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METRIC CONVEXITY OF SYMMETRIC CONES

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Abstract

In this paper we introduce a general notion of a symmetric cone, valid for the finite and infinite dimensional case, and prove that one can deduce the seminegative curvature of the Thompson part metric in this general setting, along with standard inequalities familiar from operator theory. As a special case, we prove that every symmetric cone from a JB-algebra satisfies a certain convexity property for the Thompson part metric: the distance function between points evolving in time on two geodesics is a convex function. This provides an affirmative answer to a question of Neeb [22].

1. Introduction

Let $\mathbb{A}$ be a unital C^*-algebra with identity e, and let $\mathbb{A}^+$ be the set of positive invertible elements of $\mathbb{A}$. It is known that $\mathbb{A}^+$ is an open convex cone in the space $\mathcal{H}(\mathbb{A})$ of hermitian elements. The geometry of $\mathbb{A}^+$ has been studied by several authors. One approach has been to endow $\mathbb{A}^+$ with a natural Finsler structure and metric and use these for a substitute for the Riemannian geometry commonly considered in finite-dimensional examples. One particular focus in this geometry has been the study of appropriate non-positive curvature properties. One prevalent notion of non-positive curvature is a purely metric one, that of convexity of the metric. In [3], [4] and [9], Andruchow-Corach-Stojanoff and Corach-Porta-Recht have shown the convexity of the distance function along two distinct geodesics and its equivalence to the well-known Loewner-Heinz inequality. In [22], Neeb established an appropriate differential geometric notion of seminegative (equal non-positive) curvature for certain classes of Finsler manifolds.

Our approach is somewhat different from either of the preceding. We replace the differential geometric structure by the structure of a symmetric space endowed with a midpoint operation and study seminegative curvature via convexity of the metric. In [16] we obtained the convexity of the metric for symmetric spaces with weaker metric assumptions than those enjoyed by the Finsler metric on $\mathbb{A}^+$.

The Finsler distance or length metric on $\mathbb{A}^+$ used in the earlier referenced papers agrees with the Thompson metric, which is widely known and has many applications in general convex cones of normed spaces ([27], [24]). The geodesic line passing through

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a and b is given by $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$ and the Thompson metric is defined by $d(a, b) = \max\{\log\|a^{-1/2}ba^{-1/2}\|, \ \log\|a^{1/2}b^{-1}a^{-1/2}\|\}$. The convexity theorem states that for $a, b, c, d \in \mathbb{A}^+$, the real function $t \mapsto d(\gamma_{a,b}(t), \gamma_{c,d}(t))$ is convex. For Riemannian manifolds this convexity condition is equivalent to the manifold having non-positive curvature in the Riemannian sense. In general, however, it is a weaker notion than the more prevalent metric notion of a CAT_0-space arising from Alexandrov's metric notion of spaces of non-positive curvature; see Section II.1.18 and the following appendix in [8] (cf. [5], [10]).

The main purpose of this paper is to extend the convexity result on $\mathbb{A}^+$ to much more general cones endowed with a symmetric structure that appropriately interacts with the conal structure. A special case is the symmetric cone arising as the set of invertible squares of a Jordan-Banach algebra (JB-algebra). Our general results applied to this particular case provide an affirmative answer to a question raised by Neeb [22]. A subsidiary goal of the paper is to present a very general framework, that of a general notion of a symmetric cone, in which one can derive and study various inequalities, such as those familiar from operator theory.

2. Symmetric spaces with midpoints

We recall from ([14], [15]) the underlying algebraic structure with which we work and basic properties thereof. A *$\sharp$-symmetric set* (called *dyadic symsets* in [14]) consists of a binary system $(X, \bullet)$, with left translation $S_x y := x \bullet y$ representing the point symmetry through x, satisfying for all $a, b, c \in X$:
(S1) $a \bullet a = a$ $(S_a a = a)$;
(S2) $a \bullet (a \bullet b) = b$ $(S_a S_a = \mathrm{id}_X)$;
(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$ $(S_a S_b = S_{S_a b} S_a)$;
(S4) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x \in X$, called the *midpoint* or *mean* of a and b, and denoted by $a \sharp b$.
The axioms bear close resemblance to the Loos axioms for a symmetric space [21]. A binary system $(X, \bullet)$ satisfying (S1), (S2), and (S3) also satisfies (S4) if and only if it is a quasigroup. Thus the preceding structures are also referred to as *symmetric quasigroups*. Systems satisfying only Axioms (1)–(3) are called *symmetric sets* (or *involutive quandles* in knot theory circles).

A pointed $\sharp$-symmetric set is a triple $(X, \bullet, \varepsilon)$, where $(X, \bullet)$ is a $\sharp$-symmetric set and $\varepsilon \in X$ is some distinguished point, called the base point. In this setting we define

$$x^0 = \varepsilon, \quad x^{-1} := S_\varepsilon x, \quad x^2 := S_x \varepsilon, \quad x^{1/2} := \varepsilon \sharp x$$

and inductively from these definitions all dyadic powers are defined so that the following rules are satisfied:

$$(x^r)^s = x^{rs}, \quad x^r \sharp x^s = x^{(r+s)/2}.$$

If we consider the dyadic rationals $\mathbb{D}$ endowed with the $\sharp$-symmetric structure $a \bullet b = 2a - b$ (the reflection of b through a), then $a \sharp b = (a+b)/2$, the usual midpoint, and the map $t \mapsto x^t : \mathbb{D} \to X$ is both a $\bullet$-homomorphism and $\sharp$-homomorphism. From this fact the preceding rules (and others) easily follow.

The displacement group $G(X)$ (also called the transvection group) of a $\sharp$-symmetric set X is the group generated under the composition by all transformations of the form $S_x S_y$, $x, y \in X$. It follows from Axioms (S2) and (S3) that these are automorphisms and thus there is a group action $(g, x) \mapsto g.x : G(X) \times X \to X$ with $G(X)$ acting as automorphisms. If X is pointed with base point ε, then $G(X)$ is generated by all $S_x S_\varepsilon$ and X embeds into $G(X)$ as a twisted subgroup (closed under $g \bullet h = gh^{-1}g$) via the *quadratic representation* $Q : X \to G(X)$ defined by $Q(x) = S_x S_\varepsilon$. The image $Q(X)$ is a pointed $\sharp$-symmetric set under the preceding $\bullet$-operation and the quadratic representation is an isomorphism between X and $Q(X)$. In particular, $Q(X)$ is uniquely 2-divisible and $Q(x \sharp y) = Q(x) \sharp Q(y)$, $Q(x^{1/2}) = Q(x)^{1/2}$ ([14, Theorem 5.4]). For $x, y \in X$, we write interchangeably as convenient

$$x.y = Q(x)y = Q(x)(y).$$

REMARK 2.1. The following useful calculation rules are derived in [14] or can easily be derived by the methods there:
(1) $Q(Q(x)y) = Q(x)Q(y)Q(x)$ or $(x.y).z = x.(y.(x.z))$.
(2) $(Q(x))^{-1} = Q(x^{-1})$.
(3) $(Q(x)y)^{-1} = Q(x^{-1})y^{-1}$ or $(x.y)^{-1} = x^{-1}.y^{-1}$.
(4) $S_{x^r} x^s = x^r \bullet x^s = x^{2r-s}$, $Q(x^r)x^s = x^r.x^s = x^{2r+s}$, $x^r \sharp x^s = x^{(r+s)/2}$.

Lemma 2.2 (Riccati lemma). *In a pointed $\sharp$-symmetric set X, the geometric mean $a \sharp b$ is the unique solution in X of the Riccati equation*

$$Q(x)a^{-1} = b$$

and is given by

$$a \sharp b = Q(a^{1/2})(Q(a^{-1/2})b)^{1/2} = a^{1/2}.(a^{-1/2}.b)^{1/2}$$

(*cf. the last paragraph in p.*21). *Furthermore, the geometric mean operation satisfies*
(i) $a \sharp b = b \sharp a$,
(ii) $(a \sharp b)^{-1} = a^{-1} \sharp b^{-1}$,
(iii) $g.(a \sharp b) = (g.a) \sharp (g.b)$ *for any* $g \in G(X)$.

Lemma 2.3. *In a pointed $\sharp$-symmetric set X,*

$$a \sharp Q(b)a^{-1} = b, \quad \forall a, b \in X.$$

Proof. The lemma follows from the fact that $x = Q(b)a^{-1}$ if and only if $b = a \sharp x$, which follows by the Riccati lemma. $\square$

3. Symmetric spaces with convex metrics

We now impose metric and topological structure upon $\sharp$-symmetric sets.

DEFINITION 3.1. A *pointed symmetric space with convex metric* is a pointed $\sharp$-symmetric set P equipped with a complete metric $d(\cdot, \cdot)$ satisfying for all $x, y \in P$ and $g \in G(P)$
(i) $d(g.x, g.y) = d(x, y)$,
(ii) $d(x^{-1}, y^{-1}) = d(x, y)$,
(iii) $d(x^{1/2}, y^{1/2}) \leq (1/2)\, d(x, y)$,
(iv) $x \mapsto x^2 : P \to P$ is continuous.
A *symmetric space with convex metric* is a $\sharp$-symmetric set equipped with a complete metric that is a pointed symmetric space with convex metric with respect to some pointing.

EXAMPLE 3.2. Let $\mathbb{R}$ be equipped with the standard $\sharp$-symmetric operation $x \bullet y := 2x - y$ and the usual metric. Then $x \sharp y = (x+y)/2$, the usual midpoint operation, and the metric is convex. Thus $(\mathbb{R}, \bullet, 0)$ is a pointed symmetric space with convex metric.

We recall some basic results about symmetric spaces with convex metrics from [16].

Theorem 3.3 ([16]). *Let P be a symmetric space with convex metric. Then for distinct $x, y \in P$, there exists a unique continuous homomorphism $\alpha_{x,y}$ (called an $\mathfrak{s}$-geodesic) of $\sharp$-symmetric sets from $\mathbb{R}$ into P satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Furthermore, the maps*

$$(x, y) \mapsto x \bullet y : P \times P \to P, \quad (t, x, y) \mapsto \alpha_{x,y}(t) := x \sharp_t y : \mathbb{R} \times P \times P \to P$$

are continuous.

The element $x \sharp_t y$ is called the *t-weighted mean* of x and y. Note that $x \sharp y = x \sharp_{1/2} y$.

Theorem 3.4 ([16]). *Let P be a symmetric space with convex metric. For every pair (β, γ) of $\mathfrak{s}$-geodesics, the real function*

$$t \mapsto d(\beta(t), \gamma(t))$$

is a convex function.

REMARK 3.5. We note that the unique $\mathfrak{s}$-geodesic line satisfying $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$ is

$$\alpha_{x,y}(t) = x \sharp_t y = x^{1/2}.(x^{-1/2}.y)^t$$

and $\alpha_{y,x}(1-t) = \alpha_{x,y}(t)$, $t \in \mathbb{R}$ ([16]). In particular,

$$(Q(y)x)^t = Q(y)Q(x^{1/2})(Q(x^{1/2})y^2)^{t-1}. \tag{3.1}$$

Indeed,

$$\begin{aligned} Q(y^{-1})(Q(y)x)^t &= y^{-2} \sharp_t x = x \sharp_{1-t} y^{-2} \\ &= Q(x^{1/2})(Q(x^{-1/2})y^{-2})^{1-t} \\ &= Q(x^{1/2})(Q(x^{1/2})y^2)^{t-1}. \end{aligned}$$

4. Convex cones with convex metrics

Let V be a Banach space and let Ω henceforth denote a non-empty open convex cone of V: $t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω. We further assume that Ω is a normal cone: that is, there exists a constant K with $\|x\| \le K\|y\|$ for all $x, y \in \Omega$ with $x \le y$. For a normal cone Ω, the relation

$$x \le y \quad \text{if and only if} \quad y - x \in \overline{\Omega}$$

is a partial order. We write $x < y$ if $y - x \in \Omega$.

Any member ε of Ω is an order unit for the ordered space $(V, \le)$, and the cone is normal if and only if the order unit norm determined by ε is compatible, i.e., determines the topology of V. In this case $0 \le x \le y$ implies $\|x\| \le \|y\|$ with respect to the order unit norm, that is, we may assume without loss of generality that $K = 1$. We henceforth make this assumption. In fact, for $y \in V$, $\varepsilon \pm y/M \in \Omega$ for a sufficiently large M, and hence $M\varepsilon \ge y \ge -M\varepsilon$. Moreover, $y = M[(y/M + \varepsilon) - \varepsilon] \in \Omega - \Omega$, i.e., $V = \Omega - \Omega$. By Proposition 1.1 in [24], for a normal cone $\overline{\Omega}$, the order unit norm is compatible. The normality condition of the order unit follows from its definition. See [24], [12, Section 1.2], and [29, Section 14] for more details.

A.C. Thompson [27] (cf. [23], [24]) has proved that Ω is a complete metric space with respect to the Thompson part metric defined by

$$d(x, y) = \max\left\{\log M\left(\frac{x}{y}\right), \log M\left(\frac{y}{x}\right)\right\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \le \lambda y\}$. The Thompson metric can be alternatively realized as an appropriately defined Finsler length metric. Since Ω is an open subset of

V, it carries a natural structure of (real) differentiable manifold and its tangent space $T_x\Omega$ can be identified to $V = \{x\} \times V$ (cf. [13]). For $x \in \Omega$ and $v \in V = T_x\Omega$, we define the Finsler metric by the order unit norm for v:

$$|v|_x := \inf\{t > 0 \colon -tx \leq v \leq tx\}.$$

The Thompson part metric $d(x, y)$ agrees with the Finsler distance from x to y:

$$d(x, y) = \inf\left\{\int_0^1 |\gamma'(t)|_{\gamma(t)}\, dt \colon \gamma \in S,\ \gamma(0) = x,\ \gamma(1) = y\right\} \tag{4.1}$$

where S denotes the set of piecewise C^1 maps $\gamma\colon [0, 1] \to \Omega$ ([24, Theorem 1.1]).

Lemma 4.1. *Let Ω be an open convex normal cone in a Banach space V. Suppose that there is a pointed $\sharp$-symmetric structure on Ω such that the displacements $Q(x)\colon \Omega \to \Omega$ are positively homogeneous for all $x \in \Omega$. Then $(\lambda x)^{-1} = (1/\lambda)x^{-1}$ and $\lambda^t\varepsilon = (\lambda\varepsilon)^t$ for all $\lambda > 0$ and all dyadic rationals t. Moreover $\mu x \sharp_t \lambda x = \mu^{1-t}\lambda^t x$ for all $x \in \Omega$, $\mu, \lambda > 0$ and all dyadic rationals t. Furthermore, the following conditions are equivalent*:
(i) $(\lambda x)^{1/2} = \sqrt{\lambda}x^{1/2}$ *for all $x \in \Omega$ and $\lambda > 0$*;
(ii) $(\lambda x)^t = \lambda^t x^t$ *for any dyadic rational t and $x \in \Omega$ and $\lambda > 0$*;
(iii) $Q(\lambda x) = \lambda^2 Q(x)$ *for any $x \in \Omega$ and $\lambda > 0$.*

Proof. Let A be the set of all dyadic rationals t such that $\lambda^t\varepsilon = (\lambda\varepsilon)^t$. It is immediate that $0, 1 \in A$. Set $x = \lambda\varepsilon$. Then for any dyadic rational t by homogenity of $Q(x)$ and Remark 2.1

$$x^{t+1} = Q(x^{t/2})x = Q(x^{t/2})(\lambda\varepsilon) = \lambda Q(x^{t/2})\varepsilon = \lambda x^t.$$

A simple induction then yields for any positive integer n, $x^{t+n} = \lambda^n x^t$ (for example, $x^{t+2} = Q(x^{1/2})x^{t+1} = Q(x^{1/2})(\lambda x^t) = \lambda Q(x^{1/2})x^t = \lambda x^{t+1} = \lambda^2 x^t$). It follows that $x^n = \lambda^n x^0 = \lambda^n\varepsilon$ and thus A includes all positive integers.

For a positive integer n and $t = -n$, $\varepsilon = x^{-n+n} = \lambda^n x^{-n}$. Thus $x^{-n} = \lambda^{-n}\varepsilon$ and A includes the negative integers as well.

The preceding results apply to any $\lambda > 0$, in particular to $\mu = \lambda^{1/2^m}$. Thus for $y = \mu\varepsilon$, $y^{2^m} = \mu^{2^m}\varepsilon = \lambda\varepsilon$. It follows that $(\lambda\varepsilon)^{1/2^m} = y = \lambda^{1/2^m}\varepsilon$, i.e, $1/2^m \in A$. For any integer n,

$$(\lambda\varepsilon)^{n/2^m} = ((\lambda\varepsilon)^{1/2^m})^n = (\lambda^{1/2^m}\varepsilon)^n = (\mu\varepsilon)^n = \mu^n\varepsilon = \lambda^{n/2^m}\varepsilon,$$

where the penultimate equality follows from the first two paragraphs for $\lambda = \mu$.

Suppose that μ and λ are positive real numbers. We recall from Remark 3.5 that $x \sharp_t y = x^{1/2}.(x^{-1/2}.y)^t = Q(x)^{1/2}(Q(x)^{-1/2}y)^t$. Then the preceding result implies that

$$\begin{aligned}\mu\varepsilon \sharp_t \lambda\varepsilon &= Q(\mu\varepsilon)^{1/2}(Q(\mu\varepsilon)^{-1/2}(\lambda\varepsilon))^t = Q(\mu\varepsilon)^{1/2}\left(\frac{\lambda}{\mu}Q(\mu\varepsilon)^{-1/2}(\mu\varepsilon)\right)^t\\ &= Q(\mu\varepsilon)^{1/2}\left(\frac{\lambda}{\mu}\varepsilon\right)^t = Q(\mu\varepsilon)^{1/2}\left(\frac{\lambda}{\mu}\right)^t\varepsilon = \left(\frac{\lambda}{\mu}\right)^t Q(\mu\varepsilon)^{1/2}\varepsilon\\ &= \left(\frac{\lambda}{\mu}\right)^t \mu\varepsilon = (\mu^{1-t}\lambda^t)\varepsilon\end{aligned}$$

and for $x \in \Omega$,

$$\mu x \sharp_t \lambda x = Q(x^{1/2})(\mu\varepsilon \sharp_t \lambda\varepsilon) = Q(x^{1/2})((\mu^{1-t}\lambda^t)\varepsilon) = (\mu^{1-t}\lambda^t)Q(x^{1/2})\varepsilon = (\mu^{1-t}\lambda^t)x.$$

Next, we show that

$$(\lambda x)^{-1} = \lambda^{-1}x^{-1}. \tag{4.2}$$

It follows from $(\lambda\varepsilon)^{-1} = \lambda^{-1}\varepsilon$, $\lambda x = \lambda Q(x^{1/2})\varepsilon = Q(x^{1/2})(\lambda\varepsilon)$ and Remark 2.1 that

$$(\lambda x)^{-1} = (Q(x^{1/2})(\lambda\varepsilon))^{-1} = Q(x^{-1/2})(\lambda\varepsilon)^{-1} = Q(x^{-1/2})(\lambda^{-1}\varepsilon) = \lambda^{-1}Q(x^{-1/2})\varepsilon = \lambda^{-1}x^{-1}.$$

We next prove the equivalence of the conditions in the statement. Assume $(\lambda x)^{1/2} = \sqrt{\lambda}x^{1/2}$ for $x \in \Omega$ and $\lambda > 0$. To prove (iii), we first calculate

$$\begin{aligned}x \sharp \lambda y &= x^{1/2}.(\varepsilon \sharp \lambda(x^{-1/2}.y)) = x^{1/2}.(\lambda(x^{-1/2}.y))^{1/2} = x^{1/2}.(\sqrt{\lambda}((x^{-1/2}).y)^{1/2})\\ &= \sqrt{\lambda}x^{1/2}.(\varepsilon \sharp x^{-1/2}.y) = \sqrt{\lambda}(x \sharp y).\end{aligned}$$

By the Riccati lemma, $z = Q(\lambda x)y$ implies that $\lambda x = z \sharp y^{-1}$ or $x = (1/\lambda)(z \sharp y^{-1}) = ((1/\lambda^2)z) \sharp y^{-1}$, and again by the Riccati lemma we have $(1/\lambda^2)z = Q(x)y$ or $z = \lambda^2 Q(x)y$. This shows that $Q(\lambda x) = \lambda^2 Q(x)$ for any $x \in \Omega$ and $\lambda > 0$.

(iii) implies (ii). Suppose $Q(\lambda x) = \lambda^2 Q(x)$ for $x \in \Omega$ and $\lambda > 0$. From $(\lambda x)^2 = Q(\lambda x)\varepsilon = \lambda^2 Q(x)\varepsilon = \lambda^2 x^2$ and by a simple induction $(\lambda x)^n = \lambda^n x^n$ for any positive integer n. Indeed, if $(\lambda x)^k = \lambda^k x^k$ for $k = 1, 2, \ldots, n$, then

$$(\lambda x)^{n+1} = Q(\lambda x)(\lambda x)^{n-1} = \lambda^2 Q(x)(\lambda^{n-1}x^{n-1}) = \lambda^{n+1}x^{n+1}.$$

By (4.2), $(\lambda x)^{-n} = ((\lambda x)^{-1})^n = (\lambda^{-1}x^{-1})^n = \lambda^{-n}x^{-n}$ for any positive integer n. Furthermore, $(\lambda^{1/2^m}x^{1/2^m})^{2^m} = \lambda x$ and hence $(\lambda x)^{1/2^m} = \lambda^{1/2^m}x^{1/2^m}$ for any integer m. For any integer n,

$$(\lambda x)^{n/2^m} = ((\lambda x)^{1/2^m})^n = (\lambda^{1/2^m}x^{1/2^m})^n = \lambda^{n/2^m}x^{n/2^m}.$$

Therefore (ii) follows, and the trivial implication (ii) implies (i) completes the proof. $\square$

The next theorem gives the main result of this paper. Note that all powers are computed in the given $\sharp$-symmetric structure of the cone.

Theorem 4.2. *Let Ω be an open convex normal cone in a Banach space V. Suppose that there is a pointed $\sharp$-symmetric structure on Ω satisfying*
(i) $x^{1/2} \leq (\varepsilon + x)/2$.
(ii) *the squaring map $x \mapsto x^2 = Q(x)\varepsilon$ is continuous* (*in the relative norm topology of* Ω).
(iii) *every basic displacement $Q(x)$ is continuous and linear* (*that is, additive and positively homogeneous*) *on Ω.*
Then Ω is a symmetric space with convex metric with respect to the Thompson metric that satisfies the equivalent conditions (i), (ii) *and* (iii) *of* Lemma 4.1. *Furthermore,* (i) *the order-reversing property of inversion,* (ii) *the harmonic-geometric-arithmetic mean inequality, and* (iii) *the Loewner-Heinz inequality all hold*: *for $a, b \in \Omega$,*
(i) $b^{-1} \leq a^{-1}$ *if* $a \leq b$,
(ii) $2(a^{-1} + b^{-1})^{-1} \leq a \sharp b \leq (1/2)(a + b)$, *and*
(iii) $a^{1/2} \leq b^{1/2}$ *if* $a \leq b$.

Proof. The proof proceeds in steps.

STEP 1. *Each $Q(x)$ extends to an invertible bounded linear operator on V that is an order-isomorphism.* Let $T\colon \Omega \to \Omega$ be linear (additive and positive homogeneous) and continuous. Let $\gamma\colon \Omega \times \Omega \to V$ be defined by $\gamma(x, y) = x - y$. Then γ is an open mapping that is surjective since $V = \Omega - \Omega$ (see the second paragraph of this section). One verifies directly that T extends to a map, again called T, from V to V defined by $T(x - y) := T(x) - T(y)$ and that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] & V \\ \Omega \times \Omega \arrow[u, "\gamma"] \arrow[r, "T \times T"] & \Omega \times \Omega \arrow[u, "\gamma"'] \end{tikzcd}$$

It is straightforward to verify that $T\colon V \to V$ is an additive homomorphism and homogeneous with respect to positive scalars. Since an additive homomorphism is homogeneous for the scalar -1, it follows that T is linear. Since γ is open, it is a quotient map and thus $T\colon V \to V$ is continuous, hence bounded, since $T \times T$ is continuous. For $T = Q(x)$, we conclude from hypothesis (iii) that $Q(x)$ extends (uniquely) to a bounded linear map. Since members of $G(\Omega)$ are compositions of basic displacements, the same conclusion holds for them.

Since $Q(x)^{-1} = Q(x^{-1})$, $Q(x)$ is invertible with inverse $Q(x^{-1})$ on Ω. It follows readily that their extensions are inverses. Since $Q(x)$ preserves Ω, by continuity it also preserves $\overline{\Omega}$, and thus is order-preserving on V. Since $Q(x^{-1})$ is similarly order-preserving, it follows that $Q(x)$ is an order-isomorphism. It follows that any member of $G(\Omega)$ is an order-isomorphism, being a composition of such. Hence each member of $G(\Omega)$ is an isometry for the Thompson metric, since the latter is defined from the order. Thus Axiom 3.1 (i) is satisfied.

STEP 2. *The inversion* $x \mapsto x^{-1} = S_\varepsilon x$ *is order reversing*.

Suppose that $0 < x \le y$. Then by the Riccati lemma 2.2 we have

$$Q(y \sharp x)y^{-1} = x \le y = Q(x \sharp y)x^{-1}.$$

Since $Q(y \sharp x) = Q(x \sharp y)$ is an order-isomormisphism by Step 1, the conclusion follows.

STEP 3. *The harmonic-geometric-arithmetic mean inequality holds*:

$$\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \le x \sharp y \le \frac{x+y}{2}. \tag{4.3}$$

Since each displacement is linear and order preserving on Ω and preserves the geometric mean operation by Lemma 2.2, we have

$$\begin{aligned} x \sharp y &= (Q(x^{1/2})\varepsilon) \sharp (Q(x^{1/2})Q(x^{-1/2})y) \\ &= x^{1/2}.(\varepsilon \sharp (x^{-1/2}.y)) \\ &\overset{\text{(i)}}{\le} x^{1/2}.\left(\frac{\varepsilon + x^{-1/2}.y}{2}\right) \\ &= \frac{x+y}{2}, \end{aligned}$$

the geometric-arithmetic mean inequality. The harmonic-geometric mean inequality follows from the order reversing property of inversion (Step 2) and the geometric-arithmetic mean inequality

$$\begin{aligned} x \sharp y &\overset{\text{Lemma 2.2}}{=} (x^{-1} \sharp y^{-1})^{-1} \\ &\overset{\text{Step 2}}{\ge} \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}. \end{aligned}$$

STEP 4. *The squaring map* $x \mapsto x^2$ *is continuous for the Thompson metric*. *Therefore* Axiom 3.1 (iv) *is satisfied*. Indeed, this is a consequence of the agreement of the norm topology with that of Thompson metric ([24, Proposition 1.1]).

STEP 5. *Inversion is an isometry with respect to the Thompson metric*. *Therefore* Axiom 3.1 (ii) *is satisfied*. Let $x, y \in \Omega$ and let $\lambda > 0$ such that $x \le \lambda y$. Then since $Q(y^{1/2})$ is linear and preserves the order, we have

$$y^{-1/2}.x \le y^{-1/2}.(\lambda y) = \lambda\varepsilon.$$

The order reversing property of the inversion implies that

$$(y^{-1/2}.x)^{-1} = y^{1/2}.x^{-1} \geq (\lambda\varepsilon)^{-1} \overset{\text{Lemma 4.1}}{=} \frac{1}{\lambda}\varepsilon.$$

Again by linearity,

$$x^{-1} \geq y^{-1/2}.\left(\frac{1}{\lambda}\varepsilon\right) = \frac{1}{\lambda}y^{-1}.$$

This string shows that $M(y^{-1}/x^{-1}) = M(x/y)$ and hence $d(x^{-1}, y^{-1}) = d(x, y)$.

STEP 6. *For $a \in \Omega$, define $f_a : \Omega \to \Omega$ by $f_a(x) = (1/2)(x + Q(a)x^{-1})$. Then*

$$\lim_{n\to\infty} f_a^n(x) = a, \quad \forall x \in \Omega.$$

(See [20, Theorem 7] for symmetric cones of Euclidean Jordan algebras.)

First, we observe that the map f_a is continuous (inversion, the displacements, and the linear operations are continuous). Suppose that the iteration has a limit point, say b. Then by continuity, $b = f_a(b) = (1/2)(b + Q(a)b^{-1})$ and then $b = Q(a)b^{-1}$. By the Riccati lemma, $a = b \sharp b = b$. The convergence is proved by several substeps.

(i) $f_a^n(x) \geq a$: By the G-A (geometric-arithmetic mean) inequality and Lemma 2.3,

$$f_a^n(x) = \frac{1}{2}(f_a^{n-1}(x) + Q(a) f_a^{n-1}(x)^{-1}) \geq f_a^{n-1}(x) \sharp Q(a) f_a^{n-1}(x)^{-1} = a.$$

(ii) $Q(a) f_a^n(x)^{-1} \leq a$: By linearity, the invertibility of $Q(a)$, and the equality $(Q(a)x)^{-1} = Q(a)^{-1}x^{-1}$,

$$\begin{aligned} Q(a) f_a^n(x)^{-1} &= Q(a)\left(\frac{1}{2}(f_a^{n-1}(x) + Q(a) f_a^{n-1}(x)^{-1})\right)^{-1} \\ &= \left(\frac{1}{2}(Q(a)^{-1} f_a^{n-1}(x) + f_a^{n-1}(x)^{-1})\right)^{-1} \\ &= \left(\frac{1}{2}((Q(a) f_a^{n-1}(x)^{-1})^{-1} + f_a^{n-1}(x)^{-1})\right)^{-1} \\ &\overset{\text{H-G}}{\leq} (Q(a) f_a^{n-1}(x)^{-1}) \sharp f_a^{n-1}(x) \\ &\overset{\text{Lemma 2.3}}{=} a. \end{aligned}$$

(iii) $\| f_a^n(x) - a\| \leq 1/2^{n-1}\| f_a(x) - a\|$: By (i) and (ii),

$$0 \leq f_a^n(x) - a = \frac{1}{2}(f_a^{n-1}(x) - a) - \frac{1}{2}(a - Q(a) f_a^{n-1}(x)^{-1}) \leq \frac{1}{2}(f_a^{n-1}(x) - a),$$

and then by the normality of the cone,

$$\| f_a^n(x) - a\| \leq \frac{1}{2}\| f_a^{n-1}(x) - a\|.$$

STEP 7. *The Loewner-Heinz inequality holds*:

$$0 < a \leq b \quad \text{implies} \quad a^{1/2} \leq b^{1/2}. \tag{4.4}$$

By Step 6, it is enough to show statement A_n: $f^n_{a^{1/2}}(\varepsilon) \leq f^n_{b^{1/2}}(\varepsilon)$ for all $n = 0, 1, 2, \ldots$ whenever $0 < a \leq b$ (cf. [20, Corollary 9]). However, for the induction to proceed smoothly, we prove additionally statement B_n: $Q(a^{1/2}) f^n_{a^{1/2}}(\varepsilon)^{-1} \leq Q(b^{1/2}) f^n_{b^{1/2}}(\varepsilon)^{-1}$. The statement A_0 reduces to $\varepsilon \leq \varepsilon$ and B_0 asserts that

$$a = Q(a^{1/2})(\varepsilon^{-1}) \leq Q(b^{1/2})(\varepsilon^{-1}) = b,$$

which is true by hypothesis. Suppose that A_k and B_k hold for $k = n - 1$. Then

$$\begin{aligned} f^n_{a^{1/2}}(\varepsilon) &= \frac{1}{2}\big(f^{n-1}_{a^{1/2}}(\varepsilon) + Q(a^{1/2}) f^{n-1}_{a^{1/2}}(\varepsilon)^{-1}\big) \\ &\leq \frac{1}{2}\big(f^{n-1}_{b^{1/2}}(\varepsilon) + Q(a^{1/2}) f^{n-1}_{a^{1/2}}(\varepsilon)^{-1}\big) \\ &\leq \frac{1}{2}\big(f^{n-1}_{b^{1/2}}(\varepsilon) + Q(b^{1/2}) f^{n-1}_{b^{1/2}}(\varepsilon)^{-1}\big) \\ &= f^n_{b^{1/2}}(\varepsilon), \end{aligned}$$

where the two inequalities are applications of A_{n-1} and B_{n-1} respectively.

Showing

$$Q(a^{1/2}) f^n_{a^{1/2}}(\varepsilon)^{-1} \leq Q(b^{1/2}) f^n_{b^{1/2}}(\varepsilon)^{-1} \tag{4.5}$$

is equivalent (by inverting) to showing

$$Q(a^{-1/2}) f^n_{a^{1/2}}(\varepsilon) \geq Q(b^{-1/2}) f^n_{b^{1/2}}(\varepsilon). \tag{4.6}$$

However, since

$$\begin{aligned} Q(a^{-1/2}) f^n_{a^{1/2}}(\varepsilon) &= \frac{1}{2}(Q(a^{-1/2}) f^{n-1}_{a^{1/2}}(\varepsilon) + f^{n-1}_{a^{1/2}}(\varepsilon)^{-1}) \\ Q(b^{-1/2}) f^n_{b^{1/2}}(\varepsilon) &= \frac{1}{2}(Q(b^{-1/2}) f^{n-1}_{b^{1/2}}(\varepsilon) + f^{n-1}_{b^{1/2}}(\varepsilon)^{-1}) \end{aligned}$$

and since $f^{n-1}_{a^{1/2}}(\varepsilon)^{-1} \geq f^{n-1}_{b^{1/2}}(\varepsilon)^{-1}$ (by induction hypothesis A_{n-1} and inversion), (4.6) follows from

$$Q(a^{-1/2}) f^{n-1}_{a^{1/2}}(\varepsilon) \geq Q(b^{-1/2}) f^{n-1}_{b^{1/2}}(\varepsilon) \tag{4.7}$$

or

$$Q(a^{1/2}) f^{n-1}_{a^{1/2}}(\varepsilon)^{-1} \leq Q(b^{1/2}) f^{n-1}_{b^{1/2}}(\varepsilon)^{-1}, \tag{4.8}$$

which is true by the inductive hypothesis B_{n-1}.

STEP 8. $(\lambda a)^{1/2} = \sqrt{\lambda} a^{1/2}$ *for any* $\lambda > 0$ *and* $a \in \Omega$, *and therefore the equivalent conditions of* Lemma 4.1 *are satisfied*. To prove this we use mainly the facts that $(Q(x)y)^{-1} = Q(x^{-1})y^{-1}$ (Remark 2.1) and $(\lambda x)^{-1} = \lambda^{-1}x^{-1}$ (Lemma 4.1). By Step 6, it is enough to show statement A_n: $f^n_{(\lambda a)^{1/2}}(\varepsilon) = \sqrt{\lambda} f^n_{a^{1/2}}(\lambda^{-1/2}\varepsilon)$ for all positive integers n. Set $b = (1/\sqrt{\lambda})\varepsilon$. The statement A_1 follows by a direct computation:

$$f_{(\lambda a)^{1/2}}(\varepsilon) = \frac{1}{2}(\varepsilon + Q(\lambda a)^{1/2}\varepsilon) = \frac{1}{2}(\varepsilon + \lambda a) = \frac{\sqrt{\lambda}}{2}(b + Q(a^{1/2})b^{-1}) = \sqrt{\lambda} f_{a^{1/2}}(b).$$

To proceed by induction, we need also to include the following in our induction:

$$B_n:\ Q(\lambda a)^{1/2}(\lambda f^n_{a^{1/2}}(b))^{-1} = Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1}, \quad n = 1, 2, \ldots . \tag{4.9}$$

Statement B_n is true for $n = 1$ because

$$\begin{aligned}
Q(\lambda a)^{1/2}(\lambda f_{a^{1/2}}(b))^{-1} &= \frac{2}{\lambda} Q(\lambda a)^{1/2}\left(\frac{\varepsilon}{\sqrt{\lambda}} + Q(a^{1/2})(\sqrt{\lambda}\varepsilon)\right)^{-1} = \frac{2}{\lambda} Q(\lambda a)^{1/2}\left(\frac{\varepsilon}{\sqrt{\lambda}} + \sqrt{\lambda} a\right)^{-1} \\
&= \frac{2}{\lambda}\left(\frac{(\lambda a)^{-1}}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} Q(\lambda a)^{-1/2}(\lambda a)\right)^{-1} = \frac{2}{\lambda}(\lambda^{-3/2}a^{-1} + \lambda^{-1/2}\varepsilon)^{-1} \\
&= 2(\lambda^{-1/2}a^{-1} + \lambda^{1/2}\varepsilon)^{-1} = 2Q(a^{1/2})(\lambda^{-1/2}\varepsilon + Q(a^{1/2})(\lambda^{1/2}\varepsilon))^{-1} \\
&= Q(a^{1/2})(f_{a^{1/2}}(b))^{-1}.
\end{aligned}$$

Next, suppose that B_n holds. This implies that

$$\lambda Q(a^{1/2})Q(\lambda a)^{-1/2}Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1} = Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1}. \tag{4.10}$$

Indeed, $Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1} = Q(\lambda a)^{1/2}(\lambda f^n_{a^{1/2}}(b))^{-1} = (1/\lambda)Q(\lambda a)^{1/2}(f^n_{a^{1/2}}(b))^{-1}$ implies that $\lambda Q(\lambda a)^{-1/2}Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1} = (f^n_{a^{1/2}}(b))^{-1}$. Then

$$\begin{aligned}
Q(\lambda a)^{1/2}(\lambda f^{n+1}_{a^{1/2}}(b))^{-1} &= 2Q(\lambda a)^{1/2}(\lambda f^n_{a^{1/2}}(b) + \lambda Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1})^{-1} \\
&= (Q(\lambda a)^{-1/2}(\lambda f^n_{a^{1/2}}(b)) + \lambda Q(\lambda a)^{-1/2}Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1})^{-1} \\
&= 2(Q(a^{-1/2})(f^n_{a^{1/2}}(b)) + \lambda Q(\lambda a)^{-1/2}Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1})^{-1} \\
&= 2Q(a^{1/2})(f^n_{a^{1/2}}(b) + \lambda Q(a^{1/2})Q(\lambda a)^{-1/2}Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1})^{-1} \\
&\overset{(4.10)}{=} 2Q(a^{1/2})(f^n_{a^{1/2}}(b) + Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1})^{-1} \\
&= Q(a^{1/2})(f_{a^{1/2}}(f^n_{a^{1/2}}(b)))^{-1} = Q(a^{1/2})(f^{n+1}_{a^{1/2}}(b))^{-1},
\end{aligned}$$

where the third equality follows from statement B_n and taking inverses.

Finally, suppose additionally that A_n holds, that is, $f^n_{(\lambda a)^{1/2}}(\varepsilon) = \sqrt{\lambda}\, f^n_{a^{1/2}}(b)$. Then

$$\begin{aligned} f^{n+1}_{(\lambda a)^{1/2}}(\varepsilon) &= f_{(\lambda a)^{1/2}}(f^n_{(\lambda a)^{1/2}}(\varepsilon)) = f_{(\lambda a)^{1/2}}(\sqrt{\lambda}\, f^n_{a^{1/2}}(b)) \\ &= \frac{1}{2}(\sqrt{\lambda}\, f^n_{a^{1/2}}(b) + Q(\lambda a^{1/2})(\sqrt{\lambda}\, f^n_{a^{1/2}}(b))^{-1}) \\ &= \frac{\sqrt{\lambda}}{2}(f^n_{a^{1/2}}(b) + Q(\lambda a)^{1/2}(\lambda\, f^n_{a^{1/2}}(b))^{-1}) \\ &\overset{(4.9)}{=} \frac{\sqrt{\lambda}}{2}(f^n_{a^{1/2}}(b) + Q(a^{1/2})(f^n_{a^{1/2}}(b))^{-1}) \\ &= \sqrt{\lambda}\, f_{a^{1/2}}(f^n_{a^{1/2}}(b)) = \sqrt{\lambda}\, f^{n+1}_{a^{1/2}}(b). \end{aligned}$$

STEP 9. *For* $x, y \in \Omega$,

$$d(x^{1/2}, y^{1/2}) \leq \frac{1}{2} d(x, y).$$

Therefore Axiom 3.1 (iii) *is satisfied*. It is enough to show that $M(x/y) \geq M(x^{1/2}/y^{1/2})^2$. This follows from the Loewner-Heinz inequality and Step 8:

$$\begin{aligned} M(x/y) &= \inf\{\lambda > 0 \colon x \leq \lambda y\} \\ &\geq \inf\{\lambda > 0 \colon x^{1/2} \leq (\lambda y)^{1/2} = \sqrt{\lambda} y^{1/2}\} \\ &= \inf\{t^2 \colon x^{1/2} \leq t y^{1/2}\} \\ &= M(x^{1/2}/y^{1/2})^2. \end{aligned}$$

□

5. Symmetric cones

Our earlier results motivate the following definition.

DEFINITION 5.1. Let Ω be an open normal convex cone in a Banach space V equipped with a $\sharp$-symmetric structure making it a $\sharp$-symmetric set. Then Ω is a $\sharp$-*symmetric cone* if the following conditions are satisfied:

(i) $x \sharp y \leq (x + y)/2$ for all $x, y \in \Omega$;

(ii) the following maps are continuous:

$$(x, y) \mapsto x \bullet y \colon \Omega \times \Omega \to \Omega, \quad (t, x, y) \mapsto \alpha_{x,y}(t) := x \sharp_t y \colon \mathbb{R} \times \Omega \times \Omega \to \Omega;$$

(iii) Every member of the displacement group $G(\Omega)$ extends to a bounded linear order-preserving operator on V.

(iv) $\mu x \sharp_t \lambda y = \mu^{1-t} \lambda^t (x \sharp y)$ for all $\lambda, \mu > 0$, $x, y \in \Omega$ and $t \in \mathbb{R}$.

The next result follows essentially from Theorem 4.2.

Corollary 5.2. *Let Ω be an open convex normal cone in a Banach space V. Suppose that there is a pointed $\sharp$-symmetric structure on Ω satisfying*
(i) $2x \leq \varepsilon + x^2$
(ii) *the squaring map $x \mapsto x^2 = Q(x)\varepsilon$ is continuous (in relative norm topology of Ω),*
(iii) *every basic displacement $Q(x)$ is continuous and linear (that is, additive and positively homogeneous) on Ω.*
Then Ω is a pointed symmetric space with convex metric, the Thompson metric whose metric topology agrees with the relative topology, and also a $\sharp$-symmetric cone. Conversely, a $\sharp$-symmetric cone satisfies these three conditions with respect to any pointing.

Proof. Assume conditions (i)–(iii). Note that since $x \mapsto x^2$ is a bijection, we can rewrite hypothesis (i) in the form
(i′) $x^{1/2} \leq (\varepsilon + x)/2$.
Thus the hypotheses of Theorem 4.2 are satisfied. Hence the geometric-arithmetic mean inequality holds, i.e., condition 5.1 (i) is satisfied. Property 5.1 (ii) follows directly from Theorem 4.2 and Theorem 3.3. By Step 1 of the proof of Theorem 4.2, each member of $G(\Omega)$ extends to an invertible bounded linear operator on V that is an order isomorphism, so Property 5.1 (iii) is valid. Property 5.1(iv) follows for dyadic rationals t from Step 8 of Theorem 4.2 and Lemma 4.1:

$$\begin{aligned}
\mu x \sharp_t \lambda y &= Q((\mu x)^{1/2})(Q((\mu x)^{-1/2})\lambda y)^t \\
&= Q(\mu^{1/2} x^{1/2})(\lambda Q(\mu^{-1/2} x^{-1/2})y)^t \\
&= \mu Q(x^{1/2})(\mu^{-1}\lambda Q(x^{-1/2})y)^t \\
&= \mu(\mu^{-1}\lambda)^t Q(x^{1/2})(Q(x^{-1/2})y)^t \\
&= \mu^{1-t}\lambda^t (x \sharp_t y).
\end{aligned}$$

That it holds for all $t \in \mathbb{R}$ then follows from continuity.

Step 9 of the proof of Theorem 4.2 establishes that the Thompson metric is a convex metric, and Step 4 that its metric topology agrees with the relative topology.

Conversely assume that we choose some point ε in the $\sharp$-symmetric cone Ω. Then using 5.1 (i), we have

$$2x = 2(\varepsilon \sharp x^2) \leq 2\left(\frac{\varepsilon + x^2}{2}\right) = \varepsilon + x^2,$$

and thus hypothesis (i) is satisfied. Hypotheses (ii) and (iii) follow immediately from 5.1 (ii) and 5.1 (iii) resp., since $x^2 = x \bullet \varepsilon$. $\square$

EXAMPLE 5.3. Let $\mathbb{A}$ be a unital C^*-algebra with identity e, and let $\mathbb{A}^+$ be the set of positive invertible elements of $\mathbb{A}$. It follows readily from Corollary 5.2 and standard basic facts from the theory of C^*-algebras that $\mathbb{A}^+$ is a $\sharp$-symmetric cone. To see this we need the standard basic facts that $\mathbb{A}^+$ is an open normal convex cone in the closed subspace $\mathcal{H}(\mathbb{A})$ of hermitian elements, that each element of $\mathbb{A}^+$ has a unique square root in $\mathbb{A}^+$, and that $x^2 \geq 0$ for every $x \in \mathcal{H}(\mathbb{A})$. The set $\mathbb{A}^+$ is a twisted subgroup (closed under $(x, y) \mapsto xy^{-1}x$) with unique square roots of the multiplicative group of invertible elements of $\mathbb{A}$, hence a pointed $\sharp$-symmetric set with respect to $x \bullet y = xy^{-1}x$ and distinguished point the identity e. Furthermore, the powers computed in the algebra agree with those computed in $(\mathbb{A}^+, \bullet, e)$ [14]. Hence condition (ii) of Corollary 5.2 holds. Condition 5.2 (i) is equivalent to $(e-x)^2 \geq 0$, thus valid. Since $Q(x)y = x(y^{-1})^{-1}x = xyx$, condition 5.2 (iii) holds.

The next lemma is elementary, but will prove useful for our purposes.

Lemma 5.4. *Let A be a subset of $[0, 1]$ that contains 0 and 1, is closed under the operation of taking midpoints, and is closed under sequential limits. The $A = [0, 1]$.*

Theorem 5.5 (Loewner-Heinz, [2]). *Let $\Omega \subseteq V$ be a $\sharp$-symmetric cone. If $x_1 \leq x_2$ and $y_1 \leq y_2$ for $x_1, x_2, y_1, y_2 \in \Omega$, then $x_1 \sharp_t y_1 \leq x_2 \sharp_t y_2$ for $0 \leq t \leq 1$.*

Proof. If $x_1 \sharp_t y_1 \leq x_1 \sharp_t y_2$ and $x_1 \sharp_t y_2 \leq x_2 \sharp_t y_2$, then we obtain our desired conclusion by transitivity. Thus (using commutivity of $\sharp$ and $x \sharp_t y = y \sharp_{1-t} x$) it suffices to show that $b \leq c$ implies $a \sharp_t b \leq a \sharp_t c$. By Corollary 5.2 we may choose any member of Ω for our distinguished point, so without loss of generality we assume $a = \varepsilon$. For $t = 1/2$, $\sharp_t = \sharp$, and we have by the Loewner-Heinz inequality (Theorem 4.2)

$$\varepsilon \sharp b = b^{1/2} \leq c^{1/2} = \varepsilon \sharp c,$$

so the theorem is valid for $t = 1/2$.

Let $b \leq c$ in Ω. There exists a $\bullet$-homomorphism (and hence $\sharp$-homomorphism) $\alpha_b : \mathbb{R} \to \Omega$ such that $\alpha_b(0) = \varepsilon$ and $\alpha_b(1) = b$; then by definition $b^t = \varepsilon \sharp_t b = \alpha_b(t)$. Consider the set A of all $t \in [0, 1]$ such that $b^t = \varepsilon \sharp_t b \leq \varepsilon \sharp_t c = c^t$. For $t = 0$, we have $\varepsilon \leq \varepsilon$ and for $t = 1$ we have $b \leq c$, so $0, 1 \in A$. Suppose that $t_1, t_2 \in A$. Then for $t = (t_1 + t_2)/2$,

$$b^t = b^{(t_1+t_2)/2} = b^{t_1} \sharp b^{t_2} \leq c^{t_1} \sharp c^{t_2} = c^t,$$

where the inequality follows from the case $t = 1/2$ established above. By closedness of the relation $\leq$ and continuity of $\sharp_t$, A is closed under limits of sequential limits. Thus by Lemma 5.4 $A = [0, 1]$. $\square$

Theorem 5.6. *The general harmonic-geometric-arithmetic mean inequality holds in any $\sharp$-symmetric cone Ω, that is, for $x, y \in \Omega$ and $t \in [0, 1]$:*

$$((1-t)x^{-1} + ty^{-1})^{-1} \leq x \sharp_t y \leq (1-t)x + ty.$$

Proof. Let A be the set of all $t \in [0, 1]$ for which the HGA-inequality holds. For $t = 0$ (resp. $t = 1$) it reduces to $x \leq x \leq x$ (resp. $y \leq y \leq y$) so $0, 1 \in A$. By closedness of the order and continuity of the operations, A is sequentially closed.

Let $H_t := ((1-t)x^{-1} + ty^{-1})^{-1}$, $G_t := x \sharp_t y$, and $A_t := (1-t)x + ty$ for $0 \leq t \leq 1$. Suppose that $t, s \in A$. Then by elementary computation and the standard HGA-inequality (Theorem 4.2),

$$\begin{aligned} H_{(t+s)/2} &= \left(\frac{(H_t)^{-1} + (H_s)^{-1}}{2}\right)^{-1} \\ &\overset{s,t\in A}{\leq} \left(\frac{(G_t)^{-1} + (G_s)^{-1}}{2}\right)^{-1} \\ &\overset{\text{H-G}}{\leq} G_t \sharp G_s \\ &\overset{2.1(4)}{=} G_{(t+s)/2}. \end{aligned}$$

By an analogous computation, or by taking inverses, one obtains that $G_{(t+s)/2} \leq A_{(t+s)/2}$. By Lemma 5.4 $A = [0, 1]$, yielding the theorem. □

The HGA-inequalities provide an approximation scheme for $x \sharp y$.

Lemma 5.7. *For x, y in a $\sharp$-symmetric cone Ω, $x \sharp y = H(x, y) \sharp A(x, y)$, where $H(x, y)$ is the harmonic mean and $A(x, y)$ is the arithmetic mean of x and y.*

Proof. We have

$$Q(x \sharp y)H(x, y)^{-1} = Q(x \sharp y)\frac{x^{-1} + y^{-1}}{2} = \frac{1}{2}(x \sharp y.x^{-1} + y \sharp x.y^{-1})1 = \frac{1}{2}(y + x) = A(x, y).$$

From the Riccati lemma it follows that $x \sharp y = H(x, y) \sharp A(x, y)$. □

Theorem 5.8. *For x, y in a $\sharp$-symmetric cone Ω in a Banach space V, define $H_1 = H(x, y)$, the harmonic mean, and $A_1 = A(x, y)$, the arithmetic mean. Inductively define $H_{n+1} = H(H_n, A_n)$ and $A_{n+1} = A(H_n, A_n)$. Then for each n,*

$$H_n \leq H_{n+1} \leq x \sharp y \leq A_{n+1} \leq A_n,$$

and $H_n \to x \sharp y$, $A_n \to x \sharp y$.

Proof. By Lemma 5.7 and induction, we have $H_n \sharp A_n = x \sharp y$ for each n. The asserted inequality then follows from the HGA-inequality. We fix $\varepsilon \in \Omega$ and endow V with the order-unit norm for the order unit ε; the topology of this norm agrees with that of the original Banach space norm. In this norm the arithmetic mean of two points is halfway between them in distance, and $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, hence $\|y - z\| \leq \|x - z\|$ whenever $z \leq y \leq x$. From these facts we conclude that

$$\|H_{n+1} - A_{n+1}\| \leq \|H_n - A_{n+1}\| = \frac{1}{2}\|H_n - A_n\|,$$

and thus $\|H_n - A_n\| \leq 2^{-n}\|x - y\|$ for each n. Hence $\|H_n - x \sharp y\| \leq \|H_n - A_n\| \leq 2^{-n}\|x - y\|$ for each n, so $H_n \to x \sharp y$. Similarly $A_n \to x \sharp y$. □

The previous results have been obtained in symmetric cones of Euclidean Jordan algebras [20].

Theorem 5.9 (The Furuta inequality, [11]). *Let Ω be a $\sharp$-symmetric cone in a Banach space V and let $0 < b \leq a$. If $0 \leq p, q, r \in \mathbb{R}$ satisfies $p + 2r \leq (1 + 2r)q$ and $1 \leq q$, then*

$$b^{(p+2r)/q} \leq (b^r.a^p)^{1/q}.$$

Proof. The proof is the same as given in [28] for Banach $*$-algebras with continuous involution, where the Loewner-Heinz inequality (Theorem 5.5), the order preserving property of the quadratic representations, the order reversing property of inversion (Theorem 4.2), and the equality (3.1) are applied as the main tools. □

6. JB-algebras and symmetric cones

In this section we illustrate and apply our previously results in the context of JB-algebras. A basic reference for the theory of JB-algebras, particulary the results we need in what follows, is the book of Hanche-Olsen and Størmer [12].

A Jordan algebra is a vector space Z with a commutative multiplication xy such that $x(x^2y) = x^2(xy)$ holds for $x, y \in Z$. An involution on a complex Jordan algebra Z is an antilinear involutive map $z \mapsto z^*$ with $(zw)^* = w^*z^*$ for all $z, w \in Z$. A JB-algebra V is a real Jordan algebra with unit e endowed with a complete norm $\|\cdot\|$ such that

$$\|zw\| \leq \|z\|\,\|w\|, \quad \|z^2\| = \|z\|^2, \quad \|z\|^2 \leq \|z^2 + w^2\|.$$

A JB*-algebra is a complex Banach space Z endowed with the structure of a Jordan algebra with involution $*$ such that

$$\|zw\| \leq \|z\| \cdot \|w\|, \quad \|\{zz^*z\}\| = \|z\|^3$$

for $z, w \in Z$, where $\{xy^*z\} = (xy^*)z + x(y^*z) - y^*(xz)$. There is a one-to-one correspondence between JB-algebras and JB*-algebras: For any JB*-algebra Z, the hermitian part $V := \{x \in Z\colon x^* = x\}$ is a JB-algebra under the restricted norm. Conversely, for every JB-algebra V the complexified algebra $Z := V + iV$ has a unique norm making Z equipped with the canonical involution a JB*-algebra ([30], [7], [12], [25]).

Let V be a JB-algebra. For $x \in V$ we write $L(x)(y) = xy$, the multiplication operator. We consider the set

$$\Omega := \{x \in V : \operatorname{Spec}(L(x)) \subset (0, \infty)\}.$$

Then Ω is an open convex cone of V (see [29, Section 21], particularly [29, Proposition 21.19], also [12, Section 3.3]) and is realized as

$$\Omega = \exp(V) := \{\exp(x) : x \in V\}.$$

The Banach algebra norm agrees with the order unit norm

$$|x|_e := \inf\{t > 0 : te \pm x \geq 0\},$$

or equivalently Ω is a normal cone ([1, Theorem 2.2], [12, Proposition 3.3.10], or [29, Proposition 21.19]). The quadratic representation of the Jordan algebra is defined by $P(z) = 2L(z)^2 - L(z^2)$. It is well-known that for each $z \in \Omega$, $P(z) \in G(\Omega)$, the linear automorphism group of Ω. In fact, there is a polar decomposition $G(\Omega) = P(\Omega)\operatorname{Aut}(V)$ where $\operatorname{Aut}(V)$ denotes the Jordan automorphism group of V (see [29, Corollary 22.29]). We further note ([29, Proposition 22.27]) that $\operatorname{Aut}(V) = \{g \in G(\Omega)\colon g(e) = e\}$. The basic properties

$$P(z)z^{-1} = z, \quad P(z)^{-1} = P(z^{-1}), \quad P(P(z)w) = P(z)P(w)P(z)$$

([29, Corollary 19.9 and Proposition 19.18]) yield a pointed symmetric set structure $x \bullet y = P(x)y^{-1}$ with $\varepsilon := e$ as base point on the set of invertible elements, in particular on the cone Ω (see p.67 of [21]; see also the discussion in Section 3.9 of [12]). In symmetric set notation, $P(a) = Q(a)$ and the symmetric set inverse $a^{-1} := e \bullet a$ agrees with the Jordan inverse of a.

Next, we show that the pointed symmetric space $(\Omega, \varepsilon = e)$ is $\sharp$-symmetric. Let $x, y \in \Omega$ such that $x^2 = y^2$. Then by the commutativity of Jordan products, $0 = x^2 + y^2 = L(x + y)(x - y)$. Since $L(z)$ is invertible for all $z \in \Omega$ ([29, Proposition 21.19 and Corollary 21.22], [12, Lemma 3.2.10]), $x - y = 0$. This implies that each element of Ω has a unique square root in Ω. Note that if $a = \exp(x)$, $x \in V$ then $a^{1/2} = \exp((1/2)x)$. Moreover, if $a, b \in \Omega$ then the quadratic equation

$$P(x)a^{-1} = b$$

has a unique solution in Ω. Note that $x = P(a^{1/2})(P(a^{-1/2})b)^{1/2} \in \Omega$ solves the equation (cf. [14] and [19]). Suppose that x and y are solutions in Ω. Then

$$\begin{aligned}(P(a^{-1/2})x)^2 &= P(P(a^{-1/2})x)\varepsilon = P(a^{-1/2})P(x)P(a^{-1/2})\varepsilon \\ &= P(a^{-1/2})(P(x)a^{-1}) \\ &= P(a^{-1/2})b = P(a^{-1/2})(P(y)a^{-1}) \\ &= P(a^{-1/2})P(y)P(a^{-1/2})\varepsilon \\ &= (P(a^{-1/2})y)^2\end{aligned}$$

and hence $P(a^{-1/2})x = P(a^{-1/2})y$, so $x = y$. We conclude that the open convex cone Ω is a $\sharp$-symmetric set under the operation $x \bullet y = P(x)y^{-1}$. In this case the dyadic power a^t of $a = \exp(x)$ agrees with $\exp(tx)$ and the geometric mean $a \sharp b$ of a and b is

$$a \sharp b = P(a^{1/2})(P(a^{-1/2})b)^{1/2}.$$

Corollary 6.1. *Let V be a JB-algebra and let Ω be the associated symmetric cone. Then Ω is a symmetric space with convex metric with respect to the Thompson metric. In particular, the harmonic-geometric-arithmetic mean inequality* (4.3) *and the Loewner-Heinz inequality* (4.4) *hold and the distance function between points evolving in time on two geodesics is a convex function.*

Proof. Let $x \in \Omega$. Since the subalgebra generated by e and x is isometrically (order and algebra)-isomorphic to $C(X)$ for some compact Hausdorff space ([1, Proposition 2.3]), the inequality

$$\varepsilon \sharp x = x^{1/2} \leq \frac{e + x}{2}$$

holds. The squaring map $x \mapsto x^2$ is continuous (Banach algebra product). The quadratic representation $P(x)$ is obviously a bounded linear operator. This shows that the pointed $\sharp$-symmetric set Ω satisfies the three conditions of Theorem 4.2. The last assertion then follows from Theorem 3.4. □

The last assertion of the preceding corollary provides a positive answer to a question of Neeb [22]. Neeb considers a JB*-algebra Z and the associated symmetric cone Ω in the real JB-subalgebra $V = \{z \in Z : z^* = z\}$ with the Finsler structure on Ω given by $|v|_{e^x} = \|e^{-L(x)}v\|$ for $x, v \in V$ ([22, Example 6.6]). The geodesic line passing through $\gamma(0) = e^x$ and $\gamma(1) = e^y$ is given by $\gamma(t) = e^{L(x)}(e^{-L(x)}e^y)^t$. One of Neeb's questions concerns the convexity of the real function with respect to the Finsler metric distance

$$t \mapsto d(x, \gamma(t)) \tag{6.1}$$

where $x \in \Omega$ and γ is a geodesic. From $P(\exp u) = \exp 2L(u)$ for $u \in V$ ([29, Corollary 22.8]), we see that for $x, y \in \Omega$,

$$|v|_x = |v|_{e^{\log x}} = \|e^{-L(\log x)}v\| = \|e^{2L(\log x^{-1/2})}v\| = \|P(x^{-1/2})v\|$$

($\log x$ is well-defined for any $x \in \Omega$ from the fact that the closed unital subalgebra generated by x is an abelian C^*-algebra, [29, Lemma 20.33]) and the geodesic line passing through x and y is

$$\gamma(t) = P(x^{1/2})(P(x^{-1/2})y)^t = x \sharp_t y.$$

Since the JB*-algebra norm agrees with the order unit norm,

$$|v|_x = \inf\{t > 0 \colon -te \leq P(x^{-1/2})v \leq te\} = \inf\{t > 0 \colon -tx \leq v \leq tx\}.$$

This implies that the Finsler distance is exactly the Thopmson part metric fromd (4.1) and hence the function (6.1) is convex.

REMARK 6.2. The harmonic-geometric-arithmetic mean inequality and the Loewner-Heinz inequality with applications to the Finsler geometry of finite dimensional symmetric cones are studied in [17], [18], [19] and [20]. It has recently been discovered by Bhatia [6] that the non-positive curvature property of the convex cone of positive definite matrices holds for metrics inherited from symmetric gauge functions.

EXAMPLE 6.3. The hermitian elements $x = x^*$ of any C^*-algebra form a JB-algebra with respect to the symmetric product $x \circ y := (xy + yx)/2$. In this case Ω is the cone of positive elements of Example 5.3. Spin factors [12, Chapter 6], which arose in the study of anticommutation relations in physics, provide another type of example. Given a real Hilbert space H, let $A = H \oplus \mathbb{R}1$ have the norm $\|a + \lambda 1\| = \|a\| + |\lambda|$ and define a product in A by

$$(a + \lambda 1) \circ (b + \mu 1) = ((\mu a + \lambda b) + (\langle a, b\rangle + \lambda\mu)1.$$

Then A is a JB-algebra, and hence its corresponding cone Ω satisfies the hypotheses, and hence conclusions of Theorem 4.2.

7. Hermitian Banach $*$-algebras

DEFINITION 7.1. Let Z be a unital Banach algebra Z with a continuous involution $*$ and let X consist of the self-adjoint elements of Z. Let e denote the unit element of Z. The unital Banach algebra Z is called *hermitian* if $\sigma(x) \subset \mathbb{R}$ and $\|x\| = \sup|\sigma(x)|$ for every $x = x^*$. We let

$$\Omega := \{x = x^* \colon \sigma(x) \subset (0, \infty)\}.$$

We note that by the Shirali-Ford theorem [26] $\sigma(zz^*) \subset [0, \infty)$ for every $z \in Z$. If z is invertible then $\sigma(zz^*) \subset (0, \infty)$.

Theorem 7.2. *Let Z be a hermitian Banach algebra. Then Ω is a $\sharp$-symmetric cone of X.*

Proof. It is shown in [29, Corollary 14.16] that Ω is an open convex cone of X and the order unit norm with respect to e coincides with the given norm $\|\cdot\|$, which implies the normality of Ω.

For $a \in \Omega$, we denote $a^{1/2} := \exp((1/2)\log a)$ where log denotes the principal branch of the complex logarithm. Then $(a^2)^{1/2} = a$ ([28, Lemma 6]) and therefore each element in Ω has a unique square root in Ω. Moreover, if $a, b \in \Omega$ then $ab^{-1}a = (ab^{-1/2})(ab^{-1/2})^*$ is contained in the cone Ω since it is invertible and $\sigma(zz^*) \subset [0, \infty)$. This shows that Ω is a uniquely 2-divisible twisted subgroup of $G(Z)$, the group of invertible elements of Z.

The conditions of Corollary 5.2 hold; the verification is similar to the case for C^*-algebras (Example 5.3). □

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