Bouhennache, T.
Osaka J. Math.
37 (2000), 577–601

SPECTRAL ANALYSIS OF AN ISOTROPIC STRATIFIED ELASTIC STRIP AND APPLICATIONS

TARK BOUHENNACHE

(Received September 21, 1998)

1. Introduction

The characteristics of an elastic isotropic stratified strip $\Omega = \{\mathbf{x} = (x_1, x_2); x_2 \in (0, L)\} \subset \mathbb{R}^2$ are the density $\boldsymbol{\rho}$ and Lamé coefficients $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ that we assume to be measurable functions, depending on x_2 only, and bounded from below and above by two positive constants. We shall derive a limiting absorption principle (LAP) and a division theorem for the selfadjoint operator $(\mathcal{D}(A), A)$ (see (2)) associated with Ω with Dirichlet and free surface conditions on $\{x_2 = L\}$ and $\{x_2 = 0\}$, respectively. These boundary conditions come from a model of a seismic problem. For other studies dealing with elasticity in different situations see for instance [7] and [12].

Roughly speaking, a LAP means that the resolvent operator $z \longrightarrow R_A(z) := (A - zI_d)^{-1}$ can be extended continuously to the essential spectrum (a part of the real axis) in suitable topologies. It is an important stage in scattering theory (cf. [1]). A division theorem enables to deal with a perturbed or a multistratified strip (cf. [3]).

A multiplication operator in $\oplus^\infty L^2(\mathbb{R}) := \{(f^n)_{n\geq 1}; \sum^\infty \|f^n\|^2_{L^2(\mathbb{R})} < \infty\}$ by a family of functions μ_n, $n \geq 1$, is defined by

$$(1) \qquad \begin{cases} \mathcal{D}(M) & = \{(f^n)_{n\geq 1} \in \oplus^\infty L^2(\mathbb{R}); (\mu_n f^n)_{n\geq 1} \in \oplus^\infty L^2(\mathbb{R})\} \\ M(f^n)_{n\geq 1} & = (\mu_n f^n)_{n\geq 1}. \end{cases}$$

As we will see A is unitarily equivalent to M with μ_n being the dispersion curves λ_n of A. Such a result holds for other differential operators, in particular for the acoustic operator B (cf. (9)) studied in [4] and [5]. But it is the first time that the following original phenomena are proved (cf. [3], ch.5 and ch.7). They do not take place in the acoustic case.

$$(\mathcal{P}) \qquad \begin{cases} \bullet \text{ The functions } \lambda_n \text{ are not necessarily monotonic on } \mathbb{R}_+. \\ \bullet \text{ One can have } \lambda_n''(0) = \lambda_n'''(0) = 0 \text{ in addition to } \lambda_n'(0) = 0. \end{cases}$$

One of our objectives is to show the spectral consequences of these phenomena and illustrate the fundamental difference between the elastic and the acoustic cases.

Unlike the acoustic case the main difficulty is that there is little information about the functions λ_n so deriving a LAP and a division theorem was carried out by proving only some general properties for these functions. This suggests to study M in the abstract framework taking these properties as hypotheses. This can then be directly used in dealing with other differential operators, which is another objective of this paper. In this framework functions μ_n are allowed to be non-monotonic on $\mathbb{R}_+$ and μ_n' can have roots with arbitrary orders. The richness of the elastic case, as $(\mathcal{P})$ shows, justifies this framework and proves that it applies to concrete problems.

1.1. Definition of A and dispersion curves λ_n, $n \geq 1$ The differential operator of linear elasticity in an isotropic medium in $\mathbb{R}^2$ is given by $(\mathcal{A}\mathbf{u})_i = -(1/\rho)\sum_{j=1}^{2}\partial_{x_j}\sigma_{ij}(\mathbf{u})$ where

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}), \quad \text{with} \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \leq i,\ j \leq 2,$$

and $\mathbf{u} = (u_1, u_2)$ is the displacement field. The function $\mathbf{u}$ belongs to $H^1(\Omega, \mathbb{C}^2)$, so that $\sigma_{ij}(\mathbf{u}) \in L^2(\Omega)$, and $\mathcal{A}\mathbf{u}$ can then be understood in the sense of distributions.

It is well-known that the trace $\mathbf{u}_{|x_2=L}$ belongs to $H^{1/2}(\mathbb{R}, \mathbb{C}^2)$. If in addition $\mathcal{A}\mathbf{u} \in L^2(\Omega, \mathbb{C}^2)$ then $(\sigma_{i1}(\mathbf{u}), \sigma_{i2}(\mathbf{u})) \in H(\Omega, \mathrm{div})$, $i = 1, 2$, (cf. [6], ch.9, p. 239) so $\sigma_{i2}(\mathbf{u})_{|x_2=0}$ makes sense in $H^{-1/2}(\mathbb{R})$. Denote $V = \{\mathbf{u} \in H^1(\Omega, \mathbb{C}^2); \mathbf{u}_{|x_2=L} = 0\}$ and H the Hilbert space $L^2(\Omega, \mathbb{C}^2)$ with the inner product $\langle \mathbf{u}, \mathbf{v}\rangle_H = \int_\Omega \sum_1^2 u_i \overline{v_i}\rho\, d\mathbf{x}$. The operator A is defined by

$$(2) \qquad \begin{cases} \mathcal{D}(A) = \{\mathbf{u} \in V; \mathcal{A}\mathbf{u} \in H, \sigma_{12}(\mathbf{u})_{|x_2=0} = \sigma_{22}(\mathbf{u})_{|x_2=0} = 0\}, \\ A\mathbf{u} = \mathcal{A}\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A). \end{cases}$$

It is selfadjoint in H. Indeed, taking the boundary conditions satisfied by $\mathbf{u} \in \mathcal{D}(A)$ into account and integrating by parts we obtain

$$(3) \qquad \langle A\mathbf{u}, \mathbf{v}\rangle_H = \mathbf{a}(\mathbf{u}, \mathbf{v}) := \int_\Omega \left[\lambda(\nabla\cdot\mathbf{u})\overline{(\nabla\cdot\mathbf{v})} + 2\mu\sum_{ij}\varepsilon_{ij}(\mathbf{u})\overline{\varepsilon_{ij}(\mathbf{v})}\right] d\mathbf{x}, \quad \forall \mathbf{v} \in V.$$

One can build, as in [11], a suitable extension mapping from V into $H^1(\mathbb{R}^2, \mathbb{C}^2)$ and deduce a Korn inequality:

$$\exists c > 0 \quad \text{such that} \quad \mathbf{a}(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_H^2 \geq c\|\mathbf{u}\|_V^2, \quad \forall \mathbf{u} \in V.$$

This, in other words, means that the bounded hermitian form $\mathbf{a}(\cdot,\cdot)$ is coercive over V. Using (3) we prove that A is the selfadjoint operator associated with $\mathbf{a}(\cdot,\cdot)$ (see [9], ch.6, 2.1; or [6], ch.6, p. 1205).

The partial Fourier transform $\mathcal{F}$ is defined, for all $\mathbf{u} \in H$, by

$$\mathcal{F}\mathbf{u}(p, x_2) = \text{l.i.m.}(2\pi)^{-1/2} \int_{\mathbb{R}} \mathbf{u}(x_1, x_2) e^{-ipx_1} dx_1, \quad \text{a.e. } (p, x_2) \in \Omega, \tag{4}$$

and is unitary from H onto H. Since $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ depend only on x_2, the operator $\widehat{A} := \mathcal{F} A \mathcal{F}^{-1}$, which is unitarily equivalent to A, is the direct sum, as in [7], of the field of reduced operators: $\widehat{A} = \int_{\mathbb{R}}^{\oplus} A_p \, dp$. The operators A_p are defined as follows. Denote

$$\mathcal{A}_p \mathbf{u} = -\frac{1}{\boldsymbol{\rho}} \begin{pmatrix} -p^2(\boldsymbol{\lambda} + 2\boldsymbol{\mu})u_1 + ip\boldsymbol{\lambda}\dfrac{du_2}{dx_2} + \dfrac{d}{dx_2}\left(\boldsymbol{\mu}(ipu_2 + \dfrac{d}{dx_2}u_1)\right) \\ -p^2\boldsymbol{\mu}u_2 + ip\boldsymbol{\mu}\dfrac{du_1}{dx_2} + \dfrac{d}{dx_2}\left((\boldsymbol{\lambda} + 2\boldsymbol{\mu})\dfrac{du_2}{dx_2} + ip\boldsymbol{\lambda}u_1\right) \end{pmatrix},$$

$\overline{V} = \{\mathbf{u} \in H^1((0, L), \mathbb{C}^2); \mathbf{u}(L) = 0\}$ and $\overline{H}$ the Hilbert space $L^2((0, L), \mathbb{C}^2)$ with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{\overline{H}} = \int_0^L \sum_1^2 u_i \overline{v_i} \boldsymbol{\rho} \, dx_2$. We have

$$\begin{cases} \mathcal{D}(A_p) = \left\{\mathbf{u} \in \overline{V}; \mathcal{A}_p\mathbf{u} \in \overline{H}, \boldsymbol{\mu}\left(\dfrac{du_1}{dx_2} + ipu_2\right)_{|0} = 0, \left(i\boldsymbol{\lambda}pu_1 + (\boldsymbol{\lambda} + 2\boldsymbol{\mu})\dfrac{du_2}{dx_2}\right)_{|0} = 0\right\}, \\ A_p\mathbf{u} = \mathcal{A}_p\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A_p). \end{cases}$$

As for A we prove that A_p is selfadjoint. In fact, for $\mathbf{u} \in \mathcal{D}(A_p)$ we have

$$\begin{aligned} \langle A_p\mathbf{u}, \mathbf{v} \rangle_{\overline{H}} &= a_p(\mathbf{u}, \mathbf{v}) \\ &:= \int_0^L \Bigg[\boldsymbol{\lambda}\left(ipu_1 + \frac{du_2}{dx_2}\right)\overline{\left(i\overline{p}v_1 + \frac{dv_2}{dx_2}\right)} + 2\boldsymbol{\mu}(ipu_1)\overline{(i\overline{p}v_1)} \\ &\quad + 2\boldsymbol{\mu}\frac{du_2}{dx_2}\overline{\frac{dv_2}{dx_2}} + \boldsymbol{\mu}\left(ipu_2 + \frac{du_1}{dx_2}\right)\overline{\left(i\overline{p}v_2 + \frac{dv_1}{dx_2}\right)}\Bigg] dx_2, \quad \forall \mathbf{v} \in \overline{V}. \end{aligned} \tag{5}$$

Using inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ we verify

$$\exists c > 0, \quad \|\mathbf{u}\|_{\overline{V}}^2 \leq c(1 + p^2)\|\mathbf{u}\|_{\overline{H}}^2 + a_p(\mathbf{u}, \mathbf{u}), \quad \forall \mathbf{u} \in \overline{V}. \tag{6}$$

Thus $a_p(\cdot, \cdot)$ is coercive and we check, using (5), that A_p is the associated selfadjoint operator. The compactness of embedding $\overline{V} \hookrightarrow \overline{H}$ implies that the resolvent $(A_p - zI_d)^{-1}$ is compact from $\overline{H}$ into $\overline{H}$.

Thanks to (5) we extend the definition of $a_p(\cdot, \cdot)$ to all $p \in \mathbb{C}$ and obtain a selfadjoint holomorphic family of type (a) (cf. [9], ch.7). The associated family of operators A_p, $p \in \mathbb{C}$, is thus selfadjoint holomorphic of type (B). Applying Remark 4.22 of ch.7 in [9] we infer the existence of two families of analytic functions on $\mathbb{R}$, namely $p \longrightarrow \lambda_n(p) \in \mathbb{R}$, $p \longrightarrow W_n(p) \in \overline{H}$, $n \geq 1$, such that, for all $p \in \mathbb{R}$, $\{\lambda_n(p); n \geq 1\}$ are the repeated eigenvalues of A_p and $\{W_n(p); n \geq 1\}$ a corresponding orthonormal basis of eigenvectors.

For all $\mathbf{u} \in \mathcal{D}(A_p)$ we then have

$$A_p\mathbf{u} = \sum_{n\geq 1} \lambda_n(p)\langle \mathbf{u}, W_n(p)\rangle_{\overline{H}} W_n(p), \tag{7}$$

and A is recovered this way since $\mathcal{F}A\mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} A_p\, dp$. In fact, consider the transformation $F : H \longrightarrow \oplus^{\infty}\mathrm{Ł}^2(\mathbb{R})$ defined by $F\mathbf{u} = (f^n)_{n\geq 1}$ where

$$f^n(p) = (2\pi)^{-1/2}\,\mathrm{l.i.m.} \int_{\Omega} \sum_{j=1}^{2} u_j(x_1, x_2)\overline{W_j^r(p, x_2)}e^{-ipx_1}\boldsymbol{\rho}d\mathbf{x}, \quad \text{a.e. } p \in \mathbb{R}, \tag{8}$$

with $W_n(p) = (W_n^1(p,\cdot), W_n^2(p,\cdot))$. Using the fact that $\mathcal{F}$ is unitary and that $W_n(p)$, $n \geq 1$, is an orthonormal basis we check that F is unitary. On the other hand from (7) we verify that $A = F^{-1}MF$, with $\mu_n = \lambda_n$ in (1). For details see [3], ch.4.

1.2. Comparison with the acoustic case and consequences of properties ($\mathcal{P}$) The approach is similar for the operator B associated with the acoustic strip:

$$\begin{cases} \mathcal{D}(B) = \{u \in H^1(\Omega);\, \nabla\cdot C^2\nabla u \in L^2(\Omega),\, u_{|x_2=L} = 0,\, C^2\partial_{x_2}u_{|x_2=0} = 0\}, \\ Bu = -\nabla\cdot C^2\nabla u, \quad \forall u \in \mathcal{D}(B). \end{cases} \tag{9}$$

We verify that for every p, the reduced operator B_p is the selfadjoint operator associated with the hermitian form

$$a_{p,B}(u, v) := \int_0^L C^2 \frac{du}{dx_2}\overline{\frac{dv}{dx_2}}\, dx_2 + p^2 \int_0^L C^2 u\bar{v}\, dx_2.$$

The eigenvalues and eigenvectors of the operators B_p are represented by two families of analytic functions: $(\lambda_{n,B})_{n\geq 1}$ and $(W_{n,B})_{n\geq 1}$, and we have (cf. [9], ch.7, 4.7)

$$\frac{d\lambda_{n,B}}{dp}(p) = a'_{p,B}(W_{n,B}(p), W_n(p)) := 2p\int_0^L C^2|W_{n,B}|^2\, dx_2, \quad \forall p \in \mathbb{R}.$$

Here $a'_{p,B}(\mathbf{u},\mathbf{u})$ is obtained by taking the derivative of $p \longrightarrow a_{p,B}(\mathbf{u},\mathbf{u})$ for every fixed $\mathbf{u}$. It is thus clear that $\lambda'_{n,B}(p) \geq 0$ for $p \geq 0$, and that $\lambda''_{n,B}(0) \neq 0$ for all n. In other words, ($\mathcal{P}$) does not hold in the acoustic case. In elasticity

$$\frac{d\lambda_n}{dp}(p) = a'_p(W_n(p), W_n(p)), \tag{10}$$

with

$$a'_p(\mathbf{u},\mathbf{u}) = 2\int_0^L \mathrm{Re}\left(\boldsymbol{\lambda}\frac{du_2}{dx_2}\overline{iu_1} + \boldsymbol{\mu}\frac{du_1}{dx_2}\overline{iu_2}\right) dx_2 + 2p\int_0^L \left((\boldsymbol{\lambda}+2\boldsymbol{\mu})|u_1|^2 + \boldsymbol{\mu}|u_2|^2\right) dx_2,$$

and one can see that we have not necessarily $\lambda'_n(p) \geq 0$, for $p \geq 0$, nor $\lambda''_n(0) \neq 0$. In fact, we prove ($\mathcal{P}$) in the homogeneous case (i.e. $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ constants) with the help of a mathematical formal-calculus software. For instance, with $L = \rho = \mu = 1$ and $\lambda = 7$ there exists $n \geq 1$ such that $\lambda_n'(0) = -10$ so λ_n is not monotonic on $\mathbb{R}_+$ since, as we will see, $\lambda_n(p) \to +\infty$, as $p \to +\infty$.

In general we prove, see Theorem 2.5, a "global symmetry" property: for every n there exists n' such that $\lambda_n(-p) = \lambda_{n'}(p)$ for all p. One can easily check that the eigenvalues $\lambda_n(0)$, $n \geq 1$, of A_0 are at most double. If $\lambda_n(0)$ is simple then λ_n is symmetric so $\lambda_n'(0) = 0$. For double eigenvalues we have $\lambda_n'(0) = -\lambda_{n'}'(0)$. If $\lambda_n'(0) \neq 0$ then one of the functions λ_n or $\lambda_{n'}$ is not monotonic on $\mathbb{R}_+$. In the homogeneous case we prove that this is the case if $\lambda/(\lambda + 2\mu) \neq 1/4$. Clearly, the associated dispersion curves intersect at $p = 0$.

The non-monotonicity is not only related to double eigenvalues of A_0. In fact, we prove numerically that λ_n may be non-monotonic on $\mathbb{R}_+$ even though the eigenvalue $\lambda_n(0)$ is simple, and that the dispersion curves can intersect for $p \neq 0$. For more details see [3], ch.5 and ch.7.

In the acoustic case the thresholds were defined as the eigenvalues $\lambda_{n,B}(0)$, $n \geq 1$, of the reduced operator B_0. They are values where a LAP for B is not valid if the resolvent $(B - zI_d)^{-1}$ is considered as an element of $\mathcal{B}(L^2_{s_1}(\Omega), L^2_{-s_2}(\Omega))$ (cf. [4]), where $L^2_s(\Omega)$ denotes the weighted space with $\|u\|_{L^2_s(\Omega)} = \|(1 + x_1^2)^{s/2} u\|_{L^2(\Omega)}$. On the other hand, the set of thresholds in the elastic case, i.e the set of critical values for obtaining a LAP, is

$$\Gamma = \left\{\lambda_0 \in \mathbb{R}; \exists p_0 \in \mathbb{R} \text{ and } n \geq 1 \text{ such that } \lambda_0 = \mu_n(p_0) \text{ and } \mu_n'(p_0) = 0\right\} \tag{11}$$

with $(\mu_n) = (\lambda_n)$. A first consequence of $(\mathcal{P})$ is that Γ does not coincide with the set of eigenvalues of A_0 unlike the acoustic case. Indeed, double eigenvalue of A_0 are not always thresholds, and if $\lambda_n'(p_0) = 0$, with $p_0 \neq 0$, then $\lambda_n(p_0)$ is not necessarily an eigenvalue of A_0.

The second consequence of $(\mathcal{P})$ concerns a LAP at thresholds for the selfadjoint operators $A^{\mathrm{SA}}_{\Omega_k}$ and $A^{\mathrm{AS}}_{\Omega_k}$ (see (21) and (20)), needed in dealing with a multistratified strip (cf. [3], ch.4 and ch.6), associated with the half-strip

$$\Omega_k := \{\mathbf{x} \in \Omega;\ (-1)^k x_1 > 0\}, \quad k = 1, 2. \tag{12}$$

In fact, for the acoustic case the LAP, which is not valid at thresholds for the stratified strip, becomes valid at all thresholds for some multistratified strips (see [4] and [2]). This is the main result in [2] and the proof there uses the validity of a LAP at thresholds for the half acoustic strip, in which the property $\lambda_{n,B}''(0) \neq 0$ is essential. In elasticity we prove the validity of a LAP at only some particular thresholds for $A^{\mathrm{SA}}_{\Omega_k}$ and $A^{\mathrm{AS}}_{\Omega_k}$, and the non-validity at the whole set of thresholds unlike the acoustic case.

1.3. Abstract framework We shall say that a mapping $z \longrightarrow \widetilde{R(z)}$, defined from $\overline{\mathbb{C}^{\pm}} \setminus \widetilde{\Gamma}$ into a Banach space $\widetilde{Y}$, where $\widetilde{\Gamma} \subset \mathbb{R}$, is locally δ-Hölder continuous

(δ will always be a positive number) if for all compact $K \subset \mathbb{C} \setminus \widetilde{\Gamma}$ there exists a constant $c > 0$ such that $\|\widetilde{R(z')} - \widetilde{R(z)}\|_{\widetilde{Y}} \leq c|z' - z|^{\delta}$, for all $z', z \in K \cap \overline{\mathbb{C}^{\pm}}$.

For $\mathbf{f} = (f^n)_{n\geq 1}$, $\mathbf{g} = (g^n)_{n\geq 1}$ and a family of real functions μ_n, $n \geq 1$, put

$$\mathbf{b}_n(z) = \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - z}\, dp. \tag{13}$$

In deriving a LAP we shall prove a local Hölder continuous estimate for $z \longrightarrow R_M(z) := (M - zI_d)^{-1}$. Since $\langle R_M(z)\mathbf{f}, \mathbf{g}\rangle = \sum_{n=1}^{\infty} \mathbf{b}_n(z)$, it amounts to deriving the Hölder continuous estimate for this infinite sum. To be able to use this part of the work in dealing with other operators it is advisable to carry it out in an abstract framework. The latter consists in supposing that

1. $(\mu_n)_{n\geq 1}$ is a family of functions, not necessarily $(\lambda_n)_{n\geq 1}$, satisfying certain hypothesis (**H1**).
2. **f** and **g** belong to suitable spaces Y^s, where the family Y^s, $s \geq 0$, satisfies another hypothesis (**H2**).

In the applications we are interested (see the last section) in a class of selfadjoint differential operators with analytic and real dispersion curves, this is why we assume that μ_n are real and regular. On the other hand, assuming that $|\mu_n|$ goes uniformly to infinity as $n \to \infty$, permits to treat separately the terms $\mathbf{b}_n(z)$ instead of their infinite sum. Also, supposing that, for every n, $|\mu_n(p)| \to \infty$, as $p \to \infty$, and that the roots of μ_n' is a discrete set implies that Γ is also discrete. It is then natural to put

$$(\mathbf{H1}) \quad \begin{cases} \bullet \ \lim_{n\to\infty} (\inf_{p\in\mathbb{R}} |\mu_n(p)|) = \infty; \text{ and } \lim_{p\to\infty} (|\mu_n(p)|) = \infty, \ \forall n \geq 1. \\ \bullet \ \forall n \geq 1, \text{ the roots of } \mu_n' \text{ have finite orders and form a discrete set.} \end{cases}$$

The fact that the roots of μ_n' have finite orders permits to define appropriate spaces for obtaining a LAP at thresholds. Hypothesis (**H1**) is obviously fulfilled in the acoustic case knowing that $\lambda_{n,B}(p) \geq \lambda_{n,B}(0)$ for all p, and $\lambda_{n,B}(0) \to +\infty$, as $n \to +\infty$. The proof, as we will see, is much less easy in the elastic case.

For the local Hölder continuous estimate of (13) we need local Hölder continuity of $p \longrightarrow f^n(p)\overline{g^n(p)}$. This is why we introduce the second hypothesis below which enables to use some interesting properties of the elements of Sobolev spaces $H^s(\mathbb{R})$ such as Hölder continuity.

$$(\mathbf{H2}) \quad \begin{cases} \bullet \ \forall s' \geq s \geq 0, \text{ the embeddings } Y^{s'} \hookrightarrow Y^s \hookrightarrow \oplus^{\infty} L^2(\mathbb{R}) \\ \quad \text{are continuous with dense ranges.} \\ \bullet \ \forall \phi \in C_0^{\infty}(\mathbb{R}), \ n \geq 1, \text{ the mappings} \\ \qquad (f_k)_{k\geq 1} \longmapsto \phi f_n \quad \text{and} \quad f \longmapsto (\delta_n^k \phi f)_{k\geq 1}, \\ \quad \text{defined respectively from } Y^s \text{ into } H^s(\mathbb{R}), \text{ and from} \\ \quad H^s(\mathbb{R}) \text{ into } Y^s, \text{ are continuous.} \end{cases}$$

In the proof we will indicate more precisely how (**H1**) and (**H2**) are needed.

Denote H_s the weighted space with the norm $\|\mathbf{u}\|_{H_s} := \|(1 + x_1^2)^{s/2}\mathbf{u}\|_H$. For the elasticity Y^s will be $F(H_s)$ (cf. (8)), with $\|\mathbf{f}\|_{Y^s} := \|F^{-1}\mathbf{f}\|_{H_s}$, and $(\mu_n) = (\lambda_n)$. We will see that **(H1)** and **(H2)** are satisfied, so the Hölder continuous estimate for $z \longrightarrow R_A(z)$ follows since $\|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}, H_{-s_2})} = \|R_M(z') - R_M(z)\|_{\mathcal{B}(Y_{s_1}, Y_{-s_2})}$.

This paper is organized as follows.

In Section 2 we prove the validity of **(H1)** and **(H2)** in elasticity, and give a "global symmetry" property satisfied by λ_n and W_n, $n \geq 1$. We study in Section 3 the operator M in an abstract framework. In Section 4 we apply these results to A and propose a procedure to study other differential operators. We finally prove the validity (at some thresholds) of a LAP for $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$, as well as its failure.

ACKNOWLEDGEMENT. This paper is based on a part of the author's doctoral thesis. The author would like to express thanks and appreciation to his advisor Yves Dermenjian.

2. Validity of hypotheses (H1) and (H2) in the elastic case and some symmetry properties

In Theorem 2.2 (resp. Theorem 2.3) we prove that the family $(\lambda_n)_{n\geq 1}$ (resp. $F(H_s)$, $s \geq 0$,) satisfies **(H1)** (resp. **(H2)**). Then we define the operators $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$ and give a property in Theorem 2.4. In Theorem 2.5 we prove a symmetry property satisfied by $(\lambda_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$.

We need the following lemma.

Lemma 2.1. *If a continuous function f satisfies*

$$f(p) \leq ap^2 + bp + \varepsilon \int_0^p f(\tau)\,d\tau, \quad \forall p \in \mathbb{R}_+, \tag{14}$$

where $a, b, \varepsilon \geq 0$, then

$$f(p) \leq \frac{b}{\varepsilon}(e^{\varepsilon p} - 1) + 2a\left(\frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2}(e^{\varepsilon p} - 1)\right), \quad \forall p \in \mathbb{R}_+. \tag{15}$$

The proof uses Gronwall's lemma.

Theorem 2.2. *The family $(\lambda_n)_{n\geq 1}$ satisfies* **(H1)**.

Proof. We first prove that there exists a constant $c_{R-} > 0$ such that $\lambda_n(p) \geq c_{R-}^2\, p^2$ for all $p \in \mathbb{R}$, $n \geq 1$, and after that $\inf_{p\in\mathbb{R}} \lambda_n(p) \to +\infty$, as $n \to +\infty$. This will complete the proof knowing that λ_n is analytic.

We use spectral properties of an elastic homogeneous half-space in $\mathbb{R}^2$ (with constant density $\boldsymbol{\rho}_0$ and Lamé coefficients $\boldsymbol{\lambda}_0$ and $\boldsymbol{\mu}_0$) with a free surface condition on $x_2 = 0$. The corresponding reduced operators $\widetilde{A}_p$, $p \in \mathbb{R}$, are defined exactly as A_p: it suffices to replace L by $+\infty$ and $(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ by $(\boldsymbol{\rho}_0, \boldsymbol{\lambda}_0, \boldsymbol{\mu}_0)$. For fixed p, $\widetilde{A}_p$ is the selfadjoint operator, in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{C}^2, \boldsymbol{\rho}_0 dx_2)$ equipped with $\langle \mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{R}_+,\mathbb{C}^2,\boldsymbol{\rho}_0 dx_2)} = \int_{\mathbb{R}_+} \sum_1^2 u_i \overline{v_i} \boldsymbol{\rho}\, dx_2$, associated with a hermitian and coercive form $\widetilde{a}_p(\cdot,\cdot)$ defined on $H^1(\mathbb{R}_+, \mathbb{C}^2)$. The spectrum of $\widetilde{A}_p$ (cf. [7]) is $\{c^2_{R,\boldsymbol{\rho}_0,\boldsymbol{\lambda}_0,\boldsymbol{\mu}_0} p^2\} \cup [\boldsymbol{\mu}_0 p^2/\boldsymbol{\rho}_0, +\infty[$ where $c_{R,\boldsymbol{\rho}_0,\boldsymbol{\lambda}_0,\boldsymbol{\mu}_0} > 0$ is the propagation velocity of Rayleigh waves.

The extension $\widetilde{\mathbf{u}}$ of $\mathbf{u} \in \overline{V}$ by zero belongs to $H^1(\mathbb{R}_+, \mathbb{C}^2)$ and we have $a_p(\mathbf{u},\mathbf{u}) \geq \widetilde{a}_{p,\boldsymbol{\lambda}_-,\boldsymbol{\mu}_-}(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}})$ and $\|\mathbf{u}\|_{\overline{H}} \leq \|\widetilde{\mathbf{u}}\|_{L^2(\mathbb{R}_+,\mathbb{C}^2,\boldsymbol{\rho}_+ dx_2)}$, where $\boldsymbol{\rho}_+ = \operatorname{ess\,sup} \boldsymbol{\rho}$, $\boldsymbol{\lambda}_- = \operatorname{ess\,inf} \boldsymbol{\lambda}$ and $\boldsymbol{\mu}_- = \operatorname{ess\,inf} \boldsymbol{\mu}$. Denoting $c_{R-} := c_{R,\boldsymbol{\rho}_+,\boldsymbol{\lambda}_-,\boldsymbol{\mu}_-}$ we have

$$\frac{a_p(\mathbf{u},\mathbf{u})}{\|\mathbf{u}\|^2_{\overline{H}}} \geq \frac{\widetilde{a}_{p,\boldsymbol{\lambda}_-,\boldsymbol{\mu}_-}(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}})}{\|\widetilde{\mathbf{u}}\|^2_{L^2(\mathbb{R}_+,\mathbb{C}^2,\boldsymbol{\rho}_+ dx_2)}} \geq c^2_{R-}\, p^2,$$

hence

$$\lambda_n(p) \geq \inf_{\mathbf{u}\in\overline{V},\mathbf{u}\neq 0} \frac{a_p(\mathbf{u},\mathbf{u})}{\|\mathbf{u}\|^2_{\overline{H}}} \geq c^2_{R-}\, p^2, \quad \forall p \in \mathbb{R}, n \geq 1. \tag{16}$$

It remains to prove that

$$\forall m > 0, \quad \exists N_m \in \mathbb{N} \text{ such that } n \geq N_m \Rightarrow \lambda_n(p) > m, \quad \forall p \in \mathbb{R}. \tag{17}$$

We shall use (10). Thanks to inequality $2ab \leq \varepsilon|a|^2 + 1/\varepsilon|b|^2$ we verify that $|a'_p(\mathbf{u},\mathbf{u})| \leq \varepsilon\|\mathbf{u}\|^2_{\overline{V}} + (c|p| + c'_\varepsilon)\|\mathbf{u}\|^2_{\overline{H}}$. Moreover, from (6) and inequality (16) we get $\|\mathbf{u}\|^2_{\overline{V}} \leq c_1\|\mathbf{u}\|^2_{\overline{H}} + c_2 a_p(\mathbf{u},\mathbf{u})$. The combination of the last two inequalities (we suppose $c_2 = 1$ since ε can be replaced by ε/c_2) yields $|a'_p(\mathbf{u},\mathbf{u})| \leq \varepsilon a_p(\mathbf{u},\mathbf{u}) + (c|p| + c_\varepsilon)\|u\|^2_{\overline{H}}$. Replacing $\mathbf{u}$ by $W_n(p)$ in (10) we obtain $\left|\lambda'_n(p)\right| \leq c|p| + c_\varepsilon + \varepsilon\lambda_n(p)$ so that

$$\begin{aligned} |\lambda_n(p) - \lambda_n(0)| &\leq \int_0^p \left|\frac{d\lambda_n}{d\tau}(\tau)\right| d\tau \\ &\leq \frac{c}{2}p^2 + (c_\varepsilon + \varepsilon\lambda_n(0))\, p + \varepsilon \int_0^p |\lambda_n(\tau) - \lambda_n(0)|\, d\tau. \end{aligned}$$

Applying Lemma 2.1 to $f(p) = |\lambda_n(p) - \lambda_n(0)|$, with $a = c/2$ and $b = c_\varepsilon + \varepsilon\lambda_n(0)$, we get

$$|\lambda_n(p) - \lambda_n(0)| \leq \lambda_n(0)\left(e^{\varepsilon p} - 1\right) + \frac{c_\varepsilon}{\varepsilon}\left(e^{\varepsilon p} - 1\right) + c\left(\frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2}\left(e^{\varepsilon p} - 1\right)\right). \tag{18}$$

In view of (16) we can suppose, in order to prove (17), that $0 \leq p \leq P_0 := m^{1/2}/c_{R-} + 1$. Take $\varepsilon > 0$ small enough that $e^{\varepsilon P_0} - 1 < 1/2$. We have

$$\left|\frac{c_\varepsilon}{\varepsilon}\left(e^{\varepsilon p} - 1\right) + c\left(\frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2}\left(e^{\varepsilon p} - 1\right)\right)\right| \leq m', \quad \forall p \in [0, P_0].$$

Then inequality (18) gives

$$\lambda_n(p) \geq \lambda_n(0) - |\lambda_n(p) - \lambda_n(0)| \geq \frac{\lambda_n(0)}{2} - m', \quad \forall p \in [0, P_0],$$

which ends the proof since $\lambda_n(0) \to +\infty$, as $n \to +\infty$. □

Theorem 2.3. *The family $F(H_s)$, $s \geq 0$, satisfies* (**H2**).

Proof. We will only prove the continuity for the mapping $\mathbf{f} = (f^k)_{k\geq 1} \longmapsto \phi f^n$. Let $\mathbf{f} = F\mathbf{u}$ with $\mathbf{u} = \sum_{i=1}^N g_i(x_1)\mathbf{u}_i(x_2) \in \mathcal{S} \otimes \overline{H}$ where $\{\mathbf{u}_i; i = 1, \ldots, N\}$, is an orthonormal family in the Hilbert space $\overline{H}$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the Schwartz class. Clearly

$$\|\mathbf{f}\|_{F(H_s)} = \|\mathbf{u}\|_{H_s} = \left(\sum_1^N \|g_i\|^2_{L^2_s(\mathbb{R})}\right)^{1/2} = \left(\sum_1^N \|\mathcal{F}g_i\|^2_{H^s(\mathbb{R})}\right)^{1/2}. \tag{19}$$

For all $n \geq 1$, we have

$$f^n(p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \sum_{i=1}^N g_i(x_1)\langle \mathbf{u}_i, W_n(p)\rangle_{\overline{H}} e^{-ipx_1}\, dx_1 = \sum_{i=1}^N \langle \mathbf{u}_i, W_n(p)\rangle_{\overline{H}} (\mathcal{F}g_i)(p).$$

It is well-known (cf. [13], Proposition 25.1) that for all $\psi \in \mathcal{S}$ and $h \in H^s(\mathbb{R})$ the product ψh belongs to $H^s(\mathbb{R})$ and we have, with $k = E(s) + 2$ and $E(s)$ denoting the smallest integer such that $E(s) \leq s \leq E(s) + 1$, the following inequality $\|\psi h\|_{H^s(\mathbb{R})} \leq c_s \|\psi\|_{H^k(\mathbb{R})} \|h\|_{H^s(\mathbb{R})}$. Combining this inequality and equation (19), with $\psi = \phi\langle \mathbf{u}_i, W_n(p)\rangle_{\overline{H}}$ and $h = \mathcal{F}g_i$, yields

$$\begin{aligned}\|\phi f^n\|_{H^s(\mathbb{R})} &\leq \sum_{i=1}^N \|\phi\langle \mathbf{u}_i, W_n(p)\rangle_{\overline{H}} \mathcal{F}g_i\|_{H^s(\mathbb{R})} \\ &\leq c_s \underbrace{\left(\sum_{i=1}^N \|\phi\langle \mathbf{u}_i, W_n(p)\rangle_{\overline{H}}\|^2_{H^k(\mathbb{R})}\right)^{1/2}}_{c} \underbrace{\left(\sum_{i=1}^N \|\mathcal{F}g_i\|^2_{H^s(\mathbb{R})}\right)^{1/2}}_{\|\mathbf{f}\|_{F(H_s)}}.\end{aligned}$$

Using the fact that $\mathbf{u}_i$, $i = 1, \ldots, N$, is orthonormal we verify that c is independent from $\mathbf{f}$. Thanks to the density of $\mathcal{S} \otimes \overline{H}$ in H_s the last estimate is valid for all $\mathbf{f} \in F(H_s)$. □

Denote S^{SA} and S^{AS} the unitary transformations on $\overline{H}$ defined by $S^{\text{SA}}(u_1, u_2) = (u_1, -u_2)$ and $S^{\text{AS}} = -S^{\text{SA}}$. Denote also H^{SA} (resp. H^{AS}) the subspace of functions (u_1, u_2) such that u_1 is symmetric (resp. anti-symmetric) and u_2 is anti-symmetric (resp. symmetric). The orthogonal projections $\mathbf{u}^1$ on H^{SA} and $\mathbf{u}^2$ on H^{AS} of $\mathbf{u}$ are given by

$$\mathbf{u}^1(x_1,\cdot)=\frac{\mathbf{u}(x_1,\cdot)+S^{\mathrm{SA}}\mathbf{u}(-x_1,\cdot)}{2},\quad \text{and}\quad \mathbf{u}^2(x_1,\cdot)=\frac{\mathbf{u}(x_1,\cdot)-S^{\mathrm{SA}}\mathbf{u}(-x_1,\cdot)}{2}.$$

The symmetry of our problem makes that A is reduced by H^{SA} and H^{AS} (cf. [9], ch.5, 3.9), and it is thus the direct sum of the operators A^{SA} and A^{AS} defined as follows:

$$A^{\mathrm{SA}}\mathbf{u}=A\mathbf{u},\quad \forall\mathbf{u}\in\mathcal{D}(A^{\mathrm{SA}})=H^{\mathrm{SA}}\cap\mathcal{D}(A),$$
$$A^{\mathrm{AS}}\mathbf{u}=A\mathbf{u},\quad \forall\mathbf{u}\in\mathcal{D}(A^{\mathrm{AS}})=H^{\mathrm{AS}}\cap\mathcal{D}(A).$$

Thus for all $z\in\mathbb{C}^{\pm}$ we have

$$\|R_A(z)\mathbf{u}\|_H^2=\|R_{A^{\mathrm{SA}}}(z)\mathbf{u}^1\|_H^2+\|R_{A^{\mathrm{AS}}}(z)\mathbf{u}^2\|_H^2.$$

We shall now see that A^{SA} and A^{AS} are, respectively, unitarily equivalent to the operators $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$ defined as follows. Recall that Ω_k is the half-strip defined in (12). Denote

$$V_{\Omega_k}^{\mathrm{SA}}=\{\mathbf{u}=(u_1,u_2)\in H^1(\Omega_k,\mathbb{C}^2);\mathbf{u}_{|x_2=L}=u_{2|x_1=0}=0\},$$
$$V_{\Omega_k}^{\mathrm{AS}}=\{\mathbf{u}=(u_1,u_2)\in H^1(\Omega_k,\mathbb{C}^2);\mathbf{u}_{|x_2=L}=u_{1|x_1=0}=0\}.$$

One sets

$$\text{(20)}\quad\begin{cases}\mathcal{D}(A_{\Omega_k}^{\mathrm{SA}})=\{\mathbf{u}\in V_{\Omega_k}^{\mathrm{SA}};\mathcal{A}\mathbf{u}\in L^2(\Omega_k,\mathbb{C}^2),\sigma_{12}(\mathbf{u})_{|x_2=0}=\sigma_{22}(\mathbf{u})_{|x_2=0}=\sigma_{11}(\mathbf{u})_{|x_1=0}=0\},\\ A_{\Omega_k}^{\mathrm{SA}}\mathbf{u}=\mathcal{A}\mathbf{u},\quad\forall\mathbf{u}\in\mathcal{D}(A_{\Omega_k}^{\mathrm{SA}}),\end{cases}$$

$$\text{(21)}\quad\begin{cases}\mathcal{D}(A_{\Omega_k}^{\mathrm{AS}})=\{\mathbf{u}\in V_{\Omega_k}^{\mathrm{AS}};\mathcal{A}\mathbf{u}\in L^2(\Omega_k,\mathbb{C}^2),\sigma_{12}(\mathbf{u})_{|x_2=0}=\sigma_{22}(\mathbf{u})_{|x_2=0}=\sigma_{21}(\mathbf{u})_{|x_1=0}=0\},\\ A_{\Omega_k}^{\mathrm{AS}}\mathbf{u}=\mathcal{A}\mathbf{u},\quad\forall\mathbf{u}\in\mathcal{D}(A_{\Omega_k}^{\mathrm{AS}}).\end{cases}$$

These operators are selfadjoint in the Hilbert space $H_{\Omega_k}=L^2(\Omega_k,\mathbb{C}^2)$ equipped with the inner product $\langle\mathbf{u},\mathbf{v}\rangle_{H_{\Omega_k}}=\int_{\Omega_k}\sum_1^2 u_i\overline{v_i}\rho\,d\mathbf{x}$.

Theorem 2.4. *The operator $A_{\Omega_k}^{\mathrm{SA}}$ (resp. $A_{\Omega_k}^{\mathrm{AS}}$) is unitarily equivalent to A^{SA} (resp. A^{AS}).*

Proof. For a given function $\mathbf{u}$ defined on Ω_k denote by $\mathcal{U}_k^{\mathrm{SA}}\mathbf{u}$ (resp. $\mathcal{U}_k^{\mathrm{AS}}\mathbf{u}$) the function equal to $\mathbf{u}/\sqrt{2}$ on Ω_k, and to $S^{\mathrm{SA}}\mathbf{u}(-x_1,x_2)/\sqrt{2}$ (resp. $S^{\mathrm{AS}}\mathbf{u}(-x_1,x_2)/\sqrt{2}$) on $\Omega_{k'}$ with $k'=3-k$. We verify that $\mathcal{U}_k^{\mathrm{SA}}$ (resp. $\mathcal{U}_k^{\mathrm{AS}}$) is a unitary transformation defined from H_{Ω_k} onto H^{SA} (resp. H^{AS}) and that

$$\text{(22)}\qquad A_{\Omega_k}^{\mathrm{SA}}=(\mathcal{U}_k^{\mathrm{SA}})^{-1}A^{\mathrm{SA}}\mathcal{U}_k^{\mathrm{SA}},\quad\left(\text{resp. }A_{\Omega_k}^{\mathrm{AS}}=(\mathcal{U}_k^{\mathrm{AS}})^{-1}A^{\mathrm{AS}}\mathcal{U}_k^{\mathrm{AS}}\right).\qquad\square$$

Let us now give the consequence on $(\lambda_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ of the symmetry of our differential problem. The operators A_p and A_{-p} are unitarily equivalent, in fact

$$(23) \qquad \begin{cases} \mathcal{D}(A_{-p}) = S^{\text{SA}}\mathcal{D}(A_p) = S^{\text{AS}}\mathcal{D}(A_p) \\ A_{-p} = S^{\text{SA}}A_p(S^{\text{SA}})^{-1} = S^{\text{AS}}A_p(S^{\text{AS}})^{-1}. \end{cases}$$

One can verify that the eigenvalues of A_p, in particular for $p = 0$, are at most double (for the details see [3], ch.5, 4.2). Denote $\overline{H}^{\text{SA}}$ (resp. $\overline{H}^{\text{AS}}$) the subspace of the functions $\mathbf{u} = (u_1, u_2) \in \overline{H}$ such that $u_2 = 0$ (resp. $u_1 = 0$). These subspaces reduce A_0 which is then the direct sum of the corresponding operators A_0^{SA} and A_0^{AS}. We check that A_0^{SA} and A_0^{AS} have simple eigenvalues and that an eigenvalue $\lambda_n(0)$ of A_0 is simple if and only if it is an eigenvalue of only one of these operators.

Theorem 2.5. *For all n there exists n' such that $\lambda_n(-p) = \lambda_{n'}(p)$ for all p.*

If $\lambda_n(0)$ is a simple eigenvalue of A_0 then the function λ_n is symmetric. If $\lambda_n(0)$ is an eigenvalue of A_0^{SA} (resp. A_0^{AS}) we can choose W_n such that $W_n(-p) = S^{\text{SA}}W_n(p)$ (resp. $W_n(-p) = S^{\text{AS}}W_n(p)$), for all $p \in \mathbb{R}$. In this case, for all $\boldsymbol{u} \in H_s^{\text{AS}} := H_s \cap H^{\text{AS}}$ (resp. $\boldsymbol{u} \in H_s^{\text{SA}} := H_s \cap H^{\text{SA}}$), $s > 1/2$, $F\boldsymbol{u} = (f^k)_{k\geq 1}$ is such that f^n is anti-symmetric.

Proof. To prove the first assertion we suppose that $\lambda_n(0)$ is a simple eigenvalue. The case where $\lambda_n(0)$ is double is handled in a similar fashion.

According to Kato [9], ch.7, there exists $\varepsilon, \eta > 0$ small enough such that $\lambda_n(p)$ is the unique eigenvalue of A_p in $(\lambda_n(0)-\varepsilon, \lambda_n(0)+\varepsilon)$ for $|p| < \eta$. Denote by $\check{\lambda}_n$ the function $p \longrightarrow \lambda_n(-p)$. Since A_{-p} is unitarily equivalent to A_p, $\check{\lambda}_n(p)$ is an eigenvalue of A_p and necessarily coincides with $\lambda_n(p)$ for p sufficiently small, and thus everywhere because of the analyticity. Therefore $\lambda_n(-p) = \lambda_{n'}(p)$ for all p, where $n' = n$, and λ_n is symmetric.

Denote $Q_n(p)$ the orthogonal projection associated with the eigenvalue $\lambda_n(p)$. Since $\lambda_n(0)$ is simple and according to [9], ch.7, the mapping $p \longrightarrow Q_n(p) \in \mathcal{B}(\overline{H})$ is analytic in a neighborhood of 0, and in view of (23) we have $Q_n(-p) = S^{\text{SA}}Q_n(p)(S^{\text{SA}})^{-1} = S^{\text{AS}}Q_n(p)(S^{\text{AS}})^{-1}$. Suppose up to the end of the proof that $\lambda_n(0)$ is an eigenvalue of A_0^{SA}. Take an associated eigenvector $\mathbf{v}_0 \in H^{\text{SA}}$ with $\lambda_n(0)$ and define the function $p \longrightarrow \mathbf{v}(p) := Q_n(p)\mathbf{v}_0 \in \overline{H}$ which is analytic and does not vanish in a neighborhood of 0. Thus, in this neighborhood the function $p \longrightarrow \widetilde{W}(p) := \mathbf{v}(p)/\|\mathbf{v}(p)\|_{\overline{H}}$ is well-defined, analytic and $\|\widetilde{W}(p)\|_{\overline{H}} = 1$.

Since $\mathbf{v}_0 \in H^{\text{SA}}$ we have $(S^{\text{SA}})^{-1}\mathbf{v}_0 = \mathbf{v}_0$, therefore

$$\widetilde{W}(-p) = \frac{Q_n(-p)\mathbf{v}_0}{\|Q_n(-p)\mathbf{v}_0\|_{\overline{H}}} = \frac{S^{\text{SA}}Q_n(p)(S^{\text{SA}})^{-1}\mathbf{v}_0}{\|S^{\text{SA}}Q_n(p)(S^{\text{SA}})^{-1}\mathbf{v}_0\|_{\overline{H}}} = \frac{S^{\text{SA}}\mathbf{v}(p)}{\|S^{\text{AS}}\mathbf{v}(p)\|_{\overline{H}}} = S^{\text{SA}}\widetilde{W}(p).$$

The function $\widetilde{W}$ can be continued analytically along the real axis (cf. [9], ch.7). This continuation, denoted W_n, obviously satisfies $W_n(-p) = S^{\text{SA}}W_n(p)$ for all p.

If $\mathbf{u} \in H_s$ with $s > 1/2$ we verify (cf. (8)) that the integral

$$f^n(p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \langle \mathbf{u}(x_1, \cdot), W_n(p) \rangle_{\overline{H}} e^{-ipx_1}\, dx_1, \quad \forall p \in \mathbb{R}, \tag{24}$$

with $F\mathbf{u} = (f^k)_{k \geq 1}$, is convergent. Using the variable change $x_1 \longrightarrow -x_1$ we obtain

$$f^n(-p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \langle \mathbf{u}(-x_1, \cdot), W_n(-p) \rangle_{\overline{H}} e^{-ipx_1}\, dx_1. \tag{25}$$

Clearly, for $\mathbf{u} \in H_s^{\mathrm{AS}}$ we have $S^{\mathrm{AS}}\mathbf{u}(-x_1, \cdot) = \mathbf{u}(x_1, \cdot)$. Since $\lambda_n(0)$ is an eigenvalue of A_0^{SA} we have $S^{\mathrm{AS}} W_n(-p) = -S^{\mathrm{SA}} W_n(-p) = -W_n(p)$. Knowing that S^{SA} is unitary we thus have

$$\langle \mathbf{u}(-x_1, \cdot), W_n(-p) \rangle_{\overline{H}} = \langle S^{\mathrm{AS}}\mathbf{u}(-x_1, \cdot), S^{\mathrm{AS}} W_n(-p) \rangle_{\overline{H}} = -\langle \mathbf{u}(x_1, \cdot), W_n(p) \rangle_{\overline{H}}.$$

Therefore, in view of (24) and (25), the function f^n is anti-symmetric. □

3. Multiplication operator M in $\bigoplus^\infty L^2(\mathbb{R})$ and unitarily equivalent operators to M

We suppose that $(\mu_n)_{n \geq 1}$ (resp. Y^s, $s \geq 0$) is an arbitrary family of functions (resp. spaces) satisfying hypothesis **(H1)** (resp. **(H2)**). It is well-known that the multiplication operator M defined by (1) has an absolutely continuous spectrum (cf. [14]) and that its spectral measure E_M can be explicitly given. In this work we derive a LAP outside thresholds (cf. Theorem 3.3), and at thresholds in appropriate spaces (cf. Theorem 3.5), as well as a division theorem (cf. Theorem 3.7).

This enables to deduce similar results for any selfadjoint operator D in a Hilbert space Z^0, unitarily equivalent to such a M. More precisely, suppose there exists a unitary transformation $U : Z^0 \longrightarrow Y^0$ such that $D = U^{-1}MU$. Then D has an absolutely continuous spectrum and its spectral measure, E_D, is given by $E_D = U^{-1}E_M U$. Using the spaces $Z^s := U^{-1}(Y^s)$ a LAP for D and a division theorem is directly derived.

We need the following lemmas.

Lemma 3.1. *Let $k \geq 0$ be an integer and $s > k + 1/2$. Any $f \in H^s(\mathbb{R})$ is Hölder continuous and belongs to $C^k(\mathbb{R}) \cap L^\infty(\mathbb{R})$. More precisely, there exist two constants $c(s)$ and $c(s, \delta)$, with $0 \leq \delta < \min(1, s - 1/2)$, such that*

$$|f(p)| \leq c(s) \|f\|_{H^s(\mathbb{R})}, \quad |f(p') - f(p)| \leq c(s, \delta)|p' - p|^\delta \|f\|_{H^s(\mathbb{R})}, \quad \forall p, p' \in \mathbb{R}.$$

Let $s_1, s_2 > 1/2$ and $\mathbf{f} = (f^n)_{n \geq 1} \in Y^{s_1}$, $\mathbf{g} = (g^n)_{n \geq 1} \in Y^{s_2}$. According to **(H2)**, f^n and g^n belong locally to $H^{s_1}(\mathbb{R})$ and $H^{s_2}(\mathbb{R})$, respectively. Then by the last lemma for $\phi \in C_0^\infty(\mathbb{R})$ and $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$ there exists a constant $c > 0$ such that

$$|\phi(p') f^n(p') \overline{g^n(p')} - \phi(p) f^n(p) \overline{g^n(p)}| \leq c|p' - p|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}. \tag{26}$$

Lemma 3.2. *Let $\overline{I}$ be a compact interval of $\mathbb{R}$ and h a δ-Hölder continuous function on $\overline{I}$, $\delta \in (0,1)$:*

$$\exists C(h), \quad |h(\mu') - h(\mu)| \leq C(h)|\mu' - \mu|^{\delta}, \quad \forall \mu', \mu \in \overline{I},$$

vanishing at the end points of $\overline{I}$. Then, for all compact $K \subset \mathbb{C}$, the function

$$z \longrightarrow b(z,h) := \int_I \frac{h(\mu)}{\mu - z} \, d\mu$$

is δ-Hölder continuous on $K \cap \mathbb{C}^{\pm}$: there exists a constant c, independent of h, such that

$$|b(z',h) - b(z,h)| \leq cC(h)|z' - z|^{\delta}, \quad \forall z', z \in K \cap \mathbb{C}^{\pm}.$$

For all $\mu_0 \in I$ we have

$$\lim_{\substack{z \to \mu_0 \\ \pm \operatorname{Im} z > 0}} b(z) = \text{p.v.} \int_I \frac{h(\mu)}{\mu - \mu_0} \, d\mu \pm i\pi h(\mu_0). \tag{27}$$

For the proof we refer to Gakhov ([8], ch.1, 5) or Muskhelishvili ([10], ch.2, 22).

To apply the last lemma we shall invert locally the functions μ_n, $n \geq 1$, thanks to variable changes, and use (26). To do so we fix an enumeration p_n^j, $j \in \mathcal{J}_n \subset \mathbb{Z}$, of the roots of μ_n' which form a discrete subset of $\mathbb{R}$ by (**H1**). We set p_n^0 to be the root of μ_n', if it exists, with smallest modulus (if there are two roots with opposite signs and the same modulus we take the positive one). Then denote p_n^1 (resp. p_n^{-1}) the first root greater (resp. less) than p_n^0, if it exists, and so on. The roots are thus p_n^j, $j \in \mathcal{J}_n$. If μ_n' does not vanish, that is $\mathcal{J}_n = \emptyset$, we set $p_n^0 = -\infty$ and $p_n^1 = +\infty$.

If $j_{n,\max} := \max \mathcal{J}_n < \infty$ we set $p_n^{j_{n,\max}+1} = +\infty$. Similarly, if $j_{n,\min} := \min \mathcal{J}_n > -\infty$ we set $p^{j_{n,\min}-1} = -\infty$. Let us define $\overline{\mathcal{J}_n}$ as follows: (1) $\overline{\mathcal{J}_n} = \{0\}$ if $\mathcal{J}_n = \emptyset$, in this case $p_n^0 = -\infty$ and $p_n^1 = +\infty$. (2) $\overline{\mathcal{J}_n} = \mathcal{J}_n \cup \{j_{n,\min} - 1\}$, if $j_{n,\min} > -\infty$. (3) $\overline{\mathcal{J}_n} = \mathcal{J}_n$, otherwise. Thus, for all $j \in \overline{\mathcal{J}_n}$, $\mu_n : (p_n^j, p_n^{j+1}) \longrightarrow I_n^j := \mu_n((p_n^j, p_n^{j+1}))$ is C^∞ diffeomorphism, we denote P_n^j its inverse.

3.1. Limiting absorption principle outside thresholds For all $s > 0$, denote Y^{-s} the anti-dual of Y^s. Identifying Y^0 to its anti-dual and using (**H2**) we verify that the embeddings $Y^{s_1} \hookrightarrow Y^0 \hookrightarrow Y^{-s_2}$ are continuous and have dense ranges for all $s_1, s_2 \geq 0$. For all $z \in \mathbb{C}^{\pm}$, $R_M(z)$ can then be considered as an element of $\mathcal{B}(Y^{s_1}, Y^{-s_2})$. Likewise, $R_D(z)$ can be considered as an element of $\mathcal{B}(Z^{s_1}, Z^{-s_2})$.

Theorem 3.3. *Let $s_1, s_2 > 1/2$. The mapping $z \longrightarrow R_M(z) \in \mathcal{B}(Y^{s_1}, Y^{-s_2})$, defined on $\mathbb{C}^{\pm}$, can be extended continuously to $\overline{\mathbb{C}^{\pm}} \setminus \Gamma$. This extension, denoted $R_M^{\pm}(z)$, is locally δ-Hölder continuous, $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$.*

The precedent assertion holds when we replace M by D and Y by Z. For all $\mu_0 \in \mathbb{R} \setminus \Gamma$, $f = (f^n)_{n\geq 1} \in Y^{s_1}$ and $\mathbf{g} = (g^n)_{n\geq 1} \in Y^{s_2}$, we have

$$(28)\quad \langle R_D^{\pm}(\mu_0)U^{-1}f, U^{-1}\mathbf{g}\rangle = \langle R_M^{\pm}(\mu_0)f, \mathbf{g}\rangle = \sum_{n=1}^{\infty} \text{p.v.} \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - \mu_0}\, dp$$

$$(29)\quad \pm i\pi \sum_{\{(p_0,n);n\geq 1,\mu_n(p_0)=\mu_0\}} \frac{f^n(p_0)\overline{g^n(p_0)}}{|\mu_n'(p_0)|}.$$

Proof. Let $K \subset \mathbb{C}^{\pm} \setminus \Gamma$ be a compact set, we shall prove that there exists $c = c_K$ such that

$$(30)\quad |\langle R_M(z')\mathbf{f} - R_M(z)\mathbf{f}, \mathbf{g}\rangle| \leq c|z' - z|^{\delta} \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}$$

for all $\mathbf{f} \in Y^{s_1}$, $\mathbf{g} \in Y^{s_2}$ and $z, z' \in K \cap \mathbb{C}^{\pm}$.

The assertion of the theorem will follow since

$$(31)\quad \begin{aligned} &\langle R_D^{\pm}(z)\mathbf{u}, \mathbf{v}\rangle_{Z^{-s_2}\times Z^{s_2}} = \langle R_M^{\pm}(z)U\mathbf{u}, U\mathbf{v}\rangle_{Y^{-s_2}\times Y^{s_2}},\\ &\|R_D^{\pm}(z') - R_D^{\pm}(z)\|_{\mathcal{B}(Z^{s_1}, Z^{-s_2})} = \|R_M^{\pm}(z') - R_M^{\pm}(z)\|_{\mathcal{B}(Y^{s_1}, Y^{-s_2})}.\end{aligned}$$

In view of (**H1**) and since K is compact, there exists $n_0 \geq 1$ such that $|\mu_n(p) - z|^{-1} \leq 1$ for all $p \in \mathbb{R}$, $z \in K$ and $n \geq n_0$. Set $\mathbf{b}_0(z) = \sum_{n\geq n_0} \mathbf{b}_n(z)$ (cf. (13)). Using Schwartz inequality we have

$$|\mathbf{b}_0(z') - \mathbf{b}_0(z)| \leq |z' - z| \cdot \|\mathbf{f}\|_{Y^0} \|\mathbf{g}\|_{Y^0} \leq c_0|z' - z|^{\delta} \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

To end the proof of (30) it remains to establish an analogous estimate for $\mathbf{b}_n$, $0 < n < n_0$:

$$(32)\quad |\mathbf{b}_n(z') - \mathbf{b}_n(z)| \leq c_n|z' - z|^{\delta} \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

Let $\phi_1 \in C_0^{\infty}(\mathbb{R})$ be equal to 1 on a neighborhood of $\mu_n^{-1}(K \cap \mathbb{R})$. Put $\phi_0(p) = 1 - \phi_1(p)$,

$$b_0(z) = \int_{\mathbb{R}} \frac{\phi_0(p) f^n(p)\overline{g^n(p)}}{\mu_n(p) - z}\, dp, \quad \text{and} \quad b_1(z) = \int_{\mathbb{R}} \frac{\phi_1(p) f^n(p)\overline{g^n(p)}}{\mu_n(p) - z}\, dp.$$

Due to the assumption on the support of ϕ_1 one has

$$\sup\left\{ \frac{|\phi_0(p)|}{|\mu_n(p) - z|^2};\ p \in \mathbb{R}, z \in K \right\} := d < \infty,$$

so there exists $\tilde{c_0}$ such that, for all $z', z \in K \cap \mathbb{C}^{\pm}$,

$$(33)\quad |b_0(z') - b_0(z)| \leq d|z' - z| \cdot \|f^n\|_{L^2(\mathbb{R})} \|g^n\|_{L^2(\mathbb{R})} \leq \tilde{c_0}|z' - z|^{\delta} \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

On the other hand, using the variable change $p = P_n^j(\mu)$ we get

$$b_1(z) = \sum_{j\in\overline{\mathcal{J}_n}} \int_{p_n^j}^{p_n^{j+1}} \frac{\phi_1(p) f^n(p)\overline{g^n(p)}}{\mu_n(p) - z}\, dp = \sum_{j\in\overline{\mathcal{J}_n}} b_1^j(z),$$

where

$$(34)\quad b_1^j(z) := \int_{I_n^j} \frac{h_j(\mu)}{\mu - z}\, d\mu \quad \text{and} \quad h_j(\mu) := \frac{\phi_1(P_n^j(\mu)) f^n(P_n^j(\mu))\overline{g^n(P_n^j(\mu))}}{|\mu_n'(P_n^j(\mu))|}.$$

Taking ϕ_1 with a sufficiently small support, one can assume that the support of its restriction to (p_n^j, p_n^{j+1}) is compact. So, we verify, using (26), that $\mu \longrightarrow h_j(\mu)$, which has a compact support in I_n^j, is δ-Hölder continuous: $|h_j(\mu') - h_j(\mu)| \leq c_j' \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}\ |\mu' - \mu|^\delta$, for all $\mu', \mu \in I_n^j$. Applying Lemma 3.2, it follows that $|b_1^j(z') - b_1^j(z)| \leq c^j |z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}$. Using this formula and (33) we deduce (32). □

3.2. Limiting absorption principle at thresholds In view of **(H1)**, for all $n \geq 1$ and $j \in \mathcal{J}_n$, the order of the root p_n^j of $p \longrightarrow \mu_n(p) - \mu_n(p_n^j)$, denoted $N_n^j \geq 2$, is finite. Thus there exist two functions $G_{n,1}^j, G_{n,2}^j \in C^\infty(\mathbb{R})$, such that $G_{n,1}^j(p_n^j) \neq 0$, $G_{n,2}^j(p_n^j) \neq 0$ and

$$(35)\quad \mu_n(p) - \mu_n(p_n^j) = (p - p_n^j)^{N_n^j} G_{n,1}^j(p), \quad \frac{d\mu_n}{dp}(p) = (p - p_n^j)^{N_n^j - 1} G_{n,2}^j(p).$$

Since P_n^j (resp. P_n^{j-1}) is the inverse of the restriction of μ_n to $[p_n^j, p_n^{j+1}]$ (resp. $[p_n^{j-1}, p_n^j]$) we get

$$(36)\qquad \frac{dP_n^j}{d\mu}(\mu) = \frac{1}{\mu_n'(P_n^j(\mu))} = \frac{1}{(P_n^j(\mu) - p_n^j)^{N_n^j - 1} G_{n,2}^j(P_n^j(\mu))}$$

$$(37)\qquad \frac{dP_n^{j-1}}{d\mu}(\mu) = \frac{1}{\mu_n'(P_n^{j-1}(\mu))} = \frac{1}{(P_n^{j-1}(\mu) - p_n^j)^{N_n^j - 1} G_{n,2}^j(P_n^{j-1}(\mu))}.$$

The function P_n^j (resp. P_n^{j-1}) is locally 1-Hölder continuous on I_n^j (resp. I_n^{j-1}) but not on $I_n^j \cup \{\mu_n^j\}$ (resp. $I_n^{j-1} \cup \{\mu_n^j\}$), where it is locally $(1/N_n^j)$-Hölder continuous. In fact, using the left hand side equation in (35), one can verify that

$$(38)\qquad |P_n^j(\mu') - P_n^j(\mu)| \leq c|\mu' - \mu|^{1/N^j}, \quad \forall \mu', \mu \in I_n^j \cup \{\mu_n^j\},$$

$$(39)\qquad |P_n^{j-1}(\mu') - P_n^{j-1}(\mu)| \leq c|\mu' - \mu|^{1/N^j}, \quad \forall \mu', \mu \in I_n^{j-1} \cup \{\mu_n^j\}.$$

In view of **(H2)** and Lemma 3.1, for all $n \geq 1$, $k \geq 1$, $j \in \mathcal{J}_n$ and $s > k - 1/2$, the subspace defined below is closed in Y^s. For $k = 0$ we set $NY_n^s(j, 0) := Y^s$.

$$NY_n^s(j,k) := \left\{ \mathbf{f} = (f^m)_{m\geq 1} \in Y^s; \frac{d^l f^n}{dp^l}(p_n^j) = 0, l = 0, \dots, k-1 \right\}.$$

Lemma 3.4. *Let $k \geq 0$ be an integer and $s > k + 1/2$. For all $f \in H^s(\mathbb{R})$ satisfying $d^j f/dp^j(0) = 0$, $j = 0, \dots, k$, we have $v := f/p^{k+1} \in H^{s-(k+1)}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R})$, and there exists a constant $c = c(k, s)$ such that $\|v\|_{H^{s-(k+1)}(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}$.*

The proof for $k = 0$ was given by Agmon [1], it can be generalized by recurrence to all k (for details see [3], annex C, Corollary C. 7). The case $k = 1$ was proved in [5].

Let μ_0 be a threshold. Set

$$\begin{aligned} \mathcal{J}^{\mu_0} &:= \{(n, j) \in \mathbb{N} \times \mathbb{Z}; \mu_n(p_n^j) = \mu_0\} \subset \mathbb{N} \times \mathbb{Z}, \\ n^{\mu_0} &:= \{n \geq 1; \exists j \in \mathbb{Z} \text{ such that } (n, j) \in \mathcal{J}^{\mu_0}\} \subset \mathbb{N}. \end{aligned}$$

A multi-index α is defined by its values $\alpha_{n,j} \in \mathbb{N}$, $(n, j) \in \mathcal{J}^{\mu_0}$. Denote $|\alpha| = \max\{\alpha_{n,j}; (n, j) \in \mathcal{J}^{\mu_0}\}$. Recall that N_n^j is the order of the root p_n^j of $p \longrightarrow \mu_n(p) - \mu_n(p_n^j)$, and denote $\mathbf{N}_{\mu_0}$ the multi-index $\{N_n^j; (n, j) \in \mathcal{J}^{\mu_0}\}$. For all multi-index α and $s > |\alpha| - 1/2$, let us define the following closed subspace of Y^s:

$$NY^s(\mu_0, \alpha) := \bigcap_{\{(n,j)\in\mathcal{J}^{\mu_0}\}} NY_n^s(j, \alpha_{n,j}).$$

Denote $NY^{-s}(\mu_0, \alpha)$ its anti-dual. We also define $NZ^s(\mu_0, \alpha)$ by $U^{-1}(NY^s(\mu_0, \alpha))$ and denote its anti-dual $NZ^{-s}(\mu_0, \alpha)$.

We verify that $NY^s(\mu_0, \alpha)$ is dense in Y^0, which enables to identify any $\mathbf{f} \in Y^0$ to the anti-linear form $\mathbf{g} \longrightarrow \langle \mathbf{f}, \mathbf{g}\rangle_{Y^0}$ on $NY^s(\mu_0, \alpha)$. With this identification, if α^1, α^2 are two multi-indexes and $s_1 > |\alpha^1| + 1/2$, $s_2 > |\alpha^2| + 1/2$, the embeddings $NY^{s_1}(\mu_0, \alpha^1) \hookrightarrow Y^0 \hookrightarrow NY^{-s_2}(\mu_0, \alpha^2)$ are continuous. The resolvent $R_M(z)$ can thus be considered as an element of $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$. Likewise $R_D(z) \in \mathcal{B}(NZ^{s_1}(\mu_0, \alpha^1), NZ^{-s_2}(\mu_0, \alpha^2))$. As in (31) we have

$$\begin{aligned} (40) \qquad & \|R_D^{\pm}(z') - R_D^{\pm}(z)\|_{\mathcal{B}(NZ^{s_1}(\mu_0,\alpha^1), NZ^{-s_2}(\mu_0,\alpha^2))} \\ &= \|R_M^{\pm}(z') - R_M^{\pm}(z)\|_{\mathcal{B}(NY^{s_1}(\mu_0,\alpha^1), NY^{-s_2}(\mu_0,\alpha^2))}. \end{aligned}$$

Theorem 3.5. *Assume that $\alpha^1_{n,j} + \alpha^2_{n,j} = N_n^j - 1$, for all $(n, j) \in \mathcal{J}^{\mu_0}$. The mapping $z \longrightarrow R_M(z) \in \mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$, defined on $\mathbb{C}^{\pm}$, can be extended continuously to $\{\mu_0\} \cup (\overline{\mathbb{C}^{\pm}} \setminus \Gamma)$. This extension, denoted $R_M^{\pm}(z)$, is locally $(\delta/|\mathbf{N}_{\mu_0}|)$-Hölder continuous, $\delta < \min(1, s_1 - |\alpha^1| - 1/2, s_2 - |\alpha^2| - 1/2)$.*

In view of (40) *the precedent assertion holds when we replace M by D and Y by Z.*

For all $f \in NY^{s_1}(\mu_0, \alpha^1)$ *and* $g \in NY^{s_2}(\mu_0, \alpha^2)$, *we have*

$$(41)\quad \langle R_D^{\pm}(\mu_0)U^{-1}f, U^{-1}g\rangle = \langle R_M^{\pm}(\mu_0)f, g\rangle$$
$$= \sum_{n\geq 1} \text{p.v.} \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p)-\mu_0}\,dp \pm i\pi \sum_{\{(p_0,n);n\geq 1,\mu_n(p_0)=\mu_0 \text{ and } \mu_n'(p_0)\neq 0\}} \frac{f^n(p_0)\overline{g^n(p_0)}}{|\mu_n'(p_0)|}$$

$$(42)\quad \pm i\pi \sum_{(n,j)\in\mathcal{J}^{\mu_0}} \frac{\epsilon(N_n^j)}{\left|\mu_n^{(N_n^j)}(p_n^j)\right|} \frac{(N_n^j-1)!}{\alpha_{n,j}^1!\alpha_{n,j}^2!} \frac{d^{\alpha_{n,j}^1} f^n}{dp^{\alpha_{n,j}^1}}(p_n^j) \frac{d^{\alpha_{n,j}^2}\overline{g^n}}{dp^{\alpha_{n,j}^2}}(p_n^j),$$

where $\mu_n^{(N_n^j)}$ *is the derivative of* μ_n *of order* N_n^j, *and* $\epsilon(\cdot)$ *the mapping, defined on* $\mathbb{N}$, *which takes the value* 0 *if its argument is even and* 1 *otherwise.*

Proof. As in the proof of Theorem 3.3, the same estimate holds for $\mathbf{b}_0$, b_0 and $b_1^{j'}$ if $\mu_n(p_n^{j'})$ and $\mu_n(p_n^{j'+1})$ are not equal to μ_0. It only suffices to prove a similar estimate for

$$(43)\quad \mathbf{b}_1^j(z) = b_1^{j-1}(z) + b_1^j(z) = \int_{p_n^{j-1}}^{p_n^{j+1}} \frac{\phi_1(p)f^n(p)\overline{g^n(p)}}{\mu_n(p)-z}\,dp,$$

where we suppose that supp $\phi_1 \subset (p_n^{j-1}, p_n^{j+1})$.

Since $\mathbf{f} \in NY^{s_1}(\mu_0, \alpha^1)$ and $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$ we have $f^n(p) = (p-p_n^j)^{\alpha_{n,j}^1} h_1(p)$ and $g^n(p) = (p-p_n^j)^{\alpha_{n,j}^2} h_2(p)$ where, according to Lemma 3.4, h_l, $l = 1, 2$, is δ^l-Hölder continuous, with $\delta^l < \min(1, s_l - |\alpha^l| - 1/2)$. Setting $h = h_1\overline{h}_2$, we then verify that

$$(44)\quad |h(p') - h(p)| \leq c|p'-p|^{\delta}\|\mathbf{f}\|_{Y^{s_1}}\|\mathbf{g}\|_{Y^{s_2}}, \quad \text{with} \quad \delta = \min(\delta^1, \delta^2).$$

Since $\alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j - 1$ we have

$$(45)\quad f^n(p)\overline{g^n(p)} = (p-p_n^j)^{N_n^j-1}h(p).$$

Using the variable changes $p = P_n^{j-1}(\mu)$ and $p = P_n^j(\mu)$, and taking (36) and (37) into account, we get

$$\mathbf{b}_1^j(z) = \int_{I_n^j} \frac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{(\mu-z)|G_{n,2}^j(P_n^j(\mu))|}\,d\mu - (-1)^{N_n^j} \int_{I_n^{j-1}} \frac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{(\mu-z)|G_{n,2}^j(P_n^{j-1}(\mu))|}\,d\mu.$$

Assume that $\mu_n^{j+1} > \min(\mu_n^j, \mu_n^{j-1})$ and define the function $\mathcal{H}(\cdot)$ as follows:

- if N_n^j is even (here we suppose in addition that $\mu_n^j \leq \mu_n^{j-1}$):

$$\mathcal{H}(\mu) = \begin{cases} \dfrac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|} - \dfrac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{|G_{n,2}^j(P_n^{j-1}(\mu))|}, & \text{if } \mu \in [\mu_n^j, \mu_n^{j-1}], \\ \dfrac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|}, & \text{if } \mu \in [\mu_n^{j-1}, \mu_n^{j+1}], \\ 0 & \text{otherwise,} \end{cases}$$

- if N_n^j is odd:

$$\mathcal{H}(\mu) = \begin{cases} \dfrac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{|G_{n,2}^j(P_n^{j-1}(\mu))|}, & \text{if } \mu \in [\mu_n^{j-1}, \mu_n^j], \\ \dfrac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|}, & \text{if } \mu \in [\mu_n^j, \mu_n^{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Knowing that $\mathcal{H}(\cdot)$ vanishes at μ_n^j if N_n^j is even and that it is continuous at the same point if N_n^j is odd, we verify, using (44) and (38)–(39) that:

$$\exists c > 0, \quad |\mathcal{H}(\mu') - \mathcal{H}(\mu)| \le c\|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}} |\mu' - \mu|^{\delta/N_n^j}, \quad \forall \mu', \mu \in \mathbb{R}.$$

The desired estimate for $\mathbf{b}_1^j(z)$ follows by invoking Lemma 3.2 since $\mathbf{b}_1^j(z) = \int_{\mathbb{R}} \mathcal{H}(\mu)/(\mu - z)\, d\mu$. Formula (41)–(42) is a consequence of (27) since $\mathcal{H}(\mu_n^j) = 0$ if N_n^j is even. □

Other variant of the LAP at thresholds: let α^1, α^2 such that

(46) $$\alpha_{n,j}^1, \alpha_{n,j}^2 \ge 1, \quad \alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j, \quad \forall (n, j) \in \mathcal{J}^{\mu_0}.$$

For $s_1 > |\alpha^1| + 1/2$ and $s_2 > |\alpha^2| + 1/2$ the previous theorem gives a LAP with $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$. One can in fact prove a similar result only with:

(47) $$s_1 > |\alpha^1| - 1/2, \quad s_2 > |\alpha^2| - 1/2 \quad \text{and} \quad s_1 + s_2 > |\alpha^1| + |\alpha^2|.$$

This will be used below in the case where the function μ_n is symmetric, and applied later to obtain a LAP at some thresholds for the operators associated with a half-strip (see Theorem 4.1). The conditions on s_1 and s_2 are thus improved compared to applying Theorem 3.5.

The difference from the proof of Theorem 3.5 lies in the treatment of $\mathbf{b}_1^j(z)$ (cf. (43)). Indeed, for $\mathbf{f} \in NY^{s_1}(\mu_0, \alpha^1)$ and $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$ we have, according to Lemma 3.4, $f^n(p) = (p - p_n^j)^{\alpha_{n,j}^1 - 1} h_1(p)$ and $\overline{g^n(p)} = (p - p_n^j)^{\alpha_{n,j}^2 - 1} h_2(p)$ where, h_l, $l = 1, 2$, are locally Hölder continuous. Given (46) and (47) there exist r_1 and r_2 such that $0 < r_1 < s_1 - |\alpha^1| + 1/2$, $0 < r_2 < s_2 - |\alpha^2| + 1/2$ and $r_1 + r_2 = 1$. Verifying that

$h_l(p_n^j) = 0$, we check that $h_1(p)/|p - p^j|^{r_1}$ and $h_2(p)/|p - p_n^j|^{r_2}$ are locally δ_1 and δ_2 Hölder continuous, respectively, for $\delta_1 < s_1 - |\alpha^1| + 1/2 - r_1$ and $\delta_2 < s_2 - |\alpha^2| + 1/2 - r_2$. The function h defined below is then δ-Hölder continuous with $\delta = \min(\delta_1, \delta_2)$.

$$h(p) := \pm \frac{h_1(p)}{|p - p_n^j|^{r_1}} \frac{h_2(p)}{|p - p_n^j|^{r_2}}, \quad \text{if} \quad \pm(p - p_n^j) > 0.$$

Since $\alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j$ and $r_1 + r_2 = 1$ we have, as in (45), $f^n(p)\overline{g^n(p)} = (p - p_n^j)^{N_n^j - 1} h(p)$. We then proceed as above by using the variable changes $p = P_n^{j-1}(\mu)$ and $p = P_n^j(\mu)$, and applying Lemma 3.2. Hence $z \longrightarrow R_M^{\pm}(z)$ is locally $(\delta/|\mathbf{N}_{\mu_0}|)$-Hölder on $\{\mu_0\} \cup (\overline{\mathbb{C}^{\pm}} \setminus \Gamma)$.

Considering the upper-bound over the set of (r_1, r_2) one verifies that $\delta > 0$ could be any real such that

$$\delta < \min(1, s_1 - |\alpha^1| + 1/2, s_2 - |\alpha^2| + 1/2, (s_1 + s_2 - |\alpha^1| - |\alpha^2|)/2).$$

Optimality: let s_1 and s_2 be sufficiently large. The previous results concerning the LAP at thresholds require the condition $\alpha_j^1 + \alpha_j^2 \geq N^j - 1$ for all $(j, n) \in \mathcal{J}^{\mu_0}$. One can prove its optimality (in some cases). More precisely, if $\alpha_{n,j}^1 + \alpha_{n,j}^2 < N_n^j - 1$, and if $\alpha_{n,j}^1 + \alpha_{n,j}^2$ is even or $\alpha_{n,j}^1 + \alpha_{n,j}^2$ and N_n^j are odd then we do not have a LAP with $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$. Indeed, for all $s_1 > N_n^j - 3/2$ and $\mathbf{f} \in Y^{s_1}$ such that $f^n \in NY_n^{s_1}(j, \alpha_{n,j}^1) \setminus NY_n^{s_1}(j, \alpha_{n,j}^1 + 1)$ we prove (cf. [3]) that there exists $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$, with $s_2 > \alpha_{n,j}^2 + 1/2$, such that

$$\lim_{\varepsilon \to 0} |\langle R_D(\mu_0 \pm i\varepsilon) U^{-1}\mathbf{f}, U^{-1}\mathbf{g}\rangle| = \lim_{\varepsilon \to 0} |\langle R_M(\mu_0 \pm i\varepsilon)\mathbf{f}, \mathbf{g}\rangle| = +\infty.$$

3.3. Symmetry property and the limiting absorption principle at thresholds

Let $\mu_0 = \mu_{n_0}(0)$ be a threshold such that: (1) μ_{n_0} is a symmetric function, (2) $\mu_0 = \mu_n(p_0)$ with $\mu_n'(p_0) = 0$ implies $n = n_0$ and $p_0 = 0$, (3) $\mu_{n_0}''(0) \neq 0$, that is to say $N_n^0 = 2$. Denote

(48) $$Y_{\mu_0,\mathrm{AS}}^s := \{\mathbf{f} \in Y^s; f^{n_0}(-p) = -f^{n_0}(p)\} \quad \text{and} \quad Z_{\mu_0,\mathrm{AS}}^s = U^{-1} Y_{\mu_0,\mathrm{AS}}^s.$$

According to Theorem 3.5 there is a LAP at μ_0 with $\mathcal{B}(NY_{n_0}^{s'}(0,1), Y^{-s})$ and $\mathcal{B}(Y^s, NY_{n_0}^{-s'}(0,1))$, where $s > 1/2$, $s' > 3/2$. We also remarked that a LAP holds with:

(49) $$\mathcal{B}(NY_{n_0}^{s_1}(0,1), NY_{n_0}^{-s_2}(0,1)) \text{ where } s_1, s_2 > 1/2 \text{ and } s_1 + s_2 > 2.$$

Taking the symmetry property into account the following theorem improves this result.

Theorem 3.6. *Let $s_1, s_2 > 1/2$ with $s_1 + s_2 > 2$. The mapping $z \longrightarrow R_M(z) \in \mathcal{B}(Y_{\mu_0,\mathrm{AS}}^{s_1}, Y^{-s_2})$, defined on $\mathbb{C}^{\pm}$, can be extended continuously to $\{\mu_0\} \cup (\overline{\mathbb{C}^{\pm}} \setminus \Gamma)$. This* *extension is locally* $(\delta/2)$*-Hölder continuous,* $\delta < \min(1, s_1 - 1/2, s_2 - 1/2, (s_1 + s_2 - 2)/2)$.

The precedent assertion holds when we replace, respectively, M *by* D *and* Y *by* Z.

Proof. The key idea is the construction of a linear continuous mapping $Q : Y^s \longrightarrow NY^s_{n_0}(0, 1)$ such that $\mathrm{Ran}(I_d - Q)$ is a subset of $(Y^0_{\mu_0,\mathrm{AS}})^{\perp}$ which is the orthogonal subspace to $Y^0_{\mu_0,\mathrm{AS}}$ in the Hilbert space Y^0.

Take a symmetric function $\phi \in C_0^{\infty}(\mathbb{R})$ such that $\phi(0) = 1$. Thanks to **(H2)**, $(\phi\delta^k_{n_0})_{k\geq 1}$ belongs to Y^s for all $s \geq 0$, and a suitable mapping Q is given by

$$Q\mathbf{f} = \mathbf{f} - f^{n_0}(0)(\phi\delta^k_{n_0})_{k\geq 1}, \quad \forall \mathbf{f} \in Y^s.$$

For all $\mathbf{f} \in Y^{s_1}_{\mu_0,\mathrm{AS}}$, $\mathbf{g} \in Y^{s_2}$ and $z \in \mathbb{C}^{\pm}$, we have $R_M(z)\mathbf{f} \in Y^0_{\mu_0,\mathrm{AS}}$ and $\mathbf{g} - Q\mathbf{g} \in (Y^0_{\mu_0,\mathrm{AS}})^{\perp}$, therefore

$$\langle R_M(z)\mathbf{f}, \mathbf{g}\rangle = \langle R_M(z)\mathbf{f}, Q\mathbf{g}\rangle.$$

Since $\mathbf{f}$ and $Q\mathbf{g}$ belong to $NY^{s_1}_{n_0}(0, 1)$ and $NY^{s_2}_{n_0}(0, 1)$, respectively, it suffices to use the LAP with (49). □

3.4. Division theorem We have the following division theorem outside thresholds. One can prove a similar result at thresholds.

Theorem 3.7. *Let* $\mu_0 \in \mathbb{R}\backslash\Gamma$ *and* $s > 1/2$. *Set* $\tilde{s} = \max(0, 1-s)$. *If* $\mathbf{f} \in Y^s$ *verifies one of the following three equivalent conditions*: (1) *for all* $n \geq 1$ *and* p_0 *such that* $\mu_n(p_0) = \mu_0$, *we have* $f^n(p_0) = 0$, (2) $R^+_M(\mu_0)\mathbf{f} = R^-_M(\mu_0)\mathbf{f}$, (3) $\mathrm{Im}\langle R^{\pm}_M(\mu_0)\mathbf{f}, \mathbf{f}\rangle = 0$, *then* $R^{\pm}_M(\mu_0)\mathbf{f} \in Y^{-\tilde{s}}$. *In particular,* $R^{\pm}_M(\mu_0)\mathbf{f} \in Y^0$ *if* $s \geq 1$.

Set $\mathbf{u} = U^{-1}\mathbf{f}$. *We can replace the last two conditions by* $R^+_D(\mu_0)\mathbf{u} = R^-_D(\mu_0)\mathbf{u}$ *and* $\mathrm{Im}\langle R^{\pm}_D(\mu_0)\mathbf{u}, \mathbf{u}\rangle = 0$, *respectively. In this case* $R^{\pm}_D(\mu_0)\mathbf{u} \in Z^{-\tilde{s}}$.

Proof. We verify, using (28)–(29), that the three conditions of the theorem are equivalent and that in such a case

$$\langle R^{\pm}_D(\mu_0)\mathbf{u}, \mathbf{v}\rangle_{Z^{-s}\times Z^s} = \langle R^{\pm}_M(\mu_0)\mathbf{f}, \mathbf{g}\rangle_{Y^{-s}\times Y^s} = \sum_{n\geq 1}\int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - \mu_0}\, dp,$$

where $\mathbf{v} = U^{-1}\mathbf{g}$. To prove the theorem it suffices to show that

$$|\langle R^{\pm}_M(\mu_0)\mathbf{f}, \mathbf{g}\rangle_{Y^{-s}\times Y^s}| \leq c\|\mathbf{g}\|_{Y^{\tilde{s}}}, \quad \forall \mathbf{g} \in Y^s \subset Y^{\tilde{s}}.$$

Let us go back to the proof of Theorem 3.3 (where K is now $\{\mu_0\}$, and $\phi_1 = \psi^2$

with $\psi \in C_0^\infty((p_n^j, p_n^{j+1}))$ and remark that it suffices to prove

$$|b_1^j(\mu_0)| \le c_0' \|\mathbf{g}\|_{Y^{\tilde{s}}}. \tag{50}$$

In fact, similar results obviously hold for $b_0(\mu_0)$ and $\mathbf{b}_0(\mu_0)$. Recall (cf. (34)) that

$$b_1^j(\mu_0) = \int_{p_n^j}^{p_n^{j+1}} \frac{\phi_1(p) f^n(p) \overline{g^n(p)}}{\mu_n(p) - \mu_0} dp = \int_{\mathbb{R}} \frac{\psi(p) f^n(p)}{p - p_0} \frac{(p - p_0)\psi(p)}{\mu_n(p) - \mu_0} \overline{g^n(p)}\, dp. \tag{51}$$

Denote p_0 the unique root of $p \longrightarrow \mu_n(p) - \mu_0$ in (p_n^j, p_n^{j+1}). The function ψf^n vanishes at p_0 and belongs to $H^s(\mathbb{R})$ by (**H2**). According to Lemma 3.4 (in which we consider the case $k = 0$) we have $\psi(p) f^n(p) = (p - p_0)\psi(p) v(p)$, where ψv belongs to $H^{s-1}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R})$ and there exists a constant c_1 such that

$$\left| \int_{\mathbb{R}} \psi(p) v(p) \overline{\phi(p)}\, dp \right| \le c_1' \|\phi\|_{H^{1-s}(\mathbb{R})} \le c_1 \|\phi\|_{H^{\tilde{s}}(\mathbb{R})}, \quad \forall \phi \in C_0^\infty(\mathbb{R}). \tag{52}$$

This inequality is still valid, using the density of $C_0^\infty(\mathbb{R})$ in $H^s(\mathbb{R})$, for all $\phi \in H^s(\mathbb{R})$. On the other hand, since p_0 is a simple root of $p \longrightarrow \mu_n(p) - \mu_0$ (because $\mu_0 \notin \Gamma$) we have

$$\frac{(p - p_0)\psi(p)}{\mu_n(p) - \mu_0} \in C_0^\infty(\mathbb{R}),$$

thus

$$\phi(p) := \frac{(p - p_0)\psi(p)}{\mu_n(p) - \mu_0} \overline{g^n(p)} \in H^s(\mathbb{R}) \subset H^{\tilde{s}}(\mathbb{R}),$$

and, in view of (**H2**), $\|\phi\|_{H^{\tilde{s}}(\mathbb{R})} \le c_1'' \|\mathbf{g}\|_{Y^{\tilde{s}}(\mathbb{R})}$. Combining this inequality with (52) we obtain (50), with $c_0' = c_1 c_1''$. □

4. Application to the elastic strip and generalization to other differential operators

4.1. The elastic strip case Let us go back to the elastic strip. Now $(\mu_n) = (\lambda_n)$ and $Y^s := F(H_s)$, $s \ge 0$. In view of Theorems 2.2 and 2.3 hypotheses (**H1**) and (**H2**) are satisfied and we are then within the framework of Section 3. More precisely, the conditions of application of this section are satisfied with $D = A$, $U = F$ and $Z^s = H_s$.

Thus A has an absolutely continuous spectrum (in particular A has no eigenvalues), and there is a LAP outside thresholds (cf. Theorem 3.3), and at thresholds in appropriate spaces (cf. Theorem 3.5), as well as a division theorem (cf. Theorem 3.7).

4.2. Validity of a LAP for $A_{\Omega_k}^{\text{SA}}$ and $A_{\Omega_k}^{\text{AS}}$ at some thresholds The simple eigenvalues $\lambda_n(0)$ belong to Γ (cf. (11)) since, in view of Theorem 2.5, λ_n is symmetric so that $\lambda_n'(0) = 0$. According to $(\mathcal{P})$ the order $N_n^0 \ge 2$ of the root $p = 0$ of

$p \longrightarrow \lambda_n(p) - \lambda_n(0)$ may be such that $N_n^0 > 2$. In the following definitions of the disjoint subsets Γ^{SA} and Γ^{AS} of Γ we only consider the case $N_n^0 = 2$.

$$\Gamma^{\mathrm{SA}}(\text{resp. } \Gamma^{\mathrm{AS}}) = \big\{\lambda_0 = \lambda_n(0);\ \lambda_0 \text{ is an eigenvalue of } A_0^{\mathrm{AS}}(\text{resp. } A_0^{\mathrm{SA}}),\ N_n^0 = 2,$$
$$\text{and there are no } p \neq 0 \text{ and } n' \geq 1 \text{ such that } \lambda_0 = \lambda_{n'}(p) \text{ and } \lambda'_{n'}(p) = 0\big\}.$$

According to Theorem 2.5, if $\lambda_0 \in \Gamma^{\mathrm{SA}}$ (resp. Γ^{AS}) we have $H_s^{\mathrm{SA}} \subset F^{-1}Y^s_{\lambda_0,\mathrm{AS}}$ (cf. (48)) (resp. $H_s^{\mathrm{AS}} \subset F^{-1}Y^s_{\lambda_0,\mathrm{AS}}$). Applying Theorem 3.6 we deduce that $z \longrightarrow R_A(z) \in \mathcal{B}(H_{s_1}^{\mathrm{SA}}, H_{-s_2})$ (resp. $\mathcal{B}(H_{s_1}^{\mathrm{AS}}, H_{-s_2})$) can be extended continuously to $\Gamma^{\mathrm{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$ (resp. $\Gamma^{\mathrm{AS}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$). This extension is locally $(\delta/2)$-Hölder continuous where

$$\delta < \min(1, s_1 - 1/2, s_2 - 1/2, (s_1 + s_2 - 2)/2). \tag{53}$$

The following theorem gives a LAP for $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$ (cf. (20) and (21)). Denote $H_{\Omega_k,s}$ the weighted space equipped with the norm $\|\mathbf{u}\|_{H_{\Omega_k,s}} := \|(1 + x_1^2)^{s/2}\mathbf{u}\|_{H_{\Omega_k}}$. One can verify that the transformation $\mathcal{U}_k^{\mathrm{SA}}$ (resp. $\mathcal{U}_k^{\mathrm{AS}}$), used in (22), is also unitary from $H_{\Omega_k,s}$ onto H_s^{SA} (resp. H_s^{AS}).

Theorem 4.1. *The operator $A_{\Omega_k}^{\mathrm{SA}}$ has an absolutely continuous spectrum (in particular it has no eigenvalues). Let s_1, $s_2 > 1/2$. The mapping $z \longrightarrow R_{A_{\Omega_k}^{\mathrm{SA}}}(z) \in \mathcal{B}(H_{\Omega_k,s_1}, H_{\Omega_k,-s_2})$ can be continuously extended to $\overline{\mathbb{C}^\pm} \setminus \Gamma$ (resp. $\Gamma^{\mathrm{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$ if in addition $s_1 + s_2 > 2$). This extension is locally δ-Hölder continuous, $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$, (resp. $(\delta/2)$-Hölder continuous with (53)).*

The precedent assertion holds when we replace $A_{\Omega_k}^{\mathrm{SA}}$ by $A_{\Omega_k}^{\mathrm{AS}}$ and Γ^{SA} by Γ^{AS}.

Proof. Since A is the direct sum of A^{SA} and A^{AS} the latter have, as A, an absolutely continuous spectrum. Therefore, this also holds for $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$ since they are, respectively, unitarily equivalent to the latter (cf. (22)).

We will restrict ourselves to proving a LAP for $A_{\Omega_k}^{\mathrm{SA}}$ on $\Gamma^{\mathrm{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$. From (22) follows that $R_{A_{\Omega_k}^{\mathrm{SA}}}(z) = (\mathcal{U}_k^{\mathrm{SA}})^{-1} R_{A^{\mathrm{SA}}}(z)\mathcal{U}_k^{\mathrm{SA}}$. Therefore, and this will end the proof, for any $\mathbf{u} \in H_{\Omega_k,s_1}$

$$\|R_{A_{\Omega_k}^{\mathrm{SA}}}(z')\mathbf{u} - R_{A_{\Omega_k}^{\mathrm{SA}}}(z)\mathbf{u}\|_{H_{\Omega_k,-s_2}} = \|R_{A^{\mathrm{SA}}}(z')\mathcal{U}_k^{\mathrm{SA}}\mathbf{u} - R_{A^{\mathrm{SA}}}(z)\mathcal{U}_k^{\mathrm{SA}}\mathbf{u}\|_{H_{-s_2}}$$
$$\leq \|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}^{\mathrm{SA}}, H_{-s_2})}\|\mathcal{U}_k^{\mathrm{SA}}\mathbf{u}\|_{H_{s_1}} = \|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}^{\mathrm{SA}}, H_{-s_2})}\|\mathbf{u}\|_{H_{\Omega_k,s_1}}. \quad \square$$

4.3. Non validity of a LAP for $A_{\Omega_k}^{\mathrm{SA}}$ and $A_{\Omega_k}^{\mathrm{AS}}$ at all thresholds In the acoustic case the set of thresholds is $\{\lambda_{n,B}(0);\ n \geq 1\}$, and for each n the function $\lambda_{n,B}$ is symmetric and $\lambda''_{n,B} \neq 0$. Then a similar result to the previous theorem gives a LAP at all thresholds for the operators associated with a half-strip. The difference with the elastic case is that Theorem 4.1 does not provide such a result. The reason is that, due to $(\mathcal{P})$, $\Gamma^{\mathrm{SA}} \cup \Gamma^{\mathrm{AS}}$ does not necessarily coincide with Γ.

Theorem 4.2. *Assume that the eigenvalue $\lambda_0 = \lambda_n(0)$ is simple and that $\lambda_n'(0) = \lambda_n''(0) = \lambda_n'''(0) = 0$. The limits of $R_{A^{\mathrm{SA}}_{\Omega_k}}(z)$ and $R_{A^{\mathrm{AS}}_{\Omega_k}}(z)$, as $z \to \lambda_0$ with $\pm \operatorname{Im} z > 0$, do not exist for the topology of $\mathcal{B}(H_{\Omega_k,s_1}, H_{\Omega_k,-s_2})$, where s_1 and s_2 are sufficiently large.*

Proof. We will prove the non-existence of the first limit and suppose that λ_0 is an eigenvalue of A_0^{AS}. The other cases can be treated in a similar fashion.

Let $\phi \in C_0^\infty(\mathbb{R})$ be an anti-symmetric function with a small enough support such that $\phi'(0) \neq 0$. We claim that the following function $\mathbf{u}$ belongs to H_s^{SA} with s sufficiently large.

$$\mathbf{u}(x_1, x_2) = \int_{\mathbb{R}} \phi(p) W_n(p, x_2) e^{ipx_1}\, dp.$$

Indeed, in view of Theorem 2.5 and since λ_0 is an eigenvalue of A_0^{AS} we have $S^{\mathrm{SA}} W_n(-p, \cdot) = -W_n(p, \cdot)$ for all p. Thus, using variable change $p \longrightarrow -p$ we obtain

$$\begin{aligned} S^{\mathrm{SA}}\mathbf{u}(x_1, x_2) &= \int_{\mathbb{R}} \phi(-p) S^{\mathrm{SA}} W_n(-p, x_2) e^{-ipx_1}\, dp \\ &= \int_{\mathbb{R}} \phi(p) W_n(p, x_2) e^{ip(-x_1)}\, dp = \mathbf{u}(-x_1, x_2). \end{aligned}$$

We also verify that $\mathbf{u} \in H_s$, for all s, and, using (8), that $F\mathbf{u} = (\phi \delta_n^k)_{k \geq 1}$.

Denote $\mathbf{u}' = \sqrt{2}\mathbf{u}_{|\Omega_k}$, so

$$\langle R_{A^{\mathrm{SA}}_{\Omega_k}}(z)\mathbf{u}', \mathbf{u}' \rangle_{H_{\Omega_k}} = \langle R_A(z)\mathbf{u}, \mathbf{u} \rangle_H = \int_{\mathbb{R}} \frac{|\phi(p)|^2}{\lambda_n(p) - z}\, dp,$$

and put $I^\varepsilon := \langle R_{A^{\mathrm{SA}}_{\Omega_k}}(\lambda_0 + i\varepsilon)\mathbf{u}', \mathbf{u}' \rangle_{H_{\Omega_k}}$. Let us prove that $|I^\varepsilon| \to \infty$, as $\varepsilon \to 0$, which will end the proof. Since $\lambda_n(p) - \lambda_0 = p^N G(p)$, where $N = N_n^0$, and $G(p) = G_{n,1}^j(p)$ (see (35)) we have

$$I^\varepsilon = \int_{\mathbb{R}} \frac{p^N G(p) |\phi(p)|^2}{p^{2N} G^2(p) + \varepsilon^2}\, dp + i \int_{\mathbb{R}} \frac{\varepsilon |\phi(p)|^2}{p^{2N} G^2(p) + \varepsilon^2}\, dp.$$

It then suffices to prove that the real part I_1^ε of I^ε goes to infinity. Since $\phi'(0) \neq 0$ and N is even and greater or equal to 4, there exist $r > 0$ and c_r such that

$$|I_1^\varepsilon| \geq c_r \int_{-r}^{r} \frac{|p|^{N+2}}{p^{2N} + \varepsilon^2}\, dp \to c_r \int_{-r}^{r} \frac{1}{|p|^{N-2}}\, dp = +\infty.$$

The convergence above is a consequence of the monotonic convergence theorem. □

4.4. Generalization to other differential operators Our objective here is not to state a general result concerning a precise class of differential operators but rather to propose a path to follow in which we use the general results of Section 3. It remains that in any particular case some steps have to be verified.

For a given differential selfadjoint operator D on a stratified strip Ω, or more generally on $\Omega' \times \mathbb{R}$ where Ω' is a regular and bounded open set in $\mathbb{R}^n$, $n \geq 1$, we propose to follow the following steps. Using the partial Fourier transform defined in (4) it is expected to obtain a family of reduced operators whose eigenvalues and eigenvectors are represented by two families of analytic real and vectorial functions denoted, respectively, $\lambda_{n,D}$ and $W_{n,D}$, $n \geq 1$. We have to verify that D is unitarily equivalent to the multiplication operator in $\bigoplus^{\infty} L^2(\mathbb{R})$ by $(\lambda_{n,D})_{n\geq 1}$, via the unitary transformation defined by (8) where W_n have been replaced by $W_{n,D}$. The proof of Theorem 2.3 is still valid if we replace W_n by $W_{n,D}$ so that (**H2**) is satisfied. It only remains to verify (**H1**). The main difficulty is to prove $\inf_{p\in\mathbb{R}} |\lambda_{n,D}(p)| \to +\infty$, as $n \to \infty$. One can follow the proof of Theorem 2.2 and the idea of utilizing (10).

It is easy to verify this procedure when $D = B$ and recover the well-known results of the acoustic case obtained in [4] and [5].

For some differential operators allowing separation of variables, such as B, we have $\lambda_{n,D}(p) = \lambda(p) + c_n$, where $c_n \to +\infty$, as $n \to +\infty$ and $\lambda(p) \to +\infty$, as $p \to \infty$; or $\lambda_{n,D}(p) \geq \lambda_{n,D}(0)$ with $\lambda_{n,D}(0) \to +\infty$, as $n \to +\infty$. In both cases (**H1**) is satisfied.

References

[1] S. Agmon: *Spectral properties of schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **IV(2)** (1975), 151–218.

[2] M. Ben-Artzi: *On spectral properties of the acoustic propagator in a layered band*, J. Diff. Equ. **136** (1997), 115–135.

[3] T. Bouhennache: Étude D'une Bande Élastique Multistratifée: Étude Spectrale et Principe D'absorption Limite, Thèse de doctorat, Université de Provence, décembre 1997.

[4] E. Croc and Y. Dermenjian: *Analyse spectrale d'une bande acoustic multistratifiée: Principe d'absorption limite pour une stratification simple*, SIAM J. Math. Anal. **26(4)** July (1995), 880–924.

[5] E. Croc and Y. Dermenjian: *Asymptotic behaviour of acoustic waves in a stratified strip*, SIAM J. Math. Anal. **27(6)** November (1996), 1631–1652.

[6] R. Dautray and J.L. Lions: Analyse mathématique et calcul numérique pour les sciences et les techniques, Masson, 1985.

[7] Y. Dermenjian and J.C. Guillot: *Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle*, Math. Meth. appl. Sci. **10** (1992), 87–124.

[8] F.D. Gakhov: Boundary value problems, Pergamon Press, 1966.

[9] T. Kato: Perturbation theory for linear operators, Springer-Verlag, 1966.

[10] N.I. Muskhelishvili: Singular Integral Equations, J.R.M. Radok, ed., Noordhoff International Publishing, 1977 (English translation).

[11] J.A. Nitsche: *On korn's second inequality*, R.A.I.R.O. Analyse numérique/ Numerical Analysis, **15(3)** (1981), 237–248.

[12] S. Shimizu: *Eigenfunction expansions for elastic wave propagation problems in stratified media* $\mathbb{R}^3$, Tsukuba J. Math. **18(2)** (1994), 283–350.

[13] F. Treves: Basic linear partial differential equations, Academic press, 1975.

[14] J. Weidmann: Linear Operators in Hilbert Spaces, Springer-Verlag, 1980.

C.M.I, 39 rue Joliot Curie
13453 Marseille Cedex 13
France