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SOME NOTES ON THE RADICAL OF A FINITE GROUP RING II

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1. Introduction

Throughout this paper, p is a fixed prime number and G is a p-solvable group of order $|G|=p^a h$, $(p, h)=1$. Let k be a field of characteristic p and let $\mathfrak{N}$ be the Jacobson radical of the group ring kG. We denote by $t(G)$ the smallest integer t such that $\mathfrak{N}^t=0$.

We know that $a(p-1)+1 \leqq t(G) \leqq p^a$. Let P be a Sylow p-subgroup of G. In the previous paper [9], we have shown that the second equality $t(G)=p^a$ holds (if and) only if P is cyclic. Here we shall show

Theorem 1. *Assume that P is a regular p-group. Then if $t(G)=a(p-1)+1$, P is elementary.*

A p-group P is called *regular*, if for any $x, y \in P$, it holds that $x^p y^p=(xy)^p \prod_i z_i^p$ with some $z_i \in \langle x, y \rangle'$. There are various examples of regular p-groups (see *e.g.* Huppert [7], III Satz 10.2).

To prove Theorem 1, it is sufficient to show that G has p-length one. Indeed if this were shown, then we have that $t(G)=t(P)$ (Clarke [2]). However, for a p-group, our assertion is clear by Jennings [5] (without the assumption of regularity).

Of course Theorem 1 does not hold in general. A counter example is known for $p=2$. However, since a regular 2-group is necessarily abelian (hence $t(G)=t(P)$), Theorem 1 tells nothing new for $p=2$. Our proof is almost group theoretical. We owe heavily to a recent result of Gagola [3], a special case of which is quoted in Lemma 2 of the next section.

2. Perliminary results

For convenience of later arguments, we shall here provide a proof of the following result.

Theorem 2 (Wallace [10]). $t(G) \geqq a(p-1)+1$.

Proof. First of all, we remark that if $\mathfrak{N}_0$ is the radical of the group ring kP

and $t=t(P)$, then $\mathfrak{R}_0^{t-1}=k\sigma=(0:\mathfrak{R}_0)$, where $\sigma=\sum_{x\in p} x$. This follows easily from the fact that kP is completely primary.

We shall prove the Theorem by the induction on the order of G. We may assume that $O_{p'}(G)=1$. Let V be a (non-trivial) normal p-subgroup of G. For a p-group, our assertion is clear by Jennings [5], so that if $|V|=p^b$, then $t(V)\geqq b(p-1)+1$. By the induction hypothesis, we have $t(G/V)\geqq(a-b)(p-1)+1$.

We recall that $\mathfrak{R}_1=\{\sum_x a_x(x-1)\,|\,x\in V,\ a_x\in k\}$ is the radical of kV. Furthermore, since $G\triangleright V$, $kG\mathfrak{R}_1$ is a nilpotent two sided ideal of kG, which coincides with the kernel of the natural map $kG\rightarrow k\bar{G}$, where $\bar{G}=G/V$. In particular, we have that $\mathfrak{R}^{(a-b)(p-1)}\not\subset kG\mathfrak{R}_1$. Hence, if we put $\sigma=\sum_{x\in V} x$, then $\mathfrak{R}^{(a-b)(p-1)}\sigma\neq 0$. Since we have $\sigma\in\mathfrak{R}_1^{b(p-1)}\subset\mathfrak{R}^{b(p-1)}$ as remarked above, we conclude that $\mathfrak{R}^{a(p-1)}\neq 0$. This completes the proof of Theorem 2.

Lemma 1. *Assume that $O_{p'}(G)=1$ and $U=O_p(G)$ is abelian. Let V be a minimal normal p-subgroup of G.*

If $O_{p'}(G/V)\neq 1$, then there is a normal p-subgroup W of G such that $U=V\times W$.

Proof. Let $O_{p'}(G/V)=TV/V$, where T is a p'-subgroup of G. Then $[TV, U]=[T, U]$ is a normal subgroup of G, which is contained in V. If $[T, U]=1$, then $T\subset C_G(U)$, a contradiction, since $C_G(U)\subset U$ by Hall and Higman [6]. Therefore we have $V=[T, U]$.

On the other hand, from the well-known Theorem on relatively prime automorphisms, we get $U=[T, U]\times C_U(T)=V_U\times C_U(T)$ (see *e.g.* Gorenstein [4] Chap. 5). Since $C_U(T)=C_U(TV)$ is normal in G, we have the desired conclusion by letting $W=C_U(T)$.

The following lemma is a special case of a result of Gagola [3].

Lemma 2. *Assume that $O_{p'}(G)=1$ and that $U=O_p(G)$ is minimal. Then U has a complement in G.*

Proof. Let F be the prime field of characteristic p and let $\bar{G}=G/U$. If the irreducible $F\bar{G}$-module U belongs to the principal p-block of $\bar{G}$, then $O_{p'}(G)$ acts trivially on U by Theorem 1 of Brauer [1]. But this is impossible, since $C_G(U)\subset U$. Therefore U has a complement by a theorem of Gagola [3].

3. Proof of Theorem 1

We proceed by the induction on the order of G. We may assume that $O_{p'}(G)=1$. Let V be any non-trivial normal p-subgroup of G and let $|V|=p^b$. From the proof of Theorem 2, we see that $t(V)=b(p-1)+1$ and $t(G/V)=(a-b)(p-1)+1$. This implies that V is elementary by Jennings [5]. Also P/V

is elementary by the induction hypothesis.

If G has distinct minimal normal subgroups V and W, then G can be embedded in $G/V \times G/W$ and the result is clear by induction hypothesis. Hence we may assume that G has a unique minimal normal p-subgroup, say V.

Assume that $O_p(G) > V$. If $O_{p'}(G/V) \neq 1$, we have a contradiction by Lemma 1. If $O_{p'}(G/V) = 1$, then using that P/V is abelian, we conclude that $G/V \triangleright P/V$, namely $G \triangleright P$. Then the assertion is clear. Thus we may assume that $U = O_p(G)$ is minimal. Then by Lemma 2, there is a subgroup H of G such that $G = HU$ and $H \cap U = 1$. If Q is a Sylow p-subgroup of H, then Q is elementary and $P = QU$. Using now that P is regular, we have easily that P has exponent p. Then G has p-length one by Hall and Higman [6]. As is remarked in the introduction, this completes the proof of Theorem 1.

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1. Soon after the earlier work [9] was completed, the author was informed from S. Koshitani that the assertion "(3)⇒(1)" of Theorem 4 [9] as well as the result of Clarke [2] (mentioned in the introduction) is direct from a result of Morita [8]. The author expresses his thanks to S. Koshitani.

2. The author expresses his thanks also to H. Matsuyama, who gives a direct proof of Lemma 2 as the following.

By the Schur and Zassenhaus Theorem, we may put $O_{p,p'}(G) = HU$, where H is a p'-subgroup and it is uniquely determined up to U-conjugates. Hence by the Frattini argument, it is easily shown that $N_G(H)$ is a desired complement of U.

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