$ comes from the natural Hodge filtration F defined on each complex $\Omega^\bullet_{X_i}$. In the same direction, Guillen, Puerta, Aznar, Gainza studied the construction of a particular kind of proper (smooth) hyper-coverings $X_\bullet \to X$ of a singular variety X, called the cubic hyper-resolutions of X: these are characterized by a control on the dimension of each term X_i ([7]).

Another approach to this problem could consist in the tentative of characterizing the mixed Hodge structure of an algebraic singular scheme X only in terms of its own structural geometry, without introducing resolutions of singularities or hyper-coverings of X; one can analyze a particular setting of algebraic schemes, endowed with a "richer" structure, which can furnish more informations about the singularities of the scheme. In this direction, in more recent years, the notion of scheme and the properties of schemes have been generalized by the introduction of logarithmic geometry. Briefly, a logarithmic scheme X is a classical scheme, endowed with a further structure which consists of a sheaf of commutative monoids M_X on the etale (or Zariski) site X_{et} (or X_{Zar}) of X, together with a monoid homomorphism $\alpha\colon M_X \to \mathcal{O}_X$, satisfying a certain condition (Definition 1.1). Therefore, logarithmic geometry can be thought as an extension of the classical theory of algebraic schemes. Classical examples of logarithmic schemes are a smooth variety X, with log structure induced by a normal crossing divisor D, which is exactly the case analized by Deligne in [2]. Another interesting example comes from the toroidal geometry; indeed, a singular toric variety X, endowed with log structure induced by the complement D of the torus ([16], [22]) is a good example of log scheme. Indeed, we have a strict connection between log schemes over $\mathbb{C}$ and toroidal embeddings or semi-toroidal varieties, as we show in [5]. In the previous two cases, the logarithmic de Rham complex $\Omega^\bullet_X(\log D)$, which is a locally free $\mathcal{O}_X$-module, plays a central role in the construction of the canonical functorial mixed Hodge structure on the cohomology groups $H^n(X - D, \mathbb{C})$, $n \in \mathbb{N}$. Later, Steenbrink ([24], [25], [26]) described his limit mixed Hodge structure of a projective family of manifolds with semi-stable reduction by means of log theory. These are only simple examples of logarithmic structures, but the full formalism and language of logarithmic schemes was introduced by J.M. Fontaine, L. Illusie, and K. Kato ([19]). They presented a general formulation of logarithmic structures, not only in characteristic zero. Successively, the theory of logarithmic schemes was developed by F. Kato [17], [18], C. Nakayama [21], L. Illusie [12], [13], O. Hyodo [10], K. Kato [11], [20], A. Ogus [23], and others. In this "log" setting, a singular scheme X over $\operatorname{Spec}\mathbb{C}$, if endowed with a particular structure, can be regarded as "smooth" in the logarithmic sense, i.e. log smooth in the category of log schemes.

The most important fact here, is that, for a log scheme X, it is possible to define a meaningful de Rham complex with "formal" log poles, which are determined by the monoid sheaf M_X. Indeed, when a log scheme X satisfies the condition of being *log smooth* or *ideally log smooth* (see the notion of idealized log schemes in §1.4) over the base scheme, then its log de Rham complex is formed by locally free $\mathcal{O}_X$-modules. Therefore, we are able to describe this log complex, even if we have no information about the classical de Rham complex $\Omega^\bullet_X$ associated to the possibly singular scheme X. In this sense, the theory of log schemes was also developed in order to furnish a singular variety X with a "natural de Rham complex".

Therefore, in this present work we analyze and compare the possible filtrations defined on the log de Rham complex $\omega^\bullet_X$ associated to a log smooth log scheme X over $\operatorname{Spec}\mathbb{C}$. We mainly refer to the articles of Ogus ([23]), Danilov ([1]), Ishida ([16]). In this context, we first briefly recall the main definitions and notions about log schemes and idealized log schemes (§1). Then, we start to consider the decreasing Ogus filtration $\tilde{L}^\bullet$ on $\omega^\bullet_X$ (Definition 2.1). We prove that the opposite Ogus filtration $\tilde{L}^{-\bullet}$ coincides with the Deligne weight filtration $W_\bullet$ in the case of a smooth scheme X, with log structure given by a normal crossing divisor (Lemma 2.2). Therefore, $\tilde{L}^{-\bullet}$ extends the Deligne weight filtration to the case of a log smooth log scheme whose underlying scheme is not smooth over $\mathbb{C}$.

Then, given a log smooth log scheme X over $\operatorname{Spec}\mathbb{C}$, if X_{triv} denotes the trivial locus for the log structure and $D = X - X_{\text{triv}}$, we recall the definition of the Danilov de Rham complex $\Omega^\bullet_X(\log D)$ with logarithmic poles along D (Definition 2.9), and we prove that this complex coincides with the log de Rham complex $\omega^\bullet_X$ (Proposition 2.10). Then, by using the definition of Ishida complex $\tilde{\Omega}^\bullet_X$ associated to the toroidal embedding (X_{triv}, X), we introduce an increasing filtration $I_\bullet$ on $\omega^\bullet_X$, which we call the Ishida filtration (Definition 2.7). So, by using local descriptions, we show that this filtration is a sort of globalization of a particular weight filtration $\mathcal{W}_\bullet$ on $\Omega^\bullet_X(\log D) = \omega^\bullet_X$, introduced by Danilov ([1, §15.6]) in the toric case. Finally, we prove that $I_\bullet$ coincides with the opposite Ogus filtration $\tilde{L}^{-\bullet}$ (Lemma 2.11 and Proposition 2.12). We conclude by describing the graded terms of the Ogus filtration for log smooth log schemes whose underlying schemes are proper and quasi-smooth. In this case, the (Deligne) mixed Hodge structure of X_{triv} (the trivial locus for the log structure on X) can be calculated by using the bifiltered log de Rham complex $(\omega^\bullet_X, \tilde{L}^{-\bullet}, F^\bullet)$ ($F^\bullet$ being the Hodge filtration).

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1. The language of log schemes

In this first section we briefly recall the definitions and main results about logarithmic schemes which will be useful in the following. We use the Zariski or the etale topology.

1.1. General notions.

DEFINITION 1.1. Let X be a scheme. A pre-log structure on it is a sheaf of monoids M_X on X, together with a monoid homomorphism $\alpha\colon M_X \to \mathcal{O}_X$ (where $\mathcal{O}_X$ is considered as a multiplicative monoid). The pre-log structure (M_X, α) is said to be a log structure if α induces an isomorphism of $\alpha^{-1}(\mathcal{O}_X^*)$ onto $\mathcal{O}_X^*$. In this case we identify $\alpha^{-1}(\mathcal{O}_X^*)$ with $\mathcal{O}_X^*$, and suppose $\mathcal{O}_X^*$ to be a submonoid of M_X. A log scheme X is a scheme endowed with a log structure.

We denote the log scheme X, endowed with the log structure M by (X, M), or simply by X. The trivial structure on a scheme X (denoted by *triv*) is the log structure equal to $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$. The inclusion functor from the category of log structures on a scheme X into the category of the pre-log structures on X has a left adjoint which sends a pre-log structure (M, α) into (M^a, α^a), where $M^a =: \mathcal{O}_X^* \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} M$, and $\alpha^a =: inc \oplus \alpha$, with *inc* the inclusion of $\mathcal{O}_X^*$ inside $\mathcal{O}_X$. This log structure (M^a, α^a) is called the log structure associated to the pre-log structure (M, α). A morphism $f\colon (X, M) \to (Y, N)$ is a morphism of log schemes if f is a morphism of underlying schemes and there is a morphism of sheaves of monoids $g\colon f^{-1}N \to M$ such that the diagram

$$\begin{tikzcd} f^{-1}N \arrow[r, "g"] \arrow[d] & M \arrow[d] \\ f^{-1}(\mathcal{O}_Y) \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

is commutative. Therefore, the log schemes form a category, denoted by **LSch**, and the functor from the category of schemes **Sch** to **LSch**, which sends a scheme into itself endowed with the trivial structure, is fully faithful. The category **LSch** has *finite inverse limits* ([19, (1.6)]).

DEFINITION 1.2. Let $f\colon (X, M) \to (Y, N)$ be a morphism of log schemes (with $\alpha\colon M \to \mathcal{O}_X$, $\beta\colon N \to \mathcal{O}_Y$). The pre-log structure on Y defined by the fiber product $\mathcal{O}_Y \times_{f_*\mathcal{O}_X} f_*M$ and the homomorphism into $\mathcal{O}_Y$ induced by the projection, is a log structure on Y, which is called the direct image of (M, α), and which is denoted by $(f_*M, f_*\alpha)$. The log structure on X associated to the pre-log structure defined by $f^{-1}N$ and the composed homomorphism $f^{-1}N \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is called the inverse image of (N, β), and it is denoted by $(f^*N, f^*\beta)$.

We have this identification

$$\mathrm{Hom}((N,\beta),(f_*M,\ f_*\alpha)) \cong \mathrm{Hom}((f^*N,\ f^*\beta),(M,\alpha))$$

DEFINITION 1.3. A morphism $f: X \to Y$ of log schemes is called strict iff $f^*M_Y \to M_X$ is an isomorphism.

Every morphism of log schemes $f: X \to Y$ uniquely factorizes into $X \xrightarrow{i} \overline{X} \xrightarrow{\bar{f}} Y$, where i is the identity on the underlying schemes, and $\bar{f}$ is strict.

DEFINITION 1.4. A monoid P is said to be integral iff the canonical morphism $P \to P^{\mathrm{gp}}$ is injective. It is equivalent to say that P satisfies the following condition: for any $p,q,x \in P$ such that $xp = xq$, then $p = q$. Moreover, a monoid P is said to be saturated iff it is integral and, for each $p \in P^{\mathrm{gp}}$, $p \in P$ iff there exists an integer $n \geq 1$ such that $p^n \in P$.

DEFINITION 1.5. A log structure M is called integral (resp. saturated) if M is a sheaf of integral (resp. saturated) monoids.

1.2. Charts. The notion of chart is introduced by K. Kato in [19, Definition 2.9]: it gives a (local) model for the logarithmic structure.

DEFINITION 1.6. Let X be a log scheme, and P be a monoid. A strict morphism $c: X \to \operatorname{Spec} \mathbb{Z}[P]$ is said to be a chart of X (relative to the monoid P).

The datum of a chart of (X,M) is equivalent to the datum of an homomorphism $P_X \to M$, where P_X is the constant sheaf of monoids on X of value P, inducing an isomorphism $(P_X \to \mathcal{O}_X)^a \xrightarrow{\cong} (M \xrightarrow{\alpha} \mathcal{O}_X)$. Therefore, the log structure on X is equal to the inverse image of the canonical log structure $P \to \mathbb{Z}[P]$.

If $\varphi: U \to X$ is an etale map, where U is a log scheme endowed with the log structure $\alpha_U: M_U \to \mathcal{O}_U$ induced by M, then a chart of X over U is a chart of (U,M_U).

DEFINITION 1.7. A log structure M on a scheme X is said to be quasi-coherent (resp. coherent) if, etale locally on X, there exists a chart of X relative to a monoid (resp. finitely generated monoid) P. Moreover, M is called fine (resp. fs) if it is coherent and integral (resp. coherent and saturated) (Definition 1.5). We say a log scheme X is fine (resp. fs) if it is endowed with a fine (resp. fs) log structure.

We denote with $\mathbf{LSch}^{\mathbf{f}}$ (resp. $\mathbf{LSch}^{\mathbf{fs}}$) the category of fine (resp. fine and saturated) log schemes over $\mathbb{C}$. They are full subcategories of $\mathbf{LSch}$.

DEFINITION 1.8. Let $f\colon X \to Y$ be a morphism of log schemes, and let $u\colon P \to Q$ be a monoid morphism. Then a chart of f (relative to u) is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & \operatorname{Spec} \mathbb{Z}[Q] \arrow[d] \\ Y \arrow[r, "b"] & \operatorname{Spec} \mathbb{Z}[P] \end{tikzcd}$$

with a, b two charts.

Proposition 1.9 ([12, Corollaire 3.11]). *Let $f\colon X \to Y$ be a morphism of fine (resp. fs) log schemes, and let x be a geometric point of X. Then, there exists an etale neighbourhood of x, and a chart of f relative to a monoid morphism $u\colon P \to Q$, with P, Q fine monoids, and P^{gp}, Q^{gp} without torsion (resp. with P, Q toric monoids).*

1.3. Log smooth log schemes and differentials.

DEFINITION 1.10. A morphism of log schemes $i\colon (X, M) \hookrightarrow (Y, N)$ is called a closed immersion (resp. an exact closed immersion) if the underlying map of schemes is a closed immersion, and $i^*N \to M$ is surjective (resp. is an isomorphism).

The notions of smoothness and etaleness are extended to the category **LSch** ([19, (3.3)]).

DEFINITION 1.11. A morphism $f\colon (X,M) \to (Y,N)$ of fine log schemes is called log smooth (resp. log etale) if its underlying morphism is locally of finite presentation, and, for any commutative diagram of fine log schemes

$$\begin{tikzcd} (T_0, L_0) \arrow[r, "g"] \arrow[d, "i"'] & (X, M) \arrow[d, "f"'] \\ (T, L) \arrow[r, "h"] & (Y, N) \end{tikzcd}$$

with i an exact closed immersion, and T_0 defined in T by an ideal I such that $I^2 = (0)$, there exists, etale locally on T (resp. there exists globally on T) an unique $\varphi\colon (T,L) \to (X, M)$ such that $\varphi i = g$ and $f\varphi = h$.

We have a characterization of log smoothness (resp. log etaleness) ([19, Theorem (3.5)]), which is the following

Theorem 1.12. *Let $f\colon (X, M) \to (Y, N)$ be a morphism of fine log schemes, and let $Y \to \operatorname{Spec} \mathbb{Z}[Q]$ be a chart of Y relative to Q. Then the following conditions are equivalent*

(1) *f is log smooth*;
(2) *Etale locally on X, there exists a chart $\{P_X \to M,\ Q_Y \to N,\ u\colon Q \to P\}$ of f, extending the chart of Y, such that*:

($\star$) *the kernel and the torsion part of the cokernel* (*resp. the kernel and the cokernel*) *of $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ are finite groups of orders invertible on X*;

($\star\star$) *the induced morphism $X \to Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is etale.*

(*We note that we can replace the etaleness in* ($\star\star$), *by the smoothness, without changing the conclusions*).

REMARK 1.13. We recall that, when $f\colon (X,M) \to (Y,N)$ is a morphism of fine log schemes such that $f^*N \cong M$, then f is log smooth (resp. log etale) iff the underlying morphism of schemes is smooth (resp. etale) ([19, Proposition (3.8)]).

Let $f\colon X \to Y$ be a morphism of log schemes, where $\alpha\colon M \to \mathcal{O}_X$, $\beta\colon N \to \mathcal{O}_Y$ are the two log structures.

DEFINITION 1.14. The $\mathcal{O}_X$-module $\Omega^1_{X/Y}(\log M/N)$, which we simply denote by $\omega^1_{X/Y}$, is the quotient $\left(\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})\right)\big/H$, where $\Omega^1_{X/Y}$ is the classical sheaf of relative 1-differential forms, and H is the $\mathcal{O}_X$-submodule generated by the local sections of the following forms:
(1) $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$, with $m \in M$;
(2) $(0, 1 \otimes m)$, with $m \in \operatorname{Im}\{f^{-1}N \to M\}$.
The class of $(0, 1 \otimes m)$, for $m \in M$, in $\omega^1_{X/Y}$, is usually denoted by $\operatorname{dlog} m$. Let now $\omega^p_{X/Y} =: \bigwedge^p \omega^1_{X/Y}$, for each $0 \le p \le \dim_Y X$: we get a complex $\omega^{\bullet}_{X/Y}$ of $f^{-1}\mathcal{O}_Y$-modules in the natural way.

In the category **Sch** of classical schemes, the property for a map $f\colon X \to Y$ of being smooth implies that $\Omega^1_{X/Y}$ is locally free $\mathcal{O}_X$-module. Similarly, in the category of log schemes, we have the following

Proposition 1.15 ([19, Proposition (3.10)]). *Let $f\colon (X,M) \to (Y,N)$ be a log smooth morphism of fine log schemes. Then the $\mathcal{O}_X$-module $\omega^1_{X/Y}$ is locally free of finite type.*

1.4. Idealized log schemes. The notion of log schemes was generalized even further by Ogus in [23]. He introduced the notion of idealized log scheme, which consists on a log scheme together with a fixed sheaf of ideals of the log structure. We recall some facts about this theory which will be useful for the following. Concerning the theory of commutative monoids, we refer to [20], [5].

DEFINITION 1.16. Let P be a commutative monoid. A subset I of a monoid P is said to be an ideal of P if $PI \subset I$. Moreover, an ideal $\mathfrak{p}$ of P is called a prime ideal if its complement $P \setminus \mathfrak{p}$ in P is a submonoid of P.

DEFINITION 1.17. Let P be a monoid. A face F of P is a subset of P which is the complement of a prime ideal $\mathfrak{p}$ of P, i.e. it is a submonoid of P whose complement is an ideal of P. Therefore, we can regard a prime ideal of a monoid P as the complement of a face of P.

Let now (X, M) be a log scheme over $\mathbb{C}$ and let $\beta\colon P \to M$ be a fine chart for M. If I (resp. F) is an ideal (resp. a face) of P, let $\tilde{I}$ (resp. $\tilde{F}$) denote the sheaf associated to the presheaf which is equal to the ideal of $M(U)$ generated by the image of I (resp. F) in $M(U)$, for every open set U of X. Then $\tilde{I}$ (resp. $\tilde{F}$) is a sheaf of ideals (resp. faces) in the sheaf of monoids M ([23, §2]).

DEFINITION 1.18 ([23, Definition 2.8]). An idealized log scheme is a log scheme (X, M) together with a sheaf of ideals $K_X \subseteq M$ such that $\alpha\colon M \to \mathcal{O}_X$ sends K_X to $\{0\} \subset \mathcal{O}_X$. A morphism $f\colon (X, M) \to (Y, N)$ of idealized log schemes is a morphism of log schemes such that the map $f^{-1}N \to M$ sends $f^{-1}(K_Y)$ into K_X.

We say that f is ideally strict if K_X is generated by the image of $f^{-1}K_Y \to M$, f is log strict if the natural map $f^*N \to M$ is an isomorphism and f is strict if both of these conditions are satisfied.

We denote by (X, K_X) the idealized log scheme (X, M) with the sheaf of ideals K_X. We denote by **ILSch** the category of idealized log schemes.

REMARK 1.19. The category of log schemes **LSch** is a full subcategory of **ILSch**, via the fully faithful functor $(X, M) \mapsto (X, \varnothing)$, where $\varnothing$ is the empty sheaf of ideals in M.

DEFINITION 1.20 ([23, §2]). A log thickening of idealized log schemes is a strict closed immersion $i\colon T' \hookrightarrow T$ defined by a nil ideal $\mathscr{I} \subseteq \mathcal{O}_T$. We say that the thickening is nilpotent if the ideal $\mathscr{I}$ is.

If $f\colon X \to Y$ is a morphism of idealized log schemes, then a thickening over X/Y is a diagram

$$\begin{tikzcd} T' \arrow[r, hook, "i"] \arrow[d, "g'"'] & T \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with i a log thickening.

REMARK 1.21. If $i\colon T' \hookrightarrow T$ is a log thickening, since $\overline{M}_T = M_T/\mathcal{O}_T^*$ is isomorphic to $\overline{M}_{T'}$, there is a natural bijection between the ideals in M_T and the ideals in $M_{T'}$. If $g\colon T \to X$ is such that $g \circ i = g'$, then $(g^*K_X)_{T'} = (g'^*K_X)_{T'}$.

DEFINITION 1.22 ([23, §2]). A morphism $f\colon X \to Y$ of idealized log schemes is ideally log smooth if, for every log thickening as in Definition 1.20, locally on T, there exists a map $g\colon T \to X$ such that $g \circ i = g'$ and $f \circ g = h$.

We can give the notions of ideally log unramified and ideally log etale morphisms of idealized log schemes using the lifting properties with respect to strict nilpotent log thickenings, as in Definition 1.22. Let now $f\colon X \to Y$ be an ideally log smooth map of fine idealized log schemes. We recall the etale local properties of f (Theorem 1.24). To this aim, we need the following

DEFINITION 1.23 ([23, Definition 2.9]). A chart for an idealized log scheme (X, K_X) is a morphism $(P, \tilde{K}) \to (M_X, K_X)$, where P is the constant sheaf of monoids on X of value P, K is an ideal of P, the map $P \to M_X$ is a chart (Definition 1.6), and the induced map $\tilde{K} \to K_X$ is an isomorphism (with $\tilde{K}$ the sheaf of ideals associated to K).

Theorem 1.24 ([23, Theorem 2.23]). *Let $f\colon X \to Y$ be an ideally log smooth map of fine idealized log schemes. We suppose to have a chart $\gamma\colon (Q, \tilde{J}) \to (M_Y, K_Y)$. Then, etale locally on X, we can extend γ to a chart for f*

$$\begin{array}{ccc} X & \longrightarrow & S_{P,I} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle S_\theta} \\ Y & \longrightarrow & S_{Q,J} \end{array}$$

where $\theta\colon Q \to P$, and $S_{P,I}$, $S_{Q,J}$ are the idealized log schemes $\operatorname{Spec}\mathbb{C}[P - I]$, $\operatorname{Spec}\mathbb{C}[Q - J]$, *with canonical log structures $P \to \mathbb{C}[P - I]$, and $Q \to \mathbb{C}[Q - J]$, respectively. This chart satisfies the following properties*:

1. *The map θ^{gp} is injective, and the torsion part of $(P^{\mathrm{gp}}/Q^{\mathrm{gp}})$ has invertible order in $\mathcal{O}_X$.*
2. *The map $h\colon X \to Y \times_{S_{Q,J}} S_{P,I}$, induced from the above diagram, is etale and strict* ([23, *Theorem* 2.23]).

EXAMPLE 1.25 ([23, Example 2.17]). We consider now the following case: if $(X, \alpha\colon M_X \to \mathcal{O}_X)$ is a fine log scheme and K is a coherent sheaf of ideals in M_X, let X_K be the closed subscheme defined by $\alpha(K)\mathcal{O}_X$, with the induced log structure.

Then the map of idealized log schemes

$$j : (X_K, j^*K) \to (X, \varnothing_X)$$

is ideally etale.

We consider Example 1.25 in the case of a fine monoid P with $P^* = 0$, and the log scheme $S_P = \operatorname{Spec}\mathbb{C}[P]$, with log structure $P \hookrightarrow \mathbb{C}[P]$, which is log smooth over $\operatorname{Spec}\mathbb{C}$. We take the closed log subscheme ξ_P of S_P defined by the ideal $P^+ = P - \{0\}$, which is equal to the log scheme $\operatorname{Spec}(\mathbb{C}[P]/P^+)$, endowed with log structure $P \to \mathbb{C}[P]/P^+$, where the map $P \to \mathbb{C}$ sends every element of P^+ to 0. Then the map of idealized log schemes

$$(\xi_P, P^+) \to (S_P, \varnothing)$$

is ideally log etale. Therefore, taking the composition map $(\xi_P, P^+) \to (S_P, \varnothing) \to (\operatorname{Spec}\mathbb{C}, \varnothing)$, we can conclude that the idealized log scheme (ξ_P, P^+) is ideally log smooth over $\operatorname{Spec}\mathbb{C}$, even if it is not log smooth over $\operatorname{Spec}\mathbb{C}$ in the category **LSch**.

Now, if we consider an ideally log smooth log scheme Y over $\operatorname{Spec}\mathbb{C}$, etale locally on Y, we can write $f : Y \to \operatorname{Spec}\mathbb{C}$ as a composite map

$$Y = X_K \overset{i}{\hookrightarrow} X \to \operatorname{Spec}\mathbb{C}$$

where X is a log smooh log scheme over $\operatorname{Spec}\mathbb{C}$ and $X_K \overset{i}{\hookrightarrow} X$ is a closed immersion defined by a coherent sheaf of ideals K of the log structure on X. So, an etale local model for Y is given by $\operatorname{Spec}\mathbb{C}[P - I] = \operatorname{Spec}(\mathbb{C}[P]/I\mathbb{C}[P])$, with P a toric monoid and I an ideal of P, endowed with the log structure $P \to \mathbb{C}[P]/I\mathbb{C}[P]$ which sends I to $\{0\}$. In this etale local model, X is represented by $\operatorname{Spec}\mathbb{C}[P]$ and its log structure $\alpha\colon M_X \to \mathcal{O}_X$ by $P \to \mathbb{C}[P]$.

We also have another equivalent characterization of ideally log smooth log schemes over Spec $\mathbb{C}$, which is the following

DEFINITION 1.26 ([14, Definition (1.5)]). Let X be a log scheme over Spec $\mathbb{C}$, with fs log structure M. Then, X is ideally log smooth over Spec $\mathbb{C}$ if, etale locally on X, there exist a toric monoid P, and an ideal I of P, a scheme U over $\mathbb{C}$, and etale morphisms $\varphi\colon U \to X$, $\psi\colon U \to \operatorname{Spec}(\mathbb{C}[P]/(I))$, where U is endowed with the log structure φ^*M, $\operatorname{Spec}(\mathbb{C}[P]/(I))$ is endowed with the log structure $P \to \mathbb{C}[P]/(I)$, which sends I to $\{0\}$, and φ^*M coincides with the log structure associated to $P \to \mathcal{O}_U$.

2. Ogus, Danilov and Ishida weight filtrations

From now on, let $\operatorname{Spec}\mathbb{C}$ be endowed with the trivial log structure. Let X be a log scheme over $\mathbb{C}$, endowed with an fs log structure M. We suppose that X is log smooth over $\mathbb{C}$, and its underlying scheme is proper. If X is also smooth in the classical sense over $\mathbb{C}$, then the log structure is given by a normal crossing divisor Y, and the increasing Deligne weight filtration $W_\bullet$, on the complex $\omega_X^\bullet$, is defined as follows ([2, (3.1.5)]),

$$W_i\omega_X^j = 0, \text{ if } i < 0; \quad W_i\omega_X^j = \Omega_X^{j-i} \wedge \omega_X^i, \text{ if } 0 \le i \le j; \quad W_i\omega_X^j = \omega_X^j, \text{ if } i \ge j.$$

When the underlying scheme of X is not smooth over $\mathbb{C}$, we consider another filtration on the log de Rham complex, and we prove that this reduces to $W_\bullet$ when X is smooth in the classical sense. To this aim, we recall the definition of the decreasing filtration $\tilde{L}^\bullet$, introduced by Ogus ([23, Definition 1.2]) on the log de Rham complex $\omega_X^\bullet$.

DEFINITION 2.1 ([23, Definition 1.2]). In the previous notations, let $F \subseteq M$ be a sheaf of faces of M. Let $L^i(F)\omega_X^j$ be the subsheaf of abelian groups generated by the local sections of the form $\alpha(m)\,\mathrm{dlog}\, m_1 \wedge \cdots \wedge \mathrm{dlog}\, m_j$, where $m, m_1, \ldots, m_j \in M$ and there exist $k \in \mathbb{N}$, $f \in F$ with $km + f - (m_1 + \cdots + m_i) \in M$, using the additive notation in M (where $\alpha\colon M \to \mathcal{O}_X$ is the log structure on X).

We write $\tilde{L}^i(F)\omega_X^j$ for $L^{i+j}(F)\omega_X^j$ and $L^\bullet$ for $L^\bullet(\mathcal{O}_X^*)$. In particular, $\tilde{L}^0\omega_X^j = L^j\omega_X^j$, for each j. We note that $L^i(F)\omega_X^j$ is a subgroup of ω_X^j which is stable under multiplication by sections of $\mathcal{O}_X^*$. In [23, Proof. of Lemma 2.15] Ogus proved that it is also stable under multiplication by any section of $\mathcal{O}_X$, i.e. it is an $\mathcal{O}_X$-submodule. Moreover, for $F = \mathcal{O}_X^*$, $L^i\omega_X^j$ is quasi-coherent, for each i, j ([23, Lemma 2.15, 3]).

We can see $\tilde{L}^\bullet$ as an extension of the weight filtration to the case of a general log smooth log scheme over $\mathbb{C}$ (for example, to the case of complex toric varieties endowed with canonical log structure). Indeed, we prove the following

Proposition 2.2. *Let X be a smooth scheme over $\mathbb{C}$, with log structure given by a normal crossing divisor Y. Then, the opposite Ogus filtration $\tilde{L}^{-\bullet}$ on $\omega_X^\bullet$ reduces to the Deligne weight filtration $W_\bullet$.*

Proof. Since X is smooth over $\mathrm{Spec}\,\mathbb{C}$, we can choose a local system of coordinates $\{x_1, \ldots, x_n\}$ for it ($n = \dim X$). We can suppose that the normal crossing divisor Y is given by the local equation $x_1 \cdot \cdots \cdot x_r = 0$. Then, $\{\mathrm{dlog}\, x_1, \ldots, \mathrm{dlog}\, x_r, dx_{r+1}, \ldots, dx_n\}$ is a local basis for ω_X^1. Moreover, the log structure M is locally represented by $\mathbb{N}^r$, and the log structure $\alpha\colon M \to \mathcal{O}_X$ is locally given by $\mathbb{N}^r \to \mathbb{C}[x_1, \ldots, x_n]\colon e_i \mapsto x_i$, where e_i are the elements of the canonical basis of $\mathbb{N}^r$. Now, each element m of $\mathbb{N}^r$ is equal to $m = k_1e_1 + \cdots + k_re_r$, for $k_j \ge 0$, for each j. So, let $\omega = \alpha(m)\,\mathrm{dlog}\, m_1 \wedge \cdots \wedge \mathrm{dlog}\, m_j$ an element of $L^i\omega_X^j$, where $m, m_1, \ldots, m_j \in \mathbb{N}^r$, i.e. $m = a_1e_1 + \cdots + a_re_r$, and $m_s = k_{1s}e_1 + \cdots + k_{rs}e_r$, for each $1 \le s \le j$ (where all the coefficients are in $\mathbb{N}$). The condition $km - (m_1 + \cdots + m_i) \in \mathbb{N}^r$ is equal to

$$\begin{aligned} &k(a_1 e_1 + \cdots + a_r e_r) - [(k_{11} e_1 + \cdots + k_{r1} e_r) + \cdots + (k_{1i} e_1 + \cdots + k_{ri} e_r)] \\ &= \left[ka_1 - \left(\sum_{t=1}^{i} k_{1t}\right)\right] e_1 + \cdots + \left[ka_r - \left(\sum_{t=1}^{i} k_{rt}\right)\right] e_r \in \mathbb{N}^r \end{aligned}$$

i.e. the coefficients are ≥ 0. Since $-\left(\sum_{t=1}^{i} k_{1t}\right) \leq 0$, it follows then, or $a_1 \geq 0$, or, if $a_1 = 0$, then all the terms $k_{11}, \ldots, k_{1i}$ must be equal to 0 too (because all of them are ≥ 0). In the first case, the term $x_1^{a_1} \operatorname{dlog} x_1 = x_1^{a_1 - 1} dx_1$ compares in the expression of the element ω, and it is a classical differential; in the second case $\operatorname{dlog} x_1$ does not compare in ω. The same arguments hold for $a_2, \ldots, a_r$. We can conclude that each local section of $L^i \omega_X^j = \tilde{L}^{i-j} \omega_X^j$ is equal to

$$\begin{aligned} &g \cdot x_{s_1}^{a_1} \cdot \cdots \cdot x_{s_i}^{a_i} \operatorname{dlog} e_{s_1} \wedge \cdots \wedge \operatorname{dlog} e_{s_i} \wedge \operatorname{dlog} e_{s_{i+1}} \cdots \wedge \operatorname{dlog} e_{s_j} \\ &= g \cdot x_{s_1}^{a_1 - 1} \cdot \cdots \cdot x_{s_i}^{a_i - 1} de_{s_1} \wedge \cdots \wedge de_{s_i} \wedge \operatorname{dlog} e_{s_{i+1}} \cdots \wedge \operatorname{dlog} e_{s_j} \end{aligned}$$

with $k \geq 1$, and $g \in \mathcal{O}_X^*$. Finally, since each local section of $W_{j-i} \omega_X^j$ is of the type $h \cdot x_{s_1}^{k_1} \cdot \cdots \cdot x_{s_i}^{k_i} \operatorname{dlog} e_{s_1} \wedge \cdots \wedge \operatorname{dlog} e_{s_i} \wedge \operatorname{dlog} e_{s_{i+1}} \cdots \wedge \operatorname{dlog} e_{s_j}$, with $h \in \mathcal{O}_X$, $k_1, \ldots, k_i \geq 1$, it is easy to see that $\tilde{L}^{-(j-i)} \omega_X^j$ coincides with $W_{j-i} \omega_X^j$. $\square$

The Ogus filtration $L^\bullet(F)$ on $\omega_X^\bullet$ admits a local graded description. Indeed, since X is log smooth over $\mathbb{C}$, it is etale locally equal to $\operatorname{Spec} \mathbb{C}[P]$, with P a toric monoid. We consider the P-graded $\mathbb{Z}$-algebra $\mathbb{C}[P] = \bigoplus_{p \in P} \mathbb{C}\, e(p)$, with $e \colon P \hookrightarrow \mathbb{C}[P]$. We recall that the sheaf ω_X^j is etale locally represented by $\omega_{\mathbb{C}[P]}^j := \mathbb{C}[P] \otimes_{\mathbb{Z}} \bigwedge^j P^{\mathrm{gp}}$ [23, §3]. This has a natural structure of a P-graded $\mathbb{C}[P]$-module; its component in degree p is just $\mathbb{C}\, e(p) \otimes_{\mathbb{Z}} \bigwedge^j P^{\mathrm{gp}}$.

For each $p \in P$, let $L_p \bigwedge^j P^{\mathrm{gp}}$ be a $\mathbb{Z}$-submodule of $\bigwedge^j P^{\mathrm{gp}}$. Let us suppose that this is such that, for $p \geq q$, $L_p \bigwedge^j P^{\mathrm{gp}} \subseteq L_q \bigwedge^j P^{\mathrm{gp}}$. Ogus calls such a collection of submodules a *P-filtration* of $\bigwedge^j P^{\mathrm{gp}}$. Under these assumptions, it is easy to see that the image of $\bigoplus_{p \in P} \mathbb{C}\, e(p) \otimes_{\mathbb{Z}} \left(L_p \bigwedge^j P^{\mathrm{gp}}\right)$ inside $\mathbb{C}[P] \otimes_{\mathbb{Z}} \bigwedge^j P^{\mathrm{gp}}$ is a P-graded $\mathbb{C}[P]$-submodule. We consider the following P-filtration,

DEFINITION 2.3 ([23, Definition 3.2]). Let F be a face of P. For each $p \in P$, let $L_p^i(F) \bigwedge^j P^{\mathrm{gp}}$ be the subgroup of $\bigwedge^j P^{\mathrm{gp}}$ generated by all the elements of the form $\operatorname{dlog} p_1 \wedge \cdots \wedge \operatorname{dlog} p_j$, such that $p_1, \ldots, p_j \in P$ and there exist $k \in \mathbb{N}$, $f \in F$, such that $kp + f - (p_1 + \cdots + p_i) \in P$.

In the previous definition, we can take $F = \{0\}$ and consider $L_p^i(\{0\}) \bigwedge^j P^{\mathrm{gp}}$: the condition becomes $kp - (p_1 + \cdots + p_i) \in P$. Let us denote by $L^i(F) \omega_{\mathbb{C}[P]}^j$ the P-

graded $\mathbb{C}[P]$-submodule which is the image of $\bigoplus_{p\in P}\mathbb{C}e(p)\otimes_{\mathbb{Z}}\left(L^i_p(F)\bigwedge^j P^{\mathrm{gp}}\right)$ inside $\mathbb{C}[P]\otimes_{\mathbb{Z}}\bigwedge^j P^{\mathrm{gp}}$. Then, we have the following

Lemma 2.4 ([23, Lemma 3.3]). *If $\tilde{F}$ is the sheaf of faces on $X = \operatorname{Spec}\mathbb{C}[P]$ corresponding to the face F of P, then the quasi-coherent sheaf on X associated to the $\mathbb{C}[P]$-submodule $L^i(F)\omega^j_{\mathbb{C}[P]}$ of $\omega^j_{\mathbb{C}[P]}$ is equal to $L^i(\tilde{F})\omega^j_X$. In particular, $L^i\omega^j_X$ is the sheaf on X associated to $L^i(\{0\})\omega^j_{\mathbb{C}[P]}$.*

By using the filtration $\tilde{L}^\bullet$ on the log de Rham complex, Ogus proved the following two results,

Proposition 2.5 ([23, Theorem 5.6]). *There are natural isomorphisms in the filtered derived category of sheaves on X^{an}:*

$$(\omega^\bullet_{X^{\mathrm{an}}},\tilde{L}^\bullet)\to(u_*\omega^\bullet_{U^{\mathrm{an}}},\tilde{T})\to(\mathbb{R}u_*\mathbb{C}_{U^{\mathrm{an}}},\tilde{T})$$

where $U = X_{\mathrm{triv}} := \{x\in X$ such that $M_x = \mathcal{O}^*_{X,x}\}$ is the trivial locus for the log structure M on X, $u\colon U = X_{\mathrm{triv}}\hookrightarrow X$ is the corresponding open immersion, and $\tilde{T}$ is equal to the Deligne canonical filtration $\tau_{\leq\bullet}$ ([2, (1.4.6)]).

Proposition 2.6 ([23, Theorem 1.4]). *In the previous notations, there exists a natural isomorphism*

$$\mathbb{H}^\bullet(X,\tilde{L}^0\omega^\bullet_X)\cong H^\bullet(X^{\mathrm{an}},\mathbb{C})$$

By Proposition 2.2, we can say that the previous Propositions extended the theory of Deligne ([2, Proposition (3.1.8)]) to a log smooth log scheme whose underlying scheme is not smooth over $\mathbb{C}$.

Now, in [5] we have analyzed the connections between the notion of ideally log smooth log schemes (resp. log smooth log schemes) over $\operatorname{Spec}\mathbb{C}$ and that of filtered semi-toroidal variety (resp. toroidal embeddings) in the sense of [16, Definition 5.2]. In this direction, by using Definition 1.26, we have proved that the underlying scheme of a log smooth log scheme (resp. an ideally log smooth log scheme) over $\operatorname{Spec}\mathbb{C}$ is in fact a toroidal embedding (resp. a filtered semi-toroidal variety) over $\operatorname{Spec}\mathbb{C}$ ([5, Propositions 5.12 and 5.13]). In particular, if X is log smooth over $\operatorname{Spec}\mathbb{C}$, it can be shown that the pair $(U = X_{\mathrm{triv}}, X)$ is a toroidal embedding in the sense of [22, II, Definition 1] or [16, Definition 5.2].

Then, we can consider the complex $\tilde{\Omega}^\bullet_X$ introduced by Ishida in [16]; this is one of the most important tools in toric geometry, because its cohomology is connected with the cohomological groups of the constant sheaf $\mathbb{C}$ on the singular analytic space X^{an}. We refer to [16, §6] for the definition of $\tilde{\Omega}^\bullet_X$ and to [6, Definition 1.3] for the equivalent construction of it in the logarithmic setting, as a subcomplex of $\omega^\bullet_X$. Moreover, let $V = X_{\text{smooth}} = X - \operatorname{Sing} X$ be the open subscheme which is the smooth locus (in the classical sense) of X, and let $v: V \hookrightarrow X$ be the corresponding open immersion. Ishida proved that there exists an isomorphism of complexes between $\tilde{\Omega}^{\bullet}_X$ and $v_* \Omega^{\bullet}_V$ ([16, Proposition 3.11]). Therefore, since the construction of the Ishida complex is too long to explain, in this article we prefer to use this comparison theorem and identify $\tilde{\Omega}^{\bullet}_X$ with $v_* \Omega^{\bullet}_V$.

By using the Ishida comlex $\tilde{\Omega}^{\bullet}_X$, we now introduce another filtration on $\omega^{\bullet}_X$.

DEFINITION 2.7. Under the previous notations, we define the following increasing filtration on $\omega^{\bullet}_X$, denoted by $I_{\bullet}$,

(1) $$I_i \omega^j_X = 0, \text{ if } i < 0; \quad I_i \omega^j_X = \tilde{\Omega}^{j-i}_X \wedge \omega^i_X, \text{ if } 0 \le i \le j; \quad I_i \omega^j_X = \omega^j_X, \text{ if } i \ge j.$$

We call it the Ishida filtration.

In the case of a smooth scheme X over $\operatorname{Spec} \mathbb{C}$, with log structure given by a normal crossing divisor Y, the Ishida complex coincides with the classical de Rham complex $\Omega^{\bullet}_X$ (being $V = X$), and $I.$ coincides with the Deligne weight filtration $W_{\bullet}$.

A local description of the graded terms for the filtration $I_{\bullet}$ was given by Danilov (see the filtration $W_{\bullet}$ given in [1, §15.6] in the toric case). Indeed, given a toroidal pair (X, D) (with D a divisor inside X), he defined ([1, 15.2]) a de Rham complex with logarithmic poles along a divisor D, $\Omega^{\bullet}_X(\log D)$, endowed with a particular weight filtration, which we call the Danilov weight filtration and denote here by $\mathcal{W}_{\bullet}$ (to distinguish it from the Deligne weight filtration $W_{\bullet}$).

When X is a log smooth log scheme over $\operatorname{Spec} \mathbb{C}$, we can define the log de Rham complex $\omega^{\bullet}_X$ of X and also the Danilov de Rham complex $\Omega^{\bullet}_X(\log D)$ associated to the toroidal pair (X, D), where $D = X - X_{\text{triv}}$. So, we now show that the Danilov complex $\Omega^{\bullet}_X(\log D)$ is in fact isomorphic to the log de Rham complex $\omega^{\bullet}_X$, and the filtration $I_{\bullet}$, which etale locally coincides with the Danilov weight filtration $\mathcal{W}_{\bullet}$, is equal to the opposite Ogus filtration $\tilde{L}^{-\bullet}$.

To this aim, we first note that, since X is log smooth over $\mathbb{C}$, then by [20, Proposition (8.3)] it is log regular in the sense of Kato [20, Definition (2.1)]. Therefore, by [20, Theorem (4.1)] its underlying scheme is Cohen-Macaulay and normal, so $\operatorname{codim}_X(X - V) \ge 2$. We always have an inclusion $U \subset V$. Let D_V be the divisor $D \cap V$. Then,

Lemma 2.8. *In the previous notations, D_V is a smooth divisor inside V.*

Proof. Since X is Cohen-Macaulay, namely regular in codimension ≤ 1, then V is an open subscheme of codimension ≤ 1 inside X. This means that, for each point $x \in V$, $\dim \mathcal{O}_{X,x} \le 1$, and then, by definition of log regularity [20, Definition (2.1), (2)], $M_x/\mathcal{O}^*_{X,x} \cong \mathbb{N}$ or $M_x/\mathcal{O}^*_{X,x} \cong \{1\}$. Therefore, the log structure M, restricted to the smooth locus V, is etale locally given by the monoid $\mathbb{N}$, and this exactly means that $V \cap D$ is a smooth divisor inside V. □

DEFINITION 2.9 ([1, §15.2]). In the previous notations, for each integer p, $0 \leq p \leq \dim_{\mathbb{C}} X$, the sheaf $\Omega^p_X(\log D)$ is defined as

$$\Omega^p_X(\log D) := v_*(\Omega^p_V(\log D_V))$$

This is called the sheaf of germs of p-differentials on X with logarithmic poles along D. The differential map $d^p\colon \Omega^p_X(\log D) \to \Omega^{p+1}_X(\log D)$ is induced from that of $\Omega^p_V(\log D_V)$, for every p.

Now, we analyze the local structure of the sheaves $\tilde{\Omega}^p_X$ and $\Omega^p_X(\log D)$, for every p. The etale local models for both of them are given by Danilov ([1, Propositions 4.3 and 15.5]), using the toric geometry. We briefly recall these local descriptions.

Let σ be an n-dimensional cone in $H \otimes_{\mathbb{Z}} \mathbb{R}$, where H is an n-dimensional lattice; we suppose σ generates $H_{\mathbb{Q}}$. Let $A = \mathbb{C}[\sigma \cap H]$. For each face τ of σ, Danilov defines the following subspaces V_τ of $H \otimes_{\mathbb{Z}} \mathbb{C}$. If τ is of codimension 1, then $V_\tau := (\tau + (-\tau)) \otimes_{\mathbb{Q}} \mathbb{C}$; in general, $V_\tau := \bigcap_{\tau \subset \theta} V_\theta$, where θ ranges over the faces of σ of codimension 1 which contain τ. Let $V = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Danilov introduces the following H-graded A-module

$$\Omega^p_A = \bigoplus_{h \in \sigma \cap H} \Omega^p_A(h) \tag{2}$$

where he sets, for each $h \in \sigma \cap H$, $\Omega^p_A(h) = \bigwedge^p \left(\bigcap_{\theta;\, h \in \theta} V_\theta\right) \cdot x^h$ ([1, §15.4]).

Let now I be a set of codimension 1 faces of σ. For each face $\theta \prec \sigma$, $\operatorname{codim}\theta = 1$, Danilov sets $V_\theta(\log) := V_\theta$, if $\theta \notin I$, and $V_\theta(\log) := V$, if $\theta \in I$. Then, he introduces he following H-graded A-module

$$\Omega^p_A(\log) = \bigoplus_{h \in \sigma \cap H} \Omega^p_A(\log)(h) \tag{3}$$

where he sets, for each $h \in \sigma \cap H$, $\Omega^p_A(\log)(h) = \bigwedge^p \left(\bigcap_{\theta; h \in \theta} V_\theta(\log)\right) \cdot x^h$ ([1, §15.4]).

In [1, Propositions 4.3 and 15.5], Danilov proves that, etale locally on $X = \operatorname{Spec} \mathbb{C}[\sigma \cap H]$, if $D = \bigcup_{\theta \in I} X_\theta$, the sheaves of $\mathcal{O}_X$-modules $\tilde{\Omega}^p_X$ and $\Omega^p_X(\log D)$ are the sheaves associated to the A-modules Ω^p_A and $\Omega^p_A(\log)$ respectively.

Proposition 2.10. *In the previous context, the complex $\Omega^\bullet_X(\log D)$, defined by Danilov, is isomorphic to the log de Rham complex $\omega^\bullet_X$.*

Proof. We have a global map of complexes $\omega^\bullet_X \to v_*(\Omega^\bullet_V(\log D_V)) =: \Omega^\bullet_X(\log D)$. Moreover, etale locally on X, the local models for ω^p_X and $\Omega^p_X(\log D)$ coincide. Indeed, let $\mathbb{C}[P]\otimes_{\mathbb{Z}}\bigwedge^p P^{\mathrm{gp}}$ be the local model for ω_X^p. Then, in the local description (3) of $\Omega_X^p(\log D)$, we can take $H=P^{\mathrm{gp}}$, $A=\mathbb{C}[P]$, σ to be the cone generated by P inside $H\otimes_{\mathbb{Z}}\mathbb{R}$, and I contains all the codimension 1 faces of σ. Then, for each $h\in P^{\mathrm{gp}}=H$, it is easy to see that $\Omega_A^p(\log)(h)$ is isomorphic to $\mathbb{C}e(h)\otimes_{\mathbb{Z}}\bigwedge^p P^{\mathrm{gp}}$ ($e\colon P\hookrightarrow\mathbb{C}[P]$). □

Let now $P\to M$ be a local chart for the log structure M on X, where P is a toric monoid. Then X is etale locally equal to $\operatorname{Spec}A$, where $A=\mathbb{C}[P]$. Working etale locally on X, Danilov defined a weight filtration $\mathcal{W}_\bullet$ as a H-graded filtration on $\Omega_A^j(\log)$

$$0\subset\mathcal{W}_0\Omega_A^j(\log)\subset\mathcal{W}_1\Omega_A^j(\log)\subset\cdots\subset\mathcal{W}_j\Omega_A^j(\log)=\Omega_A^j(\log)$$

where $\Omega_A^j(\log)$ is the local description (3) for the sheaf $\Omega_X^j(\log D)$. On the h-homogeneous component, this is given by

$$\mathcal{W}_k\Omega_A^j(\log)(h)=\Omega_A^{j-k}(h)\wedge\Omega_A^k(\log)(h)$$

with $\Omega_A^{j-k}(h)$ the h-homogeneous component of the A-module Ω_A^{j-k}, $h\in H$. Therefore, from (2) and (3), $\mathcal{W}_k\Omega_A^j(\log)=\Omega_A^{j-k}\wedge\Omega_A^k(\log)$. Moreover, by [1, Proposition 4.3], the Ishida sheaf $\tilde{\Omega}_X^j$ is isomorphic to the sheaf $\tilde{\Omega}_A^j$ associated to the A-module Ω_A^j described above. So, the Ishida filtration $I_\bullet$ etale locally coincides with $\mathcal{W}_\bullet$ and we can say that $I_\bullet$ is a sort of globalization of the Danilov weight filtration.

We have now two filtrations $I_\bullet$, $\tilde{L}^\bullet$ on $\omega_X^\bullet$, which we want to compare. To this aim, we first prove the following result, which gives us an interpretation of the Ishida complex in terms of the Ogus filtration.

Lemma 2.11. *In the previous notations, the Ishida complex $\tilde{\Omega}_X^\bullet$ is isomorphic to $\tilde{L}^0\omega_X^\bullet$.*

Proof. We first construct a global map $\psi^\bullet\colon\tilde{L}^0\omega_X^\bullet\to\tilde{\Omega}_X^\bullet=v_*\Omega_V^\bullet$. The construction of it is equivalent to that of a map $\Psi^\bullet\colon v^*\tilde{L}^0\omega_X^\bullet\to\Omega_V^\bullet$, by adjunction. We note that, on the smooth open V of X, the log structure is given by the smooth divisor $D_V=D\cap V$ (Lemma 2.8), and so $v^*\tilde{L}^0\omega_X^\bullet=\tilde{L}^0(v^*\omega_X^\bullet)=\tilde{L}^0\omega_V^\bullet=\Omega_V^\bullet$ (because, by Lemma 2.2, $\tilde{L}^0\omega_V^\bullet=W_0\omega_V^\bullet=\Omega_V^\bullet\hookrightarrow\omega_V^\bullet$). Therefore, the map $\Psi^\bullet$ is exactly the identity map.

Now, we want to prove that the adjoint map $\psi^\bullet$ of $\Psi^\bullet$ is etale locally an isomorphism. So, we can suppose that $X=\operatorname{Spec}\mathbb{C}[P]$, with P a toric monoid, where the log structure M on X is the canonical one $e\colon P\hookrightarrow\mathbb{C}[P]$.

We have to compare the $\mathbb{C}[P]$-submodule $\tilde{L}^0\omega_X^\bullet=\operatorname{Im}\{\bigoplus_{p\in P}\mathbb{C}e(p)\otimes_{\mathbb{Z}}L_p^j\bigwedge^j P^{\mathrm{gp}}\to\mathbb{C}[P]\otimes_{\mathbb{Z}}\bigwedge^j P^{\mathrm{gp}}\}$ of $\omega_{\mathbb{C}[P]}^j$ with the $\mathbb{C}[P]$-module $\tilde{\Omega}_{\mathbb{C}[P]}^j=\bigoplus_{p\in P}\mathbb{C}e(p)\otimes_{\mathbb{Z}}\bigwedge^j P^{\mathrm{gp}}[\rho(p)]$, where $\rho(p)=\pi\cap p^\perp$ is a face of the cone π, with $\pi^\vee$ the cone generated by P. We note that, for each $p\in P$, if $\langle p,F\rangle$ is the face of P generated by p and F, then $L_p^i(F)$ is just the image of the natural map $\bigwedge^i\langle p,F\rangle^{\mathrm{gp}}\otimes\bigwedge^{j-i}P^{\mathrm{gp}}\to\bigwedge^j P^{\mathrm{gp}}$. In our case,

$i = j$ and $F = \{0\}$, so $L_p^j(\{0\})$ is the image of the map $\bigwedge^j \langle p\rangle^{\mathrm{gp}} \to \bigwedge^j P^{\mathrm{gp}}$, which is equal to $\bigwedge^j \langle p\rangle^{\mathrm{gp}}$. Therefore, for each $p \in P$, the image of $\mathbb{C}e(p) \otimes L_p^j(\{0\}) \bigwedge^j P^{\mathrm{gp}}$ inside $\mathbb{C}e(p) \otimes \bigwedge^j P^{\mathrm{gp}}$ is just $\mathbb{C}e(p) \otimes \bigwedge^j \langle p\rangle^{\mathrm{gp}}$. On the other hand, we consider $\mathbb{C}$ $e(p) \otimes \bigwedge^j P^{\mathrm{gp}}[\rho(p)]$ and we note that, for each $p \in P$, $P^{\mathrm{gp}} \cap (p^{\perp} \cap \pi)^{\perp} = P^{\mathrm{gp}} \cap \langle p\rangle^{\mathrm{gp}} = \langle p\rangle^{\mathrm{gp}}$, if $p \in \pi^{\perp}$, and $P^{\mathrm{gp}} \cap \{0\}^{\perp} = P^{\mathrm{gp}}$, if $p \in \mathrm{int}(\pi^{\vee})$. So, if we consider the global image $\mathrm{Im}\left\{\bigoplus_{p\in P} \mathbb{C}e(p) \otimes_{\mathbb{Z}} L_p^j \bigwedge^j P^{\mathrm{gp}} \to \mathbb{C}[P] \otimes_{\mathbb{Z}} \bigwedge^j P^{\mathrm{gp}}\right\}$, we see that it coincides with the $\mathbb{C}[P]$-module $\tilde{\Omega}^j_{\mathbb{C}[P]}$. □

Proposition 2.12. *In the previous notations, the filtration $I.$ coincides with the opposite Ogus filtration $\tilde{L}^{-\bullet}$.*

Proof. Let $P \to M$ be a local chart for the log structure on X. Then X is etale locally equal to $\mathrm{Spec}\,\mathbb{C}[P]$, and the Ishida sheaf $\tilde{\Omega}^j_X$ is represented by $\tilde{\Omega}^j_{\mathbb{C}[P]}$. We have just seen that $\mathcal{W}_k\Omega^j_X(\log D) = \tilde{\Omega}^{j-k}_X \wedge \omega^k_X = I_k\omega^j_X$ and, by Lemma 2.11, $\tilde{L}^0\omega^{j-k}_X \cong \tilde{\Omega}^{j-k}_X$. By definition, $\tilde{L}^0\omega^{j-k}_X = L^{j-k}\omega^{j-k}_X$, and $\omega^k_X = L^0\omega^k_X$, so, from [23, Lemma 2.15, 2], the exterior multiplication takes an element in $L^{j-k}\omega^{j-k}_X \otimes L^0\omega^k_X$ into $L^{j-k}\omega^j_X = \tilde{L}^{-k}\omega^j_X$. Moreover, it is easy to see that each element in $\tilde{L}^{-k}\omega^j_X$ belongs to $L^{j-k}\omega^{j-k}_X \wedge L^0\omega^k_X$. Therefore, we can conclude that we have an identification, for every k, j,

$$\tilde{L}^{-k}\omega^j_X = L^{j-k}\omega^{j-k}_X \wedge L^0\omega^k_X = \tilde{L}^0\omega^{j-k}_X \wedge \omega^k_X = I_k\omega^j_X$$ □

Danilov conjectured that, in the case when X is a complete toroidal variety over $\mathrm{Spec}\,\mathbb{C}$, the mixed Hodge structure on $H^k(X - D, \mathbb{C})$ is induced by the bifiltered complex $(\Omega^\bullet_X(\log D), F^\bullet, \mathcal{W}_\bullet)$, where $F^\bullet$ is the Hodge filtration on the Danilov de Rham complex, defined by $F^p\Omega^\bullet_X(\log D) := \Omega^{\cdot \geq p}_X(\log D)$. This structure is also computed by Steenbrink for a complete V-manifold X and a divisor D with V-normal crossing ([25, Definition (1.16)]): he proved that, in this case, the Danilov weight filtration induces the Deligne weight filtration of the Hodge structure on $H^\bullet(X - D, \mathbb{C})$ ([25, Theorem (1.12), Lemma (1.19)]).

Therefore, in the log smooth case, by Proposition 2.12, we can conjecture that the mixed Hodge structure on $H^\bullet(X_{\mathrm{triv}}, \mathbb{C})$ is induced by the bifiltered complex $(\omega^\bullet_X, F^\bullet, \tilde{L}^{-\bullet})$. This conjecture is true for particular log smooth log schemes over $\mathrm{Spec}\,\mathbb{C}$, which are generalizations of smooth schemes with normal crossing divisors, as we discuss in the following.

2.1. The Ogus filtration in the quasi-smooth case. Let X be a log smooth log scheme over $\mathrm{Spec}\,\mathbb{C}$. We suppose that the underlying scheme of X is proper and quasi-smooth ([1, Definition 14.1]), namely all the local models for it are associated with simplicial cones. Moreover, we suppose that the divisor D consists of quasi-smooth components $D_1, \ldots, D_N$, those intersect quasi-transversally. An example can be given by the proper scheme $\mathcal{C}$ which is the projectivization of the affine cone $X = V(xy - z^2) \subset \mathbb{A}^3_{\mathbb{C}}$. This is a singular scheme with a toric singularity at the origin O. Locally at O, it is the toric variety associated to the cone $\sigma = \langle a_1, a_2 \rangle \subset \mathbb{R}^2$, generated by $a_1 = (1, 2)$ and $a_2 = (1, 0)$, i.e. $X = X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap P^{\mathrm{gp}}] = \operatorname{Spec} \mathbb{C}[P]$, where P is the toric monoid associated to σ, and $\sigma^\vee$ is the dual cone of σ, which is generated by $m_1 = (0, 1)$ and $m_2 = (2, -1)$ in $\mathbb{R}^2$. The log structure on $\mathcal{C}$ is given, locally at the origin, by $\alpha\colon P \hookrightarrow \mathbb{C}[P]$.

From [1, §15.7] and by Proposition 2.12, for such kind of log smooth log schemes, we have an explicit description for the graded terms of the Ogus filtration $\operatorname{Gr}^k_{\tilde{L}} \omega^j_X$, which is the following

$$\operatorname{Gr}^k_{\tilde{L}} \omega^j_X \cong \bigoplus_{0 \le i_1, \dots, i_k \le N} \tilde{\Omega}^{j-k}_{D_{i_1} \cap \dots \cap D_{i_k}} \tag{4}$$

where $\tilde{\Omega}^{j-k}_{D_{i_1} \cap \dots \cap D_{i_k}}$ are the $(j-k)$-differential forms of the Ishida complex on the semi-toroidal varieties $D_{i_1} \cap \dots \cap D_{i_k}$ (which are normal closed subvarieties of D). In other words, if we denote by $D_{\underline{k}}$ the normalization of the union of all the intersections $D_{i_1} \cap \dots \cap D_{i_k}$, i.e. $D_{\underline{k}} = \bigoplus_{i_1 < \dots < i_k} D_{i_1} \cap \dots \cap D_{i_k}$, and $\pi_k\colon D_{\underline{k}} \to X$, then

$$\operatorname{Gr}^k_{\tilde{L}} \omega^j_X \cong \pi_{k*} \tilde{\Omega}^j_{D_{\underline{k}}}[-k] \tag{5}$$

Lemma 2.13. *The spectral sequence of the hyper-cohomology of the filtered complex* $\operatorname{Gr}^k_{\tilde{L}} \omega^\bullet_X$*, with the induced Hodge filtration, degenerates at the* E_1*-terms.*

Proof. The Hodge filtration $F^\bullet$ on $\omega^\bullet_X$ induces the bete filtration on $\operatorname{Gr}^k_{\tilde{L}} \omega^\bullet_X$, and the spectral sequence of $\operatorname{Gr}^k_{\tilde{L}} \omega^\bullet_X$ can be deduced from the spectral sequence

$$E_1^{pq} = H^q\big(D_{\underline{k}}, \tilde{\Omega}^p_{D_{\underline{k}}}\big) \Longrightarrow H^{p+q}(D_{\underline{k}}, \mathbb{C}) \tag{6}$$

and, since $D_{\underline{k}}$ is a also a proper quasi-smooth toric variety, by [1, Theorem 12.5], the spectral sequence (6) degenerates at the E_1-terms. □

Now, Danilov showed that there exists the following spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D)) \Longrightarrow H^{p+q}(X - D, \mathbb{C})$$

(see [1, Theorem 15.9]). Moreover, he noted that, in the quasi-smooth case, the above sequence degenerates at the E_1-terms and converges to the Hodge filtration on $H^n(X - D, \mathbb{C})$; so, his weight filtration $\mathcal{W}_\bullet$ on $\Omega^\bullet_X(\log D)$ induces the Deligne weight filtration $W_\bullet$ of the Hodge structure on $H^n(X - D, \mathbb{C})$, $n \geq 0$.

Then, in the logarithmic context, for a log smooth log scheme X over $\operatorname{Spec} \mathbb{C}$, whose underlying scheme satisfies the previous assumptions, by using the Danilov's results and Proposition 2.12, we can describe the (Deligne) mixed Hodge structure of $H^\bullet(X_{\mathrm{triv}}, \mathbb{C})$ by using the bifiltered log de Rham complex $(\omega_X^\bullet, F^\bullet, \tilde{L}^{-\bullet})$.

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