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ALMOST-HOMOGENEOUS KÄHLER MANIFOLDS WITH HYPERSURFACE ORBITS

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1. Introduction

Let X be a connected compact complex manifold, and assume that a closed complex subgroup G of the group of holomorphic automorphisms, Aut(X), has an open orbit Ω in X. Then Ω is a dense open connected complex submanifold of X and its complement $E:=X\backslash\Omega$ is a proper analytic subset of X, possibly empty. Such manifolds are called *almost-homogeneous* and they arise quite naturally in many different settings. For example, if a manifold possesses enough holomorphic vector fields to span the tangent space at some point, then it is almost-homogeneous. Equivariant compactifications of complex homogeneous manifolds form another important example of this class of manifolds. Recently, A. Borel [9] has shown that every compact symmetric manifold[1)] is almost-homogeneous; in fact, the automorphism group has only finitely many orbits!

In this paper we are interested in almost-homogeneous manifolds which are *Kähler*. In this case, the albanese map $X\rightarrow A(X)$ of X into a compact complex torus is actually a surjective, locally trivial fiber bundle whose fiber, F, is a simply-connected almost-homogeneous projective algebraic manifold, [37], [34]. With a further assumption on the exceptional set E, we can give a more precise description (Theorem 5.2):

If E is a connected complex hypersurface orbit of G, then

(1) *F is a projective rational manifold which fibers equivariantly $F\overset{M}{\rightarrow}Q$ over a homogeneous projective rational manifold Q with fiber $M\cong \boldsymbol{P}^n$, the n-dimensional quadric Q^n, the Grassmann manifold $G_{2,2n}$, or the exceptional manifold EIII (see* Table 2.6).

(2) *One of the following holds*:

(2.1) $X\cong F\times A(X)$.

(2.2) *There exist equivariant* 2-*to*-1 *coverings $T\rightarrow A(X)$ and $\tilde{X}\rightarrow X$ such that $\tilde{X}\cong F\times T$. In this case $M\cong Q^n$.*

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1) A manifold X is *symmetric* if every point of X is an isolated fixed point of some involution of X.

(2.3) *$X \cong Q \times B$ where Q is a homogeneous projective rational manifold and B is an almost-homogeneous $\boldsymbol{P}^1$-bundle over $A(X)$ with structure group $\boldsymbol{C}$. In this case $F \cong \boldsymbol{P}^1 \times Q$.*

This theorem can be viewed as an analogue of the Borel-Remmert theorem for the homogeneous compact Kähler case, [10].

Note that one can always equivariantly modify an arbitrary compact almost-homogeneous manifold so that E has pure codimension 1 [22], and then pass to an equivariant desingularization [16]. This shows that the important assumption on E is that it is also *homogeneous* with respect to G. It implies, for instance, that equivariant meromorphic maps of X are holomorphic (Lemma 2.3), and that equivariant projective algebraic compactifications of Ω are unique (Lemma 2.5).

The above theorem also gives a good description of the compact homogeneous Cauchy-Riemann Hypersurfaces[2] which can be equivariantly imbedded in a compact Kähler manifold, since these manifolds are almost-homogeneous and can always be modified to contain a complex hypersurface orbit (Theorem 6.2).

We note in passing that one can consider the more general question of classifying $\Omega = G/H$ where a maximal compact subgroup of the complex Lie group G has real hypersurface orbits. These hypersurface orbits can be thought of as providing a natural "homogeneous" exhaustion for the homogeneous manifold Ω. The only case in which Ω *cannot* be equivariantly compactified is when the normalizer fibration $G/H \to G/N_G(H^0)$ realizes Ω as a compact torus bundle over an *algebraic* example where again a maximal compact subgroup has real hypersurface orbits. The question is whether such a bundle extends to the natural equivariant compactification of the base. The treatment of this question, however, goes beyond the scope of this paper. Even when a compactification exists, there are complicated problems arising in the non-Kähler case.

The contents of this paper is as follows:

Notations and definitions are collected in §2, along with some useful lemmas. General references to this material are [35], [21], [23].

In §3 we classify those almost-homogeneous compact Kahler manifolds whose exceptional set is not connected (Theorem 3.2). These manifolds are actually linked to special cases studied in later sections.

The important case of almost-homogeneous projective algebraic manifolds whose exceptional sets are complex hypersurface orbits (i.e. the albanese fiber) is the subject of §4. Similar results in this algebraic setting were recently

2) Here we must assume that the hypersurface is homogeneous with respect to a compact Lie group.

announced by Ahiezer [2] during the period in which the present paper was being prepared. The reader should note that a more detailed description of the algebraic groups involved can be found there.

We put the pieces together in §5, showing that the complex hypersurface orbit assumption on E implies the albanese fibration has the restricted structure mentioned in the above theorem.

In §6 we show how any compact Kähler manifold with a real hypersurface orbit can be modified to satisfy the conditions of §5. We also collect several of the preceding results to show that the Remmert-van de Ven conjecture is true in several special situations.

Although most of our results are proven for *manifolds*, it is primarily a technical matter to adjust them to apply to irreducible complex spaces. For example, if $(X, G)_{\mathcal{O}}$ is an almost-homogeneous irreducible compact Kähler space whose exceptional set E is a connected complex hypersurface orbit of G, then the equivariant normalization $\hat{X}$ of X, ν: $(\hat{X}, G)_{\mathcal{O}} \to (X, G)_{\mathcal{O}}$, must be an almost-homogeneous compact Kähler *manifold* whose exceptional set $\hat{E}=\nu^{-1}(E)$ has at most two components, each of which is a complex hypersurface orbit of G. Thus, either

1) $\hat{E} \cong E$ and the singular set of X is *exactly* E (i.e. X is "pinched" along E), or

2) $\hat{E}$ is two disjoint copies of E and $\hat{X}$ is a $\boldsymbol{P}^1$-bundle over $A(X)\times Q$ with structure group $\boldsymbol{C}^*$ (see Theorem 3.2). In this case, X is obtained from $\hat{X}$ by identifying the zero and infinity sections.

2. Preliminaries

Let X be a complex space and let G be a Lie group. We say that G acts on X if there exists a real analytic map

$$\mu\colon G\times X \to X\,, \quad g(x)\colon =\mu(g, x);\ x\in X\,, \quad g\in G\,,$$

which induces a continuous homomorphism $G\to \mathrm{Aut}(X)$. Here $\mathrm{Aut}(X)$ denotes the topological group of biholomorphic maps of X onto itself with the usual compact-open topology. We write (X, G) to denote such a real analytic action. If G is a complex Lie group and if μ is a holomorphic map, then we write $(X, G)_{\mathcal{O}}$. Finally, if X is an algebraic variety, G an algebraic group, and μ a morphism of varieties, then we write $(X, G)_{\mathcal{A}}$. In most cases it will be clear what type of group action is under discussion and we will simply say that G acts on X or that X is a G-space. For any point $x\in X$, we always have a natural identification (in the appropriate category) of the *orbit* of x, $G(x)\colon =\{g(x)\,|\,g\in G\}$, with the coset space G/G_x where G_x denotes the *isotropy subgroup* of x, $G_x\colon =\{g\in G\,|\,g(x)=x\}$. The group G is said to act *transitively* on X if $G(x)=X$ for all $x\in X$, and we say that X is *homogeneous* with respect to G.

Let $(X, G)_{\mathcal{O}}$ be an irreducible complex space. If G has an open orbit in X, then we say that X is *almost-homogeneous* with respect to G. We usually denote the open orbit by $\Omega=G(x)$ for some $x\in X$. Its complement, denoted $E:=X\backslash\Omega$, is called the *exceptional set* of X. Since X is irreducible it is easy to see that Ω is connected and dense, and that E is a proper (not necessarily connected) analytic subvariety of X.

A holomorphic (or meromorphic) map $f:(X, G)\to(Y, G')$ is said to be *equivariant* if there exists a continuous homomorphism $f_*:G\to G'$ such that the graph of f is invariant under the induced action of G on the product space $X\times Y$, $(x, y)\mapsto(g(x), f_*(g)(y))$. We reserve the special notation $(X, G)_{\mathcal{L}}$ to mean that G is an algebraic group and that there exists an equivariant imbedding $(X, G)_{\mathcal{A}}\to(\boldsymbol{P}^n, \mathrm{Aut}(\boldsymbol{P}^n))_{\mathcal{A}}$. Given a G-space Y, we say that a compact space X is a *G-equivariant compactification* of Y if there exists a G-action on X and an equivariant imbedding $i:(Y, G)\to(X, G)$ such that $i(Y)$ is an open subspace of X which intersects each component of X.

A locally trivial fiber bundle $f:(X, G)\to(Y, G)$ is called a *homogeneous bundle* when f is equivariant and G acts transitively on Y. Given a homogeneous manifold $(Y, G)_{\mathcal{O}}$ with isotropy subgroup H, a complex space F, and a continuous representation $\rho: H\to\mathrm{Aut}(F)$, then one can build a homogeneous bundle over Y with fiber F:

$$G\times_H F:=G\times F/\sim;\ (g, z)\sim(gh^{-1}, \rho(h)z).$$

The projection map $G\times_H F\to Y$ is given by $(g, z)\mapsto gH\in G/H\cong Y$. Any map of coset spaces of complex Lie groups, $G/H\to G/J$ with fiber J/H, has such a representation.

A *parabolic* subgroup P of a complex Lie group G is any subgroup of G which contains a maximal solvable subgroup of G. The quotient space G/P is always a compact simply connected projective rational manifold. Conversely, any homogeneous compact projective rational manifold is the quotient of a complex Lie group by a parabolic subgroup, [8].

If G is a real Lie group contained in a complex Lie group G', then we define the *complex hull* of G, denoted G^C, to be the smallest complex Lie subgroup of G' which contains G.

Let K be a compact Lie group and let (X, K) be an irreducible compact complex space. There exists a desingularization $\pi:\tilde{X}\to X$ of X such that $\tilde{X}$ is a K-space and π is equivariant, [16]. On the compact manifold $\tilde{X}$, K has at most a finite number of *orbit types*, that is, a finite number of conjugacy classes of isotropy subgroups (K_x) for $x\in\tilde{X}$, [21]. Thus, there exists an orbit type (K_x) for which $K(x)=K/K_x$ has maximal dimension. Such orbits are called *generic* K-orbits and their union forms a connected open and dense set in $\tilde{X}$, [21]. One of the basic tools for working with compact Lie group actions is the "Differentiable Slice Theorem" which states that for each orbit $K(x)$, $x \in \tilde{X}$, there exists a K-invariant neighborhood $U \subset \tilde{X}$ of $K(x)$ such that every orbit $K(y)$, $y \in U$, fibers equivariantly over $K(x)$.[3] Note that since π is K-equivariant, the corresponding statements also hold for X.

A useful application of these notions is the following:

Lemma 2.1. *Let K be a compact Lie group and let (X, K) be an irreducible compact complex space. Suppose there exists a K-invariant proper analytic subset E of X. Then, for a dense set of points $y \in E$, there exists a generic K-orbit in X, $K(x)$ for some $x \in X$, such that*

$$\dim_{\boldsymbol{R},y} E - \dim_{\boldsymbol{R}} K(y) < \dim_{\boldsymbol{R}} X - \dim_{\boldsymbol{R}} K(x) .$$

Proof. Let y be a manifold point of E. Choose an open K_y-invariant neighborhood U of y small enough so that we can identify U with a complex subspace of an open domain in the complex (Zariski) tangent space to X at y where the action of K_y on U is linear, [22]. Since K_y stabilizes E, the representation $K_y \to GL(T_y(X))$ reduces to $K_y \to GL(T_y(E)) + GL(V)$ where V is a complementary subspace to $T_y(E)$ in $T_y(X)$. Since y is a manifold point of E, $\dim_y E = \dim T_y(E)$, and thus $\dim U \cap V = \dim X - \dim_y E > 0$. Now, for an open set of points x in $U \cap V$ we have an equivariant fibration

$$K(x) \xrightarrow{K_y(x)} K(y) ,$$

and thus the estimate

$$\begin{aligned} \dim_{\boldsymbol{R}} K(x) &= \dim_{\boldsymbol{R}} K(y) + \dim_{\boldsymbol{R}} K_y(x) < \dim_{\boldsymbol{R}} K(y) + \dim_{\boldsymbol{R},y} U \cap V \\ &= \dim_{\boldsymbol{R}} K(y) + \dim_{\boldsymbol{R}} X - \dim_{\boldsymbol{R},y} E . \end{aligned}$$

Since the set of generic K-orbits forms an open dense subset of X, it is clear that for a dense set of manifold points y in E there will be points $x \in U \cap V$ such that $K(x)$ is a generic K-orbit. □

An immediate consequence of this lemma is

Lemma 2.2. *Let G be a connected complex Lie subgroup of* Aut(X) *and let $(X, G)_{\mathcal{O}}$ be an irreducible compact complex space. If a compact subgroup K of G has a real hypersurface orbit in X, i.e. if* $\dim_{\boldsymbol{R}} K(x) = \dim_{\boldsymbol{R}} X - 1$ *for some $x \in X$, then $(X, G)_{\mathcal{O}}$ is almost-homogeneous and K acts transitively on each connectivity component of the exceptional set of X.*

Proof. It is clear that G has an open orbit in X since $G(x)$ is a complex manifold containing $K(x)$. Also, K stabilizes the exceptional set of X, so the

3) In fact, this neighborhood U can be realized differentiably as a K-invariant neighborhood of the zero-section in the normal bundle of $K(x)$ in $\tilde{X}$, [21].

above lemma applies. □

REMARK. Since Aut(X) is a complex Lie group when X is compact [24], we need only assume that there is a compact Lie group K acting holomorphically on X with a real hypersurface orbit in the above lemma: Just define G to be K^C.

For equivariant maps and compactifications we have the following lemmas:

Lemma 2.3. *Let X be an irreducible normal complex space and let f: $(X, G)_{\mathcal{O}} \to (Y, G')_{\mathcal{O}}$ be an equivariant meromorphic map. If, for all $x \in X$, $\dim_C G(x) \geqslant \dim_C X - 1$, then f is holomorphic.*

Proof. The indeterminancy set of f has codimension at least 2 and must be stabilized by G. Since the G-orbits have at most codimension 1, the indeterminancy set must be empty and f is holomorphic. □

Lemma 2.4. *Let $(\Omega, G)_{\mathcal{A}}$ be an algebraic manifold on which G acts transitively. Then any G-equivariant compactification of $(\Omega, G)_{\mathcal{A}}$ to an irreducible projective algebraic variety is unique up to birational equivalence.*

Proof. Let $(X, G)_{\mathcal{A}}$ and $(X', G)_{\mathcal{A}}$ be two irreducible compact projective algebraic G-spaces such that Ω is biregularly equivalent to $G(x) \subset X$ and $G(x') \subset X'$ respectively. Then there is a biregular equivariant map f: $G(x) \to G(x')$ whose graph $F \subset X \times X'$ is the orbit of the point (x, x') under the algebraic action of G on the product space. Thus, F is Zariski-open in its closure $\bar{F}$, and G stabilizes $\bar{F}$. Therefore, $\bar{F}$ defines a birational G-equivariant map from X to X'. □

These two lemmas give us the following "uniqueness lemma" which will be of particular use in later proofs.

Lemma 2.5. (Uniqueness of compactification). *Let $(\Omega, G)_{\mathcal{A}}$ be an algebraic manifold on which G acts transitively, and let $(X, G)_{\mathcal{A}}$ be a G-equivariant compactification of $(\Omega, G)_{\mathcal{A}}$ to a compact projective algebraic manifold. If $X \backslash \Omega$ has pure codimension* 1, *and if the connectivity components of $X \backslash \Omega$ are homogeneous with respect to G, then $(X, G)_{\mathcal{A}}$ is unique up to G-equivariant biregular equivalence.*

It is perhaps worth noting that this lemma is *not* true if $(X, G)_{\mathcal{O}}$ is a compact projective algebraic manifold on which G acts only *holomorphically*. For example, let $\Omega = \boldsymbol{C}^* \times \boldsymbol{C}^* = G$. Then Ω can be algebraically compactified to $(\boldsymbol{P}^1 \times \boldsymbol{P}^1, G)_{\mathcal{A}}$. However, Ω also fibers equivariantly over an elliptic curve $\Omega \to T := G/\{(e^z, e^{iz}) \mid z \in \boldsymbol{C}\}$ with fiber $\boldsymbol{C}$. Therefore, Ω can be compactified holomorphically and G-equivariantly to an almost-homogeneous $\boldsymbol{P}^1$-bundle over T which is algebraic but not biregularly equivalent to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$!

In this paper we shall often be concerned with (compact) complex manifolds X on which a compact Lie group K acts with at least one real hypersurface orbit,

$H\Sigma = K(x)$ for some $x \in X$. For convenience we call such manifolds (compact) *$H\Sigma$-manifolds*. Obviously, if X is compact, the generic K-orbits are all real hypersurfaces. In fact, all but at most two K-orbits are real hypersurfaces, [31], [33]. It follows that X must be almost-homogeneous and that the exceptional set of X has at most two components, each of which must be homogeneous (Lemma 2.2). Since $H\Sigma$ is homogeneous, the Levi-curvature of $H\Sigma$ in X has constant signature. Whenever this signature is maximal (i.e. the eigenvalues all have the same sign), we write simply $H\Sigma_+$.

The $H\Sigma_+$-manifolds have been studied in various contexts. For example, in [30], Morimoto and Nagano show that a $H\Sigma_+$-manifold Ω which is *Stein* is either the ball $\boldsymbol{B}^n$, $\boldsymbol{C}^n$, or K-equivariantly diffeomorphic to the tangent bundle of a compact symmetric space A of rank 1. In this latter case, $H\Sigma$ is a unit sphere bundle over A.[4] If K^C (abstract complexification) acts holomorphically on Ω, then $\Omega \cong \boldsymbol{C}^n$, or $\Omega \cong K^C/L^C$ and $A \cong K/L$ is realized as a totally real subamanifold of Ω. In either case Ω is affine algebraic and K^C acts on Ω as a linear algebraic group. Let (X, K^C) be a compact projective algebraic manifold which is a K^C-equivariant compactification of Ω. Then, since Ω is Stein (affine algebraic), $E := X \backslash \Omega$ has complex codimension 1. By Lemma 2.2, E is homogeneous under K. Lemma 2.5 then shows that

X is unique up to K-equivariant biregular equivalence.

We list all of the possible Stein $H\Sigma_+$-manifolds M and their projective algebraic K-equivariant compactifications X in the following table. We take K to be the full connected isometry group of A (where applicable), although in some cases a smaller compact group acts transitively (cf. [2]). For this classification see [2], [19]. In [19] it is shown that the manifolds $X \backslash A$ classify all non-compact strictly pseudoconcave homogeneous manifolds (which are not homogeneous cones or $\boldsymbol{P}^n \backslash \boldsymbol{B}^n$). Note, in particular, that X is always homogeneous.

Table 2.6:

X	M	A	K
$\boldsymbol{P}^n$	$\boldsymbol{C}^n$	—	$SU(n+1)$
Q^n 1)	$Q^{(n)}$ 2)	S^n	$SO(n+1)$
$\boldsymbol{P}^n$	$\boldsymbol{P}^n \backslash Q^{n-1}$	$\boldsymbol{RP}^n$	$SO(n+1)$
$\boldsymbol{P}^n \times \boldsymbol{P}^n$	$\boldsymbol{P}^n \times \boldsymbol{P}^n \backslash E$ 3)	$\boldsymbol{P}^n_{\boldsymbol{R}}$ 4)	$\{(A, \bar{A}) \mid A \in SU(n+1)\}$
$G_{2,2n}$ 5)	$\mathrm{Sp}(n, \boldsymbol{C})/\mathrm{Sp}(n-1, \boldsymbol{C})$	$\boldsymbol{QP}^n$ 6)	$\mathrm{Sp}(n)$
$EIII$ 7)	$F_4^{\boldsymbol{C}}/\mathrm{Spin}(9, \boldsymbol{C})$	$F_4/\mathrm{Spin}(9)$ 8)	F_4

1) $Q^n = \{[z] \in \boldsymbol{P}^{n+1} \mid {}^t zz = 0\}$; 2) $Q^{(n)} = \{z \in \boldsymbol{C}^{n+1} \mid {}^t zz = 1\}$; 3) $E = \{([z], [w]) \mid {}^t zw = 0\}$; 4) $\boldsymbol{P}^n_{\boldsymbol{R}} = \{([z], [\bar{z}]) \mid [z] \in \boldsymbol{P}^n\}$; 5) Grassman manifold; 6) Quaternionic projective space; 7) $EIII = E_6/\mathrm{Spin}(10) \times SO(2)$; 8) Cayley projective plane.

4) In [30], $H\Sigma$ is assumed to be simply-connected, although one need only require that $\pi_1(H\Sigma)$ be finite, [39]. In fact, it was later proved that $\pi_1(H\Sigma)$ is *always* finite, [12].

Finally, we state a lemma which will be useful in later structure theorems.

Lemma 2.7. *Let X and Y be connected compact Kähler manifolds. If $H^1(X, \mathcal{O})=0$, then*

$$H^1(X\times Y, \mathcal{O}^*)\cong\pi_1^*H^1(X, \mathcal{O}^*)\oplus\pi_2^*H^1(Y, \mathcal{O}^*)$$

where π_1, π_2 are the natural projections.

Proof. Hodge theory and the Künneth formulas along with $H^1(X, \mathcal{O})=0$ imply that f_1, f_2 and f_3 are isomorphisms in the following diagram (cf. [14]):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^1(X\times Y, \mathcal{O}) & \xrightarrow{f_1} & H^1(X, \mathcal{O}) \oplus H^1(Y, \mathcal{O}) \\
\downarrow & & \downarrow \\
H^1(X\times Y, \mathcal{O}^*) & \longrightarrow & H^1(X, \mathcal{O}^*)\oplus H^1(Y, \mathcal{O}^*) \\
\downarrow & & \downarrow \\
H^2(X\times Y, \boldsymbol{Z}) & \xrightarrow{f_2} & H^2(X, Z) \oplus H^2(Y, \boldsymbol{Z}) \\
\downarrow & & \downarrow \\
H^2(X\times Y, \mathcal{O}) & \xrightarrow{f_3} & H^2(X, \mathcal{O}) \oplus H^2(Y, \mathcal{O})\,.
\end{array}$$

The lemma then follows by the Five-Lemma. □

3. Compact almost-homogeneous Kähler manifolds with disconnected exceptional set

Let $(X, G)_{\mathcal{O}}$ be a compact almost-homogeneous Kähler manifold. The exceptional set E of X can have at most two connectivity components, [4]. We devote this section to collecting some results for the case when E does in fact have two components.

In the algebraic setting we have the following (cf. [1], [13]).

Proposition 3.1. *If $(X, G)_{\mathcal{L}}$ is an almost homogeneous compact projective algebraic manifold with a disconnected exceptional set E, then the open orbit $\Omega=G/H$ can be realized as a principal $\boldsymbol{C}^*$-bundle over a compact homogeneous rational manifold Q,*

$$\Omega \xrightarrow{\boldsymbol{C}^*} Q\,.$$

This bundle induces an almost homogeneous $\boldsymbol{P}^1$-bundle

$$\tilde{X} \xrightarrow{\boldsymbol{P}^1} Q$$

which defines a G-equivariant projective algebraic modification of X,

$$(\tilde{X}, G)_{\mathcal{L}} \longrightarrow (X, G)_{\mathcal{L}}.$$

In addition, the two components of the exceptional set $\tilde{E}$ in $\tilde{X}$ are both isomorphic to Q and fiber equivariantly over the corresponding components of E.

Proof. Since Ω has two ends, it follows from [1], [13] that Ω is a principal $\boldsymbol{C}^*$-bundle over a compact homogeneous rational manifold Q. Let $(\tilde{X}, G)_{\mathcal{L}}$ be the natural G-equivariant algebraic compactification of this $\boldsymbol{C}^*$-bundle obtained by adding two sections. Then $\tilde{X}$ is an almost-homogeneous $\boldsymbol{P}^1$-bundle over Q, and we denote its exceptional set by $\tilde{E}$. Now, either this $\boldsymbol{P}_1$-bundle is trivial or a maximal compact subgroup of G has real hypersurface orbits in $\tilde{X}$. In either case, it follows that the two components of $\tilde{E}$ are both complex hypersurface orbits of G isomorphic to Q (see Lemma 2.2). Then, by Lemmas 2.3 and 2.4, there exists a G-equivariant *birational holomorphic* map $(\tilde{X}, G)_{\mathcal{L}} \to (X, G)_{\mathcal{L}}$, i.e. $\tilde{X}$ is a G-equivariant projective algebraic modification of X. □

REMARK. If Q is *minimal* (i.e. the quotient of a semisimple complex Lie group by a *maximal* parabolic subgroup), then either the modification map is trivial, $\tilde{X}=X$, or a component of $\tilde{E}$ is blown down to a point, because Q cannot be equivariantly fibered. In this latter case X must be $\boldsymbol{P}^n$, [36]. In all other cases nontrivial modification maps exist. The Levi-curvature of the line bundle structure of $\tilde{X}$ (equivalently, the signature of the invariant Chern form) reveals the extent to which a component of $\tilde{E}$ can be (partially) blown down. For the more general Kähler case, we make use of the *albanese map* which is a holomorphic map $\alpha\colon X \xrightarrow{F} A(X)$ of a compact Kähler manifold X into a compact complex torus $A(X)$ with $\dim_{\boldsymbol{C}} A(X) = \frac{1}{2} b_1(X)$. In addition, if $\tau\colon X \to T$ is a holomorphic map of X into a compact complex torus, then there exists a holomorphic map $\sigma\colon A(X) \to T$ such that $\tau = \sigma \cdot \alpha$. If G is a closed connected complex Lie subgroup of $\mathrm{Aut}^0(X)$, and if $(X, G)_{\mathcal{O}}$ is a compact almost-homogeneous Kähler manifold with exceptional set E, then α is a G-equivariant holomorphic fiber bundle inducing a surjective homomorphism $\alpha_*\colon G \to \mathrm{Aut}^0(A(X)) \cong A(X)$, and a surjective holomorphic map $\alpha|E\colon E \to A(X)$, [37]. Moreover, the fiber $(F, \hat{G})_{\mathcal{L}}$ is a compact almost-homogeneous simply-connected projective algebraic manifold, where $\hat{G}\colon = \ker \alpha_*$ is a linear algebraic group, [4].

Theorem 3.2. *If $(X, G)_{\mathcal{O}}$ is an almost-homogeneous compact Kähler manifold with disconnected exceptional set E, then the open orbit $\Omega = G/H$ can be realized as a principal $\boldsymbol{C}^*$-bundle over the product of a compact homogeneous rational manifold Q and the albanese torus $A(X)$ of X,*

$$\Omega \xrightarrow{\mathbf{C}^*} Q \times A(X).$$

This bundle induces an almost homogeneous $\boldsymbol{P}^1$-bundle

$$\tilde{X} \xrightarrow{\boldsymbol{P}^1} Q \times A(X)$$

which defines a G-equivariant modification of X,

$$(\tilde{X}, G)_{\mathcal{O}} \to (X, G)_{\mathcal{O}} .$$

In addition, the two components of the exceptional set $\tilde{E}$ of $\tilde{X}$ are both biholomorphic to $Q \times A(X)$ and fiber equivariantly over the corresponding components of E.

Proof. Let $\alpha: X \to A(X)$ be the albanese bundle with fiber $F_x: =\alpha^{-1}(\alpha(x))$. Since $\alpha | E: E \to A(X)$ is surjective, it follows from the equivariance of α that $E_x: =F_x \cap E$ is disconnected. Since $(F_x, \hat{G})_{\mathcal{L}}$ is a compact almost-homogeneous projective algebraic manifold with disconnected exceptional set E_x, the previous proposition implies that there exists an algebraic $\hat{G}$-equivariant modification

$$\nu: (\tilde{F}_x, \hat{G})_{\mathcal{L}} \to (F_x, \hat{G})_{\mathcal{L}}$$

where $\tilde{F}_x \to Q_x$ is the almost-homogeneous $\boldsymbol{P}^1$-bundle compactifying the principal $\boldsymbol{C}^*$-bundle $\Omega_x: =\hat{G}/H \to Q_x: =\hat{G}/P$, $\Omega_x \subset F_x$, $\tilde{F}_x$. Since ν is $\hat{G}$-equivariant, we can define a holomorphic fiber bundle space

$$\tilde{X} \xrightarrow{\tilde{F}_x} A(X) , \quad \tilde{X}: = G \times_{\hat{G}} \tilde{F}_x ;$$

and a holomorphic map

$$\tilde{X} = G \times_{\hat{G}} \tilde{F}_x \to X = G \times_{\hat{G}} F_x , \quad (g, z) \mapsto (g, \nu(z))$$

which is clearly a G-equivariant modification of X. Note that $\tilde{X}$ is also a G-equivariant almost-homogeneous $\boldsymbol{P}^1$-bundle over G/P, which is just the usual G-equivarint compactification of the $\boldsymbol{C}^*$-bundle $\Omega = G/H \to G/P$. In addition, any equivariant imbedding $(\tilde{F}_x, \hat{G})_{\mathcal{L}} \to (\boldsymbol{P}^N, \mathrm{Aut}(\boldsymbol{P}^N))_{\mathcal{L}}$ defin esan imbedding of $\tilde{X}$ into a $\boldsymbol{P}^N$-bundle over $A(X)$ which is Kähler, [25]. Therefore, $\tilde{X}$ is Kähler, and G/P being the proper image of a Kähler manifold must also be Kähler (cf. [6]). Thus, the albanese map of G/P,

$$G/P \to G/\hat{G} = A(X) \qquad \text{with fiber} \quad \hat{G}/P = Q_x ,$$

splits into a product, $G/P = Q_x \times A(X)$, [10]. Finally, since $\tilde{E}$ is just the disjoint union of two sections added to this $\boldsymbol{C}^*$-bundle, the components of E are biholomorphic to $Q_x \times A(X)$. □

We now describe the bundle structure of these manifolds.

Corollary 3.3. *There exist principal $\boldsymbol{C}^*$-bundles $L_1 \to Q$ and $L_2 \to A(X)$*

with L_2 topologically trivial such that

$$\Omega \cong \pi_1^*(L_1) \otimes \pi_2^*(L_2),$$

where π_1, π_2 are the natural projections.

Proof. By Lemma 2.7 we have $\Omega \cong \pi_1^*(L_1) \otimes \pi_2^*(L_2)$. To see that L_2 is topologically trivial we need only note that the holomorphic fibration $\Omega \to Q$ is a map of coset spaces, so that L_2 is equivalent to a homogeneous principal $\boldsymbol{C}^*$-bundle over a compact complex torus and therefore is topologically trivial, [27]. □

REMARK. Since *any* such $\boldsymbol{C}^*$-bundles $L_1 \to Q$, $L_2 \to T$ (L_2 topologically trivial) are homogeneous, they always give rise to an example, $X \xrightarrow{\boldsymbol{P}^1} Q \times T$.

From this structure theorem we easily deduce the following

Corollary 3.4. *Let E_1, E_2 be the components of E. Then*

$$A(E_1) = A(E_2) = A(X).$$

4. The algebraic case

We now restrict our attention to the case where G is a complex linear algebraic group and $(X, G)_{\mathcal{L}}$ is an almost-homogeneous projective algebraic manifold whose exceptional set E is a complex hypersurface orbit, $E = G(x_0)$. (See §3 if E is not connected.)

In this section we wish to prove a fibration theorem for such manifolds, but first we present two preparatory lemmas.

Lemma 4.1. *Let S be a reductive linear algebraic complex Lie group and H a closed algebraic subgroup of S. If S/H is not Stein, then H is contained in a proper parabolic subgroup of S, i.e. there exists a homogeneous fibration, $S/H \to S/P$, where S/P is a non-trivial compact projective rational manifold.*

Proof. If S/H is not Stein, then H is not reductive, [28], so that the unipotent radical, $R_u(H)$, of H is non-trivial. Then the increasing sequence of subgroups $N_0 \subset N_1 \subset \cdots \subset N_i \subset \cdots$ where $N_0 := N_S(R_u(H))$ and $N_i := N_S(R_u(N_{i-1}))$, must stabilize with a proper parabolic subgroup of S (see e.g. [20]). □

Lemma 4.2. *Let $(X, G)_{\mathcal{L}}$ be a compact almost-homogeneous projective algebraic manifold with $\dim X > 1$. Assume that the open orbit Ω is Stein (i.e. affine algebraic) and that the exceptional set E of X is a (necessarily connected) complex hypersurface orbit of G. Then the generic orbit of a maximal compact subgroup K of G is a real hypersurface orbit in X, i.e. X is an equivariant projective* *algebraic compactification of a Stein $H\Sigma_+$-manifold (see Table 2.6).*

Proof. Since Ω is Stein, E must be connected, [40]. Since G acts linearly, algebraically and transitively on E, E is a compact homogeneous projective rational manifold, [15]. Thus, if K is a maximal compact subgroup of G, K acts transitively on E also. Therefore, the generic K-orbits in X have real codimension 1 or 2.

If the generic K-orbit has codimension 2, then the normal (complex line) bundle of E in X is topologically trivial. This follows from the fact that one can always smoothly and K-equivariantly realize a neighborhood $N\subset X$ of E as a neighborhood of the zero section in the normal bundle of E in such a way that $K(p)\to E$ is a homogeneous fibration for $p\in N$, [21]. This fibration is a diffeomorphism because E is simply connected.

We now show that this is a contradiction. Let $(X', G)_{\mathcal{L}}$ be an equivariant compactification of the affine algebraic manifold Ω to a projective algebraic variety such that $E':=X'\backslash\Omega$ is a connected hyperplane section (see [5]). It follows from Lemmas 2.3, 2.4 that there exists a holomorphic equivariant birational map $\nu: X\to X'$, showing that E' is homogeneous under G. Equivariance also implies that ν is 1-to-1 (X is the G-equivariant normalization of X'!). If H denotes the hyperplane section bundle on X', then $\nu^*H|E$ is isomorphic to a power of the normal bundle of E in X and clearly has non-constant sections. Therefore, the normal bundle of E cannot be topologically trivial. □

Theorem 4.3. *Let $(X, G)_{\mathcal{L}}$ be an almost-homogeneous connected compact projective algebraic manifold with open orbit $\Omega=G/H$. Assume that the exceptional set $E=X\backslash\Omega$ is a connected complex hypersurface orbit of G. Then there is a G-equivariant fibration of X*

$$X \xrightarrow{M} Q$$

where $Q=G/P$ is a compact projective rational manifold, P is any minimal parabolic subgroup of G containing H, and the fiber M is biregularly equivalent to $\boldsymbol{P}^n$, Q^n, $G_{2,2n}$, or EIII (see Table 2.6).

Proof. Let P be any *minimal* parabolic subgroup of G which contains H. Then we have an equivariant fibration $\Omega\to G/P=:Q$. Let M be the P-equivariant compactification of the fiber P/H in X. By blowing up $E_M:=M\backslash(P/H)$ and passing to an equivariant desingularization of M, we may assume that M is a manifold and that E_M has pure codimension 1 (see §1). We define $X':=G\times_P M$. Then $(X', G)_{\mathcal{L}}$ is an almost-homogeneous projective algebraic manifold with open orbit Ω. Lemma 2.4 implies that X' is equivariantly birationally equivalent to X. Since $E':=X'\backslash\Omega$ has pure codimension 1, equivariance implies that the components of E' are homogeneous. Lemma 2.5

then implies that $X' \cong X$. Thus, we obtain an equivariant fibration of X, $X \xrightarrow{M} Q$. Note that the induced equivariant fibration $E \to Q$ shows that $E_M = E \cap M$ is homogeneous and connected.

If $\dim M < \dim X$, then an induction argument on dimension[4] implies that there exists an equivariant fibration of M, $M \xrightarrow{M'} Q'$, as in the statement of the theorem, where $Q' = P/P'$. By the minimality of P we have $P = P'$ and $M = M'$, and the theorem is true.

Therefore, we may assume that $M = X$, i.e. that *any minimal parabolic subgroup of G which contains H must be G itself*. In this case we claim that Ω is Stein, which by Lemma 4.2 implies that X is an equivariant compactification of a Stein $H\Sigma_+$-manifold (Table 2.6). Let K be a maximal compact subgroup of G and let $S = K^C$. Recall that the generic K-orbits in X have real codimension at most 2. We then have the following possibilities:

1) S has a compact orbit in Ω with complex codimension 1, $S(x) = K(x)$.

2) S has an open Stein orbit $S(x)$.

or 3) S has an open orbit which is *not* Stein, $S(x)$.

In case 2) we have $S(x) = \Omega$—unless $S(x) = \boldsymbol{C}^*$ and $\Omega = \boldsymbol{C}$, since a Stein manifold has only one "end" in dimensions greater than 1, [40]—showing that Ω is Stein as claimed.

Case 1) can only occur when $X = \boldsymbol{P}^1$. To see this, let $G = R_u S$ where R_u is the unipotent radical of G, [20]. Then, since G acts algebraically on X, the orbits of R_u are Zariski-open in their closures, and hence we obtain an equivariant fibration of Ω, $\Omega = G/H \xrightarrow{\rho} G/R_u H$. It follows from Lie's Theorem that, since it is solvable and acting algebraically, the R_u-orbits are holomorphically separable. Since such an orbit intersects $S(x)$ in a compact analytic set, this intersection must be finite. Thus the fibration $S(x) \to G/R_u H$ is finite, and thus the base $G/R_u H$ is a homogeneous rational manifold having the same dimension as $S(x)$. In fact, they intersect in exactly one point since $S(x)$ is a compact simply-connected projective rational manifold, and thus $G/R_u H \cong S(x)$. The above assumption on G implies that $R_u H = G$, so that $S(x)$ reduces to a point. Therefore, X, being a compact connected 1-dimensional almost-homogeneous manifold of a linear algebraic group, must be biregularly equivalent to $\boldsymbol{P}^1$.

Finally, we show that case 3) implies that $X \cong \boldsymbol{P}^n$ and $\Omega \cong \boldsymbol{C}^n$. Let $S(x) = S/S \cap H$ be the open S-orbit in X which is not Stein. There are two cases which we handle separately:

(a) $X \setminus S(x)$ is connected.

or (b) $X \setminus S(x)$ is not connected.

5) If $\dim X = 1$, the theorem is trivial.

In (a), we apply Lemma 4.1 to obtain a *proper* parabolic subgroup P_0 of S which contains $S \cap H$, and the corresponding equivariant fibration $S(x) \to S/P_0 =: Q_0$. Just as in the beginning of the proof, if M_0 denotes an equivariant compactification of the fiber to a projective algebraic manifold, then X is biregularly equivalent to the almost-homogeneous manifold $S \times_{P_0} M_0$, since E is homogeneous with respect to S and has complex codimension 1. We thus obtain a fibration of X, $X \to Q_0$, which is equivariant with respect to G since the fiber is compact and connected, [38]. Therefore, $Q_0 = G/P'$ where P' is a parabolic subgroup of G containing H. By our assumption on G, we have $G = P'$ so that $S = P_0$, contradicting the fact that P_0 is a *proper* subgroup of S. This shows that (a) does not occur.

For (b), we apply Proposition 3.1 to show there exists an S-equivariant algebraic modification of X, $\mu: \tilde{X} \to X$, where $\pi: \tilde{X} \xrightarrow{\boldsymbol{P}^1} Q'$ is an almost-homogeneous $\boldsymbol{P}^1$-bundle over a homogeneous projective rational manifold Q' with structure group $\boldsymbol{C}^*$. Let $\tilde{E} = \tilde{E}_0 \cup \tilde{E}_\infty$ be the exceptional set of $\tilde{X}$, i.e. the zero and infinity sections of the $\boldsymbol{P}^1$-bundle. By Proposition 3.1 we know $\tilde{E}_0 \cong \tilde{E}_\infty \cong Q'$ and that $\tilde{E}_\infty$ (say) is biholomorphic to E, while $\tilde{E}_0 \to \mu(\tilde{E}_0) =: Q''$ is an equivariant fibration of $\tilde{E}_0$ onto another compact homogeneous projective rational manifold $Q'' \subset X$. We now construct a holomorphic map from X to Q'' as follows:

$$X \xrightarrow{\mu^{-1}} \tilde{X} \xrightarrow{\pi} Q' \xrightarrow{\mu} Q''.$$

Note that μ^{-1} is only a meromorphic map so that $\pi' := \mu \circ \pi \circ \mu^{-1}$ is a priori only a meromorphic map. However, due to the equivariance of the maps involved, it is easy to see that π' is well-defined and continuous, and therefore holomorphic. Since the fiber is compact and connected, this map is equivariant with respect to G, [38]. Thus, $Q'' = G/P''$ where P'' is a parabolic subgroup of G containing H. Once again, this means that $G = P''$, so that Q'' reduces to a point. Therefore, X can be realized as a compact almost-homogeneous manifold (with respect to S) whose exceptional set contains an isolated fixed point. A theorem of E. Oeljeklaus [36] implies that $X \cong \boldsymbol{P}^n$ and $S(x) \cong \boldsymbol{C}^n \setminus \{0\}$. Therefore, $\Omega \cong \boldsymbol{C}^n$ as claimed.

To conclude the proof, we need only check Table 2.6 to see that, since P is minimal, the possibility that $M \cong \boldsymbol{P}^n \times \boldsymbol{P}^n$ cannot occur. □

We now list a few consequences of this theorem which further describe the properties of X.

Corollary 4.4. *$\pi_1(\Omega) = 0$ or $\boldsymbol{Z}_2$.*

Proof. This follows from the homotopy sequence $\pi_1(M \cap \Omega) \to \pi_1(\Omega) \to \pi_0(Q)$ and Table 2.6. □

Corollary 4.5. *Unless $X=\boldsymbol{P}^1\times E$ and every K-orbit is biregularly equivalent to E, the generic orbit of a maximal compact subgroup K of G is a real hypersurface orbit in X.*

Proof. By the theorem, X has a G-equivariant fibration $X\overset{M}{\to}Q$. Lemma 4.2 applied to M shows that K has real hypersurface orbits in X unless $\dim_{\boldsymbol{C}} M=1$. In this case $M=\boldsymbol{P}^1$. Now, if K does not have real hypersurface orbits in X, then the generic K-orbit must have real codimension 2, as before. These K-orbits show that the affine or line bundle structure of X is topologically trivial. Since $H^1(Q, \mathcal{O})=O$, it follows that the bundle structure is in fact holomorphically trivial and $X=\boldsymbol{P}^1\times Q=\boldsymbol{P}^1\times E$. □

Corollary 4.6. *The manifold M cannot be P-equivariantly and non-trivially fibered with positive dimensional fiber.*

Proof. If $M\to Y$ is a P-equivariant fibration of M with positive dimensional fiber Z, then the open Stein orbit P/H also fibers onto an open homogeneous submanifold of Y. Since P/H is Stein, the fiber Z must intersect $E\cap M$. By equivariance, P/H then fibers *onto* Y so that $Y=P/P'$ is a compact homogeneous projective rational manifold. By minimality of P, Y must reduce to a point. □

Corollary 4.7. *If the generic K-orbit is a real hypersurface in X, then the isotropy subgroup H has at most index 2 in $N_G(H^0)$, i.e. either $H=N_G(H^0)$ or $H\lhd N_G(H^0)$ and $N_G(H^0)/H\cong \boldsymbol{Z}_2$. This latter possibility can only occur when $M=Q^n$, a projective quadric hypersurface.*

Proof. We first show that $N_G(H^0)/H$ is finite. If the orbits of $N_G(H^0)$ are positive dimensional in G/H, then they each intersect a fixed generic real hypersurface orbit of K. Since G is acting linearly, it follows that these orbits cannot be compact. Therefore, $G/N_G(H^0)$ is compact and indeed a projective rational manifold so that $N_G(H^0)$ is parabolic. We choose P to be a minimal parabolic subgroup of G containing H which is contained in $N_G(H^0)$. Then H^0 is a normal subgroup of P and therefore fixes every point in the Stein manifold $P/H=(P/H^0)/(H/H^0)$ which is now group theoretically parallelizable. This can only happen when $P/H=\boldsymbol{C}^k$ or $\boldsymbol{C}^*$ (see Table 2.6), and the latter possibility is eliminated by our assumption that E is connected. Thus, $H=H^0$ and $P/H\cong\boldsymbol{C}^k$ is an abelian complex Lie group. But then no maximal compact subgroup of P can have real hypersurface orbits in P/H. This contradiction implies that the orbits of $N_G(H^0)$ are 0-dimensional. Thus, since $N_G(H^0)$ is an algebraic group, $N_G(H^0)/H$ is finite.

Now consider the G-equivariant *finite* covering $X\to X'$ of X onto the orbit space X' of the action of $N_G(H^0)$ on X. This map is given by $\Omega=G/H\to\Omega':=G/N_G(H^0)$ on Ω and is a biholomorphism of E onto $E':=X'\backslash\Omega'$ since E is simply connected. It is clear that K still has real hypersurface orbits in X' and that E' is a complex hypersurface orbit in X' (cf. Lemma 2.2. The construction of X' is also given by Theorem 6.1). It follows that the G-equivariant normalization $\hat{X}$ of X' is a manifold satisfying the conditions of the theorem. Therefore, there exists a parabolic subgroup P' of G containing $N_G(H^0)$. We now choose P to be a minimal parabolic subgroup containing H which is contained in P'. However, since the above map is finite, it follows that $P=P'$ and $N_G(H^0)=N_P(H^0)$. Table 2.6 shows that $N_P(H^0)=H$ unless $M=Q^n$ in which case $N_P(H^0)/H=\boldsymbol{Z}_2$. □

Corollary 4.8. *X is a projective rational manifold.*

Proof. Let B be a Borel subgroup of G. Then B has an open orbit in E isomorphic to $\boldsymbol{C}^{n-1}$ ($n=\dim X$) since E is a compact homogeneous projective rational manifold. According to [26], X is birationally equivalent to $\boldsymbol{P}^{n-1}\times V$, where V is a 1-dimensional compact projective algebraic variety. Theorem 4.3 shows that $b_1(X)=0$, and since this is a birational invariant it follows that $b_1(V)=0$, i.e. $V=\boldsymbol{P}^1$. Therefore, X is rational. □

5. The compact Kähler case

In [10], Borel-Remmert prove that the albanese fibration $\alpha\colon X\to A(X)$ of a compact homogeneous Kähler manifold X splits X into a product $X=Q\times A(X)$ where Q is a compact homogeneous projective rational manifold.

In general, this kind of splitting does not occur when X is a compact *almost-homogeneous* Kähler manifold. However, in this section we prove that if the exceptional set E of X is a connected complex hypersurface orbit, then with two exceptions the albanese fibration *does* split X into a product $X=F\times A(X)$. In any case, the complex hypersurface orbit assumption implies that $(F,\hat{G})_{\mathcal{L}}$ is always *a compact almost-homogeneous projective rational manifold as described in* §4. Of course, we must take $G\subset \mathrm{Aut}^0(X)$ in order to guarantee that $\hat{G}$ is linear algebraic (see §3).

We begin with the following

Proposition 5.1. *Let G be a closed connected complex Lie subgroup of $Aut^0(X)$ and let $(X,G)_{\mathcal{O}}$ be a compact almost-homogeneous Kähler manifold whose exceptional set E is a connected complex hypersurface orbit of G. Let $(F,\hat{G})_{\mathcal{L}}$ be the fiber of the albanese fibration $\alpha\colon X\to A(X)$. Assume that a maximal compact subgroup of $\hat{G}$ has a real hypersurface orbit in F. Then there exists a compact complex central subgroup $T\subset G$ such that either*

1) *$G\cong\hat{G}\times T$, or*

2) *$G\cong\hat{G}\times T/J$ where $J:=\{(z,z^{-1})\mid z\in\hat{G}\cap T\}$ is a finite group of order two.*

Proof. We first assume that G is the connected component of the stabilizer of E in Aut(X). Let H be the isotropy subgroup of a point x in the open G-orbit, $x\in\Omega$, and set $F_x:=\alpha^{-1}(\alpha(x))$, $\Omega_x:=\Omega\cap F_x$. Then,

$$N_G(H^0)(x)\cap\Omega_x=(N_G(H^0)/H)\cap(\hat{G}/H)=N_{\hat{G}}(H^0)/H$$

which is at most two points by Corollary 4.7. Therefore, the equivariance of the albanese fibration implies that

$$N_G(H^0)(x)\to A(X)$$

is a 1-to-1 or 2-to-1 equivariant covering map. Thus, since H acts trivially on $A(X)$, H acts trivially on the component of $N_G(H^0)(x)$ which contains x. Also, there are at most two components of $N_G(H^0)(x)$ so that H must act trivially on *all* of $N_G(H^0)(x)$. This shows that H is normal in $N_G(H^0)$ and that $T:=N_G(H^0)/H=N_G(H^0)(x)$ is a compact complex torus, perhaps with two components.

We now define a holomorphic action of T on Ω in the following way: Let $t\in T$ and $x\in\Omega$. Then $t=nH\in N_G(H^0)/H$ and $x=gH\in G/H$. Define

$$t(x):=gnH.$$

This is a well-defined holomorphic action since H is normal in $N_G(H^0)$ and T is abelian.

We wish to extend the action of T to all of X. To do this, we must inspect both the albanese fibration, $\alpha: X\to A(X)$, and the fibration

$$\beta:\Omega=G/H\to Y:=G/N_G(H^0).$$

By [17], β extends to a G-equivariant meromorphic map

$$\tilde{\beta}: X\to\bar{Y}$$

where $\bar{Y}$ is an appropriate compactification of Y to a complex space. Lemma 2.3 implies that $\tilde{\beta}$ is holomorphic. Since $\tilde{\beta}$ is also a proper map, we can find a bounded Stein neighborhood Z of $\tilde{\beta}(x_0)$, $x_0\in E$, such that $V:=\tilde{\beta}^{-1}(Z)$ is $\tilde{\beta}$-saturated, $\tilde{\beta}^{-1}(\tilde{\beta}(V))=V$. This implies that $V\backslash E$ is invariant under the action of T. Note that the restricted albanese map

$$\alpha:\Omega=G/H\to A(X)=G/\hat{G}$$

is also T-equivariant when the action of T on $A(X)$ is defined via left multiplication of cosets by elements of $N_G(H^0)$. Fix $t\in T$. By the above remarks it follows that there exists a small coordinate neighborhood U of $\alpha(x_0)$ in $A(X)$ such that $W:=V\cap\alpha^{-1}(U)$ is a bounded coordinate neighborhood of x_0 and

$$t(W\backslash E)\subset\alpha^{-1}(tU)\cap V$$

where $\alpha^{-1}(tU)\cap V$ is also a bounded coordinate neighborhood. The action of t on $W\backslash E$ is now given by bounded holomorphic functions and therefore t extends to all of W and indeed to all of X. We thus obtain a holomorphic Lie group monomorphism

$$\rho\colon T^0\rightarrow G$$

whose image we denote by T_0.

We claim that T_0 is a central subgroup of G. To see this let $t=nH\in T^0$, $n\in N_G(H^0)$; and $t_0:=\rho(t)$. Then, we have for $gH\in G/H=\Omega$

$$t_0gH=\rho(t)gH=t(gH)=gnH=gt_0H$$

since $\rho(t)H=nH$. Therefore, $t_0g=gt_0$ for all $g\in G$, $t_0\in T_0$ because G acts effectively on Ω. Consider the complex Lie group homomorphism

$$\hat{G}\times T_0\rightarrow G;\quad (g,t)\mapsto gt\,,$$

whose kernel is $J:=\{(z,z^{-1})\,|\,z\in\hat{G}\cap T_0\}$. Since $\dim T_0=\dim A(X)$, it follows that the image of this homomorphism is open and hence all of G. Now $J\cong\hat{G}\cap T_0$ and

$$\hat{G}\cap T_0=\hat{G}\cap T_0/H\cap T_0=(\hat{G}/H)\cap T_0(x)=\Omega_x\cap N_G(H^0)(x)$$

which we have already seen consists of at most two points. Therefore, $J\cong\{1\}$ or $\boldsymbol{Z}_2$.

Finally, we note that if G' is any closed subgroup of G acting transitively on Ω and E, then $G=\hat{G}\times T_0$ or $\hat{G}\times T_0/J$ as above. Let $T'=\ker(\tilde{\beta}_*|G')=G'\cap\ker\tilde{\beta}_*$. Then, since T' acts transitively on $A(X)$, it follows that $\dim A(X)\leqslant\dim T'\leqslant\dim\ker\tilde{\beta}_*=\dim T_0=\dim A(X)$. In particular, $T_0\subset T'$, so that $G'=\hat{G}'\times T_0$ or $\hat{G}'\times T_0/J'$, where $\hat{G}'=\ker(\alpha_*|G')=G'\cap\hat{G}$ and $J'=\{(z,z^{-1})\,|\,z\in\hat{G}'\cap T_0\}$. □

We now prove our main structure theorem.

Theorem 5.2. *Let $(X,G)_{\mathcal{O}}$ be a compact almost-homogeneous Kähler manifold whose exceptional set is a connected complex hypersurface orbit of G. Let $X\xrightarrow{F}A(X)$ be the albanese fibration of X. Then*

(1) *F is an almost-homogeneous compact rational manifold which fibers equivariantly, $F\xrightarrow{M}Q$, over a compact homogeneous rational manifold Q with fiber $M\cong\boldsymbol{P}^n$, Q^n, $G_{2,2n}$, or $E\ III$ (see* Table 2.6*);*

and (2) *One of the following holds*:

(2.1) *$X\cong F\times A(X)$.*

(2.2) *There exists an equivariant* 2-*to*-1 *covering of $A(X)$,*

$$T\rightarrow A(X)\,,$$

and an equivariant 2-to-1 covering of X,

$$\tilde{X} \to X$$

such that $\tilde{X} \cong F \times T$. In this case $M \cong Q''$.

(2.3) $X \cong Q \times B$, *where Q is a compact homogeneous projective rational manifold and B is an almost homogeneous $\boldsymbol{P}^1$-bundle over $A(X)$ with structure group $\boldsymbol{C}$. In this case, $F \cong \boldsymbol{P}^1 \times Q$.*

REMARK. A maximal compact subgroup of G has a real hypersurface orbit in X only in cases (2.1) and (2.2), and we have $G \cong \hat{G} \times A(X)$ and $G \cong \hat{G} \times T/J$, $J = \{(z, z^{-1}) \mid z \in \hat{G} \cap T\}$, respectively (see Proposition 5.1).

Proof. We have already noted that statement (1) is true. Let $(F, \hat{G})_{\mathcal{L}}$ be the fiber of the albanese fibration. We consider two cases: 1) A maximal compact subgroup of $\hat{G}$ has a real hypersurface orbit in F, or 2) there are no such real hypersurface orbits.

1): By the previous proposition we know $G \cong \hat{G} \times T$ or $\hat{G} \times T/J$. Consider the holomorphic map

$$\nu \colon F \times T \to X, \quad (z, t) \mapsto t(z).$$

If $G \cong \hat{G} \times T$, then this map is biholomorphic since T acts trivially on F and transitively on $A(X)$. In this case it is clear that $T \cong A(X)$, proving (2.1). If $G = \hat{G} \times T/J$, then ν defines a 2-to-1 map since every orbit of $\hat{G} \cap T$ in F consists of two points. Corollary 4.7 implies that $M = Q''$, proving (2.2).

2): If there are no real hypersurface orbits, then Corollary 4.5 implies that $F \cong \boldsymbol{P}^1 \times Q$. In fact, $\hat{G}/H \to \hat{G}/P = Q$ is a trivial $\boldsymbol{C}$-bundle and F is its compactification. The fibration

$$\Omega = G/H \xrightarrow{\boldsymbol{C}} G/P$$

is therefore an affine $\boldsymbol{C}$-bundle and X is its compactification to a $\boldsymbol{P}^1$-bundle.

Let L denote the principal $\boldsymbol{C}^*$-bundle associated to this affine $\boldsymbol{C}$-bundle. By Lemma 2.7 we have that $L = \pi_1^*(L_1) \otimes \pi_2^*(L_2)$, where $L_1 \to Q$ and $L_2 \to A(X)$ are principal $\boldsymbol{C}^*$-bundles. In addition, since the restricted affine $\boldsymbol{C}$-bundles over $Q \times \{t\}$, $t \in A(X)$, are trivial, L_1 is the trivial bundle. Thus, the structure group of the affine $\boldsymbol{C}$-bundle $\Omega \xrightarrow{\boldsymbol{C}} Q \times A(X)$ has the form $\begin{pmatrix} a(t) & b(q, t) \\ 0 & 1 \end{pmatrix}$ $q \in Q$, $t \in A(X)$. Now the restricted affine $\boldsymbol{C}$-bundles over $\{q\} \times A(X)$, $q \in Q$, are homogeneous (they are the fibers of the map of coset spaces $\Omega \to Q$). Therefore, it follows from [29] that for *fixed* q this group can be further reduced to either $\begin{pmatrix} c(q, t) & 0 \\ 0 & 1 \end{pmatrix}$ with $c(q, t) \neq 1$, or $\begin{pmatrix} 1 & d(q, t) \\ 0 & 1 \end{pmatrix}$. The first possibility is eliminated by our assumption that the exceptional set is connected (Ω has one end). This shows that L_2 is also the trivial bundle, and hence Ω is a principal $\boldsymbol{C}$-bundle. The structure group is now equivalent to $\begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix}$ $t\in A(X)$, since $H^1(Q\times A(X), \mathcal{O})\cong H^1(Q, \mathcal{O})\oplus H^1(A(X), \mathcal{O})\cong H^1(A(X), \mathcal{O})$. Thus, $\Omega\cong Q\times\Omega'$ where Ω' is a principal $\boldsymbol{C}$-bundle over $A(X)$, and $X\cong Q\times B$ as claimed. □

It is quite easy to illustrate the phenomenon of (2.2) in the above theorem: Let $T=\boldsymbol{C}^n/\Gamma$ and let $\rho: \Gamma\to\boldsymbol{Z}_2: =\{1, \sigma\}$ be a non-trivial representation. We identify σ with the involution of $Q^m\subset\boldsymbol{P}^{m+1}$, $[z_0: z_1: \cdots : z_{m+1}]\to[-z_0: z_1: \cdots : z_{m+1}]$. Then we define $X: =Q\times T/\sim$, where $(q, t)\sim(\rho(\gamma)q, t+\gamma)$ for $\gamma\in\Gamma$. This example is also presented in [4], where it is shown among other things that the structure group of the albanese fibration can *always* be reduced to a finite group when X is an almost-homogeneous compact Kähler manifold. Of course, we can construct similar examples using any equivariant fibration $F\xrightarrow{M} Q$ as in (1) with fiber $M\cong Q^n$. This is because σ commutes with the structure group of the bundle and hence acts on F.

6. Compact Kähler manifolds with real hypersurface orbits

In this section we consider a compact Kähler manifold X on which a compact Lie group K acts with at least one real hypersurface orbit, $H\Sigma=K(x)$, for some $x\in X$. Recall from §2 that such an X is called a compact (Kähler) $H\Sigma$-manifold and is almost-homogeneous with respect to $S: =K^C$ (we may as well assume that K is a closed subgroup of $\mathrm{Aut}(X)$). In addition, the connectivity components of the exceptional set E of X are homogeneous under K and S.

As usual, we begin with a proposition for the algebraic case. The proof uses the same argument as in Theorem 4.3.

Proposition 6.1. *Let $(X, S)_{\mathcal{L}}$ be a compact projective algebraic $H\Sigma$-manifold. Then there exists an equivariant algebraic modification of X,*

$$(\tilde{X}, S)_{\mathcal{L}}\to(X, S)_{\mathcal{L}},$$

such that the connectivity components of the exceptional set $\tilde{E}$ of $\tilde{X}$ are complex hypersurface orbits of S.

Proof. First note that we may assume the exceptional set E is connected, since otherwise we apply Proposition 3.1. If $S(x)=S/H$ is Stein, we refer directly to §2 and Table 2.6. If S/H is not Stein, then there exists a *proper* minimal parabolic subgroup P of S which contains H by Lemma 4.1. Thus, we obtain a non-trivial fibration $S/H\xrightarrow{P/H} S/P=: Q$. The fiber P/H has real hypersurface orbits with respect to $P\cap K$ (for appropriately chosen K in S). Therefore, $S_P: =(P\cap K)^C$ has an open orbit in P/H, say $S_P(x)$. As in the proof of Theorem 4.3, it follows from the minimality of P that $S_P(x)$ is Stein. Let M be the equivariant compactification of $S_P(x)$ in X. Then the complex hypersurface $M \backslash S_P(x)$ is an *orbit* of S_P by Lemma 2.1. Hence the equivariant normalization of M, which we again denote by M, is an almost-homogeneous projective algebraic *manifold* whose exceptional set is a complex hypersurface orbit. Define

$$\tilde{X} := S \times_P M .$$

Then $\tilde{X}$ is an almost homogeneous projective algebraic manifold whose exceptional set is a complex hypersurface orbit. Now $S(x)=S/H$ is a dense open orbit of both $\tilde{X}$ and X so that Lemmas 2.3 and 2.4 imply that there exists an equivariant holomorphic and birational map $\tilde{X} \to X$, i.e. X is an equivariant algebraic modification of $\tilde{X}$. □

We now prove the corresponding Theorem for the compact Kähler case using the albanese fibration.

Theorem 6.2. *Let $(X, S)_{\mathcal{O}}$ be a compact Kähler $H\Sigma$-manifold. Then there exists an equivariant modification of X,*

$$(\tilde{X}, S)_{\mathcal{O}} \to (X, S)_{\mathcal{O}}$$

such that $\tilde{X}$ is a compact almost-homogeneous Kähler manifold whose exceptional set is a complex hypersurface orbit of S.

Proof. Again we may assume that the exceptional set E is connected for otherwise we apply Theorem 3.2. Let $(F, \hat{S})_{\mathcal{L}}$ be the fiber of the albanese map $X \xrightarrow{F} A(X)=S/\hat{S}$. Then $K \cap \hat{S}$ has real hypersurface orbits in F. By Proposition 6.1, there exists an equivariant algebraic modification

$$\nu\colon (\tilde{F}, (K \cap \hat{S})^C)_{\mathcal{L}} \to (F, (K \cap \hat{S})^C)_{\mathcal{L}} .$$

Note that $\hat{S}=(K \cap \hat{S})^C$ since $S/\hat{S}=K/K \cap \hat{S}$ is a compact complex torus. Now,

$$\tilde{X} := S \times_{\hat{S}} \tilde{F} \to S \times_{\hat{S}} F = X; \quad (s, z) \mapsto (s, \nu(z)) ,$$

defines an S-equivariant modification of X, and the exceptional set $\tilde{E}$ of $\tilde{X}$ is a complex hypersurface orbit of S since $\tilde{E} \cap \tilde{F}$ is a complex hypersurface orbit of $\hat{S}$. The same argument as in the proof of Theorem 3.2 shows that $\tilde{X}$ is Kähler. □

Theorem 5.2 can now be used to understand any compact Kähler $H\Sigma$-manifold, and to give a classification of the real hypersurface $H\Sigma$ by means of the above theorem. The reader may wish to compare [11] for a classification of $\boldsymbol{P}^n$ as an $H\Sigma$-manifold.

As a final remark we would like to mention a conjecture attributed to Remmert and van de Ven (see [34]) that any almost-homogeneous compact Kähler manifold X with $b_1(X)=0$ should be a projective rational manifold,[6] i.e. bimeromorphically equivalent to $\boldsymbol{P}^n$. In our special case of exceptional sets as complex hypersurface orbits, we can show that this conjecture is true:

Theorem 6.3. *Let $(X, G)_{\mathcal{O}}$ be an almost homogeneous compact Kähler manifold with $b_1(X)=0$. Assume any one of the following is true:*

1) *The exceptional set of X is disconnected.*
2) *The exceptional set of X is a connected complex hypersurface orbit of G.*
3) *A maximal compact subgroup of G has a real hypersurface orbit in X.*

Then X is a projective rational manifold.

Proof. Case 1) follows from Theorem 3.2 and the fact that equivariant compactifications of homogeneous rational cones are rational (see e.g. [19]). Case 2) follows from Theorem 5.2 and case 3) from Theorems 6.2 and 5.2. □

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6) For $\dim_{\mathbf{C}} X=2$ this follows from Potter's classification [37], and was proved by Akao [3] when $\dim_{\mathbf{C}} X=3$.

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