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THE COINVARIANT ALGEBRA OF THE SYMMETRIC GROUP
AS A DIRECT SUM OF INDUCED MODULES

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Abstract

Let $R_n$ be the coinvariant algebra of the symmetric group $S_n$. The algebra has a natural gradation. For a fixed $I$ ($1 \leq I \leq n$), let $R_n(k; I)$ ($0 \leq k \leq I-1$) be the direct sum of all the homogeneous components of $R_n$ whose degrees are congruent to $k$ modulo $I$. In this article, we will show that for each $I$ there exists a subgroup $H_I$ of $S_n$ and a representation $\Psi(k; I)$ of $H_I$ such that each $R_n(k; I)$ is induced by $\Psi(k; I)$.

1. Introduction

Throughout this article, we follow [5] for fundamental terminology on partitions, Young tableaux and symmetric functions.

A partition of a positive integer $n$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of nonnegative integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We also denote the partition $\lambda$ by $(1^{m_1} 2^{m_2} \cdots n^{m_n})$, where $m_i$ is the multiplicity of $i$ in $\lambda$ for $1 \leq i \leq n$. If $\lambda$ is a partition of $n$, we simply write $\lambda \vdash n$. The Young diagram of a partition $\lambda$ is a set of points

$$Y_\lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i\},$$

in which we regard the coordinates increase from left to right, and from top to bottom. Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. A standard tableau $T$ of shape $\lambda$ is a bijection $T: Y_\lambda \rightarrow [n]$ with the condition that the assigned numbers strictly increase along both the rows and the columns in $Y_\lambda$. We illustrate the Young diagram $Y_\lambda$ and a standard tableau $T$ for $\lambda = (3, 2, 2) \vdash 7$ in the following:

$\lambda \vdash 7$

$$\begin{array}{cccc}
\ast & \ast & \ast & 1 & 3 & 4 \\
\ast & \ast & 2 & 5 \\
\ast & 6 & 7
\end{array}$$

We denote by STab($\lambda$) the set of all the standard tableaux of shape $\lambda$.

For a standard tableau $T$ of shape $\lambda \vdash n$, define the descent set $\text{Des}(T)$ by

$$\text{Des}(T) := \{i \in [n - 1] \mid i + 1 \text{ is located in a lower row than } i \text{ in } T\}.$$
We call the sum of the elements of Des($T$) the *major index* of $T$, and denote it by $\text{maj}(T)$. In the preceding example, Des($T$) = \{1, 4, 5\} and $\text{maj}(T) = 1 + 4 + 5 = 10$.

Let $S_n$ be the symmetric group of degree $n$, and

$$P_n = \mathbb{C}[x_1, x_2, \ldots, x_n]$$

denote the polynomial ring with $n$ variables over $\mathbb{C}$. As customary, $S_n$ acts on $P_n$ from the left as permutations of variables by setting

$$(wf)(x_1, x_2, \ldots, x_n) = f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(n)}),$$

where $w \in S_n$ and $f(x_1, x_2, \ldots, x_n) \in P_n$. Let $I_n = \bigoplus_{d \geq 0} I_n^d$ denote the graded $S_n$-stable ideal of $P_n$ generated by the elementary symmetric functions. Hence the quotient algebra $R_n = P_n/I_n$ is also a graded $S_n$-module. We write its homogeneous decomposition as

$$R_n = \bigoplus_{d \geq 0} R_n^d,$$

and call $R_n$ the *coinvariant algebra* of $S_n$. It is well known that the coinvariant algebra $R_n$ affords the left regular representation of $S_n$.

Let us consider, for each integer $k = 0, \ldots, n - 1$, the direct sum $R_n(k; n)$ of homogeneous components of $R$ whose degrees are congruent to $k$ modulo $n$, i.e.,

$$R_n(k; n) = \bigoplus_{d \equiv k \mod n} R_n^d.$$

Since each homogeneous component $R_n^d$ is $S_n$-invariant, these subspaces also afford representations of $S_n$, and the dimensions of these representations do not depend on $k$, i.e.,

$$\dim R_n(k; n) = (n - 1)!$$

for all $k = 0, \ldots, n - 1$.

In [4], W. Krasiński and J. Weyman consider these $S_n$-modules, and prove that each $R_n(k; n)$ is induced from a corresponding irreducible representation of a cyclic subgroup of $S_n$ (see also [2, Proposition 8.2] [6, Theorem 8.9]). Precisely, let $\gamma$ be the cyclic permutation $(12\cdots n)$, and $C_n$ the subgroup of $S_n$ generated by $\gamma$. The cyclic subgroup $C_n$ of degree $n$ has $n$ inequivalent irreducible representations

$$\psi^{(k)}: C_n \longrightarrow \mathbb{C}^\times, \quad \gamma \longmapsto \zeta_n^k,$$

where $\zeta_n$ is the primitive root of unity, and the following equivalence of $S_n$-modules holds for each $k = 0, \ldots, n - 1$:

$$R_n(k; n) \cong_{S_n} \text{ind}_{C_n}^{S_n}(\psi^{(k)}).$$
Remark. In fact, the number \( n \) by which we take modulo is the Coxeter number of \( S_n \), i.e., the order of the Coxeter elements of the Coxeter group of type \( A_{n-1} \). They also obtain similar results for Coxeter groups of type \( B_n \) and \( D_n \). Stembridge obtains more general results [8]. He treats the Complex reflection groups \( G \) and shows that the coinvariant algebra of \( G \) has the similar properties for the irreducible representation of the cyclic subgroup of \( G \) generated by a Springer’s regular element [7]. We can easily see that the Coxeter elements are regular.

They also prove that the multiplicity of an irreducible representation of \( S_n \) in \( R_n^d \) (\( d \geq 0 \)) is described by the major index of standard tableaux. It is well known that the irreducible representations of \( S_n \) are in one to one correspondence with the partitions of \( n \). For \( \lambda \vdash n \) let \( V^\lambda \) denote the corresponding irreducible representation of \( S_n \). They showed that the multiplicity \( [R_n^d : V^\lambda] \) of \( V^\lambda \) in \( R_n^d \) equals the number of standard tableaux whose major indices are \( d \):

\[
[R_n^d : V^\lambda] = \# \{ T \in STab(\lambda) \mid \text{maj}(T) = d \}.
\]

(see also [2, Theorem 8.6] [6, Theorem 8.8]). Combining these results, the multiplicities of the irreducible representation \( V^\lambda \) in the induced representations \( \psi^{(k)}_{S_n} \mapsto_{S_n} \in S_n \)

\( R_n(k; n) \) are easily obtained:

\[
[R_n(k; n) : V^\lambda] = \# \{ T \in STab(\lambda) \mid \text{maj}(T) = k \mod n \}.
\]

It should be mentioned here that a more refined result is obtained by R. Adin, F. Brenti and Y. Roichman [1] recently. For each subset \( S \subseteq [n - 1] \), they construct an \( S_n \)-module \( R_S \) satisfying

\[
R_n^d = \bigoplus_S R_n^S,
\]

where the direct sum is taken over the subsets \( S \subseteq [n - 1] \) such that \( \sum_{i \in S} i = d \), and describe the multiplicities of irreducible constituents on \( R_n^S \) as follows:

\[
[R_n^S : V^\lambda] = \# \{ T \in STab(\lambda) \mid \text{Des}(T) = S \}.
\]

They also consider an analogue of the theorem of Kraśkiewicz and Weyman for the Weyl groups of type \( B \), and obtain a result on the irreducible decompositions of the coinvariant algebras of type \( B \) finer than one already obtained by Stembridge in [8].

The aim of the present article is to achieve a generalization of these results in the following sense. Fix an integer \( I \in [n] \) and consider subspaces of \( R_n \) obtained by gathering homogeneous components whose degrees are congruent modulo \( I \). Precisely,
for each \( k = 0, \ldots, l - 1 \) we will consider
\[
R_n(k; l) = \bigoplus_{d \equiv k \mod l} R_n^d.
\]

We can see that the dimension of the space \( R_n(k; l) \) is independent of \( k \), i.e.,
\[
\dim R_n(k; l) = \frac{n!}{l}
\]
for all \( k = 0, \ldots, l - 1 \) (Proposition 4). In this article we will seek out a systematic realization of each submodule \( R_n(k; l) \) as a \( S_n \)-module induced from a subgroup of \( S_n \) that is determined by \( l \). First we settle a subgroup \( H_l \) of \( S_n \) for each \( l \in [n] \), then construct a representation \( \Psi(k; l) \) of \( H_l \) for each \( k = 0, \ldots, l - 1 \). When we write \( n = dl + r \) with \( 0 \leq r \leq l - 1 \), the subgroup \( H_l \) turns out to be isomorphic to a direct product of the cyclic group of order \( l \) and the symmetric group of degree \( r \), i.e.,
\[
H_l \cong C_l \times S_r.
\]

The representation \( \Psi(k; l) \) of \( H_l \) is not necessarily irreducible in contrast to the case \( l = n \) (Section 4). Finally, we verify that
\[
R_n(k; l) \cong \bigoplus_{\rho} \ind_{H_l}^{S_n} (\Psi(k; l))
\]
for each \( l \) and \( k \) by comparing the graded characters of \( R_n \) and \( \bigoplus_{k=0}^{l-1} \ind_{H_l}^{S_n} (\Psi(k; l)) \) as polynomials in \( q \) modulo \( q^l - 1 \) (Theorem 8).

2. Coinvariant algebra and its graded character

Let \( R_n = \bigoplus_{d \geq 0} R_n^d \) be the coinvariant algebra of \( S_n \) and its homogeneous decomposition. Let \( q \) be an indeterminate over \( \mathbb{C} \). Define the graded character of \( R_n \) by
\[
X_n(q) = \sum_{d \geq 0} q^d \chi_{n,d},
\]
where \( \chi_{n,d} \) is the character of the representation \( R_n^d \) of \( S_n \). We denote by \( X_{n,\rho}(q) \) and \( \chi_{n,\rho}^d \) the value of \( X_n(q) \) and \( \chi_{n,d} \) at elements of cycle-type \( \rho \vdash n \), respectively. Precisely, \( X_{n,\rho}(q) \) is a polynomial in \( q \) whose coefficient in \( q^d \) is \( \chi_{n,\rho}^d \). This polynomial \( X_{n,\rho}(q) \) is also known as a Green polynomial \( Q_{\rho}^{(n)}(q) \) of type A [3] [5, III.7].

The graded character of \( R_n \) has a well-known product formula ([3, Appendix], see also [2, Proposition 8.1]), that plays an essential role in the present article.

**Proposition 1.** For any partition \( \rho = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \) of \( n \), we have
\[
X_{n,\rho}(q) = \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-q^{m_1})(1-q^{2m_2}) \cdots (1-q^{nm_n})}.
\]
From the Proposition above, we can prove the following auxiliary result.

**Proposition 2.** Fix an integer \( l \in [n] \). Let \( p \) be a divisor of \( l \), \( n = ep + s \) (\( 0 \leq s \leq p - 1 \)), and \( \theta \) a primitive \( p \)-th root of unity. If \( \rho \vdash n \) satisfies

\[
X_{n, \rho}(\theta) \neq 0,
\]

then \( \rho = (1^{m_1} \cdots s^{m_s} p^e) \), where \( m_1 + 2m_2 + \cdots + sm_s = s \).

**Proof.** We apply Stembridge’s argument for the case \( l = n \) (see [2, Section 8]) to our situation. By Proposition 1, we have

\[
X_{n, \rho}(\theta) = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1 - q)^{m_1}(1 - q^{2p})^{m_2} \cdots (1 - q^{np})^{m_n}}|_{q = \theta},
\]

for \( \rho = (1^{m_1} 2^{m_2} \cdots n^{m_0}) \vdash n \). Thus \( X_{n, \rho}(\theta) \neq 0 \) implies that all the vanishing factors in the numerator are canceled by corresponding factors in the denominator. There are \( e \) vanishing factors: \( 1 - q^p, 1 - q^{2p}, \ldots, 1 - q^{np} \) in the numerator, and \( m_p + m_{2p} + \cdots + m_{ep} \) vanishing factors: \( (1 - q^p)^{m_p}, (1 - q^{2p})^{m_{2p}}, \ldots, (1 - q^{np})^{m_{ep}} \) in the denominator. Since

\[
pm_p + 2pm_{2p} + \cdots + em_{ep} \leq m_1 + 2m_2 + \cdots + nm_n = n \quad (= ep + s),
\]

we have

\[
m_p + 2m_{2p} + \cdots + em_{ep} \leq e.
\]

Therefore,

\[
e = m_p + m_{2p} + \cdots + m_{ep} \leq m_p + 2m_{2p} + \cdots + em_{ep} \leq e.
\]

Hence, we have \( m_p = e \). We also obtain \( m_i = 0 \) for \( s + 1 \leq i \leq n \) \((i \neq p)\) since \( n - pm_p = n - pe = s \). Thus, we have

\[
m_1 + 2m_2 + \cdots + sm_s = s.
\]

Let \( l \in [n] \) be a fixed integer. For each \( k = 0, 1, \ldots, l - 1 \), we define

\[
R_l(k; l) := \bigoplus_{d \equiv k \mod l} R^d_l,
\]

i.e.,

\[
R_l = \bigoplus_{k=0}^{l-1} R_l(k; l).
\]
We prove that the dimensions of the spaces $R_n(k;l)$ are independent of the choice of $k$. We first show the following lemma.

**Lemma 3.** Let $q$ be an indeterminate and $f(q) = \sum_{i \geq 0} a_i q^i \in \mathbb{C}[q]$ a polynomial in $q$. Let $l \geq 2$ be an integer and $\zeta_l$ a primitive $l$-th root of unity. Then the following conditions are equivalent:

1. $f(\zeta_l^k) = 0$ for each $k = 1, \ldots, l-1$.
2. The partial sums $c_k = \sum_{i \equiv k \mod l} a_i$ $(k = 0, 1, \ldots, l-1)$ of coefficients of the polynomial $f(q)$ are independent of the choice of $k$.

**Proof.** If the condition (b) holds, then $f(q)$ is divisible by

$$1 + q + q^2 + \cdots + q^{l-1} = \frac{1 - q^l}{1 - q},$$

and hence we have (a).

We shall prove the converse. From (a) we have

$$f(\zeta_l^k) = a_0 + a_1 \zeta_l^k + a_2 (\zeta_l^k)^2 + \cdots = 0 \quad (k = 0, 1, \ldots, l-1).$$

By the definition of $c_k$, it reduces to the linear equation system in $c_0, \ldots, c_{l-1}$:

$$
\begin{align*}
&c_0 + c_1 \zeta_l^1 + c_2 \zeta_l^2 + \cdots + c_{l-1} \zeta_l^{l-1} = 0, \\
&c_0 + c_1 \zeta_l^2 + c_2 (\zeta_l^2)^2 + \cdots + c_{l-1} (\zeta_l^2)^{l-1} = 0,
&\vdots \\
&c_0 + c_1 \zeta_l^{l-1} + c_2 (\zeta_l^{l-1})^2 + \cdots + c_{l-1} (\zeta_l^{l-1})^{l-1} = 0.
\end{align*}
$$

Since the rank of the coefficient matrix of the equation system is $l - 1$, it has an one dimensional solution space. It is clear that $(c_0, c_1, \ldots, c_{l-1}) = (1, 1, \ldots, 1)$ satisfies the equation system, hence we have $c_0 = c_1 = \cdots = c_{l-1}$.

By using the above lemma, we easily reach our aim.

**Proposition 4.** Let $l \in [n]$ be a fixed integer. Then the dimension of $R_n(k;l)$ is independent of the choice of $k = 0, 1, \ldots, l-1$, i.e., we have

$$\dim R_n(k;l) = \frac{n!}{l}$$

for all $k = 0, 1, \ldots, l$.

**Proof.** If $l = 1$, then the assertion is trivial. Suppose that $l \geq 2$. Let $\zeta_l$ be a primitive $l$-th root of unity. If we evaluate the formula in Proposition 1 at the identity ele-
ment \( e \in S_n \), then we have

\[
[X_n(q)](e) = X_{n,(\nu)}(q) = \frac{(1 - q)(1 - q^2) \cdots (1 - q^n)}{(1 - q)^n} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})
\]

It follows immediately that, for each \( k = 0, \ldots, l - 1 \),

\[
X_{n,(\nu)}(\xi_k^k) = \sum_{d \geq 0} (\dim R_n^d) q^{d} \bigg|_{q = \xi_k^k} = 0.
\]

By Lemma 3, we obtain that \( \dim R_n(k; l) = \sum_{d \equiv k \mod l} \dim R_n^d \) is independent of \( 0 \leq k \leq l - 1 \) and is equal to \( n! / l \). \( \square \)

If \( w \in S_n \), the cycle type \( \rho(w) \) of \( w \) is the partition \( \rho(w) = (1^{m_1}2^{m_2} \cdots r^{m_r}) \). For a partition \( \rho \) of \( n \), let \( C_\rho \) be the conjugacy class in \( S_n \) containing \( w \in S_n \) such that \( \rho(w) = \rho \). For any partition \( \rho = (1^{m_1}2^{m_2} \cdots r^{m_r}) \), define

\[
z_\rho = \frac{n!}{|C_\rho|} = 1^{m_1}2^{m_2} \cdots r^{m_r} \cdot m_1! m_2! \cdots m_r!.
\]

Let \( f \) and \( g \) be arbitrary class functions on \( S_n \). There is a natural scalar product of \( f \) and \( g \) defined by

\[
\langle f, g \rangle_{S_n} := \frac{1}{n!} \sum_{w \in S_n} f(w)g(w).
\]

(For a general finite group \( G \), the scalar product is defined by \( \langle f, g \rangle := (1/|G|) \times \sum_{w \in G} f(w)g(w) \), where \( \overline{g(w)} \) denotes the complex conjugate of \( g(w) \). However, we can use \( g(w) \) instead of \( g(w) \) here since all characters of \( S_n \) are rational.) Note that if \( \delta_\lambda \) (\( \lambda \vdash n \)) is the class function defined by

\[
\delta_\lambda(w) = \begin{cases} 1 & \text{if } \rho(w) = \lambda, \\ 0 & \text{otherwise}, \end{cases}
\]

then \( \langle \delta_\lambda, f \rangle_{S_n} = z_\lambda^{-1} f(\lambda) \).

If \( n = dl + r \) (\( 0 \leq r \leq l - 1 \)), then we can embed \( S_{dl} \times S_r \) in \( S_n \) by

\[
S_{dl} = \{ w \in S_r \mid w(i) = i \text{ for all } i = dl + 1, \ldots, n \},
\]

\[
S_r = \{ w \in S_r \mid w(i) = i \text{ for all } i = 1, \ldots, dl \}.
\]

We see that, if \( u \in S_{dl} \) and \( v \in S_r \), the element \( u \times v \in S_n \) has cycle-type \( \rho(u \times v) = \rho(u) \cup \rho(v) \).
Let $f$ and $g$ be characters of the representations $\phi$ of $S_{dl}$ and $\psi$ of $S_r$, respectively. Then $f \times g$ defined by

$$(f \times g)(u, v) = f(u)g(v) \quad (u \in S_{dl}, \ v \in S_r)$$

is the character of the tensor product representation $\phi \otimes \psi$ of $S_{dl} \times S_r$. We define

$$f \ast g = \text{ind}_{S_{dl} \times S_r}^{S_r}(f \times g),$$

which is a character of the induced representation $\text{ind}_{S_{dl} \times S_r}^{S_r}(\phi \otimes \psi)$ of $S_r$.

The following is a key proposition to the main result.

**Proposition 5.** Let $n$ be a positive integer, and choose an integer $l$ $(1 \leq l \leq n)$. If $n = dl + r$ $(0 \leq r < l)$, then we have

$$X_{n\rho}(q) \equiv (X_{dl}(q) \cdot X_r(q)) \mod q^l - 1.$$

**Proof.** We show that

$$(X_{dl}(\theta) \cdot X_r(\theta))_{\rho} = \begin{cases} X_{n\rho}(\theta) & \text{if } \rho = (1^{m_1} \cdots s^{m_s} p^e) \\ 0 & \text{otherwise.} \end{cases}$$

for each $\rho \vdash n$, where $(X_{dl}(q) \cdot X_r(q))_{\rho}$ is the value of $(X_{dl}(q) \cdot X_r(q))$ at elements of cycle-type $\rho$. By the Lagrange interpolation and Proposition 2, in order to verify (2.2), it is sufficient to show that

$$X_{dl}(\theta) \cdot X_r(\theta)_{\rho} = \begin{cases} 0 & \text{if } \rho = (1^{m_1} \cdots s^{m_s} p^e) \\ 1 & \text{otherwise.} \end{cases}$$

for each $\theta = \zeta_k^l$ $(k = 0, \ldots, l - 1)$, where $p$ is the multiplicative order of $\theta$. Note that $p$ divides $l$. Using the property of the class function $\delta_\rho$, we then have

$$z_{\rho^l}^{-1}(X_{dl}(\theta) \cdot X_r(\theta))_{\rho}$$

$$= \langle (X_{dl}(\theta) \cdot X_r(\theta)), \delta_\rho \rangle_{S_l}$$

$$= \left(\langle X_{dl}(\theta) \times X_r(\theta), \text{res}_{S_{dl} \times S_r}^{S_r}(\delta_\rho) \rangle_{S_{dl} \times S_r}\right)$$

(by Frobenius reciprocity)

$$= \frac{1}{(dl)!r!} \sum_{u \in S_{dl}} \sum_{v \in S_r} (X_{dl}(\theta) \times X_r(\theta))(u, v) \delta_\rho(u \times v)$$

$$= \frac{1}{(dl)!r!} \sum_{u \in S_{dl}} \sum_{v \in S_r} \sum_{\rho^l \cdot \sigma^l} X_{dl, \rho^l}(\theta) X_{r, \sigma^l}(\theta) \delta_\rho(u) \delta_\rho^l(v)$$

$$= \sum_{\rho^l \cdot \sigma^l} z_{\rho^l}^{-1} z_{\sigma^l}^{-1} X_{dl, \rho^l}(\theta) X_{r, \sigma^l}(\theta).$$
where $\rho^1 \vdash dl$ and $\rho^2 \vdash r$ are partitions such that $\rho^1 \cup \rho^2 = \rho$. Now let $n = ep + s$ and $r = fp + s$ ($0 \leq s < p$). Then $dl/p = e - f$. By Proposition 2, $X_{dl,p} X_{r,p} = 0$ unless $\rho^1 = (p^{e-f})$ and $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f)$. Hence, if $\rho$ is not of the form $(1^{m_1} \cdots s^{m_s} p^f)$ for some $(1^{m_1} \cdots s^{m_s}) \vdash s$, we have $(X_{dl}(\theta), X_r(\theta)) = 0$. On the other hand, we pick $\rho^1 = (p^{e-f})$ and $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f)$ so that $\rho = (1^{m_1} \cdots s^{m_s} p^f)$, and finally we have

$$z_\rho^{-1}(X_{dl}(\theta), X_r(\theta)) = z_{(p^{e-f})}^{-1}(1^{m_1} \cdots s^{m_s} p^f) X_{dl}(p^{e-f})(\theta) X_{r}(1^{m_1} \cdots s^{m_s} p^f)(\theta)$$

$$= z_{(p^{e-f})}^{-1}(1^{m_1} \cdots s^{m_s} p^f) \frac{(1-q) \cdots (1-q^{dl})}{(1-q^{p^{e-f}})} \frac{(1-q) \cdots (1-q^r)}{(1-q)^{m_1} \cdots (1-q^{s^{m_s}(1-q^{p^f})})}$$

$$= z_{(p^{e-f})}^{-1}(1^{m_1} \cdots s^{m_s} p^f) \frac{e^{-f}}{f} \frac{(1-q) \cdots (1-q^{dl+1}) \cdots (1-q^{dl+r})}{(1-q)^{m_1} \cdots (1-q^{s^{m_s}(1-q^{p^f})})}$$

$$= z_{(p^{e-f})}^{-1}(1^{m_1} \cdots s^{m_s} p^f) \frac{(1-q) \cdots (1-q^n)}{(1-q)^{m_1} \cdots (1-q^{s^{m_s}(1-q^{p^f})})}$$

$$= z_\rho^{-1} X_{n,\rho}(\theta)$$

Translating Proposition 2 and Proposition 5 into the language of the Green polynomials, we obtain the following formula.

**Corollary 6.** Let $n > l$ be positive integers, $p$ a divisor of $l$, and $\theta$ a primitive $p$-th root of unity. If we write $n = dl + r = ep + s$ ($0 \leq r \leq l - 1$, $0 \leq s \leq p - 1$), then

1. $Q^{(p^e)}(\theta) = 0$ unless $\rho = (1^{m_1} \cdots s^{m_s} p^f)$ and $m_1 + 2m_2 + \cdots + sm_s = s$.

2. If $\rho = (1^{m_1} \cdots s^{m_s} p^f)$,

$$Q^{(p^e)}(\rho)(q) \equiv Q^{(p^e)}_\rho(q) \text{ mod } q^l - 1,$$

where $\rho^1 = (p^{e-f}) \vdash dl$ and $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f) \vdash r$.

**3. l|n case**

In this section, we consider the case where $l$ divides $n$, and show that each $R_{n}(k;l)$ is induced from a representation of a cyclic subgroup of $S_n$.

Suppose that $l$ divides $n$, and say $d = n/l$. Let $C_l$ be the cyclic group of order $l$, and we embed $C_l$ into $S_n$ as follows:

$$C_l \cong \langle \gamma_1 \gamma_2 \cdots \gamma_d \rangle \subset S_n,$$

where $\gamma_1 = (1, 2, \ldots, l), \gamma_2 = (l + 1, l + 1, \ldots, 2l), \ldots, \gamma_d = ((d - 1)l + 1, \ldots, dl)$. The cyclic group $C_l$ has inequivalent $l$ irreducible representations $\psi^{(0)}, \ldots, \psi^{(l-1)}$, i.e.,

$$\psi^{(k)} : C_l \rightarrow \mathbb{C}^X, \quad \gamma_1 \gamma_2 \cdots \gamma_d \mapsto \zeta_l^k,$$
where \( \zeta_l \) denotes a primitive \( l \)-th root of unity. Let

\[
\tau^{(k)} := \frac{1}{l} \sum_{i=0}^{l-1} \zeta_l^{-ik} (\gamma_1 \cdots \gamma_l)^i \quad (k = 0, 1, \ldots, l - 1).
\]

We can easily check that each \( \tau^{(k)} \) is an idempotent by a direct calculation.

Let \( \mathbb{C}[S_l] \) be the group algebra of \( S_l \). Consider the representation of \( S_n \) afforded by the left ideal \( \mathbb{C}[S_n] \tau^{(k)} \), which is equivalent to the induced representation \( \text{ind}^S_G(\psi^{(k)}) \). Its character \( \chi[\mathbb{C}[S_n] \tau^{(k)}] \) is given by \( \Gamma_n \tau^{(k)} \), where \( \Gamma_n \) is an operator defined by

\[
\Gamma_n: \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n], \quad \rho \mapsto \sum_{w \in S_n} w^{-1} \rho w
\]

(see e.g., [2, Proposition 5.2] [6, Lemma 8.4]). Here we regard an element \( \rho = \sum_{w \in S_n} \rho_w w \in \mathbb{C}[S_n] \) as the function on \( S_n \) that maps \( w \in S_n \) to the coefficient \( \rho_w \):\[\text{ind}^{S_l}_G(\chi[\psi^{(k)}]) = \Gamma_n \tau^{(k)},\]

where \( \chi[\psi^{(k)}] \) stands for the \( C_l \)-character of \( \psi^{(k)} \).

We have shown in Proposition 4 that the dimension of the space

\[
R_n(k;l) = \bigoplus_{d \equiv k \mod l} R_n^d
\]

is constant with respect to \( k = 0, \ldots, l - 1 \). This fact suggests that every \( R_n(k;l) \) \((k = 0, \ldots, l - 1)\) are induced from the same dimensional representations of a certain subgroup of \( S_l \). In fact, we can verify that, for each \( k = 0, \ldots, l - 1 \), there exists an irreducible representation of \( C_l \) that yields \( R_n(k;l) \).

**Proposition 7.** Let \( n \) be a positive integer and \( l \) a divisor of \( n \). Write \( d = n/l \). For \( i = 1, 2, \ldots, d \), let \( \gamma_i \) be the cyclic permutation \(((i-1)l+1, (i-1)l+2, \ldots, il)\). Let \( C_l \) be the cyclic subgroup of \( S_n \) generated by \( \gamma_1 \cdots \gamma_d \) and \( \{ \psi^{(k)} \mid k = 0, 1, \ldots, l - 1 \} \) the set of its inequivalent irreducible representations. Then, we have an isomorphism of \( S_n \)-modules

\[
R_n(k;l) \cong S_l \text{ind}^{S_l}_{C_l}(\psi^{(k)}) \quad (k = 0, 1, \ldots, l - 1).
\]

**Proof.** We prove that

\[
X_n(q) \equiv \sum_{k=0}^{l-1} q^k \text{ind}^{S_l}_{C_l}(\chi[\psi^{(k)}]) \mod q^l - 1.
\]

Using the Lagrange interpolation again, we only have to show that the both sides of (3.1) coincide when \( q = \zeta_l^s \) \((s = 0, 1, \ldots, l - 1)\).
Recall that
\[ \text{ind}^S_{\mathcal{C}_l}(\chi[\psi^{(k)}]) = \Gamma_n\tau^{(k)} \]
for each \( k = 0, \ldots, l-1 \). Substituting \( q = \xi^s \) in the right hand side of (3.1), we obtain
\[
\sum_{k=0}^{l-1} (\xi^s)^k \text{ind}^S_{\mathcal{C}_l}(\chi[\psi^{(k)}]) = \sum_{k=0}^{l-1} \xi^k \Gamma_n\tau^{(k)} = \Gamma_n(\gamma_1 \cdots \gamma_d)^s \sum_{k=0}^{l-1} \tau^{(k)}
\]
\[
= \Gamma_n(\gamma_1 \cdots \gamma_d)^s \sum_{k=0}^{l-1} \frac{1}{l} \sum_{i=0}^{l-1} \xi^{-ik}(\gamma_1 \cdots \gamma_d)^i
\]
\[
= \Gamma_n(\gamma_1 \cdots \gamma_d)^s \frac{1}{l} \sum_{i=0}^{l-1} (1 + \xi^{-i} + \xi^{-2i} + \cdots + \xi^{-(l-1)i})(\gamma_1 \cdots \gamma_d)^i
\]
\[
= \Gamma_n(\gamma_1 \cdots \gamma_d)^s
\]
for each \( s = 0, 1, \ldots, l-1 \). Since the cycle-type of \( (\gamma_1 \cdots \gamma_d)^s \) can be written as \( (p^e) \) (\( e = n/p \)), where \( p \) is the multiplicative order of \( (\xi^n)^p = 1 \), we have
\[
\sum_{k=0}^{l-1} (\xi^s)^k \text{ind}^S_{\mathcal{C}_l}(\chi[\psi^{(k)}]) \equiv \begin{cases} \xi^{(p^e)}, & \text{if } \rho = (p^e) \\ 0, & \text{otherwise} \end{cases}
\]
for a partition \( \rho \). Hence the congruence (3.1) immediately follows from Proposition 1 and Proposition 2.

4. **Main result**

Let \( n \) be a positive integer, and choose an integer \( l = 1, 2, \ldots, n \). Suppose that \( n = dl + r \), where \( 0 \leq r \leq l-1 \). Let \( R_n \) be the coinvariant algebra of \( S_n \), and \( R_n = \bigoplus_{d \geq 0} R^d_n \) its homogeneous decomposition. For each \( k = 0, 1, \ldots, l-1 \), define
\[
R_n(k;l) := \bigoplus_{d|k, \mod l} R^d_n.
\]
Now, for each \( l = 1, 2, \ldots, n \), we define a subgroup \( H_l \) of \( S_n \) by
\[
H_l = (\gamma_1 \gamma_2 \cdots \gamma_d) \times S_r
\]
\[
\cong C_l \times S_r,
\]
where \( \gamma_l \) is the cyclic permutation \( ((i-1)l + 1, (i-1)l + 2, \ldots, il) \), and the symmetric group \( S_r \) of degree \( r \) is identified as the subgroup \( \{ w \in S_n \mid w(i) = i \text{ for all } i = 1, 2, \ldots, n-r \} \) of \( S_n \).
For each \( k = 0, 1, \ldots, l - 1 \), we construct a representation \( \Psi(k; l) \) of \( \mathcal{H}_l \) as follows:

\[
\Psi(k; l) := \bigoplus_{\lambda} \bigoplus_{T \in \text{Stab}(\lambda)} \psi^{(k, \text{maj}(T))} \otimes V^\lambda,
\]

where \( k - \text{maj}(T) = k - \text{maj}(T) \mod l \). \( \{ \psi^{(i)} \mid i = 0, \ldots, l - 1 \} \) is the set of inequivalent irreducible representation of \( C_l \), and \( V^\lambda(\lambda, l - r) \) is the irreducible representation of \( S_r \) corresponding to the partition \( \lambda \) of \( r \). Then it can be seen that the dimension of \( \Psi(k; l) \) does not depend on \( k \) and hence so does \( \text{deg} \text{ind}^{S_n}_{\mathcal{H}_l}(\Psi(k; l)) \). Actually, since \( \text{deg} V^\lambda = \# \text{Stab}(\lambda) \) and \( \sum_{\lambda, r} \# \text{Stab}(\lambda)^2 = r! \), we have

\[
\text{deg} \Psi(k; l) = \sum_{\lambda, r} \sum_{T \in \text{Stab}(\lambda)} \text{deg} \psi^{(k, \text{maj}(T))} \otimes V^\lambda = \sum_{\lambda, r} \# \text{Stab}(\lambda) = \sum_{\lambda, r} \# \text{Stab}(\lambda)^2 = r!,
\]

and \( \text{deg} \text{ind}^{S_n}_{\mathcal{H}_l}(\Psi(k; l)) = r! n!/r! l! = n!/l! \), which coincides with the dimension of \( R_n(k; l) \). Moreover, we prove that these two representations are equivalent.

**Theorem 8 (Main result).** Let \( n \) be a positive integer. Fix an integer \( l \in [n] \) and write \( n = dl + r \) (\( 0 \leq r \leq l - 1 \)). Let \( \mathcal{H}_l \cong C_l \times S_r \) be the subgroup of \( S_n \) defined above and \( \Psi(k; l) \) \( (k = 0, 1, \ldots, l - 1) \) representations of it defined by

\[
\Psi(k; l) := \bigoplus_{\lambda, r} \bigoplus_{T \in \text{Stab}(\lambda)} \psi^{(k, \text{maj}(T))} \otimes V^\lambda,
\]

where \( \psi^{(i)} \) and \( V^\lambda \) stand for the irreducible representations of \( C_l \) and \( S_r \), respectively. Then, for each \( k = 0, 1, \ldots, l - 1 \), there is an isomorphism

\[
R_n(k; l) \cong S_n \text{ind}^{S_n}_{\mathcal{H}_l}(\Psi(k; l))
\]
as an \( S_n \)-module.

**Proof.** By the definition of \( \Psi(k; l) \), it suffices to show

\[
X_n(q) \equiv \sum_{k=0}^{l-1} q^k \sum_{\lambda, r} \sum_{T \in \text{Stab}(\lambda)} \text{ind}^{S_n}_{\mathcal{H}_l}(\chi \psi^{(k, \text{maj}(T))} \otimes V^\lambda) \mod q^l - 1.
\]

Let \( S_{dl} \) and \( S_r \) be the subgroup of \( S_n \) defined in (2.1). Since \( \mathcal{H}_l \) is a subgroup
of \( S_d \times S_r \), we have
\[
\text{ind}_{H}^{S_r} \left( \psi^{(k, \text{maj}(T))} \otimes V^\lambda \right) \cong \text{ind}_{S_d \times S_r}^{S_r} \left( \text{ind}_{H}^{S_d \times S_r} \left( \psi^{(k, \text{maj}(T))} \otimes V^\lambda \right) \right)
\]
for any \( \lambda \vdash r \). Therefore, the right hand side of (4.1) equals
\[
\sum_{k=0}^{l-1} \sum_{\lambda \vdash r} \sum_{T \in \text{STab}(\lambda)} q^k \text{ind}_{S_d \times S_r}^{S_r} \left( \text{ind}_{C_T}^{S_d \times S_r} \left( \chi^{(k, \text{maj}(T))} \otimes \chi[V^\lambda] \right) \right)
\]
\[
= \sum_{k} \sum_{\lambda} \sum_{T} q^k \text{ind}_{S_d \times S_r}^{S_r} \left( \text{ind}_{C_T}^{S_d \times S_r} \left( \chi^{(k, \text{maj}(T))} \right) \times \chi[V^\lambda] \right)
\]
\[
= \text{ind}_{S_d \times S_r}^{S_r} \left( \sum_{\lambda} \sum_{T} q^{\text{maj}(T)} \text{ind}_{C_T}^{S_d \times S_r} \left( \chi^{(k, \text{maj}(T))} \right) \times q^{\text{maj}(T)} \chi[V^\lambda] \right)
\]
\[
\equiv \text{ind}_{S_d \times S_r}^{S_r} X_{dl}(q) \left( \sum_{\lambda} \sum_{T} q^{\text{maj}(T)} \chi[V^\lambda] \right) \mod q^l - 1 \text{ by (3.1)}.
\]
(4.2)

By the theorem of Krasikiewicz-Weyman, the multiplicity \([R_n^d : V^\lambda] \) of irreducible components isomorphic to \( V^\lambda \) (\( \lambda \vdash n \)) is the number of standard Young tableaux of shape \( \lambda \) whose major index equals \( d \), that is,
\[
[R_n^d : V^\lambda] = \# \{ T \in \text{STab}(\lambda) : \text{maj}(T) = d \}.
\]
Hence we have
\[
X_r(q) = \sum_{\lambda \vdash r} \sum_{T \in \text{STab}(\lambda)} q^{\text{maj}(T)} \chi[V^\lambda].
\]
(4.3)

Applying (4.3) and Proposition 5, we see that (4.2) equals
\[
\text{ind}_{S_d \times S_r}^{S_r} X_{dl}(q) \left( \sum_{\lambda \vdash r} \sum_{T \in \text{STab}(\lambda)} q^{\text{maj}(T)} \chi[V^\lambda] \right)
\]
\[
= \text{ind}_{S_d \times S_r}^{S_r} X_{dl}(q) \times X_r(q)
\]
\[
= (X_{dl}(q) : X_r(q))
\]
\[
\equiv X_n(q) \mod q^l - 1,
\]
and complete the proof. \( \square \)

When \( r = 0 \) or 1, \( H \) is a cyclic group and \( \Psi(k; l) \) is irreducible. In this case, the generator of \( H \) coincides with a regular element of \( S_n \) defined by Springer [7].

It is obvious that the multiplicity of \( V^\lambda \) in \( R_n(k; l) \) is obtained by counting the number of standard Young tableaux of shape \( \lambda \) with the major index congruent
to $k$ modulo $l$, that is, 

$$[R_k(k; l) : V^2] = 2\{T \in \text{Stab}(\lambda) \mid \text{maj}(T) \equiv k \mod l\}.$$ 

**Example.** In the case of $n = 5$ and $l = 3$, the subgroup $H_5$ is $\langle(123) \rangle \times \langle(45)\rangle$, which is isomorphic to $C_3 \times S_2$. Then we have 

$$R_k(k; 3) \cong \text{ind}^S_{H_3} \left( (\psi^{(k)} \otimes V^{(2)}) \oplus (\psi^{(k-1)} \otimes V^{(1,1)}) \right)$$

for each $k = 0, 1, 2$.

If we consider the case $n = 11$ and $l = 4$ (thus $r = 3$), then the subgroup $H_{11}$ is $\langle(1234)(5678)\rangle \times \langle(9, 10), (10, 11)\rangle$ isomorphic to $C_4 \times S_3$. Hence, for each $R_{11}(k; 4) (k = 0, 1, 2, 3)$ is isomorphic to the representation induced by

$$
\begin{align*}
\Psi(0; 4) &= (\psi^{(0)} \otimes V^{(3)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(1,1,1)}), \\
\Psi(1; 4) &= (\psi^{(1)} \otimes V^{(3)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(1,1,1)}), \\
\Psi(2; 4) &= (\psi^{(2)} \otimes V^{(3)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(1,1,1)}), \\
\Psi(3; 4) &= (\psi^{(3)} \otimes V^{(3)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(1,1,1)}),
\end{align*}
$$

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**References**


THE COINVARIANT ALGEBRA OF $S_n$

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