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INVARIANT DIFFERENTIAL OPERATORS ON THE GRASSMANN MANIFOLD $G_{2,n-1}(\mathbf{C})$

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0. Introduction. The present paper is the latter one of twin papers on invariant linear differential operators of Grassmann manifolds. In the former one [9] we determined and clarified the structure of the algebra $D(SG_{2,n-1}(\mathbf{R}))$ of invariant linear differential operators on the Grassmann manifold $SG_{2,n-1}(\mathbf{R})$ of oriented 2-planes in \mathbf{R}^{n+1} by exhibiting a set of generators with their simultaneous eigenspace decompositions.

The complex Grassmann manifold $G_{2,n-1}(\mathbf{C})$ defined as the totality of complex 2-planes passing through the origin of \mathbf{C}^{n+1} , is known to be a symmetric space of rank 2. Hence, the algebra $D(G_{2,n-1}(\mathbf{C}))$ of invariant linear differential operators acting on $C^\infty(G_{2,n-1}(\mathbf{C}), \mathbf{R})$ is generated by two differential operators, where $C^\infty(M, \mathbf{K})$ denotes the algebra of \mathbf{K} -valued C^∞ -functions defined on a complex manifold M and \mathbf{K} denotes either the real number field \mathbf{R} or the complex number field \mathbf{C} .

The aim of the present paper lies, as in [9], in exhibiting a simultaneous eigenspace decomposition of an explicit set of generators Δ_0^\wedge and Δ_1^\wedge of the algebra $D(G_{2,n-1}(\mathbf{C}))$.

Define

$$\begin{aligned} \mathbf{S}^p(\mathbf{P}_n(\mathbf{C})) &:= \sum_{k+l=p} \mathbf{S}^{k,l}(\mathbf{P}_n(\mathbf{C})) \quad (\text{direct sum}), \\ \mathbf{S}^*(\mathbf{P}_n(\mathbf{C})) &:= \sum_{p \geq 0} \mathbf{S}^p(\mathbf{P}_n(\mathbf{C})) \quad (\text{direct sum}), \\ \mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C})) &:= \sum_{k,l \geq 0} \mathbf{S}^{k,l}(\mathbf{P}_n(\mathbf{C})) \quad (\text{direct sum}), \end{aligned}$$

where $\mathbf{S}^{k,l}(\mathbf{P}_n(\mathbf{C}))$ is the $C^\infty(\mathbf{P}_n(\mathbf{C}), \mathbf{C})$ -module of complex (contravariant) symmetric tensor fields of bidegree (k, l) on the complex projective space $\mathbf{P}_n(\mathbf{C})$. $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$ is a bigraded algebra over $C^\infty(\mathbf{P}_n(\mathbf{C}), \mathbf{C})$. We obtained in [8] the following about the complex projective space $(\mathbf{P}_n(\mathbf{C}), g_0)$ with prescribed standard Riemannian metric g_0 :

(1) The eigenspace decomposition of Δ_0 restricted to $\mathbf{K}^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ is given, Where Δ_0 is the Lichnerowicz operator acting on $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$ and $\mathbf{K}^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ is the bigraded \mathbf{C} -subalgebra of $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$ defined as

$$\mathbf{K}^{**}(\mathbf{P}_n(\mathbf{C}), g_0) = \sum_{k, l \geq 0} \mathbf{K}^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0),$$

where $\mathbf{K}^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) = \mathbf{S}^{k, l}(\mathbf{P}_n(\mathbf{C})) \cap \mathbf{K}^p(\mathbf{P}_n(\mathbf{C}), g_0)$ for $p = k + l$ and $\mathbf{K}^p(\mathbf{P}_n(\mathbf{C}), g_0)$ is the \mathbf{C} -submodule in $\mathbf{S}^p(\mathbf{P}_n(\mathbf{C}))$ linearly generated by the totality of p -th symmetric tensor products of Killing vector fields on $(\mathbf{P}_n(\mathbf{C}), g_0)$.

(2) Denote by Δ_0^\wedge the Laplace-Beltrami operator on $(\mathbf{G}_{2, n-1}(\mathbf{C}), g_1)$, where g_1 is the standard metric on $\mathbf{G}_{2, n-1}(\mathbf{C})$. Then Δ_0^\wedge is related to the Lichnerowicz operator Δ_0 through the Radon transform

$$\wedge : \mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C})) \rightarrow C^\infty(\mathbf{G}_{2, n-1}(\mathbf{C}))$$

by the formula:

$$(\Delta_0 \xi)^\wedge = \Delta_0^\wedge \xi^\wedge$$

for $\xi \in \mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$.

(3) The eigenspace decomposition in (1) is transferred to that of Δ_0^\wedge by means of the Radon transform.

In the present paper a new differential operator Δ_1 on $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$ with properties analogous to (1), (2) and (3) above is constructed. Especially, it is shown that Δ_0^\wedge together with Δ_1^\wedge generates the algebra $\mathbf{D}(\mathbf{G}_{2, n-1}(\mathbf{C}))$. In the section 1 we recall the results obtained in [8] with some improvements. Δ_1 is defined at the end of the section 1. In the section 2 the eigenspace decomposition of Δ_1 restricted to $\mathbf{K}^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ is obtained. Δ_0^\wedge and Δ_1^\wedge together with their simultaneous eigenspace decomposition are studied in the section 3.

1. Fundamental operators. Let M be a complex manifold of complex dimension n . Denote by $\mathbf{E}^p(M)$ the $\mathbf{C}^\infty(M, \mathbf{C})$ -module of complex linear differential operators of order at most p . Put

$$\mathbf{E}^*(M) := \bigcup_{p \geq 0} \mathbf{E}^p(M).$$

$\mathbf{E}^*(M)$ will be abbreviated as $\mathbf{E}(M)$.

Let $\mathbf{S}^{k, l}(M)$ be the $\mathbf{C}^\infty(M, \mathbf{C})$ -module of complex symmetric tensor fields of bidegree (k, l) on M .

Define

$$\begin{aligned} \mathbf{S}^p(M) &:= \sum_{k+l=p} \mathbf{S}^{k, l}(M) \quad (\text{direct sum}), \\ \mathbf{S}^*(M) &:= \sum_{p \geq 0} \mathbf{S}^p(M) \quad (\text{direct sum}), \\ \mathbf{S}^{**}(M) &:= \sum_{k, l \geq 0} \mathbf{S}^{k, l}(M) \quad (\text{direct sum}). \end{aligned}$$

$\mathbf{S}^{**}(M)$ is a bigraded $\mathbf{C}^\infty(M, \mathbf{C})$ -algebra.

Denote the symbol operator of degree p by

$$\sigma^p : \mathbf{E}^p(M) \ni D \mapsto \sigma^p(D) \in \mathbf{S}^p(M),$$

where $\sigma^p(D)$ is the symbol tensor field of D .

Let

$$\iota^p: \mathbf{E}^{p-1}(M) \rightarrow \mathbf{E}^p(M)$$

be the canonical injection. Then we obtain a short exact sequence of $C^\infty(M, \mathbf{C})$ -modules:

$$0 \rightarrow \mathbf{E}^{p-1}(M) \xrightarrow{\iota^p} \mathbf{E}^p(M) \xrightarrow{\sigma^p} \mathbf{S}^p(M) \rightarrow 0.$$

Put

$$\mathbf{L}^*(M) := \bigcup_{q \in \mathbb{Z}} \mathbf{L}^q(M),$$

where we set

$$\mathbf{L}^q(M) := \begin{cases} \mathbf{E}^{q+1}(M) & \text{for } q \geq -1, \\ \{0\} & \text{for } q \leq -2. \end{cases}$$

$\mathbf{L}^*(M)$ is not only a filtered associative algebra over \mathbf{C} with respect to the product of operators, it is a filtered Lie algebra over \mathbf{C} (cf. [8]) for the bracket product $[D_1, D_2] := D_1 D_2 - D_2 D_1$. In fact we have

$$[\mathbf{E}^p(M), \mathbf{E}^q(M)] \subset \mathbf{E}^{p+q-1}(M).$$

$\mathbf{S}^*(M)$ is canonically \mathbf{C} -isomorphic to the associated graded Lie algebra $\mathbf{Gr}(\mathbf{L}^*(M))$:

$$\begin{aligned} \mathbf{S}^*(M) &\cong \sum_{q \geq 0} \mathbf{E}^{q-1}(M)/\mathbf{E}^{q-2}(M) \text{ (direct sum)} \\ &= \sum_{q \in \mathbb{Z}} \mathbf{L}^q(M)/\mathbf{L}^{q-1}(M) \text{ (direct sum)} = \mathbf{Gr}(\mathbf{L}^*(M)), \end{aligned}$$

as $\mathbf{S}^p(M) \cong \mathbf{E}^{p-1}(M)/\mathbf{E}^{p-2}(M)$ for $p \geq 0$. Hence, the bracket product in $\mathbf{S}^*(M)$ inherited from that of $\mathbf{L}^*(M)$ through the isomorphism $\mathbf{S}^*(M) \cong \mathbf{Gr}(\mathbf{L}^*(M))$ is given by

$$[\xi, \eta] = \sigma^{p+q-1}[D_1, D_2],$$

where $D_1 \in \mathbf{E}^p(M)$ and $D_2 \in \mathbf{E}^q(M)$ are chosen so that $\xi = \sigma^p(D_1)$ and $\eta = \sigma^q(D_2)$.

For a compact Kählerian manifold (M, g) , $\mathbf{S}^{k,l}(M)$ is equipped with a positive definite Hermitian inner product defined by

$$(1.1) \quad (\xi, \eta) = k! l! \int \langle \xi, \eta \rangle d\sigma \text{ for } \xi, \eta \in \mathbf{S}^{k,l}(M),$$

where $\langle \cdot, \cdot \rangle$ is the pointwise inner product associated with the metric g and $d\sigma$ is the canonical volume element.

Let $P = P(M, G)$ be a differentiable principal bundle on a differentiable manifold M with Lie group G as its fibre. Let $\mathbf{E}^G(P)$ be the totality of G -invariant complex linear differential operators on P . $\mathbf{E}^G(P)$ is a \mathbf{C} -subalgebra of $\mathbf{E}(P)$ if we regard $\mathbf{E}(P)$ as an algebra over \mathbf{C} .

Lemma 1.1. (cf. [5] and [7]).

$$\mathbf{E}(M) \cong \mathbf{E}^G(P)/\mathbf{J},$$

where \mathbf{J} is the two-sided ideal in $\mathbf{E}^G(P)$ generated by G -invariant vertical vector fields on P .

Applying Lemma 1.1 to the Hopf fibering

$$\varphi: S^{2n+1} \rightarrow \mathbf{P}_n(\mathbf{C})$$

with fibre S^1 , we obtain an isomorphism

$$(1.2) \quad \pi_H: \mathbf{E}^{S^1}(S^{2n+1})/\mathbf{J} \cong \mathbf{E}(\mathbf{P}_n(\mathbf{C})),$$

where \mathbf{J} is as in Lemma 1.1 the two-sided ideal in $\mathbf{E}^{S^1}(S^{2n+1})$ generated by S^1 -invariant vertical vector fields.

Lemma 1.2 ([6]). *Let M_i ($i=1, 2$) be differentiable manifolds. There are subalgebras $\tilde{\mathbf{E}}(M_i)$ ($i=1, 2$) of $\mathbf{E}(M_1 \times M_2)$ canonically isomorphic to $\mathbf{E}(M_i)$ respectively, each one of which is the centralizer of the other in $\mathbf{E}(M_1 \times M_2)$.*

Let

$$(1.3) \quad \iota: S^{2n+1} \rightarrow \mathbf{C}^{n+1} - \{0\}$$

be the canonical imbedding whose image is the unit sphere: $\{z = (z^0, z^1, \dots, z^n) \in \mathbf{C}^{n+1} - \{0\} \mid r^2 = 1\}$, where $r^2 = \sum_{a=0}^n z^a \bar{z}^a$.

$\mathbf{C}^{n+1} - \{0\}$ can be regarded as a product bundle on S^{2n+1} with \mathbf{R} as its fibre. Thus as an application of Lemma 1.2, the existence of

$$\tilde{\mathbf{E}}(S^{2n+1}) := \{D \in \mathbf{E}(\mathbf{C}^{n+1} - \{0\}) \mid [D, r^2] = 0 \text{ and } [D, \partial/\partial(r^2)] = 0\}$$

as a subalgebra of $\mathbf{E}(\mathbf{C}^{n+1} - \{0\})$ and of an isomorphism

$$\tilde{\iota}: \mathbf{E}(S^{2n+1}) \rightarrow \tilde{\mathbf{E}}(S^{2n+1})$$

is assured.

Connecting π_H in (1.2) with $\tilde{\iota}$ above, we obtain an isomorphism

$$(1.4) \quad \pi^\dagger: \mathbf{E}(\mathbf{P}_n(\mathbf{C})) \rightarrow \mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C}))/(\tau),$$

where $\mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C}))$ is a subalgebra of $\mathbf{E}(S^{2n+1})$ defined as the image of $\mathbf{E}^{S^1}(\mathbf{P}_n(\mathbf{C}))$ by the isomorphism $\tilde{\iota}$. Here (τ) is a two-sided ideal in $\mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C}))$ defined as the image of \mathbf{J} in (1.2) by $\tilde{\iota}$. (τ) is generated by the S^1 -invariant vertical vector field:

$$\tau^\dagger = \sqrt{-1}(\zeta - \bar{\zeta}) \in \mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C})),$$

where $\zeta = \sum_{a=0}^n z^a \partial/\partial z^a$ and $\bar{\zeta} = \sum_{a=0}^n \bar{z}^a \partial/\partial \bar{z}^a$ [8]. Here we have

$$\mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C})) = \bigcup_{p \geq 0} (\mathbf{E}^\dagger)^p(\mathbf{P}_n(\mathbf{C}))$$

with

$$(\mathbf{E}^\dagger)^p(\mathbf{P}_n(\mathbf{C})) = \mathbf{E}^p(\mathbf{C}^{n+1} - \{0\}) \cap \mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C})).$$

Notice that τ^\dagger is an infinitesimal generator of the S^1 -action of isometries on S^{2n+1} given by the multiplication of $z \in \mathbf{C}$ with $|z|=1$.

Put

$$\begin{aligned} (\mathbf{S}^\dagger)^p(\mathbf{P}_n(\mathbf{C})) &:= \sigma^p(\mathbf{E}^\dagger)^p(\mathbf{P}_n(\mathbf{C})), \\ (\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C})) &:= \sum_{p \geq 0} \sigma^p(\mathbf{E}^\dagger)^p(\mathbf{P}_n(\mathbf{C})) \text{ (direct sum).} \end{aligned}$$

Then we have an isomorphism

$$(1.5) \quad \pi_s^\dagger: \mathbf{S}^*(\mathbf{P}_n(\mathbf{C})) \rightarrow (\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))/(\tau)_s,$$

where $(\tau)_s$ is the two-sided ideal in $(\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$ generated by

$$\tau_s^\dagger = \sqrt{-1}(\zeta_s - \bar{\zeta}_s) \in (\mathbf{S}^\dagger)^1(\mathbf{P}_n(\mathbf{C}))$$

with $\zeta_s = \sum_{a=0}^n z^a \partial/\partial z^a$ and $\bar{\zeta}_s = \sum_{a=0}^n \bar{z}^a \partial/\partial \bar{z}^a$.

REMARK. When we regard an element $\zeta \in (\mathbf{E}^\dagger)^1(\mathbf{P}_n(\mathbf{C}))$ as an element of $(\mathbf{S}^\dagger)^1(\mathbf{P}_n(\mathbf{C}))$, we distinguish it from ζ just by putting a subscript s as ζ_s above. We need such a distinction specifically in (3) and (4) of Definition 1.3.

A representative in $\mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C}))$ of $D \in \mathbf{E}(\mathbf{P}_n(\mathbf{C}))$ under the identification (1.4) will be denoted by D^\dagger in the following. Similarly, a representative in $(\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$ of $\xi \in \mathbf{S}^*(\mathbf{P}_n(\mathbf{C}))$ will be denoted by ξ^\dagger .

Lemma 1.3. $\xi \in \mathbf{S}^*(\mathbf{C}^{n+1} - \{0\})$ belongs to $(\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$ if and only if

$$[\xi, r^2] = 0, [\xi, \zeta_s] = 0, [\xi, \bar{\zeta}_s] = 0.$$

Proof. This is obvious from the construction of $(\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$. Q.E.D.

DEFINITION 1.1. Define

$$(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C})) := \sum_{k, l \geq 0} (\mathbf{S}^\dagger)^{k+l}(\mathbf{P}_n(\mathbf{C})) \cap \mathbf{S}^{k+l}(\mathbf{C}^{n+1} - \{0\}) \text{ (direct sum).}$$

Lemma 1.4. There is a canonical isomorphism ϕ of the bigraded algebras

$$\phi: (\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C})) \rightarrow \mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C})),$$

where the map ϕ is the restriction of the inverse of the isomorphism π_s^\dagger in (1.5) to $(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$.

Proof. Both of the surjectivity and the triviality of the kernel of ϕ are proved in [8; Lemma 1.3].

Q.E.D.

From now on, every $\xi \in \mathbf{S}^{k,l}(\mathbf{C}^{n+1}-\{0\})$ with components $\xi^{a_1 \cdots a_k \bar{b}_1 \cdots \bar{b}_l} \in C^\infty(\mathbf{C}^{n+1}-\{0\})$ will be identified with a function on the cotangent bundle $T^*(\mathbf{C}^{n+1}-\{0\})$:

$$(1.6) \quad \xi = \frac{1}{k! l!} \sum \xi^{a_1 \cdots a_k \bar{b}_1 \cdots \bar{b}_l} w_{a_1} \cdots w_{a_k} \bar{w}_{b_1} \cdots \bar{w}_{b_l},$$

where w_i 's ($0 \leq i \leq n$) together with \bar{w}_j 's ($0 \leq j \leq n$) are regarded as the current coordinates in

$$T^*(\mathbf{C}^{n+1}-\{0\})_{(z, \bar{z})} = \left\{ \sum_{i=0}^n (w_i dz_i|_{(z, \bar{z})} + \bar{w}_i d\bar{z}_i|_{(z, \bar{z})}) \right\},$$

where $T^*(\mathbf{C}^{n+1}-\{0\})_{(z, \bar{z})}$ is the cotangent space at $(z, \bar{z}) \in \mathbf{C}^{n+1}-\{0\}$. Namely, we regard a contravariant symmetric tensor field of bidegree (k, l) as a homogeneous polynomial of bidegree (k, l) with respect to the variables w_i 's and \bar{w}_j 's.

Denote by $\check{\mathbf{E}}(\mathbf{C}^{n+1}-\{0\})$ the set of all linear differential operators of $4(n+1)$ variables $z^0, \dots, z^n, \bar{z}^0, \dots, \bar{z}^n, w_0, \dots, w_n, \bar{w}_0, \dots, \bar{w}_n$, the coefficients of which are C^∞ with respect to the variables z^i 's and \bar{z}^j 's on $\mathbf{C}^{n+1}-\{0\}$ and are homogeneous polynomials with respect to the variables w_i 's and \bar{w}_j 's ($0 \leq i, j \leq n$). An element of $\check{\mathbf{E}}(\mathbf{C}^{n+1}-\{0\})$ can be regarded as a linear differential operator acting on $\mathbf{S}^*(\mathbf{C}^{n+1}-\{0\})$ in virtue of the identification (1.6).

We also remark that $\mathbf{E}(\mathbf{C}^{n+1}-\{0\}) \subset \check{\mathbf{E}}(\mathbf{C}^{n+1}-\{0\})$.

EXAMPLES. ζ_s and ξ_s in (1.4) and τ_s^\dagger in (1.5) are reexpressed as follows:

$$\zeta_s = \sum_{a=0}^n z^a w_a, \quad \xi_s = \sum_{a=0}^n \bar{z}^a \bar{w}_a$$

and

$$\tau_s^\dagger = \sqrt{-1} \left(\sum_{a=0}^n z^a w_a - \sum_{a=0}^n \bar{z}^a \bar{w}_a \right).$$

Lemma 1.5. (1) $\zeta \in (\mathbf{S})^{k,l}(\mathbf{C}^{n+1}-\{0\})$ belongs to $(\mathbf{S}^\dagger)^{k,l}(\mathbf{P}_n(\mathbf{C}))$ if and only if

- (i) $\sum_{a=0}^n \bar{z}_a \partial \xi / \partial w_a = 0$,
- (ii) $\sum_{a=0}^n z^a \partial \xi / \partial \bar{w}_a = 0$,
- (iii) $\sum_{a=0}^n z^a \partial \xi / \partial z^a = k\xi$,
- (iv) $\sum_{a=0}^n \bar{z}^a \partial \xi / \partial \bar{z}^a = l\xi$.

(2) If $\xi \in (\mathbf{S}^\dagger)^{k,l}(\mathbf{P}_n(\mathbf{C}))$, then we have necessarily

$$\sum_{a=0}^n w_a \partial \xi / \partial w_a = k\xi, \quad \sum_{a=0}^n \bar{w}_a \partial \xi / \partial \bar{w}_a = l\xi.$$

Proof. (1) (i)~(iv) follow from Lemma 1.3 if $\xi \in (\mathbf{S}^\dagger)^{k,l}(\mathbf{P}_n(\mathbf{C}))$. In virtue of (1.6) we can regard $\xi \in (\mathbf{S}^\dagger)^{k,l}(\mathbf{P}_n(\mathbf{C}))$ as a homogeneous function of homogeneous degree k and l with respect to the variables w and \bar{w} , respectively. The two identities in (2) follow from this fact by Euler's theorem. Q.E.D.

DEFINITION 1.2. (1) Denote by $\check{\mathbf{I}}$ the left ideal in $\check{\mathbf{E}}(\mathbf{C}^{n+1}-\{0\})$ generated

by the following four linear differential operators:

- (i) $\zeta - \sum_{a=0}^n w_a \partial/\partial w_a$,
- (ii) $\xi - \sum_{a=0}^n \bar{w}_a \partial/\partial \bar{w}_a$,
- (iii) $(1/r^2) \sum_{a=0}^n \bar{z}^a \partial/\partial w_a$,
- (iv) $(1/r^2) \sum_{a=0}^n z^a \partial/\partial \bar{w}_a$.

(2) We denote by $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$ the normalizer of $\check{\mathbf{I}}$ in $\check{\mathbf{E}}(\mathbf{C}^{n+1} - \{0\})$ viewed as a Lie algebra, i.e.,

$$\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C})) = \{D \mid [D, \check{\mathbf{I}}] \subset \check{\mathbf{I}}\}.$$

Lemma 1.6. $D \in \mathbf{E}(\mathbf{C}^{n+1} - \{0\})$ preserves $(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$ if and only if $D \in \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$.

Proof. The assertion is obtained by expressing Lemma 1.5 (1) in terms of $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$. Q.E.D.

Put

$$\check{\mathbf{I}}_0 := \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C})) \cap \check{\mathbf{I}}.$$

Then $\check{\mathbf{I}}_0$ is easily proved to be a two-sided ideal in $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$ and

$$\mathbf{EO}(\mathbf{P}_n(\mathbf{C})) := \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))/\check{\mathbf{I}}_0$$

is regarded as an algebra of linear differential operators acting on $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$.

DEFINITION 1.3. Put

- (1) $(T^*)^\dagger := 2 \sum_{a,b=0}^n (r^2 \delta_{ab} - z^a \bar{z}^b) w_a \bar{w}_a$,
- (2) $T^\dagger := (1/2r^2) \sum_{a=0}^n \partial^2/\partial w_a \partial \bar{w}_a$,
- (3) $(\partial^*)^\dagger := 2r^2 \sum_{a=0}^n w_a \partial/\partial \bar{z}^a + 2\xi_s (\zeta - \xi)$,
- (4) $(\bar{\partial}^*)^\dagger := 2r^2 \sum_{a=0}^n \bar{w}_a \partial/\partial z^a - 2\bar{\xi}_s (\zeta - \xi)$,
- (5) $\partial^\dagger := - \sum_{a=0}^n \partial^2/\partial z^a \partial w_a - \sum_{a=0}^n (\xi_s/r^2) \partial^2/\partial \bar{w}_a \partial w_a$,
- (6) $\bar{\partial}^\dagger := - \sum_{a=0}^n \partial^2/\partial \bar{z}^a \partial \bar{w}_a - \sum_{a=0}^n (\bar{\xi}_s/r^2) \partial^2/\partial w_a \partial \bar{w}_a$,
- (7) $\kappa_{a,b} := \sqrt{-1} (z^a \partial/\partial z^b - \bar{z}^b \partial/\partial z^a)$,
- (8) $\bar{\kappa}_{a,b} := \sqrt{-1} (z^a \partial/\partial \bar{z}^b - \bar{z}^b \partial/\partial \bar{z}^a + \bar{w}^a \partial/\partial \bar{w}^b - w^b \partial/\partial w^a)$,

where $0 \leq a, b \leq n$.

Lemma 1.7. (1) $(T^*)^\dagger$ is an element of $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C})) \cap (\mathbf{S}^\dagger)^2(\mathbf{P}_n(\mathbf{C}))$. $T^\dagger, (\partial^*)^\dagger, (\bar{\partial}^*)^\dagger, \partial^\dagger, \bar{\partial}^\dagger, \zeta$ and ξ are element of $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$,
(2) $\kappa_{a,b}$ is an element of $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C})) \cap \mathbf{E}^1(\mathbf{P}_n(\mathbf{C}))$,
(3) $\bar{\kappa}_{a,b}$ is an element of $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$,
where $0 \leq a, b \leq n$.

Proof. (1) These properties can be verified immediately. (2) is an immediate consequence of Lemma 1.5 (1). (3) is verified by examining the bracket products of $\bar{\kappa}_{a,b}$ with the four generators of $\check{\mathbf{I}}$, respectively. Q.E.D.

Denote by $\kappa_{a,b}^*$ and $\check{\kappa}_{a,b}^*$ the adjoint operator of $\kappa_{a,b}$ with respect to the Hermitian inner product defined on $C^\infty(\mathbf{P}_n(\mathbf{C}))$ and the adjoint operator of $\check{\kappa}_{a,b}$ with respect to the canonical Hermitian inner product defined on $(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$, respectively.

Lemma 1.8.

$$(1) \quad \kappa_{a,b}^* = -\kappa_{b,a} \text{ and } (2) \quad \check{\kappa}_{a,b}^* = -\check{\kappa}_{b,a}.$$

Proof. These follow immediately from their definitions, respectively.

Q.E.D.

Lemma 1.9. (1) τ^\dagger can be expressed as follows :

$$\tau^\dagger = \sum_{a=0}^n \kappa_{a,a}.$$

(2) Each of $\check{\kappa}_{a,b}$ ($0 \leq a, b \leq n, a \neq b$) satisfies

$$[\kappa_{a,b}, \xi^\dagger] = \check{\kappa}_{a,b}(\xi^\dagger)$$

for $\xi^\dagger \in (\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$, where the bracket in the left-hand side is the bracket product in $(\mathbf{S}^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$.

(3) Put

$$\check{\tau} := \sum_{a=0}^n \check{\kappa}_{a,a}.$$

Then

$$\check{\tau} \in \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C})) \cap \check{\mathbf{I}}.$$

Proof. This can be verified immediately.

Q.E.D.

DEFINITION 1.4. (1) Define $\Delta_0^\dagger \in (\mathbf{E}^\dagger)^2(\mathbf{P}_n(\mathbf{C}))$ by

$$\Delta_0^\dagger = \sum_{a,b=0}^n \kappa_{a,b}^* \kappa_{a,b} + \sum_{a,b=0}^n \kappa_{a,b} \kappa_{a,b}^*.$$

(2) Define $\Delta_0^\dagger \in \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$ by

$$\Delta_0^\dagger = \sum_{a,b=0}^n \check{\kappa}_{a,b}^* \check{\kappa}_{a,b} + \sum_{a,b=0}^n \check{\kappa}_{a,b} \check{\kappa}_{a,b}^*.$$

Lemma 1.10. (1) Δ_0^\dagger in Definition 1.4(1) is a representative in $\mathbf{E}^\dagger(\mathbf{P}_n(\mathbf{C}))$ modulo (τ) of the Laplace-Beltrami operator Δ_0 on $(\mathbf{P}_n(\mathbf{C}), g_0)$.

(2) Δ_0^\dagger in Definition 1.4 (2) is a representative in $\widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$ modulo $\check{\mathbf{I}}_0$ of the Lichnerowicz operator Δ_0 acting on $(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$ (cf. [8] pp. 123~129 for the definition of the Lichnerowicz operator).

Proof. (1) By a direct calculation we have

$$\begin{aligned} \sum_{a,b=0}^n \kappa_{a,b}^* \kappa_{a,b} + \sum_{a,b=0}^n \kappa_{a,b} \kappa_{a,b}^* &= -4 \sum_{a,b=0}^n (r^2 \delta^{ab} - z^a \bar{z}^b) \partial^2 / \partial z^a \partial \bar{z}^b \\ &+ 2n \sum_{a=0}^n z^a \partial / \partial z^a + 2n \sum_{b=0}^n \bar{z}^b \partial / \partial \bar{z}^b - 2(\tau^*)^2. \end{aligned}$$

This operator satisfies the following three conditions:

(i) its symbol tensor field coincides with $-g_0^\dagger$ modulo $(\tau)_s$; (ii) it is a selfadjoint linear differential operator; (iii) it annihilates constant functions. Such an operator must be a representative of the Laplace-Beltrami operator.

(2) A representative $\Delta_0^\dagger \in \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$ of the Lichnerowicz operator Δ_0 is given by (c.f. [8] Lemma 2.13)

$$[\delta^\dagger, (\delta^*)^\dagger] + 4\{(\zeta + \bar{\zeta})(n-1) + 2\zeta^2 + 2\bar{\zeta}^2 - 2\zeta\bar{\zeta}\} - 8T^*T,$$

where $(\delta^*)^\dagger := (\partial^*)^\dagger + (\bar{\partial}^*)^\dagger$ and $\delta^\dagger := \partial^\dagger + \bar{\partial}^\dagger$ are representatives of $\delta^* := \partial^* + \bar{\partial}^*$ resp. $\delta := \partial + \bar{\partial}$. (Compare with [8] pp. 137~139 for the representatives of Δ_0 , δ^* , and δ , where representatives of these operators are treated in a slightly different manner from the present paper). By direct calculations we can verify

$$\Delta_0^\dagger = \sum_{a,b=0}^n (\check{\kappa}_{a,b}^* \check{\kappa}_{a,b} + \check{\kappa}_{a,b} \check{\kappa}_{a,b}^*) \text{ modulo } \mathbf{I}_0.$$

Q.E.D.

DEFINITION 1.5. Define

(1) an endomorphism S of bidegree $(-1, -1)$ on the bigraded algebra $(S^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$ by

$$S = \Delta_0^\dagger T^\dagger - \lambda_{k,l,1} T^\dagger + 6(T^*)^\dagger (T^\dagger)^2 - \partial^\dagger T^\dagger (\partial^*)^\dagger + (\partial^*)^\dagger T^\dagger \partial^\dagger$$

on $(S^\dagger)^{kl}(\mathbf{P}_n(\mathbf{C}))$, where in general

$$\lambda_{k,l,m} = 4\{(2k-m)n + 3k^2 + l^2 - 2kl - (m+1)(k+l) + m^2 + 2m\}$$

for $k, l, m \geq 0$, $(k, l, m \in \mathbf{Z})$,

- (2) $B_m^* := 4m(m+1)(T^*)^\dagger + 2(\partial^\dagger)^*(\bar{\partial}^\dagger)^*$ for $m \geq 1$ ($m \in \mathbf{Z}$),
- (3) $A_m^* := (\prod_{i=1}^m B_i^*)(T^\dagger)^m$ ($A_0^* = \text{id.}$) for $m \geq 1$ ($m \in \mathbf{Z}$).

DEFINITION 1.6. (1) $(\mathbf{K}^\dagger)^*(\mathbf{P}_n(\mathbf{C}), g_0)$ is the graded \mathbf{C} -subalgebra of $(S^\dagger)^*(\mathbf{P}_n(\mathbf{C}))$ generated by $\kappa_{a,b}$ ($0 \leq a, b \leq n$), i.e.,

$$(\mathbf{K}^\dagger)^*(\mathbf{P}_n(\mathbf{C}), g_0) := \sum_{p=0}^{\infty} (\mathbf{K}^\dagger)^p(\mathbf{P}_n(\mathbf{C}), g_0) \text{ (direct sum),}$$

where

$$(\mathbf{K}^\dagger)^p(\mathbf{P}_n(\mathbf{C}), g_0) := (\mathbf{K}^\dagger)^*(\mathbf{P}_n(\mathbf{C}), g_0) \cap (S^\dagger)^p(\mathbf{P}_n(\mathbf{C})).$$

(2) Define

$$(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0) := \sum_{k, l \geq 0} (\mathbf{K}^\dagger)^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) \text{ (direct sum),}$$

where

$$(\mathbf{K}^\dagger)^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) := (\mathbf{K}^\dagger)^{k+l}(\mathbf{P}_n(\mathbf{C}), g_0) \cap (\mathbf{S}^\dagger)^{k, l}(\mathbf{P}_n(\mathbf{C})).$$

$(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ is a bigraded \mathbf{C} -subalgebra of $(\mathbf{S}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}))$.

Theorem 1.1. ([8] p. 136) (1) *We have*

$$(\mathbf{K}^\dagger)^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) = \{0\}$$

for $k \neq l$.

(2) $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ is generated by $\kappa_{ab, cd} \in (\mathbf{K}^\dagger)^{1, 1}(\mathbf{P}_n(\mathbf{C}), g_0)$ ($0 \leq a, b, c, d \leq n$), where

$$\begin{aligned} \kappa_{a, \bar{b}} &:= z^a \partial / \partial \bar{z}^b - z^b \partial / \partial \bar{z}^a, \\ \kappa_{\bar{c}, d} &:= \bar{z}^c \partial / \partial z^d - \bar{z}^d \partial / \partial z^c, \\ \kappa_{ab, cd} &:= \kappa_{a, d} \kappa_{b, c} - \kappa_{a, c} \kappa_{b, d} = \kappa_{a, \bar{b}} \kappa_{\bar{c}, d}. \end{aligned}$$

DEFINITION 1.7. (1) Denote by $(T^*)_0^\dagger$ the restriction of $(T^*)^\dagger$ to $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$. Notice that $(T^*)_0^\dagger$ preserves $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$.

(2) Denote the image of $(T^*)_0^\dagger$ by $\text{Im } (T^*)_0^\dagger (\subset (\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0))$ and denote the orthogonal complement of $\text{Im } (T^*)_0^\dagger$ in $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ by $\mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$.

Thus we have

$$(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0) = \text{Im } (T^*)_0^\dagger \oplus \mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0),$$

and $\mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ has a bigradation:

$$\mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0) = \sum_{k, l=0}^{\infty} \mathbf{P}^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) \text{ (direct sum),}$$

where

$$\mathbf{P}^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0) := \mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0) \cap (\mathbf{K}^\dagger)^{k, l}(\mathbf{P}_n(\mathbf{C}), g_0).$$

Lemma 1.11. ([8] p. 147, Lemma 4.2.) (1) *The endomorphism S leaves $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ invariant.*

(2) $A_k^*(k \geq 0)$ also leaves $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$ invariant.

Denote the canonical projection by

$$\Pi_0: (\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0) \rightarrow \mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}), g_0).$$

Π_0 can be proved to be commutative with Δ_0 .

Put

$$C_m^* := \Pi_0 A_m^* (m \geq 0).$$

C_m^* 's satisfy

$$\Delta_0^\dagger C_m^* - \lambda_{k,m} C_m^* + \frac{1}{(m+1)^2} C_{m+1}^* = 0 \quad (k \geq m+1 > m \geq 0)$$

on $(\mathbf{K}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0)$,
where $\lambda_{k,m} := \lambda_{k,k,m} = 4\{(2k-m)n + 2k^2 - 2(m+1)k + m^2 + 2m\}$.

DEFINITION 1.8. (1) Define an operator

$$P_{k,m} := \frac{n+2k-2m-2}{m! (n+2k-m-2)!} \sum_{i=m}^k \frac{(-1)^{i-m}(n+2k-i-m-3)!}{2^{2i}(i!)^2 (i-m)!} C_{m+1}^*$$

$: \mathbf{K}^{**}(\mathbf{P}_n(\mathbf{C})) \rightarrow \mathbf{P}^{**}(\mathbf{P}_n(\mathbf{C}))$ for $k \geq m \geq 0$.

(2) Denote the image of the map $P_{k,m}$ by $E_{k,m}$. Notice that Δ_0^\dagger preserves $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$.

Theorem 1.2. (c.f. [8]) Let k and m be as in Definition 1.8, then

- (1) $\Delta_0 P_{k,m} = \lambda_{k,m} P_{k,m}$ on $(\mathbf{K}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_n)$.
- (2) Each $E_{k,m}$ is non-trivial under the assumption $n \geq 3$.
- (3) We have direct sum decompositions :

$$(\mathbf{K}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0) = \sum_{h=0}^k ((T^*)^\dagger)^h \mathbf{P}^{k-h, k-h}(\mathbf{P}_n(\mathbf{C}), g_0)$$

and

$$\mathbf{P}^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0) = \sum_{m=0}^k E_{k,m}.$$

(1)~(3) yield the eigenspace decomposition of the restriction of Δ_0^\dagger on $(\mathbf{K}^\dagger)^{**}(\mathbf{P}_n(\mathbf{C}), g_0)$.

DENINITION 1.9. Define

- (1) $D_{abcd} := \frac{1}{8} \sum_{e,f,g,h=0}^n \delta_{ab}^{ef} g_h^* \check{\kappa}_{e,f} \check{\kappa}_{g,h} \in \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$.
- (2) $\Delta_1^\dagger := \frac{1}{4i} \sum_{a,b,c,d=0}^n (D_{abcd}^* D_{abcd} + D_{abdc}^* D_{abdc}) \subset \widetilde{\mathbf{EO}}(\mathbf{P}_n(\mathbf{C}))$.
- (3) The linear differential operator on $\mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C}))$ corresponding to Δ_1^\dagger is denoted by Δ_1 .

Theorem 1.3.

$$(1) \quad [\check{\kappa}_{a,b}, \Delta_0^\dagger] = 0. \quad (2) \quad [\check{\kappa}_{a,b}, \Delta_1^\dagger] = 0.$$

Proof. These identities follow easily from their definitions. Q.E.D.

Theorem 1.4. Δ_0^\dagger and Δ_1^\dagger commute with the operators introduced in Definition 13 as follows :

- (1) $[(T^*)^\dagger, \Delta_1^\dagger] = 0,$
- (2) $[T^\dagger, \Delta_1^\dagger] = 0,$
- (3) $[(\partial^*)^\dagger, \Delta_1^\dagger] = 0,$
- (4) $[\partial^\dagger, \Delta_1^\dagger] = 0,$
- (5) $[(\bar{\partial}^*)^\dagger, \Delta_1^\dagger] = 0,$
- (6) $[\bar{\partial}^\dagger, \Delta_1^\dagger] = 0,$

($i=0, 1$).

Proof. From Definition 1.3 we obtain these formulae immediately (cf. [8] pp. 54~55 and p. 59). Q.E.D.

2. The simultaneous eigenspace decomposition of Δ_0 and Δ_1 on $\mathbf{K}^{}(\mathbf{P}_n(\mathbf{C}), g_0)$.** In this section we assume $n+1 \geq 4$, where n is the complex dimension of $\mathbf{P}_n(\mathbf{C})$.

Theorem 2.1. Δ_1^\dagger can be expressed as follows :

$$\begin{aligned} \Delta_1^\dagger = & -2(T^*)^\dagger T^\dagger \Delta_0^\dagger + (pn+2kl-2p)\Delta_0^\dagger - 8(T^*)^\dagger (T^\dagger)^2 \\ & - 2(T^*)^\dagger (\partial^\dagger \bar{\partial}^\dagger + \bar{\partial}^\dagger \partial^\dagger) - 2((\partial^*)^\dagger (\bar{\partial}^*)^\dagger + (\bar{\partial}^*)^\dagger (\partial^*)^\dagger) T^\dagger - \\ & 2(n+2l-2)(\bar{\partial}^*)^\dagger \bar{\partial}^\dagger - 2(n+2k-2)(\bar{\partial}^*)^\dagger \bar{\partial}^\dagger + 8((p-2)n+k^2 \\ & + l^2 - 3p + 4)(T^*)^\dagger T^\dagger - 4k(n+l-2)((k-1)n+k^2 \\ & - kl - 2k + 1) - 4l(n+k-2)((l-1)n+k^2 + l^2 - kl - 2l + 1) \\ & (p=k+l) \text{ on } (\mathbf{S}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0). \end{aligned}$$

Proof. From the definition of Δ_1^\dagger in 1 and Definition 1.3 we can obtain the required relation by direct calculations. Q.E.D.

Corollary. Restricting the action of Δ_1^\dagger to $(\mathbf{S}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}))$, we obtain the reduced form of Theorem 2.1 :

$$\begin{aligned} \Delta_1 = & -2(T^*)^\dagger T^\dagger \Delta_0^\dagger + 2k(n+2k-2)\Delta_0^\dagger - 8((T^*)^\dagger)^2 (T^\dagger)^2 - \\ & 2(T^*)^\dagger (\bar{\partial}^\dagger \partial^\dagger + \partial^\dagger \bar{\partial}^\dagger) - 2((\partial^*)^\dagger (\bar{\partial}^*)^\dagger + (\bar{\partial}^*)^\dagger (\partial^*)^\dagger) T - \\ & 2(n+2k-2)((\partial^*)^\dagger \partial^\dagger - (\bar{\partial}^*)^\dagger (\bar{\partial}^\dagger)) + 16(k-1)(n+k-2)(T^*)^\dagger T^\dagger \\ & - 8k(n+k-2)(k-1)(n+k-1). \end{aligned}$$

Lemma 2.1 Δ_1^\dagger and S satisfy

$$\begin{aligned} \Delta_1^\dagger + 4(T^*)^\dagger S = & 2(k+1)(n+k-1)\Delta_1^\dagger - 8k(n+k-1)(k+1)(n+k) \\ & - 2(n+2k-1)(\partial^\dagger (\partial^*)^\dagger + (\bar{\partial}^\dagger (\bar{\partial}^*)^\dagger)) - 2((\bar{\partial}^*)^\dagger T^\dagger (\partial^*)^\dagger \\ & + (\partial^*)^\dagger T^\dagger (\bar{\partial}^*)^\dagger) \quad \text{on } (\mathbf{S}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0). \end{aligned}$$

Proof. From Definition 1.5 (1), we can express $(T^*)^\dagger S$ in terms of fundamental operators. Eliminating the first term of the right-hand side in the formula in Theorem 2.1, we obtain the required relation. Q.E.D.

Theorem 2.2. We have

$$\Delta_1^\dagger = \sum_{m=0}^k \mu_{k,m} P_{k,m}$$

as the eigenspace decomposition of Δ_1^\dagger restricted to $(\mathbf{K}^\dagger)^{k,k}(\mathbf{P}_n(\mathbf{C}), g_0)$, where $\mu_{k,m} =$

$$8(k-m)(k+1)(n+k+1)(n+k-m-2).$$

Proof. Restricting Δ_1^\dagger on $E_{k,m}$, we obtain in virtue of Lemma 2.1

$$\begin{aligned}\Delta_1^\dagger &= 2(k+1)(n+k-1) \{ \Delta_0^\dagger - 4k(n+k) \} \\ &= 2(k+1)(n+k-1) \{ 4(2k-m)n + 2k^2 - 2(m+1)k + m^2 + 2m - 4k(n+k) \},\end{aligned}$$

which coincides with the desired value $\mu_{k,m}$. Q.E.D.

3. The Radon transform and $D(G_{2n-1}(\mathbf{C}))$. In this section we also assume $n+1 \geq 4$. Denote by $W_2(\mathbf{C}^{n+1})$ the Stiefel manifold of all 2-frames in \mathbf{C}^{n+1} and denote by $V_2(\mathbf{C}^{n+1})$ the submanifold of $W_2(\mathbf{C}^{n+1})$ defined as the totality of orthonormal 2-frames with respect to the standard Hermitian metric g on \mathbf{C}^{n+1} . $V_2(\mathbf{C}^{n+1})$ is identified with a homogeneous space:

$$U(n+1)/U(n-1).$$

Denote by $G_{2,n-1}(\mathbf{C})$ the Grassmann manifold of all complex 2-planes passing through the origin of \mathbf{C}^{n+1} . As is well known, $G_{2n-1}(\mathbf{C})$ is identified with a homogeneous space

$$U(n+1)/U(n-1) \times U(2).$$

$V_2(\mathbf{C}^{n+1})$ can be regarded as a principal bundle on the complex Grassmann manifold $G_{2,n-1}(\mathbf{C})$ with structure group $U(2)$, where the projection π_V is defined canonically.

Applying Lemma 1.1 to the principal bundle

$$\pi_V: V_2(\mathbf{C}^{n+1}) \rightarrow G_{2,n-1}(\mathbf{C})$$

with $U(2)$ as its fibre, we obtain an isomorphism:

$$(3.1)_V \quad E(G_{2,n-1}(\mathbf{C})) \cong E^{U(2)}(V_2(\mathbf{C}^{n+1}))/J,$$

where J is the two-sided ideal in $E^{U(2)}(V_2(\mathbf{C}^{n+1}))$ generated by $U(2)$ -invariant vertical vector fields. On the other hand, there is a polar decomposition of the Stiefel manifold $W_2(\mathbf{C}^{n+1})$:

$$(3.2) \quad W_2(\mathbf{C}^{n+1}) \cong H_2^+ \times V_2(\mathbf{C}^{n+1}),$$

where H_2^+ is the space of positive definite 2×2 Hermitian matrices [8]. Denote by $\pi_V: W_2(\mathbf{C}^{n+1}) \rightarrow V_2(\mathbf{C}^{n+1})$ the canonical projection to the second factor of (3.2).

Put $\rho_{\alpha\beta}^2 := \langle \mathbf{q}_\alpha, \mathbf{q}_\beta \rangle$, where $0 \leq \alpha, \beta \leq 1$, $\mathbf{q} = (\mathbf{q}_0, \mathbf{q}_1) \in W_2(\mathbf{C}^{n+1})$ and $\langle \cdot, \cdot \rangle$ denotes the pointwise inner product as introduced in (1.1). The positive definite square root matrix $(\rho_{\alpha\beta})$ of $(\rho_{\alpha\beta}^2)$ is called the *radial part* of \mathbf{q} , which can

be regarded as the H_2^+ part of q in the polar decomposition (3.2). In virtue of Lemma 1.2 the polar decomposition (3.2) assures the existence of two subalgebras, each one of which is the centralizer of the other in $\mathbf{E}(\mathbf{W}_2(\mathbf{C}^{n+1}))$ and the second one of which is canonically isomorphic to $\mathbf{E}(\mathbf{V}_2(\mathbf{C}^{n+1}))$. Thus a linear differential operator $D \in \mathbf{E}(\mathbf{V}_2(\mathbf{C}^{n+1}))$ can be represented by a linear differential operator $D^{\dagger\dagger} \in \mathbf{E}(\mathbf{W}_2(\mathbf{C}^{n+1}))$ satisfying

$$(3.3) \quad [D^{\dagger\dagger}, \rho_{\alpha\beta}^2] = 0 \text{ and } [D^{\dagger\dagger}, \frac{\partial}{\partial \rho_{\alpha\beta}^2}] = 0 \quad (0 \leq \alpha, \beta \leq 1).$$

The totality of such operators in $\mathbf{E}(\mathbf{W}_2(\mathbf{C}^{n+1}))$ is designated as $\mathbf{E}^{\dagger\dagger}(\mathbf{V}_2(\mathbf{C}^{n+1}))$.

Similarly, we have an isomorphism:

$$(3.1)_W \quad \mathbf{E}(\mathbf{G}_{2,n-1}(\mathbf{C})) \cong (\mathbf{E}^{U(2)})^{\dagger\dagger}(\mathbf{V}_2(\mathbf{C}^{n+1}))/\mathbf{J}^{\dagger\dagger},$$

where $(\mathbf{E}^{U(2)})^{\dagger\dagger}(\mathbf{V}_2(\mathbf{C}^{n+1}))$ is the subalgebra of $\mathbf{E}^{\dagger\dagger}(\mathbf{V}_2(\mathbf{C}^{n+1}))$, which is canonically isomorphic to $\mathbf{E}^{U(2)}(\mathbf{V}_2(\mathbf{C}^{n+1}))$ and $\mathbf{J}^{\dagger\dagger}$ is the ideal in $(\mathbf{E}^{U(2)})^{\dagger\dagger}(\mathbf{V}_2(\mathbf{C}^{n+1}))$ corresponding to \mathbf{J} in (3.1)_V.

To an arbitrary element $q \in \mathbf{W}_2(\mathbf{C}^{n+1})$ corresponds a linear isometric imbedding $\iota_q: (\mathbf{C}^2 - \{0\}, \iota^* g_0) \hookrightarrow (\mathbf{C}^{n+1} - \{0\}, g_0)$, where g_0 is the metric in $\mathbf{C}^{n+1} - \{0\}$ defined as

$$g_0 = \frac{1}{r^2} g,$$

where g is the canonical flat metric in \mathbf{C}^{n+1} and r^2 is as in (1.3).

Let ξ be a contravariant tensor field on a Riemannian manifold. We denote by ξ^* the corresponding covariant tensor field. Conversely, if ξ is a covariant tensor field, we denote by ξ^* the corresponding contravariant tensor field. A contravariant symmetric tensor field ξ defined on $(\mathbf{C}^{n+1} - \{0\}, g_0)$ induces a contravariant symmetric tensor field $((\iota_q)^* \xi^*)^*$ on $(\mathbf{C}^2 - \{0\}, \iota_q^* g_0)$ through the above imbedding ι_q . Fundamental differential operators of $(\mathbf{S}^{\dagger})^{**}(\mathbf{P}_1(\mathbf{C}))$ will be denoted by lower index 1, e.g., T_1 , (δ_1^*) .

DEFINITION 3.1. Define the *Radon transform*

$$\wedge: (\mathbf{S}^{\dagger})^{**}(\mathbf{P}_n(\mathbf{C})) \rightarrow C^{\infty}(\mathbf{G}_{2,n-1}(\mathbf{C}), \mathbf{C})$$

by

$$(\xi^*)^{\wedge}(\Gamma) = \begin{cases} (2^k / \text{Vol}(S^3)) \int_{S^2} (T_1^{\dagger})^k ((\iota_q)^* (\xi^*)^*)^* d\sigma & (k = l) \\ 0 & (k \neq l) \end{cases}$$

for $\xi \in (\mathbf{S}^{\dagger})^{k,l}(\mathbf{P}_n(\mathbf{C}))$, where $\Gamma = \pi_V \pi_W(g)$ for $q \in \mathbf{W}_2(\mathbf{C}^{n+1})$ and $\text{Vol}(S^3)$ is the total volume of the standard sphere S^3 . The corresponding map

$$\wedge: \mathbf{S}^{**}(\mathbf{P}_n(\mathbf{C})) \rightarrow C^{\infty}(\mathbf{G}_{2,n-1}(\mathbf{C}), \mathbf{C})$$

is also called the *Radon transform*.

Notice that behind the naturality of such a definition of the Radon transform lies a fact that the Hopf fibering φ in (1.2) is a Riemannian submersion.

Let $p = (p_0, p_1) \in V_2(\mathbf{C}^{n+1})$. Put $P^{ab} = p_0^a p_1^b - p_1^a p_0^b$, where $p_a = \sum_{a=0}^n p_a^a e_a$ for a fixed orthonormal basis (e_0, \dots, e_n) in (\mathbf{C}^{n+1}, g_0) . We can easily verify that $\sum_{a,b=0}^n P^{ab} \bar{P}^{ab} = 2$.

$\{P^{ab} \mid 0 \leq a, b \leq n, a \neq b\}$ determined by a frame $p \in V_2(\mathbf{C}^{n+1})$ is called a system of *normalized Plucker coordinates* of the 2-plane spanned by the frame p .

Theorem 3.1. (1) *The image of $(K^*)^{**}(P_n(\mathbf{C}), g_0)$ by the Radon transform is the subalgebra of $C^\infty(G_{2,n-1}(\mathbf{C}), (\mathbf{C}))$ generated by the totality of products $P^{ab} \bar{P}^{cd}$'s ($0 \leq a < b \leq n$ and $0 \leq c < d \leq n$). It is uniformly dense in $C^\infty(G_{2,n-1}(\mathbf{C}), (\mathbf{C}))$.*

(2) *The kernel of the Radon transform restricted to $K^{**}(P_n(\mathbf{C}), g_0)$ is the ideal generated by $g_0^* / 2 - 1$.*

Proof. Both of (1) and (2) were proved in [8] as the basic properties of the Radon transform (cf. p. 150~p. 153 in [8]). Q.E.D.

DEFINITION 3.2. Define

$$(1) \quad {}_w\hat{\kappa}_{a,b} := \sqrt{-1}(q_0^a \partial / \partial q_0^b - \bar{q}_0^b \partial / \partial \bar{q}_0^a + q_1^a \partial / \partial q_1^b - \bar{q}_1^b \partial / \partial \bar{q}_1^a) \\ \in \mathbf{E}^1(W_2(\mathbf{C}^{n+1})).$$

$$(2) \quad {}_w\Delta_0^\wedge := \sum_{a,b=0}^n {}_w\hat{\kappa}_{a,b} {}_w\hat{\kappa}_{a,b}^* + \sum_{a,b=0}^n {}_w\hat{\kappa}_{a,b} {}_w\hat{\kappa}_{a,b}^* \in \mathbf{E}^2(W_2(\mathbf{C}^{n+1})).$$

$$(3) \quad D_{abcd}^\wedge := \frac{1}{2^3} \sum_{e,f,g,h=0}^n \delta_{abcd}^{e,f,g,h} {}_w\hat{\kappa}_{e,f} {}_w\hat{\kappa}_{g,h}.$$

$$(4) \quad {}_w\Delta_1^\wedge := \frac{1}{4!} \sum_{a,b,c,d=0}^n \{(D_{abcd}^\wedge)^* D_{abcd}^\wedge + D_{abcd}^\wedge (D_{abcd}^\wedge)^*\} \\ \in \mathbf{E}^4(W_2(\mathbf{C}^{n+1})).$$

Lemma 3.1. ${}_w\hat{\kappa}_{a,b}$ ($0 \leq a < b \leq n$) and ${}_w\Delta_i^\wedge$ ($i=0, 1$) belong to $(\mathbf{E}^{U(2)})^{**}(V_2(\mathbf{C}^{n+1}))$ and each of them is a representative of some linear differential operator in $\mathbf{E}(G_{2,n-1}(\mathbf{C}))$.

Proof. We can verify by routine calculations that these operators satisfy the equations (3.3). Moreover, these operators can be proved to be $GL(2, \mathbf{C})$ -invariant by direct calculations. Thus our second assertions follow from (3.1)_w. Q.E.D.

DEFINITION 3.3. Denote by $\hat{\kappa}_{a,b}$ ($0 \leq a < b \leq n$) and Δ_i^\wedge ($i=0, 1$) the linear differential operators belonging to $\mathbf{E}(G_{2,n-1}(\mathbf{C}))$ whose representatives in $\mathbf{E}(W_2(\mathbf{C}^{n+1}))$ are ${}_w\hat{\kappa}_{a,b}$ and ${}_w\Delta_i^\wedge$ ($i=0, 1$), respectively. Denote by g_1 the canonical $U(n+1)$ -invariant metric on $G_{2,n-1}(\mathbf{C})$.

Notice that

(1) $\hat{\kappa}_{a,b}$ is a Killing vector field on $(G_{2,n-1}(\mathbf{C}), g_1)$. (2) Δ_0^\wedge is the Laplace-Beltrami operator on $(G_{2,n-1}(\mathbf{C}), g_1)$. In virtue of Lemma 3.1, for (1) it is suffi-

cient to show that ${}_{\omega}\hat{\kappa}_{a,b}$'s are infinitesimal generators of the action of $U(n+1)$ on $W_2(C)$ and this can be immediately checked. The second proof of them to be Killing is obtained in conjunction with (2) as follows: ${}_{\omega}\Delta_0^{\wedge}$ is expressed explicitly as

$$\begin{aligned} {}_{\omega}\Delta_0^{\wedge} = & -4 \sum_{\alpha, \beta, \gamma, \delta=0}^1 \sum_{a, b=0}^n (\delta^{ab} - \bar{q}_a^{\alpha} q_{\beta}^b (\rho^2)^{\alpha\beta}) \rho_{\gamma\delta}^2 \partial^a / \partial \bar{q}_{\gamma}^{\alpha} \partial q_{\delta}^b \\ & + 2n \sum_{\alpha=0}^1 \sum_{a=0}^n (q_{\alpha}^a \partial / \partial q_{\alpha}^a + \bar{q}_{\alpha}^a \partial / \partial \bar{q}_{\alpha}^a), \end{aligned}$$

where $((\rho^2)^{\alpha\beta})$ is the inverse matrix of $(\rho_{\alpha\beta}^2)$. ${}_{\omega}\Delta_0^{\wedge}$ can be proved to be a representative of the Laplacian on $(G_{2,n-1}(C), g_1)$, for the proof of which we refer to Lemma 3.4 of [8]. A vector field on a Riemannian manifold is Killing if and only if it commutes with the Laplace Beltrami operator (cf. [7]) and we have $[{}_{\omega}\hat{\kappa}_{a,b}, {}_{\omega}\Delta_0^{\wedge}] = 0$ by direct calculation. The second proof of (1) follows from this fact immediately.

Theorem 3.2. *The $U(n+1)$ -actions commute with the Radon transform. Namely,*

$$\wedge \rho_0 = \rho_1 \wedge,$$

where ρ_0 and ρ_1 are the natural representation of $U(n+1)$ on $(S^{\dagger})^{**}(P_n(C))$ and on $C^{\infty}(G_{2,n-1}(C), C)$, respectively.

Proof. This follows from the definition of the Radon transform obviously. Q.E.D.

Corollary.

$$\hat{\kappa}_{a,b}(\eta^{\dagger})^{\wedge} = (\check{\kappa}_{a,b}\eta^{\dagger})^{\wedge},$$

where $\eta^{\dagger} \in (S^{\dagger})^{**}(P_n(C))$ is the unique representative of $\eta \in S^{**}(P_n(C))$.

Proof. In virtue of Lemma 1.9 (2) our assertion is the infinitesimal version of Theorem 3.2. Note that the uniqueness of the representative follows from Lemma 1.4. Q.E.D.

Theorem 3.3. *Let $\eta^{\dagger} \in (S^{\dagger})^{**}(P_n(C))$ be a representative of $\eta \in S^{**}(P_n(C))$ and $(\eta^{\dagger})^{\wedge}$ the Radon transform of η^{\dagger} . Then*

- (1) $((\Delta_0 \eta)^{\dagger})^{\wedge} = \Delta_0^{\wedge} (\eta^{\dagger})^{\wedge}.$
- (2) $((\Delta_1 \eta)^{\dagger})^{\wedge} = \Delta_1^{\wedge} (\eta^{\dagger})^{\wedge}.$

Proof. In virtue of Definition 3.1, Definition 3.2 and Definition 3.3, the assertions follow from Corollary above. Q.E.D.

DEFINITION 3.4. (1) Denote by $E_{k,m}^{\wedge}$ the image of $E_{k,m}$ by the Radon transform.

(2) Denote by $\mathbf{D}(G_{2,n-1}(\mathbf{C}))$ the algebra of the totality of $U(n+1)$ -invariant differential operators acting on $C^\infty(G_{2,n-1}(\mathbf{C}), \mathbf{R})$ [3].

The eigenvalues of Δ_0^\wedge and Δ_1^\wedge restricted to $E_{k,m}^\wedge$ coincide with $\lambda_{k,m}$ and $\mu_{k,m}$, respectively. These are direct consequences of Theorem 3.3.

Main theorem. (1) Δ_0^\wedge , together with Δ_1^\wedge generates $\mathbf{D}(G_{2,n-1}(\mathbf{C}))$. (2) Each $E_{k,m}^\wedge (k \geq m \geq 0)$ is a $U(n+1)$ -irreducible representation subspace of $C^\infty(G_{2,n-1}(\mathbf{C}), \mathbf{R})$.

Proof. Notice first that $\Delta_i^\wedge (i=0, 1)$ preserve $C^\infty(G_{2,n-1}(\mathbf{C}), \mathbf{R})$ and they can be regarded as elements belonging to $\mathbf{D}(G_{2,n-1}(\mathbf{C}))$.

(1) It is known that $\mathbf{D}(G_{2,n-1}(\mathbf{C}))$ is generated by two invariant linear differential operators of order 2 and 4, respectively (cf. [3]). It remains to show that $\Delta_i^\wedge (i=1, 2)$ are algebraically independent over the field of real numbers.

Now suppose that

$$f(\Delta_0^\wedge, \Delta_1^\wedge) = 0,$$

where $f(x, y)$ is an irreducible real polynomial in two variables. Then we have

$$f(\Delta_0^\wedge, \Delta_1^\wedge)\xi = f(\lambda_{k,m}, \mu_{k,m}) = 0,$$

where ξ is a non-trivial element of $E_{k,m}^\wedge$. Therefore we have

$$f(\lambda_{k,m}, \mu_{k,m}) = 0, \quad k \geq m \geq 0 \quad (k, m \in \mathbf{Z}).$$

We can deduce from this that the left-hand side of the equality above vanishes as a polynomial of two real variables k and m . By the chain rule, we obtain

$$\frac{\partial \lambda_{k,m}}{\partial k} \frac{\partial f}{\partial x}(\lambda_{k,m}, \mu_{k,m}) + \frac{\partial \mu_{k,m}}{\partial k} \frac{\partial f}{\partial y}(\lambda_{k,m}, \mu_{k,m}) = 0,$$

$$\frac{\partial \lambda_{k,m}}{\partial m} \frac{\partial f}{\partial x}(\lambda_{k,m}, \mu_{k,m}) + \frac{\partial \mu_{k,m}}{\partial m} \frac{\partial f}{\partial y}(\lambda_{k,m}, \mu_{k,m}) = 0,$$

As we can prove the non-vanishing of the determinant of the coefficient matrix of the simultaneous equations above for sufficiently large values of the indices k and m by direct calculations, we can conclude that there exist k_0 and m_0 such that

$$\frac{\partial f(\lambda_{k,m}, \mu_{k,m})}{\partial x} = 0 \text{ and } \frac{\partial f(\lambda_{k,m}, \mu_{k,m})}{\partial y} = 0$$

for $k \geq k_0$ and $m \geq m_0$. This means that the real algebraic curve defined by $f(x, y) = 0$ has an infinite number of singular points in virtue of the following lemma. This is a contradiction.

In order to prove (2) we also need the following

Lemma 3.2. $\lambda_{k,m} = \lambda_{k',m'}$ and $\mu_{k,m} = \mu_{k',m'}$ if and only if

$$k = k' \text{ and } m = m'.$$

Proof. Assume that $\lambda_{k,m} = \lambda_{k',m'}$ and $\mu_{k,m} = \mu_{k',m'}$,
Put $\varphi(t) = t^2 + (n-2)t$, Then

$$\lambda_{k,m} = 4(\varphi(k-m) + k(n+k))$$

and

$$\mu_{k,m} = \varphi(k-m)\varphi(k+1) = \varphi(k+1)\{4\lambda_{k,m} - k(n+k)\}.$$

If we substitute this into

$$\mu_{k,m} = \mu_{k',m'},$$

we obtain

$$(k-k')(n+k+k')\{4\lambda_{k',m'} - \varphi(k+1) - k'(n+k')\} = 0.$$

From this we can verify the assertion.

Q.E.D.

Proof of (2) in Main theorem. From Lemma 3.2, $E_{k,m}^\wedge$ is concluded to be a maximal simultaneous eigenspace with eigenvalue $\lambda_{k,m}$ and $\mu_{k,m}$ of Δ_0^\wedge and Δ_1^\wedge , respectively. The irreducibility of the space follows from a known result (cf. [3]).

Q.E.D.

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