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THE COINVARIANT ALGEBRA OF THE SYMMETRIC GROUP AS A DIRECT SUM OF INDUCED MODULES

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Abstract

Let R_n be the coinvariant algebra of the symmetric group S_n. The algebra has a natural gradation. For a fixed l $(1 \leq l \leq n)$, let $R_n(k;l)$ $(0 \leq k \leq l-1)$ be the direct sum of all the homogeneous components of R_n whose degrees are congruent to k modulo l. In this article, we will show that for each l there exists a subgroup H_l of S_n and a representation $\Psi(k;l)$ of H_l such that each $R_n(k;l)$ is induced by $\Psi(k;l)$.

1. Introduction

Throughout this article, we follow [5] for fundamental terminology on partitions, Young tableaux and symmetric functions.

A *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of nonnegative integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We also denote the partition λ by $(1^{m_1} 2^{m_2} \cdots n^{m_n})$, where m_i is the multiplicity of i in λ for $1 \leq i \leq n$. If λ is a partition of n, we simply write $\lambda \vdash n$. The *Young diagram* of a partition λ is a set of points

$$Y_\lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i\},$$

in which we regard the coordinates increase from left to right, and from top to bottom. Let $[n]$ denote the set of integers $\{1, 2, \dots, n\}$. A *standard tableau* T of shape λ is a bijection $T : Y_\lambda \to [n]$ with the condition that the assigned numbers strictly increase along both the rows and the columns in Y_λ. We illustrate the Young diagram Y_λ and a standard tableau T for $\lambda = (3, 2, 2) \vdash 7$ in the following:

$$Y_\lambda = \begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{matrix}\ , \quad T = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \end{matrix}\ .$$

We denote by $\mathrm{STab}(\lambda)$ the set of all the standard tableaux of shape λ.

For a standard tableau T of shape $\lambda \vdash n$, define the *descent set* $\mathrm{Des}(T)$ by

$$\mathrm{Des}(T) := \{i \in [n-1] \mid i+1 \text{ is located in a lower row than } i \text{ in } T\}.$$

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We call the sum of the elements of $\mathrm{Des}(T)$ the *major index* of T, and denote it by $\mathrm{maj}(T)$. In the preceding example, $\mathrm{Des}(T) = \{1, 4, 5\}$ and $\mathrm{maj}(T) = 1 + 4 + 5 = 10$.

Let S_n be the symmetric group of degree n, and

$$P_n = \mathbb{C}[x_1, x_2, \ldots, x_n]$$

denote the polynomial ring with n variables over $\mathbb{C}$. As customary, S_n acts on P_n from the left as permutations of variables by setting

$$(wf)(x_1, x_2, \ldots, x_n) = f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(n)}),$$

where $w \in S_n$ and $f(x_1, x_2, \ldots, x_n) \in P_n$. Let $I_n = \bigoplus_{d\geq 0} I^d$ denote the graded S_n-stable ideal of P_n generated by the elementary symmetric functions. Hence the quotient algebra $R_n = P_n/I_n$ is also a graded S_n-module. We write its homogeneous decomposition as

$$R_n = \bigoplus_{d\geq 0} R_n^d,$$

and call R_n the *coinvariant algebra* of S_n It is well known that the coinvariant algebra R_n affords the left regular representation of S_n.

Let us consider, for each integer $k = 0, \ldots, n-1$, the direct sum $R_n(k; n)$ of homogeneous components of R whose degrees are congruent to k modulo n, i.e.,

$$R_n(k; n) = \bigoplus_{d\equiv k \bmod n} R_n^d.$$

Since each homogeneous component R_n^d is S_n-invariant, these subspaces also afford representations of S_n, and the dimensions of these representations do not depend on k, i.e.,

$$\dim R_n(k; n) = (n-1)!$$

for all $k = 0, \ldots, n-1$.

In [4], W. Kraśkiewicz and J. Weyman consider these S_n-modules, and prove that each $R_n(k; n)$ is induced from a corresponding irreducible representation of a cyclic subgroup of S_n (see also [2, Proposition 8.2] [6, Theorem 8.9]). Precisely, let γ be the cyclic permutation $(12 \cdots n)$, and C_n the subgroup of S_n generated by γ. The cyclic subgroup C_n of degree n has n inequivalent irreducible representations

$$\psi^{(k)} \colon C_n \longrightarrow \mathbb{C}^\times, \quad \gamma \longmapsto \zeta_n^k,$$

where ζ_n is the primitive root of unity, and the following equivalence of S_n-modules holds for each $k = 0, \ldots, n-1$:

$$R_n(k; n) \cong_{S_n} \mathrm{ind}_{C_n}^{S_n}(\psi^{(k)}).$$

REMARK. In fact, the number n by which we take modulo is the *Coxeter number* of S_n, i.e., the order of the Coxeter elements of the Coxeter group of type A_{n-1}. They also obtain similar results for Coxeter groups of type B_n and D_n. Stembridge obtains more general results [8]. He treats the Complex reflection groups G and shows that the coinvariant algebra of G has the similar properties for the irreducible representation of the cyclic subgroup of G generated by a *Springer's regular element* [7]. We can easily see that the Coxeter elements are regular.

They also prove that the multiplicity of a irreducible representation of S_n in R_n^d ($d \geq 0$) is described by the major index of standard tableaux. It is well known that the irreducible representations of S_n are in one to one correspondence with the partitions of n. For $\lambda \vdash n$ let V^λ denote the corresponding irreducible representation of S_n. They showed that the multiplicity $[R_n^d : V^\lambda]$ of V^λ in R_n^d equals the number of standard tableaux whose major indices are d:

$$[R_n^d : V^\lambda] = \sharp\{T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) = d\}.$$

(see also [2, Theorem 8.6] [6, Theorem 8.8].) Combining these results, the multiplicities of the irreducible representation V^λ in the induced representations $\psi^{(k)}\uparrow_{C_n}^{S_n} \cong_{S_n} R_n(k; n)$ are easily obtained:

$$[R_n(k;n) : V^\lambda] = \sharp\{T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) \equiv k \bmod n\}.$$

It should be mentioned here that a more refined result is obtained by R. Adin, F. Brenti and Y. Roichman [1] recently. For each subset $S \subseteq [n-1]$, they construct an S_n-module R_S satisfying

$$R_n^d = \bigoplus_S R_n^S,$$

where the direct sum is taken over the subsets $S \subseteq [n-1]$ such that $\sum_{i \in S} i = d$, and describe the multiplicities of irreducible constituents on R_n^S as follows:

$$[R_n^S : V^\lambda] = \sharp\{T \in \mathrm{STab}(\lambda) \mid \mathrm{Des}(T) = S\}.$$

They also consider an analogue of the theorem of Kraśkiewicz and Weyman for the Weyl groups of type B, and obtain a result on the irreducible decompositions of the coinvariant algebras of type B finer than one already obtained by Stembridge in [8].

The aim of the present article is to achieve a generalization of these results in the following sense. Fix an integer $l \in [n]$ and consider subspaces of R_n obtained by gathering homogeneous components whose degrees are congruent modulo l. Precisely, for each $k = 0, \dots, l-1$ we will consider

$$R_n(k;l) = \bigoplus_{d \equiv k \bmod l} R_n^d.$$

We can see that the dimension of the space $R_n(k;l)$ is independent of k, i.e.,

$$\dim R_n(k;l) = \frac{n!}{l}$$

for all $k = 0, \dots, l-1$ (Proposition 4). In this article we will seek out a systematic realization of each submodule $R_n(k;l)$ as a S_n-module induced from a subgroup of S_n that is determined by l. First we settle a subgroup H_l of S_n for each $l \in [n]$, then construct a representation $\Psi(k;l)$ of H_l for each $k = 0, \dots, l-1$. When we write $n = dl + r$ with $0 \leq r \leq l-1$, the subgroup H_l turns out to be isomorphic to a direct product of the cyclic group of order l and the symmetric group of degree r, i.e.,

$$H_l \cong C_l \times S_r.$$

The representation $\Psi(k;l)$ of H_l is not necessarily irreducible in contrast to the case $l = n$ (Section 4). Finally, we verify that

$$R_n(k;l) \cong_{S_n} \operatorname{ind}_{H_l}^{S_n}(\Psi(k;l))$$

for each l and k by comparing the graded characters of R_n and $\bigoplus_{k=0}^{l-1} \operatorname{ind}_{H_l}^{S_n}(\Psi(k;l))$ as polynomials in q modulo $q^l - 1$ (Theorem 8).

2. Coinvariant algebra and its graded character

Let $R_n = \bigoplus_{d \geq 0} R_n^d$ be the coinvariant algebra of S_n and its homogeneous decomposition. Let q be an indeterminate over $\mathbb{C}$. Define the graded character of R_n by

$$X_n(q) = \sum_{d \geq 0} q^d \chi^{n,d},$$

where $\chi^{n,d}$ is the character of the representation R_n^d of S_n. We denote by $X_{n,\rho}(q)$ and $\chi_\rho^{n,d}$ the value of $X_n(q)$ and $\chi^{n,d}$ at elements of cycle-type $\rho \vdash n$, respectively. Precisely, $X_{n,\rho}(q)$ is a polynomial in q whose coefficient in q^d is $\chi_\rho^{n,d}$. This polynomial $X_{n,\rho}(q)$ is also known as a *Green polynomial* $Q_\rho^{(1^n)}(q)$ of type A [3] [5, III.7].

The graded character of R_n has a well-known product formula ([3, Appendix]. see also [2, Proposition 8.1]), that plays an essential role in the present article.

Proposition 1. *For any partition $\rho = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ of n, we have*

$$X_{n,\rho}(q) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^n)^{m_n}}.$$

From the Proposition above, we can prove the following auxiliary result.

Proposition 2. *Fix a integer $l \in [n]$. Let p be a divisor of l, $n = ep+s$ $(0 \le s \le p-1)$, and θ a primitive p-th root of unity. If $\rho \vdash n$ satisfies*

$$X_{n,\rho}(\theta) \neq 0,$$

then $\rho = (1^{m_1} \cdots s^{m_s} p^e)$, where $m_1 + 2m_2 + \cdots + sm_s = s$.

Proof. We apply Stembridge's argument for the case $l = n$ (see [2, Section 8]) to our situation. By Proposition 1, we have

$$X_{n,\rho}(\theta) = \left. \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^n)^{m_n}} \right|_{q=\theta},$$

for $\rho = (1^{m_1}2^{m_2}\cdots n^{m_n}) \vdash n$. Thus $X_{n,\rho}(\theta) \neq 0$ implies that all the vanishing factors in the numerator are canceled by corresponding factors in the denominator. There are e vanishing factors: $1-q^p$, $1-q^{2p}, \dots, 1-q^{ep}$ in the numerator, and $m_p+m_{2p}+\cdots+m_{ep}$ vanishing factors: $(1-q^p)^{m_p}$, $(1-q^{2p})^{m_{2p}}, \dots, (1-q^{ep})^{m_{ep}}$ in the denominator. Since

$$pm_p + 2pm_{2p} + \cdots + epm_{ep} \le m_1 + 2m_2 + \cdots + nm_n = n \ (= ep + s),$$

we have

$$m_p + 2m_{2p} + \cdots + em_{ep} \le e.$$

Therefore,

$$e = m_p + m_{2p} + \cdots + m_{ep} \le m_p + 2m_{2p} + \cdots + em_{ep} \le e.$$

Hence, we have $m_p = e$. We also obtain $m_i = 0$ for $s+1 \le i \le n$ $(i \neq p)$ since $n - pm_p = n - pe = s$. Thus, we have

$$m_1 + 2m_2 + \cdots + sm_s = s. \qquad \square$$

Let $l \in [n]$ be a fixed integer. For each $k = 0, 1, \dots, l-1$, we define

$$R_n(k;l) := \bigoplus_{d \equiv k \bmod l} R_n^d,$$

i.e.,

$$R_n = \bigoplus_{k=0}^{l-1} R_n(k;l).$$

We prove that the dimensions of the spaces $R_n(k;l)$ are independent of the choice of k. We first show the following lemma.

Lemma 3. *Let q be an indeterminate and $f(q) = \sum_{i\geq 0} a_i q^i \in \mathbb{C}[q]$ a polynomial in q. Let $l \geq 2$ be an integer and ζ_l a primitive l-th root of unity. Then the following conditions are equivalent*:
(1) $f(\zeta_l^k) = 0$ *for each* $k = 1, \dots, l-1$,
(2) *The partial sums* $c_k = \sum_{i\equiv k \bmod l} a_i$ $(k = 0, 1, \dots, l-1)$ *of coefficients of the polynomial* $f(q)$ *are independent of the choice of* k.

Proof. If the condition (b) holds, then $f(q)$ is divisible by

$$1 + q + q^2 + \cdots + q^{l-1} = \frac{1-q^l}{1-q},$$

and hence we have (a).

We shall prove the converse. From (a) we have

$$f(\zeta_l^k) = a_0 + a_1\zeta_l^k + a_2(\zeta_l^k)^2 + \cdots = 0 \quad (k = 0, 1, \dots, l-1).$$

By the definition of c_k, it reduces to the linear equation system in $c_0, \dots, c_{l-1}$:

$$\begin{cases} c_0 + c_1\zeta_l + c_2\zeta_l^2 + \cdots + c_{l-1}\zeta_l^{l-1} = 0, \\ c_0 + c_1\zeta_l^2 + c_2(\zeta_l^2)^2 + \cdots + c_{l-1}(\zeta_l^2)^{l-1} = 0, \\ \qquad \vdots \\ c_0 + c_1\zeta_l^{l-1} + c_2(\zeta_l^{l-1})^2 + \cdots + c_{l-1}(\zeta_l^{l-1})^{l-1} = 0. \end{cases}$$

Since the rank of the coefficient matrix of the equation system is $l-1$, it has an one dimensional solution space. It is clear that $(c_0, c_1, \dots, c_{l-1}) = (1, 1, \dots, 1)$ satisfies the equation system, hence we have $c_0 = c_1 = \cdots = c_{l-1}$. $\square$

By using the above lemma, we easily reach our aim.

Proposition 4. *Let $l \in [n]$ be a fixed integer. Then the dimension of $R_n(k;l)$ is independent of the choice of $k = 0, 1, \dots, l-1$, i.e., we have*

$$\dim R_n(k;l) = \frac{n!}{l}$$

for all $k = 0, 1, \dots, l$.

Proof. If $l = 1$, then the assertion is trivial. Suppose that $l \geq 2$. Let ζ_l be a primitive l-th root of unity. If we evaluate the formula in Proposition 1 at the identity element $e \in S_n$, then we have

$$\begin{aligned}[X_n(q)](e) &= X_{n,(1^n)}(q) \\ &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n} \\ &= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})\end{aligned}$$

It follows immediately that, for each $k = 0, \dots, l-1$,

$$X_{n,(1^n)}(\zeta_l^k) = \sum_{d\ge 0}(\dim R_n^d)q^d|_{q=\zeta_l^k} = 0.$$

By Lemma 3, we obtain that $\dim R_n(k;l) = \sum_{d\equiv k \bmod l} \dim R_n^d$ is independent of $0 \le k \le l-1$ and is equal to $n!/l$. □

If $w \in S_n$, the cycle type $\rho(w)$ of w is the partition $\rho(w) = (1^{m_1}2^{m_2}\cdots n^{m_n})$. For a partition ρ of n, let C_ρ be the conjugacy class in S_n containing $w \in S_n$ such that $\rho(w) = \rho$. For any partition $\rho = (1^{m_1}2^{m_2}\cdots n^{m_n})$, define

$$z_\rho = \frac{n!}{|C_\rho|} = 1^{m_1}m_1!2^{m_2}m_2!\cdots n^{m_n}m_n!.$$

Let f and g be arbitrary class functions on S_n. There is a natural scalar product of f and g defined by

$$\langle f, g\rangle_{S_n} := \frac{1}{n!}\sum_{w\in S_n} f(w)g(w).$$

(For a general finite group G, the scalar product is defined by $\langle f, g\rangle := (1/|G|)\times \sum_{w\in G} f(w)\overline{g(w)}$, where $\overline{g(w)}$ denotes the complex conjugate of $g(w)$. However, we can use $g(w)$ instead of $\overline{g(w)}$ here since all characters of S_n are rational.) Note that if δ_λ $(\lambda \vdash n)$ is the class function defined by

$$\delta_\lambda(w) = \begin{cases} 1 & \text{if } \rho(w) = \lambda \\ 0 & \text{otherwise,}\end{cases}$$

then $\langle \delta_\lambda, f\rangle_{S_n} = z_\lambda^{-1} f(\lambda)$.

If $n = dl + r$ $(0 \le r \le l-1)$, then we can embed $S_{dl} \times S_r$ in S_n by

$$\begin{aligned} S_{dl} &= \{w \in S_n \mid w(i) = i \text{ for all } i = dl+1, \dots, n\}, \\ S_r &= \{w \in S_n \mid w(i) = i \text{ for all } i = 1, \dots, dl\}. \end{aligned} \tag{2.1}$$

We see that, if $u \in S_{dl}$ and $v \in S_r$, the element $u \times v \in S_n$ has cycle-type $\rho(u \times v) = \rho(u) \cup \rho(v)$.

Let f and g be characters of the representations ϕ of S_{dl} and ψ of S_r, respectively. Then $f \times g$ defined by

$$(f \times g)(u, v) = f(u)g(v) \quad (u \in S_{dl},\ v \in S_r)$$

is the character of the tensor product representation $\phi \otimes \psi$ of $S_{dl} \times S_r$. We define

$$f \,.\, g = \mathrm{ind}_{S_{dl} \times S_r}^{S_n}(f \times g),$$

which is a character of the induced representation $\mathrm{ind}_{S_{dl} \times S_r}^{S_n}(\phi \otimes \psi)$ of S_n.

The following is a key proposition to the main result.

Proposition 5. *Let n be a positive integer, and choose an integer l $(1 \leq l \leq n)$. If $n = dl + r$ $(0 \leq r < l)$, then we have*

$$X_n(q) \equiv (X_{dl}(q) \,.\, X_r(q)) \quad \bmod q^l - 1.$$

Proof. We show that

$$X_{n,\rho}(q) \equiv (X_{dl}(q) \,.\, X_r(q))_\rho \quad \bmod q^l - 1 \tag{2.2}$$

for each $\rho \vdash n$, where $(X_{dl}(q) \,.\, X_r(q))_\rho$ is the value of $(X_{dl}(q) \,.\, X_r(q))$ at elements of cycle-type ρ. By the Lagrange interpolation and Proposition 2, in order to verify (2.2), it is sufficient to show that

$$(X_{dl}(\theta) \,.\, X_r(\theta))_\rho = \begin{cases} X_{n,\rho}(\theta) & \text{if } \rho = (1^{m_1} \cdots s^{m_s} p^e) \\ 0 & \text{otherwise.} \end{cases}$$

for each $\theta = \zeta_l^k$ $(k = 0, \ldots, l-1)$, where p is the multiplicative order of θ. Note that p divides l. Using the property of the class function δ_ρ, we then have

$$\begin{aligned}
&z_\rho^{-1}(X_{dl}(\theta) \,.\, X_r(\theta))_\rho \\
&= \langle (X_{dl}(\theta) \,.\, X_r(\theta)), \delta_\rho \rangle_{S_n} \\
&= \left\langle (X_{dl}(\theta) \times X_r(\theta)), \mathrm{res}_{S_{dl} \times S_r}^{S_n}(\delta_\rho) \right\rangle_{S_{dl} \times S_r} \qquad \text{(by Frobenius reciprocity)} \\
&= \frac{1}{(dl)!\, r!} \sum_{u \in S_{dl}} \sum_{v \in S_r} \big(X_{dl}(\theta) \times X_r(\theta)\big)(u, v)\delta_\rho(u \times v) \\
&= \frac{1}{(dl)!\, r!} \sum_{u \in S_{dl}} \sum_{v \in S_r} \sum_{\rho^1, \rho^2} X_{dl,\rho(u)}(\theta)\, X_{r,\rho(v)}(\theta)\delta_{\rho^1}(u)\delta_{\rho^2}(v) \\
&= \sum_{\rho^1, \rho^2} z_{\rho^1}^{-1} z_{\rho^2}^{-1} X_{dl,\rho^1}(\theta)\, X_{r,\rho^2}(\theta),
\end{aligned}$$

where $\rho^1 \vdash dl$ and $\rho^2 \vdash r$ are partitions such that $\rho^1 \cup \rho^2 = \rho$. Now let $n = ep + s$ and $r = fp + s$ $(0 \le s < p)$. Then $dl/p = e - f$. By Proposition 2, $X_{dl,\rho^1} X_{r,\rho^2} = 0$ unless $\rho^1 = (p^{e-f})$ and $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f)$. Hence, if ρ is not of the form $(1^{m_1} \cdots s^{m_s} p^e)$ for some $(1^{m_1} \cdots s^{m_s}) \vdash s$, we have $(X_{dl}(\theta) . X_r(\theta)) = 0$. On the other hand, we pick $\rho^1 = (p^{e-f})$ and $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f)$ so that $\rho = (1^{m_1} \cdots s^{m_s} p^e)$, and finally we have

$$\begin{aligned}
&z_\rho^{-1}(X_{dl}(\theta) . X_r(\theta))_\rho \\
&= z_{(p^{e-f})}^{-1} z_{(1^{m_1} \dots s^{m_s} p^f)}^{-1} X_{dl,(p^{e-f})}(\theta)\, X_{r,(1^{m_1} \dots s^{m_s} p^f)}(\theta) \\
&= z_{(p^{e-f})}^{-1} z_{(1^{m_1} \dots s^{m_s} p^f)}^{-1} \frac{(1-q)\cdots(1-q^{dl})}{(1-q^p)^{e-f}} \frac{(1-q)\cdots(1-q^r)}{(1-q)^{m_1}\cdots(1-q^s)^{m_s}(1-q^p)^f}\bigg|_{q=\theta} \\
&= z_{(p^{e-f})}^{-1} z_{(1^{m_1} \dots s^{m_s} p^f)}^{-1} \binom{e}{f}^{-1} \frac{(1-q)\cdots(1-q^{dl})(1-q^{dl+1})\cdots(1-q^{dl+r})}{(1-q)^{m_1}\cdots(1-q^s)^{m_s}(1-q^p)^e}\bigg|_{q=\theta} \\
&= z_\rho^{-1} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^{m_1}\cdots(1-q^s)^{m_s}(1-q^p)^e}\bigg|_{q=\theta} \\
&= z_\rho^{-1} X_{n,\rho}(\theta) \qquad \square
\end{aligned}$$

Translating Proposition 2 and Proposition 5 into the language of the Green polynomials, we obtain the following formula.

Corollary 6. *Let $n > l$ be positive integers, p a divisor of l, and θ a primitive p-th root of unity. If we write $n = dl + r = ep + s$ $(0 \le r \le l-1,\ 0 \le s \le p-1)$, then*
(1) $Q_\rho^{(1^n)}(\theta) = 0$ *unless* $\rho = (1^{m_1} \cdots s^{m_s} p^e)$ *and* $m_1 + 2m_2 + \cdots + sm_s = s$.
(2) *If* $\rho = (1^{m_1} \cdots s^{m_s} p^e)$,

$$Q_\rho^{(1^n)}(q) \equiv Q_{\rho^1}^{(1^{dl})}(q)\, Q_{\rho^2}^{(1^r)}(q) \bmod q^l - 1,$$

where $\rho^1 = (p^{e-f}) \vdash dl$ *and* $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f) \vdash r$.

3. $l|n$ case

In this section, we consider the case where l divides n, and show that each $R_n(k;l)$ is induced from a representation of a cyclic subgroup of S_n.

Suppose that l divides n, and say $d = n/l$. Let C_l be the cyclic group of order l, and we embed C_l into S_n as follows:

$$C_l \cong \langle \gamma_1\gamma_2\cdots\gamma_d \rangle \subset S_n,$$

where $\gamma_1 = (1, 2, \dots, l), \gamma_2 = (l+1, l+1, \dots, 2l), \dots, \gamma_d = ((d-1)l+1, \dots, dl)$. The cyclic group C_l has inequivalent l irreducible representations $\psi^{(0)}, \dots, \psi^{(l-1)}$, i.e.,

$$\psi^{(k)}\colon C_l \longrightarrow \mathbb{C}^\times, \quad \gamma_1\gamma_2\cdots\gamma_d \longmapsto \zeta_l^k,$$

where ζ_l denotes a primitive l-th root of unity. Let

$$\tau^{(k)} := \frac{1}{l}\sum_{i=0}^{l-1} \zeta_l^{-ik}(\gamma_1\cdots\gamma_d)^i \quad (k = 0, 1, \ldots, l-1).$$

We can easily check that each $\tau^{(k)}$ is an idempotent by a direct calculation.

Let $\mathbb{C}[S_n]$ be the group algebra of S_n. Consider the representation of S_n afforded by the left ideal $\mathbb{C}[S_n]\tau^{(k)}$, which is equivalent to the induced representation $\mathrm{ind}_{C_l}^{S_n}(\psi^{(k)})$. Its character $\chi[\mathbb{C}[S_n]\tau^{(k)}]$ is given by $\Gamma_n\tau^{(k)}$, where Γ_n is an operator defined by

$$\Gamma_n\colon \mathbb{C}[S_n] \longrightarrow \mathbb{C}[S_n], \quad \rho \longmapsto \sum_{w\in S_n} w^{-1}\rho w$$

(see e.g., [2, Proposition 5.2] [6, Lemma 8.4]). Here we regard an element $\rho = \sum_{w\in S_n}\rho_w w \in \mathbb{C}[S_n]$ as the function on S_n that maps $w \in S_n$ to the coefficient ρ_w:

$$\mathrm{ind}_{C_l}^{S_n}(\chi[\psi^{(k)}]) = \Gamma_n\tau^{(k)},$$

where $\chi[\psi^{(k)}]$ stands for the C_l-character of $\psi^{(k)}$.

We have shown in Proposition 4 that the dimension of the space

$$R_n(k;l) = \bigoplus_{d\equiv k \bmod l} R_n^d$$

is constant with respect to $k = 0, \ldots, l-1$. This fact suggests that every $R_n(k;l)$ $(k = 0, \ldots, l-1)$ are induced from the same dimensional representations of a certain subgroup of S_n. In fact, we can verify that, for each $k = 0, \ldots, l-1$, there exists an irreducible representation of C_l that yields $R_n(k;l)$.

Proposition 7. *Let n be a positive integer and l a divisor of n. Write $d = n/l$. For $i = 1, 2, \ldots, d$, let γ_i be the cyclic permutation $((i-1)l+1, (i-1)l+2, \ldots, il)$. Let C_l be the cyclic subgroup of S_n generated by $\gamma_1\cdots\gamma_d$ and $\{\psi^{(k)} \mid k = 0, 1, \ldots, l-1\}$ the set of its inequivalent irreducible representations. Then, we have an isomorphism of S_n-modules*

$$R_n(k;l) \cong_{S_n} \mathrm{ind}_{C_l}^{S_n}(\psi^{(k)}) \quad (k = 0, 1, \ldots, l-1).$$

Proof. We prove that

$$X_n(q) \equiv \sum_{k=0}^{l-1} q^k\, \mathrm{ind}_{C_l}^{S_n}(\chi[\psi^{(k)}]) \mod q^l - 1. \tag{3.1}$$

Using the Lagrange interpolation again, we only have to show that the both sides of (3.1) coincide when $q = \zeta_l^s$ $(s = 0, 1, \ldots, l-1)$.

Recall that

$$\mathrm{ind}_{C_l}^{S_n}(\chi[\psi^{(k)}]) = \Gamma_n \tau^{(k)}$$

for each $k = 0, \ldots, l-1$. Substituting $q = \zeta_l^s$ in the right hand side of (3.1), we obtain

$$\begin{aligned}
\sum_{k=0}^{l-1} (\zeta_l^s)^k \,\mathrm{ind}_{C_l}^{S_n}(\chi[\psi^{(k)}]) &= \sum_{k=0}^{l-1} \zeta_l^{ks} \Gamma_n \tau^{(k)} = \Gamma_n(\gamma_1 \cdots \gamma_d)^s \sum_{k=0}^{l-1} \tau^{(k)} \\
&= \Gamma_n(\gamma_1 \cdots \gamma_d)^s \sum_{k=0}^{l-1} \frac{1}{l} \sum_{i=0}^{l-1} \zeta_l^{-ik} (\gamma_1 \cdots \gamma_d)^i \\
&= \Gamma_n(\gamma_1 \cdots \gamma_d)^s \frac{1}{l} \sum_{i=0}^{l-1} (1 + \zeta_l^{-i} + \zeta_l^{-2i} + \cdots + \zeta_l^{-(l-1)i})(\gamma_1 \cdots \gamma_d)^i \\
&= \Gamma_n(\gamma_1 \cdots \gamma_d)^s
\end{aligned}$$

for each $s = 0, 1, \ldots, l-1$. Since the cycle-type of $(\gamma_1 \cdots \gamma_d)^s$ can be written as (p^e) $(e = n/p)$, where p is the multiplicative order of $(\zeta_l^s)^p = 1$, we have

$$\sum_{k=0}^{l-1} (\zeta_l^s)^k \,\mathrm{ind}_{C_l}^{S_n}(\chi[\psi^{(k)}])_\rho = \begin{cases} z_{(p^e)}, & \text{if } \rho = (p^e) \\ 0, & \text{otherwise} \end{cases}$$

for a partition ρ. Hence the congruence (3.1) immediately follows from Proposition 1 and Proposition 2. $\square$

4. Main result

Let n be a positive integer, and choose an integer $l = 1, 2, \ldots, n$. Suppose that $n = dl + r$, where $0 \leq r \leq l-1$. Let R_n be the coinvariant algebra of S_n, and $R_n = \bigoplus_{d \geq 0} R_n^d$ its homogeneous decomposition. For each $k = 0, 1, \ldots, l-1$, define

$$R_n(k; l) := \bigoplus_{d \equiv k \bmod l} R_n^d.$$

Now, for each $l = 1, 2, \ldots, n$, we define a subgroup H_l of S_n by

$$\begin{aligned}
H_l &= \langle \gamma_1 \gamma_2 \cdots \gamma_d \rangle \times S_r \\
&\cong C_l \times S_r,
\end{aligned}$$

where γ_i is the cyclic permutation $((i-1)l+1, (i-1)l+2, \ldots, il)$, and the symmetric group S_r of degree r is identified as the subgroup $\{w \in S_n \mid w(i) = i \text{ for all } i = 1, 2, \ldots, n-r\}$ of S_n.

For each $k = 0, 1, \ldots, l-1$, we construct a representation $\Psi(k;l)$ of H_l as follows:

$$\Psi(k;l) := \bigoplus_{\lambda\vdash r} \bigoplus_{T\in\mathrm{STab}(\lambda)} \psi^{(\underline{k-\mathrm{maj}(T)})} \otimes V^{\lambda},$$

where $\underline{k-\mathrm{maj}(T)} = k-\mathrm{maj}(T) \mod l$, $\{\psi^{(i)} \mid i = 0, \ldots, l-1\}$ is the set of inequivalent irreducible representation of C_l, and V^{λ} $(\lambda \vdash r)$ is the irreducible representation of S_r corresponding to the partition λ of r. Then it can be seen that the dimension of $\Psi(k;l)$ does not depend on k and hence so does $\deg \mathrm{ind}_{H_l}^{S_n}(\Psi(k;l))$. Actually, since $\deg V^{\lambda} = \sharp\ \mathrm{STab}(\lambda)$ and $\sum_{\lambda\vdash r} \sharp\,\mathrm{STab}(\lambda)^2 = r!$, we have

$$\begin{aligned}
\deg \Psi(k;l) &= \sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} \deg \psi^{(\underline{k-\mathrm{maj}(T)})} \otimes V^{\lambda} \\
&= \sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} \sharp\,\mathrm{STab}(\lambda) \\
&= \sum_{\lambda\vdash r} \sharp\,\mathrm{STab}(\lambda)^2 \\
&= r!,
\end{aligned}$$

and $\deg \mathrm{ind}_{H_l}^{S_n}(\Psi(k;l)) = r!\,n!/r!l = n!/l$, which coincides with the dimension of $R_n(k;l)$. Moreover, we prove that these two representations are equivalent.

Theorem 8 (Main result). *Let n be a positive integer. Fix an integer $l \in [n]$ and write $n = dl + r$ $(0 \leq r \leq l-1)$. Let $H_l \cong C_l \times S_r$ be the subgroup of S_n defined above and $\Psi(k;l)$ $(k = 0, 1, \ldots, l-1)$ representations of it defined by*

$$\Psi(k;l) := \bigoplus_{\lambda\vdash r} \bigoplus_{T\in\mathrm{STab}(\lambda)} \psi^{(k-\mathrm{maj}(T))} \otimes V^{\lambda},$$

where $\psi^{(i)}$ and V^{λ} stand for the irreducible representations of C_l and S_r, respectively. Then, for each $k = 0, 1, \ldots, l-1$, there is an isomorphism

$$R_n(k;l) \cong_{S_n} \mathrm{ind}_{H_l}^{S_n}(\Psi(k;l)).$$

as an S_n-module.

Proof. By the definition of $\Psi(k;l)$, it suffices to show

$$X_n(q) \equiv \sum_{k=0}^{l-1} q^k \sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} \mathrm{ind}_{H_l}^{S_n}\left(\chi\left[\psi^{(k-\mathrm{maj}(T))} \otimes V^{\lambda}\right]\right) \mod q^l - 1. \tag{4.1}$$

Let S_{dl} and S_r be the subgroup of S_n defined in (2.1). Since H_l is a subgroup of $S_{dl} \times S_r$, we have

$$\mathrm{ind}_{H_l}^{S_n} \left(\psi^{\underline{(k-\mathrm{maj}(T))}} \otimes V^{\lambda}\right) \cong_{S_n} \mathrm{ind}_{S_{dl}\times S_r}^{S_n} \left(\mathrm{ind}_{H_l}^{S_{dl}\times S_r} \left(\psi^{\underline{(k-\mathrm{maj}(T))}} \otimes V^{\lambda}\right)\right)$$

for any $\lambda \vdash r$. Therefore, the right hand side of (4.1) equals

$$\begin{aligned}
&\sum_{k=0}^{l-1} \sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} q^k \, \mathrm{ind}_{S_{dl}\times S_r}^{S_n} \left(\mathrm{ind}_{C_l\times S_r}^{S_{dl}\times S_r} \left(\chi\left[\psi^{\underline{(k-\mathrm{maj}(T))}} \otimes V^{\lambda}\right]\right)\right) \\
&= \sum_k \sum_\lambda \sum_T q^k \, \mathrm{ind}_{S_{dl}\times S_r}^{S_n} \left(\mathrm{ind}_{C_l}^{S_{dl}} \left(\chi\left[\psi^{\underline{(k-\mathrm{maj}(T))}}\right]\right) \times \chi[V^{\lambda}]\right) \\
&= \mathrm{ind}_{S_{dl}\times S_r}^{S_n} \left(\sum_k \sum_\lambda \sum_T q^{k-\mathrm{maj}(T)} \, \mathrm{ind}_{C_l}^{S_{dl}} \left(\chi\left[\psi^{\underline{(k-\mathrm{maj}(T))}}\right]\right) \times q^{\mathrm{maj}(T)} \chi[V^{\lambda}]\right)
\end{aligned}$$

$$(4.2) \qquad \equiv \mathrm{ind}_{S_{dl}\times S_r}^{S_n} X_{dl}(q) \left(\sum_\lambda \sum_T q^{\mathrm{maj}(T)} \chi[V^{\lambda}]\right) \quad \mathrm{mod}\ q^l - 1 \text{ by (3.1).}$$

By the theorem of Kraśkiewicz-Weyman, the multiplicity $[R_n^d : V^{\lambda}]$ of irreducible components isomorphic to V^{λ} $(\lambda \vdash n)$ is the number of standard Young tableaux of shape λ whose major index equals d, that is,

$$[R_n^d : V^{\lambda}] = \sharp\{T \in \mathrm{STab}(\lambda) : \mathrm{maj}(T) = d\}.$$

Hence we have

$$(4.3) \qquad X_r(q) = \sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} q^{\mathrm{maj}(T)} \chi[V^{\lambda}].$$

Applying (4.3) and Proposition 5, we see that (4.2) equals

$$\begin{aligned}
&\mathrm{ind}_{S_{dl}\times S_r}^{S_n} X_{dl}(q) \left(\sum_{\lambda\vdash r} \sum_{T\in\mathrm{STab}(\lambda)} q^{\mathrm{maj}(T)} \chi[V^{\lambda}]\right) \\
&= \mathrm{ind}_{S_{dl}\times S_r}^{S_n} X_{dl}(q) \times X_r(q) \\
&= (X_{dl}(q) \,.\, X_r(q)) \\
&\equiv X_n(q) \quad \mathrm{mod}\ q^l - 1,
\end{aligned}$$

and complete the proof. □

When $r = 0$ or 1, H_l is a cyclic group and $\Psi(k; l)$ is irreducible. In this case, the generator of H_l coincides with a regular element of S_n defined by Springer [7].

It is obvious that the multiplicity of V^{λ} in $R_n(k; l)$ is obtained by counting the number of standard Young tableaux of shape λ with the major index congruent to k modulo l, that is,

$$[R_n(k;l) : V^\lambda] = \sharp\{T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) \equiv k \mod l\}.$$

EXAMPLE. In the case of $n = 5$ and $l = 3$, the subgroup H_3 is $\langle(123)\rangle \times \langle(45)\rangle$, which is isomorphic to $C_3 \times S_2$. Then we have

$$R_5(k;3) \cong_{S_5} \mathrm{ind}_{H_3}^{S_5}\Big(\big(\psi^{(k)} \otimes V^{(2)}\big) \oplus \big(\psi^{(\underline{k-1})} \otimes V^{(1,1)}\big)\Big)$$

for each $k = 0, 1, 2$.

If we consider the case $n = 11$ and $l = 4$ (thus $r = 3$), then the subgroup H_4 is $\langle(1234)(5678)\rangle \times \langle(9,10),(10,11)\rangle$ isomorphic to $C_4 \times S_3$. Hence, for each $R_{11}(k;4)$ ($k = 0, 1, 2, 3$) is isomorphic to the representation induced by

$$\begin{aligned}
\Psi(0;4) &= (\psi^{(0)} \otimes V^{(3)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(1,1,1)}),\\
\Psi(1;4) &= (\psi^{(1)} \otimes V^{(3)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(1,1,1)}),\\
\Psi(2;4) &= (\psi^{(2)} \otimes V^{(3)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(1,1,1)}),\\
\Psi(3;4) &= (\psi^{(3)} \otimes V^{(3)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(1,1,1)}).
\end{aligned}$$

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