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<th>Some notes on the radical of a finite group ring. II</th>
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<tr>
<td>Author(s)</td>
<td>Tsushima, Yukio</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 16(1) P.35-P.38</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1979</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/4833">https://doi.org/10.18910/4833</a></td>
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<td>DOI</td>
<td>10.18910/4833</td>
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SOME NOTES ON THE RADICAL OF A
FINITE GROUP RING II

YUKIO TSUSHIMA

(Received October 31, 1977)

1. Introduction

Throughout this paper, \( p \) is a fixed prime number and \( G \) is a \( p \)-solvable group of order \( |G| = p^ah \), \((p, h) = 1\). Let \( k \) be a field of characteristic \( p \) and let \( \mathfrak{N} \) be the Jacobson radical of the group ring \( kG \). We denote by \( t(G) \) the smallest integer \( t \) such that \( \mathfrak{N}^t = 0 \).

We know that \( a(p-1)+1 \leq t(G) \leq p^a \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \). In the previous paper [9], we have shown that the second equality \( t(G) = p^a \) holds (if and) only if \( P \) is cyclic. Here we shall show

**Theorem 1.** Assume that \( P \) is a regular \( p \)-group. Then if \( t(G) = a(p-1)+1 \), \( P \) is elementary.

A \( p \)-group \( P \) is called *regular*, if for any \( x, y \in P \), it holds that \( x^p y^p = (xy)^p \prod z_i^t \) with some \( z_i \in \langle x, y \rangle \). There are various examples of regular \( p \)-groups (see e.g. Huppert [7], III Satz 10.2).

To prove Theorem 1, it is sufficient to show that \( G \) has \( p \)-length one. Indeed if this were shown, then we have that \( t(G) = t(P) \) (Clarke [2]). However, for a \( p \)-group, our assertion is clear by Jennings [5] (without the assumption of regularity).

Of course Theorem 1 does not hold in general. A counter example is known for \( p=2 \). However, since a regular 2-group is necessarily abelian (hence \( t(G) = t(P) \)), Theorem 1 tells nothing new for \( p=2 \). Our proof is almost group theoretical. We owe heavily to a recent result of Gagola [3], a special case of which is quoted in Lemma 2 of the next section.

2. Preliminary results

For convenience of later arguments, we shall here provide a proof of the following result.

**Theorem 2** (Wallace [10]). \( t(G) \geq a(p-1)+1 \).

Proof. First of all, we remark that if \( \mathfrak{N}_e \) is the radical of the group ring \( kP \)
and \( t=t(P) \), then \( \mathfrak{R}_0^{-1}=k\sigma=(0: \mathfrak{R}_0) \), where \( \sigma=\sum_{r\in P} a_r \). This follows easily from the fact that \( kP \) is completely primary.

We shall prove the Theorem by the induction on the order of \( G \). We may assume that \( O_{p'}(G)=1 \). Let \( V \) be a (non-trivial) normal \( p \)-subgroup of \( G \). For a \( p \)-group, our assertion is clear by Jennings [5], so that if \( |V|=p^4 \), then \( t(V)\geq b(p-1)+1 \). By the induction hypothesis, we have \( t(G/V)\geq(a-b)(p-1)+1 \).

We recall that \( \mathfrak{N}_p=\{\sum a_x(x-1)\mid x\in V, a_x\in k\} \) is the radical of \( kV \). Furthermore, since \( G> V, kG\mathfrak{N}_p \) is a nilpotent two sided ideal of \( kG \), which coincides with the kernel of the natural map \( kG\rightarrow k\overline{G}, \) where \( \overline{G}=G/V \). In particular, we have that \( \mathfrak{N}^{(a-b)(p-1)}\subset kG\mathfrak{N}_p \). Hence, if we put \( \sigma=\sum_{r\in V} a_r \), then \( \mathfrak{N}^{(a-b)(p-1)}\sigma\neq 0 \).

Since we have \( \sigma\in \mathfrak{N}^{(p-1)}\subset \mathfrak{N}^{(p-1)} \) as remarked above, we conclude that \( \mathfrak{N}^{(p-1)}\neq 0 \). This completes the proof of Theorem 2.

**Lemma 1.** Assume that \( O_{p'}(G)=1 \) and \( U=O_p(G) \) is abelian. Let \( V \) be a minimal normal \( p \)-subgroup of \( G \).

If \( O_{p'}(G/V)\neq 1 \), then there is a normal \( p \)-subgroup \( W \) of \( G \) such that \( U=V\times W \).

**Proof.** Let \( O_{p'}(G/V)=TV/V \), where \( T \) is a \( p' \)-subgroup of \( G \). Then \( [TV, U]=[T, U] \) is a normal subgroup of \( G \), which is contained in \( V \). If \( [T, U]=1 \), then \( T\subset C_G(U) \), a contradiction, since \( C_G(U)\subset U \) by Hall and Higman [6]. Therefore we have \( V=[T, U] \).

On the other hand, from the well-known Theorem on relatively prime automorphisms, we get \( U=[T, U]\times C_v(T)=V_u\times C_v(T) \) (see e.g. Gorenstein [4] Chap. 5). Since \( C_v(T)=C_v(TV) \) is normal in \( G \), we have the desired conclusion by letting \( W=C_v(T) \).

The following lemma is a special case of a result of Gagola [3].

**Lemma 2.** Assume that \( O_{p'}(G)=1 \) and that \( U=O_p(G) \) is minimal. Then \( U \) has a complement in \( G \).

**Proof.** Let \( F \) be the prime field of characteristic \( p \) and let \( G=\overline{G}/U \). If the irreducible \( FG \)-module \( U \) belongs to the principal \( p \)-block of \( \overline{G} \), then \( O_{p'}(G) \) acts trivially on \( U \) by Theorem 1 of Brauer [1]. But this is impossible, since \( C_G(U)\subset U \). Therefore \( U \) has a complement by a theorem of Gagola [3].

3. **Proof of Theorem 1**

We proceed by the induction on the order of \( G \). We may assume that \( O_{p'}(G)=1 \). Let \( V \) be any non-trivial normal \( p \)-subgroup of \( G \) and let \( |V|=p^4 \). From the proof of Theorem 2, we see that \( t(V)=b(p-1)+1 \) and \( t(G/V)=(a-b)(p-1)+1 \). This implies that \( V \) is elementary by Jennings [5]. Also \( P/V \)
is elementary by the induction hypothesis.

If $G$ has distinct minimal normal subgroups $V$ and $W$, then $G$ can be embedded in $G/V \times G/W$ and the result is clear by induction hypothesis. Hence we may assume that $G$ has a unique minimal normal $p$-subgroup, say $V$.

Assume that $O_p(G) > V$. If $O_{p'}(G/V) = 1$, we have a contradiction by Lemma 1. If $O_{p'}(G/V) = 1$, then using that $P/V$ is abelian, we conclude that $G/V > P/V$, namely $G > P$. Then the assertion is clear. Thus we may assume that $U = O_p(G)$ is minimal. Then by Lemma 2, there is a subgroup $H$ of $G$ such that $G = HU$ and $H \cap U = 1$. If $Q$ is a Sylow $p$-subgroup of $H$, then $Q$ is elementary and $P = QU$. Using now that $P$ is regular, we have easily that $P$ has exponent $p$. Then $G$ has $p$-length one by Hall and Higman [6]. As is remarked in the introduction, this completes the proof of Theorem 1.

Acknowledgement

1. Soon after the earlier work [9] was completed, the author was informed from S. Koshitani that the assertion "(3)⇒(1)" of Theorem 4 [9] as well as the result of Clarke [2] (mentioned in the introduction) is direct from a result of Morita [8]. The author expresses his thanks to S. Koshitani.

2. The author expresses his thanks also to H. Matsuyama, who gives a direct proof of Lemma 2 as the following.

By the Schur and Zassenhaus Theorem, we may put $O_{p,p'}(G) = HU$, where $H$ is a $p'$-subgroup and it is uniquely determined up to $U$-conjugates. Hence by the Frattini argument, it is easily shown that $N_G(H)$ is a desired complement of $U$.

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