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SOME NOTES ON THE RADICAL OF A FINITE GROUP RING II

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1. Introduction

Throughout this paper, p is a fixed prime number and G is a p -solvable group of order $|G|=p^a h$, $(p, h)=1$. Let k be a field of characteristic p and let \mathfrak{R} be the Jacobson radical of the group ring kG . We denote by $t(G)$ the smallest integer t such that $\mathfrak{R}^t=0$.

We know that $a(p-1)+1 \leq t(G) \leq p^a$. Let P be a Sylow p -subgroup of G . In the previous paper [9], we have shown that the second equality $t(G)=p^a$ holds (if and) only if P is cyclic. Here we shall show

Theorem 1. *Assume that P is a regular p -group. Then if $t(G)=a(p-1)+1$, P is elementary.*

A p -group P is called *regular*, if for any $x, y \in P$, it holds that $x^p y^p = (xy)^p \prod_i z_i^p$ with some $z_i \in \langle x, y \rangle$. There are various examples of regular p -groups (see e.g. Huppert [7], III Satz 10.2).

To prove Theorem 1, it is sufficient to show that G has p -length one. Indeed if this were shown, then we have that $t(G)=t(P)$ (Clarke [2]). However, for a p -group, our assertion is clear by Jennings [5] (without the assumption of regularity).

Of course Theorem 1 does not hold in general. A counter example is known for $p=2$. However, since a regular 2-group is necessarily abelian (hence $t(G)=t(P)$), Theorem 1 tells nothing new for $p=2$. Our proof is almost group theoretical. We owe heavily to a recent result of Gagola [3], a special case of which is quoted in Lemma 2 of the next section.

2. Preliminary results

For convenience of later arguments, we shall here provide a proof of the following result.

Theorem 2 (Wallace [10]). $t(G) \geq a(p-1)+1$.

Proof. First of all, we remark that if \mathfrak{R}_0 is the radical of the group ring kP

and $t=t(P)$, then $\mathfrak{N}_0^{t-1}=k\sigma=(0:\mathfrak{N}_0)$, where $\sigma=\sum_{x\in P}x$. This follows easily from the fact that kP is completely primary.

We shall prove the Theorem by the induction on the order of G . We may assume that $O_{p'}(G)=1$. Let V be a (non-trivial) normal p -subgroup of G . For a p -group, our assertion is clear by Jennings [5], so that if $|V|=p^b$, then $t(V)\geq b(p-1)+1$. By the induction hypothesis, we have $t(G/V)\geq(a-b)(p-1)+1$.

We recall that $\mathfrak{N}_1=\{\sum_x a_x(x-1) \mid x\in V, a_x\in k\}$ is the radical of kV . Furthermore, since $G \triangleright V$, $kG\mathfrak{N}_1$ is a nilpotent two sided ideal of kG , which coincides with the kernel of the natural map $kG \rightarrow k\bar{G}$, where $\bar{G}=G/V$. In particular, we have that $\mathfrak{N}^{(a-b)(p-1)} \not\subset kG\mathfrak{N}_1$. Hence, if we put $\sigma=\sum_{x\in V}x$, then $\mathfrak{N}^{(a-b)(p-1)}\sigma \neq 0$. Since we have $\sigma \in \mathfrak{N}_1^{b(p-1)} \subset \mathfrak{N}^{b(p-1)}$ as remarked above, we conclude that $\mathfrak{N}^{a(p-1)} \neq 0$. This completes the proof of Theorem 2.

Lemma 1. *Assume that $O_{p'}(G)=1$ and $U=O_p(G)$ is abelian. Let V be a minimal normal p -subgroup of G .*

If $O_{p'}(G/V) \neq 1$, then there is a normal p -subgroup W of G such that $U=V \times W$.

Proof. Let $O_{p'}(G/V)=TV/V$, where T is a p' -subgroup of G . Then $[TV, U]=[T, U]$ is a normal subgroup of G , which is contained in V . If $[T, U]=1$, then $T \subset C_G(U)$, a contradiction, since $C_G(U) \subset U$ by Hall and Higman [6]. Therefore we have $V=[T, U]$.

On the other hand, from the well-known Theorem on relatively prime automorphisms, we get $U=[T, U] \times C_U(T)=V_U \times C_U(T)$ (see e.g. Gorenstein [4] Chap. 5). Since $C_U(T)=C_U(TV)$ is normal in G , we have the desired conclusion by letting $W=C_U(T)$.

The following lemma is a special case of a result of Gagola [3].

Lemma 2. *Assume that $O_{p'}(G)=1$ and that $U=O_p(G)$ is minimal. Then U has a complement in G .*

Proof. Let F be the prime field of characteristic p and let $\bar{G}=G/U$. If the irreducible $F\bar{G}$ -module U belongs to the principal p -block of \bar{G} , then $O_{p'}(G)$ acts trivially on U by Theorem 1 of Brauer [1]. But this is impossible, since $C_G(U) \subset U$. Therefore U has a complement by a theorem of Gagola [3].

3. Proof of Theorem 1

We proceed by the induction on the order of G . We may assume that $O_{p'}(G)=1$. Let V be any non-trivial normal p -subgroup of G and let $|V|=p^b$. From the proof of Theorem 2, we see that $t(V)=b(p-1)+1$ and $t(G/V)=(a-b)(p-1)+1$. This implies that V is elementary by Jennings [5]. Also P/V

is elementary by the induction hypothesis.

If G has distinct minimal normal subgroups V and W , then G can be embedded in $G/V \times G/W$ and the result is clear by induction hypothesis. Hence we may assume that G has a unique minimal normal p -subgroup, say V .

Assume that $O_p(G) > V$. If $O_{p'}(G/V) \neq 1$, we have a contradiction by Lemma 1. If $O_{p'}(G/V) = 1$, then using that P/V is abelian, we conclude that $G/V \triangleright P/V$, namely $G \triangleright P$. Then the assertion is clear. Thus we may assume that $U = O_p(G)$ is minimal. Then by Lemma 2, there is a subgroup H of G such that $G = HU$ and $H \cap U = 1$. If Q is a Sylow p -subgroup of H , then Q is elementary and $P = QU$. Using now that P is regular, we have easily that P has exponent p . Then G has p -length one by Hall and Higman [6]. As is remarked in the introduction, this completes the proof of Theorem 1.

Acknowledgement

1. Soon after the earlier work [9] was completed, the author was informed from S. Koshitani that the assertion “(3) \Rightarrow (1)” of Theorem 4 [9] as well as the result of Clarke [2] (mentioned in the introduction) is direct from a result of Morita [8]. The author expresses his thanks to S. Koshitani.

2. The author expresses his thanks also to H. Matsuyama, who gives a direct proof of Lemma 2 as the following.

By the Schur and Zassenhaus Theorem, we may put $O_{p,p'}(G) = HU$, where H is a p' -subgroup and it is uniquely determined up to U -conjugates. Hence by the Frattini argument, it is easily shown that $N_G(H)$ is a desired complement of U .

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References

- [1] R. Brauer: *Some applications of the theory of blocks of characters of finite groups* I, J. Algebra **1** (1964), 152–167.
- [2] R.J. Clarke: *On the radical of the group ring of a p -nilpotent group*, J. Australian Math. Soc. **13** (1972), 119–123.
- [3] S.M. Gagola: *A splitting condition using block theory*, Michigan Math. J. **23** (1976), 203–206.
- [4] D. Gorenstein: *Finite groups*, Harper and Row, New York, 1968.
- [5] S.A. Jennings: *The structure of the group ring of a p -group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.
- [6] P. Hall and G. Higman: *On the p -length of p -soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. **6** (1956), 1–42.
- [7] B. Huppert: *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg, 1967.
- [8] K. Morita: *On group rings over modular field which possess radicals expressible as*

- principal ideals*, Sci. Report of T.B.D. **4** (1951), 177–194.
- [9] Y. Tsushima: *Some notes on the radical of a finite group ring I*, Osaka J. Math, **15** (1978), 647–653.
- [10] D.A. Wallace: *Lower bounds for the radical of the group algebra of a finite p -soluble group*, Proc. Edinburgh Math. Soc. **16** (1968/69), 127–134.