Quark-Hadron matter in two color Nambu-Jona-Lasinio model

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Abstract

Quark-hadron matter at finite temperature and density is studied using a two color and two flavor (Polyakov loop extended) Nambu-Jona-Lasinio model with scalar diquark channel. A hadronic effective Lagrangian is derived by bosonization of the quark fields and renormalized using the Eguchi method. Although the bosonization technique has been used only for mesons, we can apply the technique not only for mesons (quark-antiquark) but also baryons (diquark) in two color scheme. Since baryons construct a matter, the bosonization of diquark-baryons is important to understand the behavior of the chiral symmetry in medium. We find the derived Lagrangian can be identified as an extended linear sigma model with meson and diquark-baryon fields. Hence our Lagrangian can describe both the quark and hadron dynamics.

The derived hadron effective Lagrangian is applied to thermodynamics by dropping the interaction terms (Gaussian approximation). We introduce the Polyakov loop and its effective potential to investigate the quark confinement kinematics in finite temperature. Since the diquark-baryons are bosons in two color scheme, Bose-Einstein condensation arises at finite density at a certain chemical potential. The order parameters which are the chiral condensate, the diquark condensate and the expectation value of the Polyakov loop are studied as functions of temperature and density in the mean field approximation. Masses of mesons and diquark-baryons are investigated at finite temperature and density. At finite temperature, the behaviors of the masses are discussed with and without the Polyakov loop effect. For a finite density system, the diquark mass goes to zero and becomes the Nambu-Goldstone boson which comes from the baryon number symmetry breaking. We investigate the equation of state of quark-hadron matter by taking into account the contributions of mesons and diquark-baryons in addition to the quark quasi-particles. We describe the baryon number density and compare our results with lattice QCD simulations. We find the Gaussian approximation for the hadron Lagrangian is not enough to reproduce the lattice results and the hadron Lagrangian should be treated fully by including the interaction terms using a non-perturbative method.

Finally, we review the Gaussian functional method as a non-perturbative treatment. We have not applied this method to our extended linear sigma model yet, but we report the features of the Gaussian function approach.

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Contents

Chapter 1

Introduction

Nucleus is one of most important ingredients in our universe. A collection of nucleons form a many body system (nucleus or nuclear matter) by strong nuclear interaction and realize various non-trivial physical phenomena. Nucleon is known as one of strongly interacting particles called hadrons. Hadrons such as nucleons are classified into baryons which construct matter and mesons such as pions which intermediate interaction among nucleons. The hadron interaction is about 100 times stronger than the electromagnetic interaction and has to be treated non-perturbatively.

It is considered that the origin of long range attractive nuclear force comes from pion exchange between nucleons. The middle range attractive force also can be described with light mesons. The light mass mesons such as pions are considered as generated from spontaneously breaking of chiral symmetry as Nambu-Goldstone (NG) particle. The concept of spontaneous chiral symmetry breaking (SCSB) is introduced by Y. Nambu and G. Jona-Lasinio [1,2]. When we consider massless particles (nucleons) in a chiral invariant model, their masses are generated by the SCSB and the massless particles appear as a bound state of nucleaons. Since the real pion has a small mass, it is considered the fundamental particles have a small mass (current mass). The SCSB realizes the light mass pions which intermediate the interaction between nucleaons, then combine the nucleons and eventually make nucleus and nuclear matter (hadron matter). The chiral symmetry and its spontaneously breaking mechanism are important ingredients of hadron many body physics. It is believed that the chiral symmetry is spontaneously broken by the effect of many body correlation and the symmetry is restored at high temperature and/or density. Hence it is very interesting to study the behavior of the chiral symmetry in medium and the property of hadron matter.

Although the hadron many body theory itself requires a complicated non-perturbative treatment with chiral symmetry, hadrons themselves furthermore have internal structure in terms of quarks and gluons. The quarks and gluons have a quantum number of color charge and its dynamics is described by quantum chromodynamics (QCD). The quarks are fermions and construct a matter and the gluons are the gauge bosons and provide interaction among quarks. QCD is a non-Abelian gauge theory belonging $\mathrm{SU}(3)$ color group.

The chiral symmetry is nowadays understood as a fundamental symmetry of QCD. The QCD Lagrangian is invariant under global chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$ when all the N_f quarks are massless. The $\mathrm{U}(1)_A$ axial symmetry is exact only at the classical level, and quantum corrections break the $\mathrm{U}(1)_A$ axial symmetry. At low energy scale, the chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ is spontaneously broken to the vector symmetry

SU$(N_f)_V$ which is characterized by a non-vanishing chiral condensate $\langle \bar{q}q \rangle$. Finally, the vector symmetry U$(1)_V$ corresponds to the baryon number which is an exact symmetry.

QCD is further characterized by two important features. The first one is asymptotic freedom that the interaction strength becomes weaker at high energies, or equivalently small distances and hence the perturbation theory can be applied at some energy scale much larger than the so-called QCD scale $\Lambda_{QCD} \sim 200$ [MeV]. The second one is color confinement, i.e. the phenomenon that a colored object, such as a single quark and gluon, does not exist as a single isolated body and they make colorless composites, the hadrons. *Therefore the hadron physics should be described from quarks and gluons based on QCD*. However, since QCD is a strong coupling theory at low energies, almost no hadron phenomena are derived analytically from QCD directly yet and are described by lattice QCD simulation numerically.

It is believed that QCD quark-gluon matter at finite baryon density ρ_B and temperature T has a rich phase structure characterized by the chiral symmetry and confinement. The order parameter of chiral symmetry breaking corresponds to the chiral condensate. At low temperature and density, the quarks and the gluons are confined inside the hadrons and chiral symmetry is spontaneously broken. When the temperature and/or density is increased, the chiral symmetry is restored at certain temperature and/or density. The confinement is also an important property. To discuss the hadron physics from quarks and gluons, we need the confinement mechanism, which, however, is not described from QCD analytically yet. At high temperature and/or density it is believed that the quarks and gluons are de-confined and become active degrees of freedom. At finite temperature, Polyakov loop Φ can be treated as an order parameter of the confinement [3–5]. Thus, hadron matter described from QCD or in terms of quarks and gluons at various temperature and density should have these behaviors, chiral symmetry breaking and color confinement.

In the theoretical approach there are usually two different strategies to investigate the strongly interacting matter. The first one is the numerical simulation by using lattice gauge theory, which is called lattice QCD. Lattice QCD simulations can handle strongly interacting matter at finite temperature and at very small baryon chemical potential. As a consequence of the sign problem in lattice QCD at finite density, it is extremely difficult, however, to deal with the QCD quark-gluon matter at finite density covering broad ranges of real baryon chemical potential. The other is a model calculation based on fundamental symmetries of QCD called effective model or effective theory. Effective model approaches have successfully explained many experimental results of low energy hadron phenomena historically. In this thesis we adopt the second approach, because our main interest is the hadron phase in QCD matter and at finite density which cannot be dealt with in the present lattice QCD simulation. Our study has several steps:

1. Start from an effective theory in terms of quarks and gluons having the SCSB and the kinematical confinement mechanisms.

2. Construct mesons and baryons and their dynamics from the effective theory.

3. Apply the quark and hadron dynamics to finite density and temperature and investigate the matter properties.

The comparison with the lattice QCD at finite temperature and our analysis is important because the *ab initio* calculation can be considered as “experimental data” to check our approach. Then we want to explore the quark-hadron matter property at finite density.

The Nambu-Jona-Lasinio (NJL) model [1, 2, 6–8] is often used as an effective model of QCD, schematic approach to strongly interacting matter [9–20], based on chiral symmetry and its spontaneous breaking. The NJL model is further generalized by introducing the Polyakov loop to account for important thermodynamical aspects of color confinement [21–23], which is called Polyakov loop extended Nambu-Jona-Lasinio (PNJL) model. This PNJL model is by now widely used for the discussion of quark-hadron matter at finite temperature and density [23–41]. The PNJL model results can be compared directly with lattice QCD at finite temperature and zero baryon density. There is, however, a basic conceptual problem when applying the PNJL model at finite baryon density: color-singlet baryon formation is not accounted for. While color-nonsinglet degrees of freedom are suppressed in the "baryonic" phase by the Polyakov loop, three quarks coupled to a color-singlet are still delocalized and spread over all space instead of being confined in localized baryonic clusters.

A remarkable method for the derivation of hadrons from the quark model (NJL model) is discussed by Eguchi and several authors [42–44] who derive an effective meson model (linear sigma model) from NJL model using the bosonization method. The quark bilinear form such as $\bar{q}q$ in the Lagrangian in the partition function is replaced by an auxiliary variable σ to integrate out the quark fields, *leading to an effective Lagrangian for σ as a composite of quark fields*. The corresponding bosonic Lagrangian can be constructed by quark loop expansion. The derived meson Lagrangian properties are described by the original quark dynamics [45–47]. A most important aspect of Eguchi method is that the derived effective meson (hadron) Lagrangian is renormalizable, although the NJL model is unrenormalizable and momentum cut-off Λ has to be introduced to regularize quantum corrections. The quark loop expansion terms for the auxiliary fields can be classified into divergent integration terms U_{div} and convergent integration terms U_{conv} in the limit of the cut-off $\Lambda \to \infty$. Eguchi found that the divergent terms U_{div} appear up to fourth order of meson fields and correspond to the radiative corrections of the linear sigma model. The renormalization prescriptions for the linear sigma model can be applied to the divergent terms U_{div}, which are absorbed into renormalization parameters for linear sigma model although the divergent terms are calculated using quark propagators. Hence, the divergent U_{div} are renormalized and the obtained Lagrangian can be identified as the linear sigma model, which is renormalizable. This method gives us a big advantage because our purpose is to describe hadron theory from quark level, which should be renormalizable. We thus have a desire to include not only mesons but also baryons in the prescription. Baryons are described, however, the three body composite in contrast with mesons. The bosonization technique cannot be applied directly for baryons yet.

These difficulties lead us to the simpler two color $\mathrm{SU}(2)_c$ QCD scheme to study finite density quark-hadron matter. In this theory quarks carry baryon number $\frac{1}{2}$ and baryons emerge as diquarks, i.e. spin singlet or triplet bosons. The resulting physics is qualitatively different from "real" QCD with $N_c = 3$. Nonetheless, exploring the $N_c = 2$ theory and designing (P)NJL type models in which both mesons (quark-antiquark modes) and baryons (diquarks) emerge as active hadronic degrees of freedom, can teach important issues about the thermodynamics of strongly interacting quark-hadron matter at non-zero baryon chemical potential. While the two color QCD is not real, the corresponding lattice QCD approach is able to avoid the sign problem and the simulations can be performed at any baryon chemical potential. Several lattice QCD studies for color $\mathrm{SU}(2)$ are available [48–53] and provide equations of state and hadron masses at both finite temperature and density. It is then an interesting question to what extent these lattice QCD results can be understood and interpreted in terms of models, such as the

PNJL approach, in order to identify leading mechanisms and basic symmetry breaking patterns. Our work is aimed in that direction as the first step of our interest. Theoretical studies based on symmetries [54–63] and on a color $\mathrm{SU}(2)$ NJL model [64–69] have already given insights concerning the matter properties and hadron masses.

However, most studies are focused on the quark matter and phase diagram in the mean field approximation. The properties of hadrons, especially baryons in medium, are not discussed in full length. For example, meson-baryon interaction should be described from quark. We would like to connect the quarks to hadrons, not only mesons but also (diquark-)baryons, using the Eguchi method. By using the Eguchi method, we can obtain an effective meson-baryon Lagrangian, which consists of composite hadrons made of the quarks and is renormalizable. We are then able to take into account the effect of hadrons by solving a many body problem with the meson-baryon Lagrangian. At the same time, the two color NJL model may provide a method to handle the quark-gluon confinement and contribute to the discussion in the color $\mathrm{SU}(3)$ many body physics.

This thesis is organized as follows. In Chap. 2, we briefly review the NJL model in two color system and introduce the auxiliary fields as spinless hadron fields to integrate out the quark fields. We compute the NJL model Lagrangian in the mean field approximation and express the mass-gap equations in terms of divergent integrals including quark propagators, which have the information of chiral and diquark phase transition. In Chap. 3, the bosonization technique is manipulated in order to derive an effective hadron Lagrangian. The quark loop propagators are expanded and decomposed into the divergent and convergent parts. By retaining only the divergent integrals, we find the hadron Lagrangian, which can be identified as extended linear sigma model Lagrangian. The final hadron Lagrangian is obtained by performing renormalization following the method of Eguchi. Finally, we obtain the mixed Lagrangian, which can describe both quarks and hadrons dynamics. In Chap. 4, the quark-hadron Lagrangian is applied to thermodynamics of quark-hadron matter using the Matsubara formalism. In order to integrate the hadron fields, we introduce the Gaussian approximation dropping the interaction terms among hadrons. The Polyakov loop and its effective potential are introduced in the quark loop integrals. The Polyakov loop effect suppresses the colored particle degree of freedom and reproduces the entanglement of chiral condensate and deconfinement at finite temperature, which is well-known in the lattice calculation. We discuss the thermodynamical properties analytically. The massless NG boson appears due to the baryon number symmetry breaking at finite density and plays an important role in the matter property. Then the numerical results are compared with the lattice calculation. We reproduce most of the hadron properties at finite temperature and/or density. However, for the matter density as function of the chemical potential of quark-hadron matter, we find the Gaussian approximation for the hadron Lagrangian is not enough and the hadron Lagrangian should be treated fully including the interaction terms using a non-perturbative approach. In Chap. 5, we review the Gaussian functional method. We have not applied this method numerically to our extended linear sigma model yet, but we report the features of the Gaussian functional method. Finally, we summarize our present study on the bosonization of the NJL model in two color scheme, thermodynamical property of the hadron and the Gaussian functional method in Chap. 6, together with the discussion and the outlook of the present study.

Chapter 2

Two Color Quark Model

We construct two color NJL type Lagrangian with an interaction in the scalar diquark channel [64, 70]. Since we consider $\mathrm{SU}(2)$ gauge group, a diquark qq makes color singlet. The interaction term is generated from the conserved QCD color current, hence NJL model can be regarded as an effective model of QCD [6–8]. The Lagrangian has the chiral symmetry in the chiral limit. From the discussion of the symmetry breaking pattern, we will introduce two order parameters which are the chiral condensate $\langle \bar{q}q \rangle$ and the diquark condensate $\langle qq \rangle$ [54, 55, 71]. The auxiliary fields will be introduced and treated as the composite particles for hadrons. The vacuum energy will be evaluated in the mean field approximation at finite temperature and density.

2.1 Symmetries in two color QCD

We first review symmetries and its breaking pattern of two color QCD following Refs. [54, 55, 61, 63] to construct an effective model of QCD. A theory of the strong interaction is described by QCD Lagrangian:

$$\mathcal{L}_{QCD} = \sum_{n=1}^{N_f} \bar{\psi}_n i \gamma_\mu D^\mu \psi_n - \frac{1}{2} \mathrm{tr} \left[F^{\mu\nu} F_{\mu\nu} \right], \tag{2.1}$$

with the massless N_f quarks. The field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_s [A_\mu, A_\nu]$ with the gluon fields $A_\mu = A_\mu^a \frac{t_a}{2}$ where t_a $(a = 1, 2, 3)$ is the Pauli matrices of color $\mathrm{SU}(2)$, satisfying $\mathrm{tr}[t^a t^b] = 2\delta^{ab}$, and the covariant derivative is $D^\mu = \partial^\mu - i g_s A^\mu$ where g_s is the gauge coupling constant. This Lagrangian has the global symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R \to \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$. The $\mathrm{U}(1)_A$ axial symmetry is broken by axial anomaly.

Our aim is to construct an effective theory of QCD. In this context, we focus on the quark sector of the QCD Lagrangian. The local color gauge symmetry is replaced by global symmetry. In two color QCD, it is known that the global symmetry of the theory is $\mathrm{SU}(2N_f)$ rather than the above $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)$. This can be seen explicitly by using chiral Weyl components of the Dirac spinor $\psi^T = (q_L, q_R)$ (the superscript T denotes the transpose of the matrix):

$$\mathcal{L}_{kin} = \bar{\psi} i \gamma_\mu D^\mu \psi = q_L^\dagger i \sigma_\mu D^\mu q_L + q_R^\dagger i \bar{\sigma}_\mu D^\mu q_R, \tag{2.2}$$

where the gamma matrices γ_μ are decomposed the spin Pauli matrices σ_μ and $\bar{\sigma}_\mu$ in the chiral representation. We have omitted the flavor indices n and its summation. The pseudo-real property of the generators of SU(2) (Pauli matrices):

$$t_a^* = t_a^T = -t_2 t_a t_2, \quad (a = 1, 2, 3) \tag{2.3}$$

gives the property of the Dirac operator, $D_\mu = \partial_\mu - i g_s A_\mu$ (A_μ is antihermitian SU(2) color matrices):

$$D_0^T = D_0 = t_2 D_0 t_2, \quad D_i^T = -t_2 D_i t_2. \tag{2.4}$$

We introduce

$$\tilde{q} = \sigma_2 t_2 q_R^\dagger, \quad \tilde{q}^\dagger = q_R^T t_2 \sigma_2. \tag{2.5}$$

By using this notation, the second term of Eq. (2.2) becomes

$$\begin{aligned} i q_R^\dagger \bar{\sigma}_\mu D^\mu q_R =& i q_R^\dagger \sigma_0 \partial^0 q_R - i q_R^\dagger \sigma_0 i g_s A^0 q_R + i q_R^\dagger \sigma_i \partial^i q_R - i q_R^\dagger \sigma_i i g_s A^i q_R \\ =& i q_R^\dagger t_2 \sigma_2 \sigma_0^T \sigma_2 t_2 \partial^0 q_R + i q_R^\dagger t_2 \sigma_2 \sigma_0^T i g_s (A^0)^T t_2 \sigma_2 q_R \\ &+ i q_R^\dagger t_2 \sigma_2 \sigma_i^T \sigma_2 t_2 \partial^i q_R - i q_R^\dagger \sigma_2 \sigma_i^T \sigma_2 i g_s t_2 (A^i)^T t_2 q_R \\ =& i (q_R^T t_2 \sigma_2) \sigma_0 \partial^0 (\sigma_2 t_2 q_R^\dagger) - i (q_R^T t_2 \sigma_2) \sigma_0 i g_s A^0 (\sigma_2 t_2 q_R^\dagger) \\ &- i (q_R^T t_2 \sigma_2) \sigma_i \partial^i (\sigma_2 t_2 q_R^\dagger) + i (q_R^T t_2 \sigma_2) \sigma_i i g_s A^i (\sigma_2 t_2 q_R^\dagger) \\ =& i \tilde{q}^\dagger \sigma_\mu D^\mu \tilde{q}. \end{aligned} \tag{2.6}$$

Here, the property (2.3) of Pauli matrices for both t_a of color and σ_k of spin and the anticommutativity of $\tilde{q}, \tilde{q}^\dagger$ have been used. The Lagrangian is rewritten as

$$\mathcal{L}_{kin} = q^\dagger i \sigma_\mu D^\mu q + \tilde{q}^\dagger i \sigma_\mu D^\mu \tilde{q} = \Psi^\dagger i \sigma_\mu D^\mu \Psi, \tag{2.7}$$

which now has a manifest SU$(2N_f)$ "flavor" symmetry. The field Ψ denotes a Weyl spinor which has $2N_f$ "flavor" components as

$$\Psi = \begin{pmatrix} q \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} q^1 \\ \vdots \\ q^{N_f} \\ \tilde{q}^1 \\ \vdots \\ \tilde{q}^{N_f} \end{pmatrix}, \tag{2.8}$$

where the superscript denotes the flavor indices. The field $\tilde{q}$ can be considered as the charge conjugated quarks. The $i\sigma_2$ corresponds to the charge conjugation of Weyl spinor $q_L^c = -i\sigma_2 q_R^\dagger$ (see Appendix B.3 with Weyl spinor representation) and the it_2 interchange the color and complementary color. Hence $\tilde{q}$ have opposite baryon charge to normal quark. The SU$(2N_f)$ transformation

$$\Psi \to g\Psi, \quad (g \in \mathrm{SU}(2N_f)), \tag{2.9}$$

transforms quark and antiquark at the same time. The Lagrangian (2.7) is obviously invariant under this SU$(2N_f)$ transformation, which is called Pauli-Gürsey (PG) symmetry, instead of SU$(N_f) \times$ SU$(N_f) \times$ U(1).

We next write down various useful quark bilinears in terms of $q, \tilde{q}$ and determine their transformation properties under SU$(2N_f)$. We first see the scalar:

$$\begin{aligned}\bar{\psi}\psi =& q_R^\dagger q_L + q_L^\dagger q_R \\ =& \tilde{q}^T \sigma_2 t_2 q + q^\dagger \sigma_2 t_2 (\tilde{q}^\dagger)^T \\ =& \frac{1}{2}\tilde{q}^T \sigma_2 t_2 q - \frac{1}{2}(q^T \sigma_2 t_2 \tilde{q})^T + \frac{1}{2} q^\dagger \sigma_2 t_2 (\tilde{q}^\dagger)^T - \frac{1}{2}(\tilde{q}^\dagger \sigma_2 t_2 q^*)^T \\ =& -\frac{1}{2}\left(\Psi^T \sigma_2 t_2 \Sigma_0 \Psi - \Psi^\dagger \sigma_2 t_2 \Sigma_0 \Psi^*\right), \end{aligned} \tag{2.10}$$

where the symplectic matrix

$$\Sigma_0 = \begin{pmatrix} 0 & 1_{N_f} \\ -1_{N_f} & 0 \end{pmatrix} \tag{2.11}$$

acts in the $2N_f$-dimensional extended flavor space where 1_{N_f} is $N_f \times N_f$ unit matrix. An explicit (or dynamical) quark mass $m_q\bar{\psi}\psi$, therefore, explicitly (or spontaneously) breaks the original SU$(2N_f)$ down to the compact symplectic group Sp$(2N_f)$.

The chemical potential term $\mu\bar{\psi}\gamma_0\psi$ is rewritten as

$$\begin{aligned}\bar{\psi}\gamma_0\psi =& q_L^\dagger q_L + q_R^\dagger q_R \\ =& q^\dagger q + \tilde{q}^T (\tilde{q}^\dagger)^T \\ =& q^\dagger q - \tilde{q}^\dagger \tilde{q} \\ =& \Psi^\dagger B_0 \Psi, \end{aligned} \tag{2.12}$$

with

$$B_0 = -\gamma_0 \Sigma_0 = \begin{pmatrix} 1_{N_f} & 0 \\ 0 & -1_{N_f} \end{pmatrix}. \tag{2.13}$$

It is easy to understand the meaning of $+1_{N_f}$ and -1_{N_f} in the matrix B_0 are just baryon charges of quarks and conjugate quarks. Since the matrix B_0 is block-diagonal, it is clear that the chemical potential term $\Psi^\dagger B_0 \Psi$ has SU$(N_f) \times$ SU(N_f). The U$(1)_B$, which can be thought as generated by the B_0, also preserves. Thus, the existence of the chemical potential μ explicitly breaks SU$(2N_f)$ down to SU$(N_f) \times$ SU$(N_f) \times$ U(1).

2.2 Two Color NJL model

The purpose of this work is to describe the structure of hadrons microscopically from the quarks and gluons and to understand hadronic phenomena at finite density and temperature. In this thesis, we use the Nambu-Jona-Lasinio (NJL) model as our starting microscopic theory. The primary connection of the NJL model with QCD is provided by several assumptions.

- The light quarks (up and down quarks in this thesis, but generally strange quark is also included) are considered as basic degrees of freedom.
- The gluon degrees of freedom are frozen and absorbed into a local effective interaction between quarks in the low-energy and long-wavelength limit.

- The interaction is constructed in accordance with the symmetries of QCD.

Take a current quark fields belonging to SU(2) color and SU(2) flavor. The QCD conserved color current $J^\mu_a = \bar{\psi}\gamma^\mu t_a \psi$ generates the interaction

$$\mathcal{L}_{int} = - G_c \sum_{a=1}^{3} J^\mu_a J^a_\mu, \tag{2.14}$$

where t_a is the Pauli matrices of color SU(2), the quark fields are $\psi = (u, d)^T$ and G_c is a coupling constant. This current can be represented with the Weyl spinor $\Psi^T = (q, \tilde{q})$:

$$J^\mu_a = \bar{\psi}\gamma^\mu t_a \psi = q^\dagger_L \bar{\sigma}^\mu t_a q_L + q^\dagger_R \sigma^\mu t_a q_R, \tag{2.15}$$

and

$$q^\dagger_R \sigma^0 t_a q_R = - q^\dagger_R \sigma_2 (\sigma^0)^T \sigma_2 t_2 t^T_a t_2 q_R = \tilde{q}^\dagger \sigma_0 t_a \tilde{q}, \tag{2.16}$$

$$q^\dagger_R \sigma^i t_a q_R = - \tilde{q}^\dagger \sigma^i t_a \tilde{q}, \tag{2.17}$$

where $i = 1, 2, 3$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$. Then the current can be written as

$$J^\mu_a = q^\dagger \bar{\sigma}^\mu t_a q + \tilde{q}^\dagger \bar{\sigma}^\mu t_a \tilde{q} = \Psi^\dagger \bar{\sigma}^\mu t_a \Psi, \tag{2.18}$$

which is invariant under SU(4) $(N_f = 2)$ transformation (PG symmetry). The kinetic term of the quark is written as

$$\mathcal{L}_{kin} = \bar{\psi}(i\partial\!\!\!/ - m_0 + \gamma_0 \mu)\psi, \tag{2.19}$$

with the current quark mass $m_0 = m_u = m_d$ and the quark chemical potential μ. Our model Lagrangian is written as

$$\mathcal{L}_{NJL} = \bar{\psi}(i\partial\!\!\!/ - m_0 + \gamma_0 \mu)\psi - G_c \sum_{a=1}^{3} J^\mu_a J^a_\mu. \tag{2.20}$$

In the chiral limit $(m_0 \to 0)$ and zero chemical potential $(\mu = 0)$ the Lagrangian is invariant under SU(4). The SU(4) symmetry is spontaneously broken by the standard chiral condensate $\langle\bar{\psi}\psi\rangle$ down to Sp(4). This symmetry breaking pattern generates five Nambu-Goldstone (NG) bosons ($\dim \mathrm{SU}(N) = N^2 - 1$ and $\dim \mathrm{Sp}(2N) = N(2N + 1)$), which corresponds to three pions, diquark and antidiquark. On the other hand, when the chemical potential is finite $\mu \neq 0$ in the chiral limit the symmetry is SU$(2)_L$ $\times$ SU$(2)_R$ $\times$ U$(1)_B$. We insert the current quark mass m_0, the axial symmetry is explicitly broken then SU$(2)_V$ $\times$ U$(1)_B$ is remained and the five NG bosons become pseudo-NG boson (their mass will be denote m_π). Since the diquark is treated as baryon at the same time boson, the diquark-baryon condensation can occur at the certain chemical potential as the Bose-Einstein condensation (BEC). As we will see later the critical chemical potential μ_c is half pion mass. Once the chemical potential exceeds this point, the baryon number symmetry is spontaneously broken by the diquark condensate $\langle\psi\psi\rangle$ down to Sp(2) and one NG boson is realized.

In the two flavor case there are only two order parameters, the chiral condensate $\langle\bar{\psi}\psi\rangle$ and the scalar diquark condensate, symbolically denoted by $\langle\psi\psi\rangle$. We focus on the behavior of these order parameters at finite temperature and density in this thesis and rewrite the interaction term by Fierz transformation (see Appendix C) in terms of the color singlet scalar and pseudoscalar quark-antiquark and scalar diquark channels:

$$\mathcal{L}_{int} = \frac{G_0}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] + \frac{H_0}{2}(\bar{\psi}i\gamma_5 t_2\tau_2 C\bar{\psi}^T)(\psi^T Ci\gamma_5 t_2\tau_2\psi), \tag{2.21}$$

with the charge conjugation operator for fermions $C = i\gamma_0\gamma_2$ (see Appendix B). The coupling constant for the mesonic channel G_0 and for diquark channel H_0 are uniquely fixed by Fierz transformation as

$$G_0 = H_0 = \frac{3}{2}G_c. \tag{2.22}$$

Hence mesonic channel and quark channel are transformed under $\mathrm{SU}(4)$ transformation at the same time, which corresponds to PG symmetry in hadronic level. We keep, however, the coupling constants independent so that we are able to study as well cases in which PG symmetry is not exactly realized. Our starting Lagrangian is now

$$\begin{aligned}\mathcal{L}_{NJL} =& \bar{\psi}(i\partial\!\!\!/ - m_0 + \gamma_0\mu)\psi \\ &+ \frac{G_0}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] + \frac{H_0}{2}(\bar{\psi}i\gamma_5 t_2\tau_2 C\bar{\psi}^T)(\psi^T Ci\gamma_5 t_2\tau_2\psi)\end{aligned} \tag{2.23}$$

2.3 Auxiliary Fields

The partition function of the NJL model is

$$Z = \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(i\int d^4x\mathcal{L}_{NJL}\right), \tag{2.24}$$

where $\mathcal{N}$ is normalization constant. The bosonization technique is now used to write the Lagrangian in terms of auxiliary meson fields, $\sigma \sim \bar{\psi}\psi,\ \vec{\pi} \sim \bar{\psi}i\gamma_5\vec{\tau}\psi$ and diquark fields, $\Delta \sim \psi^T Ci\gamma_5 t_2\tau_2\psi,\ \Delta^* \sim \bar{\psi}i\gamma_5 t_2\tau_2 C\bar{\psi}^T$ [42, 70, 72, 73]. We introduce

$$1 = \mathcal{N}'\int \mathcal{D}\sigma\mathcal{D}\vec{\pi}\mathcal{D}\Delta\mathcal{D}\Delta^*\exp\left(i\int d^4x[-\frac{1}{2}M_s^2(\sigma^2+\vec{\pi}^2) - \frac{1}{2}M_d^2\Delta^*\Delta]\right), \tag{2.25}$$

with the normalization constant $\mathcal{N}'$. We omit the normalization constants $\mathcal{N}$ and $\mathcal{N}'$ from here since they do not contribute physics. Obviously, the partition function (2.24) is invariant even multiplying the above constant (2.25):

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\vec{\pi}\mathcal{D}\Delta\mathcal{D}\Delta^*\exp\left(i\int d^4x\left[\mathcal{L}_{NJL} - \frac{1}{2}M_s^2(\sigma^2+\vec{\pi}^2) - \frac{1}{2}M_d^2\Delta^*\Delta\right]\right). \tag{2.26}$$

Performing Hubbard-Stranovich transformation [74] as

$$\sigma \to \sigma + \frac{g_0}{M_s^2}\bar{\psi}\psi, \tag{2.27}$$

$$\vec{\pi} \to \vec{\pi} + \frac{g_0}{M_s^2}\bar{\psi}i\gamma_5\vec{\tau}\psi, \tag{2.28}$$

$$\Delta \to \Delta + i\frac{g_d}{M_d^2}\psi^T Ci\gamma_5 t_2\tau_2\psi, \tag{2.29}$$

$$\Delta^* \to \Delta^* - i\frac{g_d}{M_d^2}\bar{\psi}i\gamma_5 t_2\tau_2 C\bar{\psi}^T, \tag{2.30}$$

the partition function can be written as

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\vec{\pi}\mathcal{D}\Delta\mathcal{D}\Delta^* \exp\left(i\int d^4x\mathcal{L}_{aux}\right) , \tag{2.31}$$

with the Lagrangian

$$\begin{aligned}\mathcal{L}_{aux} =& \frac{1}{2}\begin{pmatrix}\bar{\psi} & \psi^T C\end{pmatrix}\begin{pmatrix}S^{-1}(\mu) & g_d\gamma_5 t_2\tau_2\Delta(x)\\ -g_d\Delta^*(x)\gamma_5 t_2\tau_2 & S^{-1}(-\mu)\end{pmatrix}\begin{pmatrix}\psi\\ C\bar{\psi}^T\end{pmatrix}\\ &-\frac{1}{2}M_s^2(\sigma^2(x)+\vec{\pi}^2(x)) - \frac{1}{2}M_d^2\Delta^*(x)\Delta(x),\end{aligned} \tag{2.32}$$

with $S^{-1}(\pm\mu) = i\gamma_\mu\partial^\mu - m_0 \pm \gamma_0\mu - g_0\left(\sigma(x) \pm i\gamma_5\vec{\pi}(x)\cdot\vec{\tau}\right)$. We have introduced a meson coupling constant g_0 and a diquark coupling constant g_d together with bare scalar and diquark masses, M_s and M_d, in preparation of the standard renormalization scheme. The meson and diquark coupling constants and masses are related as

$$\frac{g_0^2}{M_s^2} = G_0, \quad \frac{g_d^2}{M_d^2} = H_0. \tag{2.33}$$

Since the original NJL coupling constants G_0 and H_0 have the mass dimension -2, the decomposed coupling constants g_0 and g_d are dimensionless and the masses M_s and M_d have mass dimension one.

Introducing a chiral order parameter σ_0 (proportional to the quark condensate $\langle\bar{\psi}\psi\rangle$) as $\sigma(x) = \sigma_0 + s(x)$ and the diquark condensates, Δ_0 and Δ_0^*, as $\Delta(x) = \Delta_0 + d(x)$ and $\Delta^*(x) = \Delta_0^* + d^*(x)$, and integrating out the quark fields, the effective Lagrangian entering the partition function (2.31) becomes

$$\begin{aligned}\mathcal{L}_{eff} =& -\frac{i}{2}\mathrm{tr}\left(\ln\hat{S}^{-1} + \ln(1+\hat{S}\hat{K})\right) - \frac{1}{2}M_s^2\sigma_0^2 - \frac{1}{2}M_s^2(s^2(x)+\vec{\pi}^2(x)) - M_s^2\sigma_0 s(x)\\ &-\frac{1}{2}M_d^2\Delta_0^*\Delta_0 - \frac{1}{2}M_d^2(\Delta_0^* d(x) + d^*(x)\Delta_0) - \frac{1}{2}M_d^2 d^*(x)d(x).\end{aligned} \tag{2.34}$$

The trace is taken over spin, momentum, flavor and color spaces. The matrices $\hat{S}^{-1}$ and $\hat{K}$ are defined as,

$$\hat{S}^{-1} = \begin{pmatrix}S_0^{-1}(\mu) & \Delta^-\\ \Delta^+ & S_0^{-1}(-\mu)\end{pmatrix}, \tag{2.35}$$

$$\hat{K} = \begin{pmatrix}-g_0\left(s(x)+i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)\right) & g_d\gamma_5 t_2\tau_2 d(x)\\ -g_d d^*(x)\gamma_5 t_2\tau_2 & -g_0(s(x)-i\gamma_5\vec{\tau}\cdot\vec{\pi}(x))\end{pmatrix}, \tag{2.36}$$

with $\Delta^- = g_d\gamma_5 t_2\tau_2\Delta_0, \Delta^+ = -g_d\Delta_0^*\gamma_5 t_2\tau_2$ and $S_0^{-1}(\pm\mu) = i\gamma_\mu\partial^\mu - m \pm \gamma_0\mu$. A dynamical quark mass is defined as $m = m_0 + g_0\sigma_0$. The Nambu-Gorkov quark propagator matrix $\hat{S}$ is determined by solving $\hat{S}\hat{S}^{-1} = 1$ and expressed as:

$$\hat{S} = \begin{pmatrix}G^+ & H^-\\ H^+ & G^-\end{pmatrix}, \tag{2.37}$$

with the components

$$\begin{aligned} G^{\pm} &= \left(S_0^{-1}(\pm\mu) - \Sigma^{\pm}\right)^{-1} \\ \Sigma^{\pm} &= \Delta^{\mp} S_0(\mp\mu)\Delta^{\pm} \\ H^{\pm} &= -\, S_0(\mp\mu)\Delta^{\pm} G^{\pm}. \end{aligned} \tag{2.38}$$

A simple form for the components of the Nambu-Gorkov propagator is found introducing the energy projectors onto states of positive and negative energy for free massive spin $1/2$ quasi-particles following Huang *et al.* [75, 76],

$$\Lambda_{\pm} = \frac{1}{2}\left(1 \pm \frac{\gamma_0(\vec{\gamma}\cdot\vec{p} + m)}{E_p}\right), \quad \tilde{\Lambda}_{\pm} = \frac{1}{2}\left(1 \pm \frac{\gamma_0(\vec{\gamma}\cdot\vec{p} - m)}{E_p}\right), \tag{2.39}$$

with $E_p = \sqrt{\vec{p}^2 + m^2}$. These operators satisfy the projection properties $\Lambda_{\pm}\Lambda_{\pm} = \Lambda_{\pm}$, $\Lambda_{\pm}\Lambda_{\mp} = 0$, $\Lambda_+ + \Lambda_- = 1$, and similar relations for $\tilde{\Lambda}$. Furthermore, the two projection operators are related as $\gamma_0\Lambda_{\pm}\gamma_0 = \tilde{\Lambda}_{\mp}$ and $\gamma_5\Lambda_{\pm}\gamma_5 = \tilde{\Lambda}_{\pm}$.

We calculate these components with the energy projection operators. First, the Dirac propagator in the momentum representation with $\tilde{p}_0 = p_0 \pm \mu$ is calculated as

$$\begin{aligned} S_0(\pm\mu) &= \frac{1}{\gamma_0\tilde{p}_0 - \vec{\gamma}\cdot\vec{p} - m} \\ &= \frac{\gamma_0\tilde{p}_0 - \vec{\gamma}\cdot\vec{p} + m}{\tilde{p}_0^2 - E_p^2} \\ &= \frac{\gamma_0\tilde{p}_0 - \vec{\gamma}\cdot\vec{p} + m}{2E_p}\left(\frac{1}{\tilde{p}_0 - E_p} - \frac{1}{\tilde{p}_0 + E_p}\right) \\ &= \frac{1}{2E_p}\left(\frac{\gamma_0 E_p - \vec{\gamma}\cdot\vec{p} + m}{p_0 - E_p^{\mp}} + \frac{\gamma_0 E_p + \vec{\gamma}\cdot\vec{p} - m}{p_0 + E_p^{\pm}}\right) \\ &= \frac{\gamma_0\tilde{\Lambda}_-}{p_0 - E_p^{\mp}} + \frac{\gamma_0\tilde{\Lambda}_+}{p_0 + E_p^{\pm}}, \end{aligned} \tag{2.40}$$

where in the third line

$$\begin{aligned} \frac{\gamma_0\tilde{p}_0}{\tilde{p}_0 - E_p} - \frac{\gamma_0\tilde{p}_0}{\tilde{p}_0 + E_p} &= \frac{2\gamma_0\tilde{p}_0 E_p}{\tilde{p}_0^2 - E_p^2} \\ &= \frac{2\gamma_0\tilde{p}_0 E_p}{2\tilde{p}_0}\left(\frac{1}{\tilde{p}_0 + E_p} + \frac{1}{\tilde{p}_0 - E_p}\right) \\ &= \frac{\gamma_0 E_p}{\tilde{p}_0 + E_p} + \frac{\gamma_0 E_p}{\tilde{p}_0 - E_p}, \end{aligned} \tag{2.41}$$

where we have defined $E_p^{\pm} = E_p \pm \mu$. Then, its inverse is worked out as

$$
\begin{aligned}
S_0^{-1}(\pm\mu) =& \left[\frac{\gamma_0\tilde{\Lambda}_-(p_0+E_p^{\pm}) + \gamma_0\tilde{\Lambda}_+(p_0+E_p^{\mp})}{(p_0-E_p^{\mp})(p_0+E_p^{\pm})} \right]^{-1} \\
=&(p_0-E_p^{\mp})(p_0+E_p^{\pm})(\tilde{\Lambda}_+ + \tilde{\Lambda}_-)[\gamma_0\tilde{\Lambda}_-(p_0+E_p^{\pm}) + \gamma_0\tilde{\Lambda}_+(p_0-E_p^{\mp})]^{-1} \\
=&(p_0-E_p^{\mp})(p_0+E_p^{\pm})\tilde{\Lambda}_+\{\gamma_0\tilde{\Lambda}_-(p_0+E_p^{\pm}) + \gamma_0\tilde{\Lambda}_+(p_0-E_p^{\mp})\}^{-1}(\tilde{\Lambda}_+^{-1}\tilde{\Lambda}_+) \\
&+ (p_0-E_p^{\mp})(p_0+E_p^{\pm})\tilde{\Lambda}_-\{\gamma_0\tilde{\Lambda}_-(p_0+E_p^{\pm}) + \gamma_0\tilde{\Lambda}_+(p_0-E_p^{\mp})\}^{-1}(\tilde{\Lambda}_-^{-1}\tilde{\Lambda}_-) \\
=&(p_0+E_p^{\pm})\tilde{\Lambda}_+(\gamma_0\tilde{\Lambda}_+)^{-1}\tilde{\Lambda}_+(\gamma_0^{-1}\gamma_0) + (p_0-E_p^{\mp})\tilde{\Lambda}_-(\gamma_0\tilde{\Lambda}_-)^{-1}\tilde{\Lambda}_-(\gamma_0^{-1}\gamma_0) \\
=&(p_0+E_p^{\pm})\tilde{\Lambda}_+\gamma_0 + (p_0-E_p^{\mp})\tilde{\Lambda}_-\gamma_0,
\end{aligned} \tag{2.42}
$$

using the property $\tilde{\Lambda}_+ + \tilde{\Lambda}_- = 1$ and $\tilde{\Lambda}_{\pm}\tilde{\Lambda}_{\mp} = 0$. Next, we have

$$
\begin{aligned}
\Sigma^{\pm} =& \Delta^{\mp} S_0(\mp\mu)\Delta^{\pm} \\
=& -|\Delta|^2\gamma_5 \left(\frac{\gamma_0\tilde{\Lambda}_-}{p_0-E_p^{\pm}} + \frac{\gamma_0\tilde{\Lambda}_+}{p_0+E_p^{\mp}} \right)\gamma_5 \\
=& g_d^2|\Delta_0|^2 \left(\frac{\gamma_0\Lambda_-}{p_0-E_p^{\pm}} + \frac{\gamma_0\Lambda_+}{p_0+E_p^{\mp}} \right),
\end{aligned} \tag{2.43}
$$

with $g_d^2|\Delta_0|^2 = |\Delta|^2$. Then we obtain

$$
\begin{aligned}
G^{\pm}(p) =& \left[(p_0+E_p^{\pm})\tilde{\Lambda}_+\gamma_0 + (p_0-E_p^{\mp})\tilde{\Lambda}_-\gamma_0 - \Delta^2\left(\frac{\gamma_0\Lambda_-}{p_0-E_p^{\pm}} + \frac{\gamma_0\Lambda_+}{p_0+E_p^{\mp}} \right) \right]^{-1} \\
=& \left[\frac{p_0^2-(E_\Delta^{\pm})^2}{p_0-E_p^{\pm}}\gamma_0\Lambda_- + \frac{p_0^2-(E_\Delta^{\mp})^2}{p_0+E_p^{\mp}}\gamma_0\Lambda_+ \right]^{-1} \\
=& \frac{p_0+E_p^{\mp}}{p_0^2-(E_\Delta^{\mp})^2}\Lambda_+(\gamma_0\Lambda_+)^{-1}\Lambda_+(\gamma_0^{-1}\gamma_0) + \frac{p_0-E_p^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\Lambda_-(\gamma_0\Lambda_-)^{-1}\Lambda_-(\gamma_0^{-1}\gamma_0) \\
=& \frac{p_0+E_p^{\mp}}{p_0^2-(E_\Delta^{\mp})^2}\Lambda_+\gamma_0 + \frac{p_0-E_p^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\Lambda_-\gamma_0,
\end{aligned} \tag{2.44}
$$

similar with the method used in Eq. (2.41). The quasi-particle energy is $E_\Delta^{\pm} = \sqrt{(E_p^{\pm})^2 + |\Delta|^2}$ with the dynamical quark mass $m = m_0 + g_0\sigma_0$ and the diquark gap Δ_0. Finally, we obtain

$$
\begin{aligned}
H^{\pm}(p) =& -S_0(\mp\mu)\Delta^{\pm}G^{\pm} \\
=& \frac{\Delta^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\tilde{\Lambda}_+ + \frac{\Delta^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\tilde{\Lambda}_-,
\end{aligned} \tag{2.45}
$$

noting that $\Delta^{\pm}$ include γ_5.

2.4 The Mean Field Approximation

Keeping only the expectation values $\langle\sigma(x)\rangle = \sigma_0$, $\langle\Delta(x)\rangle = \Delta_0$ and $\langle\Delta^*(x)\rangle = \Delta_0^*$, i.e. dropping the fluctuating meson and diquark fields $s(x)$, $\vec{\pi}(x)$, $d(x)$ and $d^*(x)$, one arrives at the mean-field Lagrangian

$$\mathcal{L}_{MF} = -\frac{i}{2}\mathrm{tr}\ln \hat{S}^{-1} - \frac{1}{2}M_s^2\sigma_0^2 - \frac{1}{2}M_d^2\Delta_0^*\Delta_0. \tag{2.46}$$

The trace for spin matrices is transformed to the determinant by using the relation $\mathrm{tr}\ln \hat{S}^{-1} = \ln\det \hat{S}^{-1}$ and the others are kept as trace. The determinant is obtained by two ways [75,76] as

$$\begin{aligned}\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} =& \det\left\{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} 1 & C^{-1}D \\ B^{-1}A & 1 \end{pmatrix}\right\} \\ =&(-CB)(1 - B^{-1}AC^{-1}D) = -CB + CAC^{-1}D \equiv D_1,\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} =& \det\begin{pmatrix} D & C \\ B & A \end{pmatrix} = \det\left\{\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}\begin{pmatrix} 1 & B^{-1}A \\ C^{-1}D & 1 \end{pmatrix}\right\} \\ =& - BC + BDB^{-1}A \equiv D_2.\end{aligned} \tag{2.48}$$

So, the determinant can be written as

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sqrt{D_1 \cdot D_2}. \tag{2.49}$$

Using this equation and Eq. (2.41),

$$\det \hat{S}^{-1} = \sqrt{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)}\ . \tag{2.50}$$

Temperature is now introduced using the Matsubara formalism (see Appendix D),

$$i\int \frac{d^4p}{(2\pi)^4} \to -T\sum_n \int \frac{d^3p}{(2\pi)^3}\ , \tag{2.51}$$

with the replacement $p_0 \to i\omega_n$ where $\omega_n = (2n+1)\pi T$ are the fermionic Matsubara frequencies. Taking the frequency sum, the thermodynamical potential becomes

$$\begin{aligned}\Omega_{MF} =& - \frac{T}{V}\ln Z_{MF} \\ =& - \mathrm{tr}\int \frac{d^3p}{(2\pi)^3}(E_\Delta^+ + E_\Delta^-) - 2\mathrm{tr}\int \frac{d^3p}{(2\pi)^3}T[\ln(1+e^{-\beta E_\Delta^+}) + \ln(1+e^{-\beta E_\Delta^-})] \\ &+ \frac{1}{2}M_s^2\sigma_0^2 + \frac{1}{2}M_d^2|\Delta_0|^2.\end{aligned} \tag{2.52}$$

Here we have written the inverse of the temperature T as $\beta = 1/T$.

The derivatives of the thermodynamical potential (2.52) with respect to σ_0 and $|\Delta_0|$ determine the chiral condensate and the diquark condensate at the minimum of Ω_{MF}:

$$\begin{aligned}\frac{\partial\Omega_{MF}}{\partial\sigma_0} =& - \mathrm{tr}\int \frac{d^3p}{(2\pi)^3}\frac{g_0 m}{E_p}\left[\frac{E_p^+}{E_\Delta^+}\left(1 - 2n_F(E_\Delta^+)\right) + \frac{E_p^-}{E_\Delta^-}\left(1 - 2n_F(E_\Delta^-)\right)\right] + M_s^2\sigma_0 \\ =&0,\end{aligned} \tag{2.53}$$

$$\begin{aligned}\frac{\partial\Omega_{MF}}{\partial|\Delta_0|} =& - |\Delta_0|\mathrm{tr}\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{E_\Delta^+}\left(1 - 2n_F(E_\Delta^+)\right) + \frac{1}{E_\Delta^-}\left(1 - 2n_F(E_\Delta^-)\right)\right] + M_d^2|\Delta_0| \\ =&0,\end{aligned} \tag{2.54}$$

with the Fermi distribution function $n_F(E) = (1+e^{\beta E})^{-1}$. The chemical potential μ is included in the definition of $E_\Delta^\pm$.

Chapter 3

Hadron Lagrangian from the Bosonization

In this chapter we discuss the derivation of the effective hadron Lagrangian by performing the bosonization technique of Eguchi [42]. The fluctuating meson and diquark fields introduced in the previous chapter will be considered term by term. Higher order terms of the hadron fields will be generated from the loop expansion of the logarithmic term. We will find the bosonization technique leads to an extended linear sigma model incorporating diquark degrees of freedom in the NJL Lagrangian.

3.1 Hadron Sector of the Lagrangian

The mean field part of the Lagrangian is discussed in the previous chapter. In this chapter let us focus on the fluctuating part which is

$$\begin{aligned}\mathcal{L}_{hadron} = &-\frac{i}{2}\mathrm{tr}\ln(1+\hat{S}(p)\hat{K}(x)) - M_s^2\sigma_0 s(x) - \frac{1}{2}M_d^2(\Delta_0^* d(x) + d^*(x)\Delta_0) \\ &-\frac{1}{2}M_s^2(s^2(x)+\vec{\pi}^2(x)) - \frac{1}{2}M_d^2 d^*(x)d(x),\end{aligned} \tag{3.1}$$

with

$$\hat{S}(p) = \begin{pmatrix} G^+(p) & H^-(p) \\ H^+(p) & G^-(p)\end{pmatrix}, \tag{3.2}$$

$$\hat{K}(x) = \begin{pmatrix} -g_0\left(s(x)+i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)\right) & g_d\gamma_5 t_2\tau_2 d(x) \\ -g_d d^*(x)\gamma_5 t_2\tau_2 & -g_0(s(x)-i\gamma_5\vec{\tau}\cdot\vec{\pi}(x))\end{pmatrix}, \tag{3.3}$$

and its components

$$G^{\pm}(p) = \frac{p_0+E_p^{\mp}}{p_0^2-(E_\Delta^{\mp})^2}\Lambda_+\gamma_0 + \frac{p_0-E_p^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\Lambda_-\gamma_0, \tag{3.4}$$

$$H^{\pm}(p) = \frac{\Delta^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\tilde{\Lambda}_+ + \frac{\Delta^{\pm}}{p_0^2-(E_\Delta^{\pm})^2}\tilde{\Lambda}_-, \tag{3.5}$$

with $\Delta^- = g_d\gamma_5 t_2\tau_2\Delta_0, \Delta^+ = -g_d\Delta_0^*\gamma_5 t_2\tau_2$. The trace is taken over spin, momentum, flavor and color spaces. We would like to extract the properties of mesons and diquark-baryons at finite temperature and density. For this purpose, we expand the logarithmic term in Eq. (3.1) as

$$-\frac{i}{2}\mathrm{tr}\ln(1+\hat{S}\hat{K}) = -\frac{i}{2}\mathrm{tr}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(\hat{S}\hat{K})^k \equiv \sum_{k=1}^{\infty}U^{(k)}\ , \tag{3.6}$$

where

$$U^{(k)} = -\frac{i(-1)^{k+1}}{2k}\mathrm{tr}(\hat{S}\hat{K})^k\ . \tag{3.7}$$

The matrix is defied as

$$\hat{S}\hat{K} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.8}$$

with elements

$$\begin{aligned}
A =& -G^{+}(p)g_0(s(x)+i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)) - H^{-}(p)g_d d^*(x)\gamma_5 t_2\tau_2\ ,\\
B =& G^{+}(p)g_d\gamma_5 t_2\tau_2 d(x) - H^{-}(p)g_0(s(x)-i\gamma_5\vec{\tau}\cdot\vec{\pi}(x))\ ,\\
C =& -H^{+}(p)g_0(s(x)+i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)) - G^{-}(p)g_d d^*(x)\gamma_5 t_2\tau_2\ ,\\
D =& H^{+}(p)g_d\gamma_5 t_2\tau_2 d(x) - G^{-}(p)g_0(s(x)-i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)).
\end{aligned} \tag{3.9}$$

With these expressions we are able to derive the Lagrangian involving the meson and diquark-baryon fields.

Since the first four therms in $U^{(k)}$ provide divergent integrals in the limit of cut-off $\Lambda \to \infty$, we classify the terms as

$$U_{div} + U_{conv}, \tag{3.10}$$

where U_{div} indicates divergent integration terms and U_{conv} convergent terms in $U^{(k)}$ $(k = 1,2,3,4)$ and all other $\sum_{k=5}^{\infty}U^{(k)}$. According to Ref. [42], the divergent terms can be absorbed into renormalization parameters, which will be introduced later, and eventually we can obtain a renormalizable hadronic Lagrangian since the coupling constants g_0 and g_d are dimensionless and the order of the auxiliary fields are up to four. We follow this prescription because we require a renormalizable hadron Lagrangian as the composite of quarks. Further, the fourth order fields Lagrangian can be used to consider the spontaneously breaking of symmetry. We will introduce the finite NJL cut-off when computing numerical results.

A diagrammatic expression of the series $U^{(k)}$ up to fourth order is shown in Fig. 3.1. The first order term is called the dangerous term which vanish for the stability. The second order term corresponds the kinetic and mass terms. The third and fourth order terms are the interaction. The evaluation of each term will be discussed in detail from next section.

3.2 The Gap Equations (k=1)

We first work out the case of $k = 1$, that is $U^{(1)} = -\frac{i}{2}\mathrm{tr}\hat{S}\hat{K}$, to derive the mass gap equations, which correspond to the minimization condition (2.53) and (2.54) of the thermodynamical po-

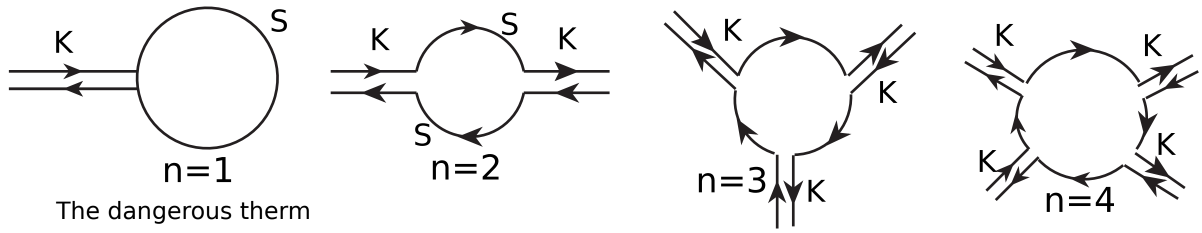


Figure 3.1: Diagrammatic expression of the series Eq. (3.7). The solid lines represent the quark propagators, and the external double lines mean auxiliary fields K and internal lines are the loop integral S.

tential in the mean-field approximation described in the previous section,

$$\begin{aligned}U^{(1)} =& -\frac{i}{2}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[g_0(-G^+(p)-G^-(p))s(x)+g_0(-G^+(p)i\gamma_5+G^-(p)i\gamma_5)\vec{\tau}\cdot\vec{\pi}(x)\right.\\ &\left.-g_dH^-(p)\gamma_5t_2\tau_2d^*(x)+g_dH^+(p)\gamma_5t_2\tau_2d(x)\right]\\ =&\int d^4x\left(\Gamma_s s(x)+\Gamma_d d(x)+\Gamma_{d^*}d^*(x)\right).\end{aligned}\tag{3.11}$$

The trace of the Dirac matrix in the pion term gives zero and we have dropped this term here. The first order of the effective Lagrangian can be written as

$$\mathcal{L}^{(1)}=\left(\Gamma_s-M_s^2\sigma_0\right)s(x)+\left(\Gamma_d-\frac{1}{2}M_d^2\Delta_0^*\right)d(x)+\left(\Gamma_{d^*}-\frac{1}{2}M_d^2\Delta_0\right)d^*(x).\tag{3.12}$$

The result for Γ_s is

$$\begin{aligned}\Gamma_s =&\frac{i}{2}g_0\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}(G^+(p)+G^-(p))\\ =&2img_0\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\frac{1}{p_0^2-(E_\Delta^+)^2}+\frac{1}{p_0^2-(E_\Delta^-)^2}+\frac{\mu}{E_p}\left(\frac{1}{p_0^2-(E_\Delta^+)^2}-\frac{1}{p_0^2-(E_\Delta^-)^2}\right)\right].\end{aligned}\tag{3.13}$$

The trace tr is taken over spin, flavor and color spaces and tr_{fc} is over flavor and color only.

The momentum space integrals (3.13) are divergent and must be treated accordingly. For the second term under the integral we write:

$$\begin{aligned}\frac{\mu}{E_p}\left(\frac{1}{p_0^2-(E_\Delta^+)^2}-\frac{1}{p_0^2-(E_\Delta^-)^2}\right)=&\frac{\mu}{E_p}\left(\frac{1}{p_0^2-(E_\Delta^-)^2-4\mu E_p}-\frac{1}{p_0^2-(E_\Delta^+)^2+4\mu E_p}\right)\\ \sim&\frac{\mu}{E_p}\left[\frac{1}{p_0^2-(E_\Delta^-)^2}\left(1+\frac{4\mu E_p}{p_0^2-(E_\Delta^-)^2}\right)\right.\\ &\left.-\frac{1}{p_0^2-(E_\Delta^+)^2}\left(1-\frac{4\mu E_p}{p_0^2-(E_\Delta^+)^2}\right)\right]\\ 2\frac{\mu}{E_p}\left(\frac{1}{p_0^2-(E_\Delta^+)^2}-\frac{1}{p_0^2-(E_\Delta^-)^2}\right)=&4\mu^2\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}\right)\\ \frac{\mu}{E_p}\left(\frac{1}{p_0^2-(E_\Delta^+)^2}-\frac{1}{p_0^2-(E_\Delta^-)^2}\right)=&2\mu^2\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}\right),\end{aligned}\tag{3.14}$$

where in the third line only those terms have been kept that are divergent when taking the momentum integral. Using this expression Γ_s can be written as

$$\Gamma_s = 2mg_0(I_2 - 2\mu^2 I_0), \tag{3.15}$$

with the divergent integrals I_2 and I_0 defined as follows:

$$I_2 = i\mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p_0^2 - (E_\Delta^+)^2} + \frac{1}{p_0^2 - (E_\Delta^-)^2} \right), \tag{3.16}$$

$$I_0 = -\, i\mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} \right). \tag{3.17}$$

The result for Γ_d is

$$\begin{aligned}\Gamma_d &= -\frac{i}{2} g_d \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} H^+(p)\gamma_5 t_2 \tau_2 \\ &= i g_d^2 \Delta_0^* \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p_0^2 - E_\Delta^{+2}} + \frac{1}{p_0^2 - E_\Delta^{-2}} \right) \\ &= g_d^2 \Delta_0^* I_2 \end{aligned} \tag{3.18}$$

while

$$\Gamma_{d^*} = g_d^2\, \Delta_0 I_2 \,. \tag{3.19}$$

The first order of the Lagrangian is required to be zero due to stability as

$$-M_s^2 \sigma_0 + 2g_0 m(I_2 - 2\mu^2 I_0) = 0., \tag{3.20}$$

and the diquark and anti-diquark terms become

$$-M_d^2 \Delta_0 + 2g_d^2 \Delta_0 I_2 = 0. \tag{3.21}$$

We recall that the relation between m and σ_0 is $m = g_0\sigma_0 + m_0$.

For later use, we write I_2 and I_0 as functions of temperature and chemical potential using the Matsubara formalism (see Appendix D.5):

$$\begin{aligned} I_2 =& -\mathrm{tr}_{fc} T \sum_n \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{(i\omega_n)^2 - (E_\Delta^+)^2} + \frac{1}{(i\omega_n)^2 - (E_\Delta^-)^2} \right) \\ =& -\mathrm{tr}_{fc} T \sum_n \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_\Delta^+} \left(\frac{1}{i\omega_n - E_\Delta^+} - \frac{1}{i\omega_n + E_\Delta^+} \right) + \frac{1}{2E_\Delta^-} \left(\frac{1}{i\omega_n - E_\Delta^-} - \frac{1}{i\omega_n + E_\Delta^-} \right) \right) \\ =& -\mathrm{tr}_{fc} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_\Delta^+} (n_F(E_\Delta^+) - n_F(-E_\Delta^+)) + \frac{1}{2E_\Delta^-} (n_F(E_\Delta^-) - n_F(-E_\Delta^-)) \right) \\ =& \mathrm{tr}_{fc} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_\Delta^+} (1 - 2n_F(E_\Delta^+)) + \frac{1}{2E_\Delta^-} (1 - 2n_F(E_\Delta^-)) \right) , \end{aligned} \tag{3.22}$$

where the Fermi distribution $n_F(E) = (e^{\beta E} + 1)^{-1}$ and the relation $n_F(-E) = 1 - n_F(E)$ have been used. To see Eqs. (3.20) and (3.21) are equivalent to the mean field equations, we further manipulate Eq. (3.13) as

$$\Gamma_s = 2ig_0 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \frac{m}{E_p} \left[\frac{E_p^+}{p_0^2 - (E_\Delta^+)^2} + \frac{E_p^-}{p_0^2 - (E_\Delta^-)^2} \right]. \tag{3.23}$$

In the Matsubara formulation of Γ_s, then one can find that this expression agrees with the mean field equations (2.53) and (2.54).

Next, we work out I_0 in the Matsubara formalism:

$$\begin{aligned}
I_0 =&\mathrm{tr}_{fc} T \sum_n \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{((i\omega_n)^2 - (E_\Delta^+)^2)^2} + \frac{1}{((i\omega_n)^2 - (E_\Delta^-)^2)^2} \right) \\
=&\mathrm{tr} \int \frac{d^3p}{(2\pi)^3} T \sum_n \left[\frac{1}{4(E_\Delta^+)^2} \left(\frac{1}{(i\omega_n - E_\Delta^+)^2} - \frac{2}{2E_\Delta^+} \left(\frac{1}{i\omega_n - E_\Delta^+} - \frac{1}{i\omega_n + E_\Delta^+} \right) + \frac{1}{(i\omega_n + E_\Delta^+)^2} \right) \right. \\
&\left. + \frac{1}{4(E_\Delta^-)^2} \left(\frac{1}{(i\omega_n - E_\Delta^-)^2} - \frac{2}{2E_\Delta^-} \left(\frac{1}{i\omega_n - E_\Delta^-} - \frac{1}{i\omega_n + E_\Delta^-} \right) + \frac{1}{(i\omega_n + E_\Delta^-)^2} \right) \right] \\
=&\mathrm{tr}_{fc} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{4(E_\Delta^+)^3} \left(-E_\Delta^+ \beta e^{\beta E_\Delta^+} n_F^2(E_\Delta^+) - (n_F(E_\Delta^+) - n_F(-E_\Delta^+)) - E_\Delta^+ \beta e^{-\beta E_\Delta^+} n_F^2(-E_\Delta^+) \right) \right. \\
&\left. + \frac{1}{4(E_\Delta^-)^3} \left(-E_\Delta^- \beta e^{\beta E_\Delta^-} n_F^2(E_\Delta^-) - (n_F(E_\Delta^-) - n_F(-E_\Delta^-)) - E_\Delta^+ \beta e^{-\beta E_\Delta^-} n_F^2(-E_\Delta^-) \right) \right] \\
=&\mathrm{tr} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{4(E_\Delta^+)^3} \left(1 - 2n_F(E_\Delta^+) - 2E_\Delta^+ \beta e^{\beta E_\Delta^+} n_F^2(E_\Delta^+) \right) \right. \\
&\left. + \frac{1}{4(E_\Delta^-)^3} \left(1 - 2n_F(E_\Delta^-) - 2E_\Delta^- \beta e^{\beta E_\Delta^-} n_F^2(E_\Delta^-) \right) \right] , \qquad (3.24)
\end{aligned}$$

where $n_F^2(-E) = (e^{-\beta E} + 1)^{-2} = e^{2\beta E} n_F^2(E)$ has been used. The divergent integrals I_0 and I_2 are calculated introducing a momentum cut-off Λ.

3.3 Mass and Kinetic Energy Terms (k=2)

The kinetic energies and the mass terms of mesons and diquark-baryons emerge from Eq. (3.7) at the order of $k = 2$:

$$U^{(2)} = \frac{i}{4} \mathrm{tr}(\hat{S}\hat{K})^2 = \int d^4y \sum_{ij} \Gamma_{ij}(x-y) \phi_i(x) \phi_j(y), \qquad (3.25)$$

where $\phi_i = \sigma, \vec{\pi}, d, d^*$. As we will see later, the sigma meson and the diquark-baryons make the mixing due to the scalar property while the pion does not. The second order of the effective Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}^{(2)} = \int d^4y \Bigg(&\left[\Gamma_s(x-y) - \frac{1}{2} M_s^2 \delta(x-y) \right] s(x)s(y) + \left[\Gamma_\pi(x-y) - \frac{1}{2} M_s^2 \delta(x-y) \right] \vec{\pi}(x)\vec{\pi}(y) \\
&+ \left[\Gamma_{dd^*}(x-y) - \frac{1}{2} M_d^2 \delta(x-y) \right] d(x)d^*(y) + \Gamma_{dd}(x-y) d(x)d(x) \\
&+ \Gamma_{d^*d^*}(x-y) d^*(x)d^*(y) + \Gamma_{sd}(x-y) s(x)d(y) + \Gamma_{sd^*}(x-y) s(x)d^*(y) \Bigg). \qquad (3.26)
\end{aligned}$$

The non-local propagator terms Γ_{ij} are expanded as

$$\begin{aligned}\Gamma(x-y) =& \int \frac{d^4q}{(2\pi)^4} e^{iq(x-y)}\Gamma(q)\\ &\sim\delta(x-y)\Gamma(0) - i\partial^\mu\delta(x-y)\partial_\mu\Gamma(q)\mid_{q=0} -\frac{1}{2}\partial^\mu\partial^\nu\delta(x-y)\partial_\mu\partial_\nu\Gamma(q)\mid_{q=0}.\end{aligned} \tag{3.27}$$

The integration variable momentum p^μ appear as $p_0^2-(E_\Delta^\pm)^2$ in the integrand. The quasi-particle energies $E_\Delta^\pm = \sqrt{(E_p^\pm)^2+|\Delta|^2}(E_p^\pm = \sqrt{\vec{p}^2+m^2}\pm\mu)$ seem to break the Lorentz covariance and hence we have to calculate the derivative with respect to q^0 and $\vec{q}$ in the $U^{(2)}$ separately.

We first note useful formulas in the limit of the cut-off $\Lambda\to\infty$ below:

$$(E_\Delta^+)^2 + (E_\Delta^-)^2 = 2(E_p^2+\mu^2+|\Delta|^2) \tag{3.28}$$

$$\begin{aligned}4p_0^2 \to& p_0^2-\vec{p}^2 = p_0^2 - E_p^2 + m^2 \sim p_0^2 - E_p^2 - \mu^2 - |\Delta|^2\\ =& \begin{cases} p_0^2-(E_\Delta^\pm)^2 \pm 2\mu E_p \\ \frac{1}{2}[(p_0^2-E_\Delta^+)^2 + (p_0^2-E_\Delta^-)^2] \end{cases}\end{aligned} \tag{3.29}$$

$$\vec{p}^2 \to -\frac{3}{4}(p_0^2-\vec{p}^2) \tag{3.30}$$

$$E_p^2 = \vec{p}^2 + m^2 \to -\frac{3}{4}(p_0^2-E_p^2) = -\frac{3}{4}(p_0^2-(E_\Delta^\pm)^2) \tag{3.31}$$

$$\frac{p^ip^j}{E_p^2} \overset{i=j}{=} 1 - \frac{m^2}{E_p^2} \tag{3.32}$$

$$\frac{p^ip^j}{E_p^4} - \frac{\delta^{ij}}{E_p^2} \overset{i=j}{=} -\frac{m^2}{E_p^4} \tag{3.33}$$

$$-4p^ip^j \overset{i=j}{=} p_0^2 - \vec{p}^2 \to p_0^2 - E_p^2 - \mu^2 - |\Delta|^2, \tag{3.34}$$

where $i,j=1,2,3$.

3.3.1 Pion

The result of the Fourier transformed propagator for the pion in Eq. (3.25) is:

$$\begin{aligned}\Gamma_\pi(q) =& \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}g_0^2\left(H^-(p+q)\gamma_5H^+(p)\gamma_5 + H^+(p+q)\gamma_5H^-(p)\gamma_5\right.\\ &\left.-G^+(p+q)\gamma_5G^+(p)\gamma_5 - G^-(p+q)\gamma_5G^-(p)\gamma_5\right),\end{aligned} \tag{3.35}$$

where the arguments p and $p+q$ denote the four momentum in the $G^\pm$ and $H^\pm$ with the internal quark momentum p and the external momentum q. To proceed the calculation, we define compact expressions

$$p' = p+q = p^\mu + q^\mu = p_0' - \vec{p}' = (p_0+q_0) - (\vec{p}+\vec{q}), \tag{3.36}$$

$$E_p' = \sqrt{(\vec{p}+\vec{q})^2+m^2}, \tag{3.37}$$

$$E_{p'}^\pm = E_p' \pm \mu, \tag{3.38}$$

$$(E_{\Delta'}^\pm)^2 = (E_{p'}^\pm)^2 + |\Delta|^2. \tag{3.39}$$

In this notation

$$
\begin{aligned}
\Gamma_\pi(q) =& \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}g_0^2(H^-(p')\gamma_5H^+(p)\gamma_5+H^+(p')\gamma_5H^-(p)\gamma_5 \\
&-G^+(p')\gamma_5G^+(p)\gamma_5-G^-(p')\gamma_5G^-(p)\gamma_5) \\
=&\frac{i}{4}g_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{(p_0'+E_{p'}^-)(p_0-E_p^+)+(p_0'-E_{p'}^-)(p_0+E_p^+)-2|\Delta|^2}{((p_0')^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)}\right.\right. \\
&\left.+\frac{(p_0'+E_{p'}^+)(p_0-E_p^-)+(p_0'-E_{p'}^+)(p_0+E_p^-)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)}\right)\left(1+\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right) \\
&+\left(\frac{(p_0'+E_{p'}^-)(p_0+E_p^-)+(p_0'-E_{p'}^-)(p_0-E_p^-)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)}\right. \\
&\left.\left.+\frac{(p_0'+E_{p'}^+)(p_0+E_p^+)+(p_0'-E_{p'}^+)(p_0-E_p^+)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)}\right)\left(1-\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\right] \\
\equiv& \frac{i}{4}g_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}(A_\pi B_\pi+C_\pi D_\pi).
\end{aligned}
\tag{3.40}
$$

The propagator is expended around $q=0$ as Eq. (3.27). We consider $\Gamma_\pi(0)$ first

$$
\begin{aligned}
\Gamma_\pi(0) =& ig_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\frac{(p_0+E_p^-)(p_0-E_p^+)+(p_0-E_p^-)(p_0+E_p^+)-2|\Delta|^2}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)} \\
=&2ig_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\frac{p_0^2-E_p^-E_p^+-|\Delta|^2}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)} \\
=&2ig_0^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{p_0^2-E_p^2+\mu^2-|\Delta|^2}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)}\right) \\
=&2ig_0^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{2}\frac{(p_0^2-(E_\Delta^+)^2)+(p_0^2-(E_\Delta^-)^2)}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)}+\frac{2\mu^2}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)}\right) \\
=&ig_0^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{p_0^2-(E_\Delta^-)^2}+\frac{1}{p_0^2-(E_\Delta^+)^2}\right) \\
&+2ig_0^2\mu^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}\right) \\
=&g_0^2I_2-2g_0^2\mu^2I_0.
\end{aligned}
\tag{3.41}
$$

Note that all the integrands are given by the divergent integral I_2 and I_0 by using Eqs. (3.28) - (3.34).

As we mentioned, the derivatives are evaluated for the temporal ∂_0 and the spatial ∂_i parts separately. We evaluate

$$
\begin{aligned}
\partial_0\Gamma_\pi(q) =& \frac{i}{4}g_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left(\partial_0(A_\pi B_\pi)+\partial_0(C_\pi D_\pi)\right) \\
=&\frac{i}{4}g_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}(\dot{A}_\pi B_\pi+A_\pi\dot{B}_\pi+\dot{C}_\pi D_\pi+C_\pi\dot{D}_\pi) \\
=&\frac{i}{4}g_0^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}(\dot{A}_\pi+\dot{C}_\pi).
\end{aligned}
\tag{3.42}
$$

Since the second therm of B_π and D_π obviously make convergent integrals (higher order of the denominator), we have dropped these therms in the last line. We calculate each terms

$$\begin{aligned}\dot{A}_\pi =& \frac{p_0 - E_p^+ + p_0 + E_p^+}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\ &- 2p_0' \frac{(p_0' + E_{p'}^-)(p_0 - E_p^+) + (p_0' - E_{p'}^-)(p_0 + E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^-)^2)^2(p_0^2 - (E_\Delta^+)^2)}, \end{aligned} \tag{3.43}$$

$$\begin{aligned}\dot{D}_\pi =& \frac{p_0 + E_p^+ + p_0 - E_p^+}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2 - (E_\Delta^+)^2)} \\ &- 2p_0' \frac{(p_0' + E_{p'}^+)(p_0 + E_p^+) + (p_0' - E_{p'}^+)(p_0 - E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^+)^2)^2(p_0^2 - (E_\Delta^+)^2)}. \end{aligned} \tag{3.44}$$

The integral over p^0 is zero due to the odd function of p^0. We further calculate the second order derivatives by dropping the odd p_0

$$\begin{aligned}\ddot{A}_\pi =& - 2\frac{(p_0' + E_{p'}^-)(p_0 - E_p^+) + (p_0' - E_{p'}^-)(p_0 + E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^-)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\ &- 2p_0' \frac{p_0 - E_p^+ + p_0 + E_p^+}{(p_0'^2 - (E_{\Delta'}^-)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\ &+ 8(p_0')^2 \frac{(p_0' + E_{p'}^-)(p_0 - E_p^+) + (p_0' - E_{p'}^-)(p_0 + E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^-)^2)^3(p_0^2 - (E_\Delta^+)^2)}, \end{aligned} \tag{3.45}$$

$$\begin{aligned}\ddot{D}_\pi =& - 2\frac{(p_0' + E_{p'}^+)(p_0 + E_p^+) + (p_0' - E_{p'}^+)(p_0 - E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^+)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\ &- 2p_0' \frac{p_0 + E_p^+ + p_0 - E_p^+}{(p_0'^2 - (E_{\Delta'}^+)^2)^2(p_0^2 - (E_\Delta^-)^2)} \\ &+ 8(p_0')^2 \frac{(p_0' + E_{p'}^+)(p_0 + E_p^+) + (p_0' - E_{p'}^+)(p_0 - E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^+)^2)^3(p_0^2 - (E_\Delta^+)^2)}. \end{aligned} \tag{3.46}$$

Taking $q = 0$,

$$\begin{aligned}(\partial_0)^2\Gamma_\pi(q)\big|_{q=0} =& - 2ig_0^2 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \right) \\ =& - ig_0^2 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} \right) \\ =& g_0^2 I_0. \end{aligned} \tag{3.47}$$

The first order space derivative is

$$\begin{aligned}\partial_i\Gamma_\pi(q)|_{q=0} =& 2ig_0^2 \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left(\frac{-p^i}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \right. \\ &+ \frac{p^i(1 - \mu(E_p)^{-1})(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2)}{(p_0^2 - (E_\Delta^-)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\ &\left. + \frac{p^i(1 + \mu(E_p)^{-1})(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2)}{(p_0^2 - (E_\Delta^+)^2)^2(p_0^2 - (E_\Delta^-)^2)} \right) \\ \to& 0 \quad \text{(odd)} \end{aligned} \tag{3.48}$$

The second order derivative is written as $A''_\pi B_\pi + 2A'_\pi B'_\pi + A_\pi B''_\pi + C''_\pi D + 2C'_\pi D'_\pi + C_\pi D''_\pi$. Then $B'_\pi = D'_\pi = D_\pi = 0, B_\pi = 2$ and $B''_\pi = -D''_\pi = p^i p^j / E_p^4 - \delta^{ij}/E_p^2$. Hence,

$$\partial_i \partial_j \Gamma_\pi(q)|_{q=0} = \frac{i}{4} g_0^2 \mathrm{tr} \int \frac{d^4 p}{(2\pi)^4} \left(2A''_\pi + (A_\pi - C_\pi) \left(\frac{p^i p^j}{E_p^4} - \frac{\delta^{ij}}{E_p^2} \right) \right). \tag{3.49}$$

We evaluate

$$\frac{p^i p^j}{E_p^4} - \frac{\delta^{ij}}{E_p^2} \stackrel{i=j}{=} \frac{1}{E_p^2} - \frac{m^2}{E_p^4} - \frac{1}{E_p^2} = -\frac{m^2}{E_p^4}, \tag{3.50}$$

so the second term of $\partial_i \partial_j \Gamma_\pi(q)|_{q=0}$ converges. Thus, we have to work out only A'' term as

$$\begin{aligned}
A''_\pi =& \frac{-4\delta^{ij}}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} + \frac{4p^i p^j}{E_p^2 (p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
&+ 4\delta^{ij}(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2) \\
&\times \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2 (p_0^2 - (E_\Delta^+)^2)} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2 (p_0^2 - (E_\Delta^-)^2)} \right) \\
&- 8p^i p^j \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2 (p_0^2 - (E_\Delta^+)^2)} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2 (p_0^2 - (E_\Delta^-)^2)} \right) \\
&+ 16 p^i p^j (p_0^2 - E_p^2 - \mu^2 - |\Delta|^2) \\
&\times \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^3 (p_0^2 - (E_\Delta^+)^2)} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^3 (p_0^2 - (E_\Delta^-)^2)} \right) \\
\stackrel{i=j}{=}& \frac{2}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{2}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{4}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
&+ \frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{2}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
&+ \frac{-3}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{-3}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{-2}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
=& \frac{4}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} = 2 \left(\frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} \right).
\end{aligned} \tag{3.51}$$

Hence, the second order derivative is written as

$$\partial_i \partial_j \Gamma_\pi(q)|_{q=0} = -g_0^2 I_0 \delta_{ij}. \tag{3.52}$$

Since the spatial components have minus sign in the Minkowski metric, the final result of the second order derivative of Γ_π can be written as

$$\partial_\mu \partial_\nu \Gamma_\pi(q)|_{q=0} = g_0^2 I_0 g_{\mu\nu}. \tag{3.53}$$

The pion part in the second order is written as

$$\mathcal{L}_\pi^{(2)} = \frac{1}{2} g_0^2 I_0 \partial^\mu \vec{\pi}(x) \partial_\mu \vec{\pi}(x) - \frac{1}{2} \left[M_s^2 - 2g_0^2 (I_2 - 2\mu^2 I_0) \right] \vec{\pi}^2(x), \tag{3.54}$$

which can be identified with the Klein-Gordon type Lagrangian if the mass term is given as

$$m_\pi^2 = M_s^2 - 2g_0^2 I_2 + 4g_0^2 \mu^2 I_0. \tag{3.55}$$

The mass term is described by the quark propagator and hence reflect the quark property and the kinetic term include the divergent integral I_0. It is notable that this pion mass form is similar to the quark mass gap equation.

3.3.2 Sigma Meson

We consider the sigma meson term:

$$
\begin{aligned}
\Gamma_s(q) =& \frac{i}{4}\mathrm{tr}\int \frac{d^4p}{(2\pi)^4} g_0^2 \left(G^+(p')G^+(p) + G^-(p')G^-(p) + H^-(p')H^+(p) + H^+(p')H^-(p)\right) \\
=& \frac{i}{4} g_0^2 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left[\left(\frac{(p_0' + E_{p'}^-)(p_0 - E_p^+) + (p_0' - E_{p'}^-)(p_0 + E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2 - (E_\Delta^+)^2)}\right.\right. \\
&+ \left.\frac{(p_0' + E_{p'}^+)(p_0 - E_p^-) + (p_0' - E_{p'}^+)(p_0 + E_p^-) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2 - (E_\Delta^-)^2)}\right)\left(1 + \frac{\vec{p}' \cdot \vec{p} - m^2}{E_{p'}E_p}\right) \\
&+ \left(\frac{(p_0' + E_{p'}^-)(p_0 + E_p^-) + (p_0' - E_{p'}^-)(p_0 - E_p^-) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2 - (E_\Delta^-)^2)}\right. \\
&+ \left.\left.\frac{(p_0' + E_{p'}^+)(p_0 + E_p^+) + (p_0' - E_{p\prime}^+)(p_0 - E_p^+) - 2|\Delta|^2}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2 - (E_\Delta^+)^2)}\right)\left(1 - \frac{\vec{p}\,' \cdot \vec{p} - m^2}{E_{p'}E_p}\right)\right] \\
\equiv& \frac{i}{4} g_0^2 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} (A_s B_s + C_s D_s).
\end{aligned}
\tag{3.56}
$$

We work out $q = 0$ case in which $B_s = 2 - \frac{2m^2}{E_p^2}$ and $D_s = \frac{2m^2}{E_p^2}$:

$$
\begin{aligned}
\Gamma_s(0) =& i g_0^2 \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left[\frac{2(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2 + 2\mu)}{(p_0^2 - E_\Delta^{-2})(p_0^2 - (E_\Delta^+)^2)}\right. \\
&+ \frac{m^2}{E_p^2}\left(\frac{p_0^2 - (E_p^-)^2 - |\Delta|^2 + 2(E_p^-)^2}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{p_0^2 - (E_p^+)^2 - |\Delta|^2 + 2(E_p^+)^2}{(p_0^2 - (E_\Delta^+)^2)^2}\right. \\
&- \left.\left.\frac{2(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2 + 2\mu^2)}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)}\right)\right] \\
=& i g_0^2 \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left[\left(\frac{1}{p_0^2 - (E_\Delta^+)^2} + \frac{1}{p_0^2 - (E_\Delta^-)^2}\right) + 2\mu^2\left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2}\right)\right. \\
&+ \left. 2m^2\left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2}\right)\right] \\
=& g_0^2 I_2 - 2g_0^2\mu^2 I_0 - 2g_0^2 m^2 I_0.
\end{aligned}
\tag{3.57}
$$

The first order time derivative is given as $(\dot{A}_s B_s + \dot{C}_s D_s)|_{q=0} = 2\dot{A}_s + (\dot{C}_s - \dot{A}_s)\frac{2m^2}{E_p^2}$.

$$
\begin{aligned}
\dot{A}_s =& \frac{2p_0}{(p_0^{2\prime} - (E_{\Delta'}^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
& - 2p_0' \frac{(p_0' + E_{p'}^-)(p_0 - E_p^+) + (p_0' - E_{p'}^-)(p_0 + E_p^+) - 2|\Delta|^2}{(p_0^{2\prime} - (E_{\Delta'}^-)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\
& + \frac{2p_0}{(p_0^{2\prime} - (E_{\Delta'}^+)^2)(p_0^2 - (E_\Delta^-)^2)} \\
& - 2p_0' \frac{(p_0' + E_{p'}^+)(p_0 - E_p^-) + (p_0' - E_{p'}^+)(p_0 + E_p^-) - 2|\Delta|^2}{(p_0^{2\prime} - (E_{\Delta'}^+)^2)^2(p_0^2 - (E_\Delta^-)^2)} \\
=& \frac{4p_0}{(p_0^2 - (E_\Delta^-)^2)(p_0^2 - (E_\Delta^+)^2)} \\
& - 4p_0 \frac{p_0^2 - E_p^2 + \mu^2 - |\Delta|^2}{(p_0^2 - (E_\Delta^-)^2)^2(p_0^2 - (E_\Delta^+)^2)} - 4p_0 \frac{p_0^2 - E_p^2 + \mu^2 - |\Delta|^2}{(p_0^2 - (E_\Delta^+)^2)^2(p_0^2 - (E_\Delta^-)^2)} \\
=& -2p_0 \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} \right),
\end{aligned}
\tag{3.58}
$$

and

$$
\begin{aligned}
\dot{C}_s =& \frac{2p_0}{(p_0^{2\prime} - (E_{\Delta'}^-)^2)(p_0^2 - (E_\Delta^-)^2)} \\
& - 2p_0' \frac{(p_0' + E_{p'}^-)(p_0 + E_p^-) + (p_0' - E_{p'}^-)(p_0 - E_p^-) - 2|\Delta|^2}{(p_0^{2\prime} - (E_{\Delta'}^-)^2)^2(p_0^2 - (E_\Delta^-)^2)} \\
& + \frac{2p_0}{(p_0^{2\prime} - (E_{\Delta'}^+)^2)(p_0^2 - (E_\Delta^+)^2)} \\
& - 2p_0' \frac{(p_0' + E_{p'}^+)(p_0 + E_p^+) + (p_0' - E_{p'}^+)(p_0 - E_p^+) - 2|\Delta|^2}{(p_0^{2\prime} - (E_{\Delta'}^+)^2)^2(p_0^2 - (E_\Delta^+)^2)} \\
=& \frac{2p_0}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{2p_0}{(p_0^2 - (E_\Delta^+)^2)^2} \\
& - 4p_0 \frac{p_0^2 + (E_p^-)^2 - |\Delta|^2}{(p_0^2 - (E_\Delta^-)^2)^3} - 4p_0 \frac{p_0^2 + (E_p^+)^2 - |\Delta|^2}{(p_0^2 - (E_\Delta^+)^2)^3} \\
=& -2p_0 \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} \right) \\
& - 8p_0 \left(\frac{(E_p^-)^2}{(p_0^2 - (E_\Delta^-)^2)^3} + \frac{(E_p^+)^2}{(p_0^2 - (E_\Delta^+)^2)^3} \right).
\end{aligned}
\tag{3.59}
$$

The result is written as

$$
\partial_0 \Gamma_s(q)|_{q=0} = -i\mathrm{tr} \int \frac{d^4p}{(2\pi)^4} p_0 \left(\frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} \right) \to 0. \tag{3.60}
$$

The second order derivative is given as $(\ddot{A}_s B_s + \ddot{C}_s D_s)|_{q=0} = 2\ddot{A}_s + (\ddot{C}_s - \ddot{A}_s)\frac{2m^2}{E_p^2}$. The second term $(\ddot{C}_s - \ddot{A}_s)\frac{2m^2}{E_p^2}$ gives finite:

$$\begin{aligned}
\ddot{A}_s =&(-8p_0^2 - 4(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2 + 2\mu^2)) \\
&\times\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2(p_0^2-(E_\Delta^+)^2)} + \frac{1}{(p_0^2-(E_\Delta^+)^2)^2(p_0^2-(E_\Delta^-)^2)}\right) \\
&+ 16p_0^2(p_0^2 - E_p^2 - \mu^2 - |\Delta|^2 + 2\mu^2) \\
&\times\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^3(p_0^2-(E_\Delta^+)^2)} + \frac{1}{(p_0^2-(E_\Delta^+)^2)^3(p_0^2-(E_\Delta^-)^2)}\right) \\
=&-2\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2} + \frac{1}{(p_0^2-(E_\Delta^-)^2)}\right).
\end{aligned} \tag{3.61}$$

Then

$$\partial_0\partial_0\Gamma_{ss}(q)|_{q=0} = g_0^2 I_0. \tag{3.62}$$

The first order space derivative is given as $(A_s'B_s + A_sB_s' + C_s'D_s + C_sD_s')|_{q=0}$. Then $B_s'(0) = \frac{2m^2p^i}{E_p^4}$, $D_s'(0) = -\frac{2m^2p^i}{E_p^4}$, hence A_sB_s' and C_sD_s' give finite. The A_s' and C_s' are written as

$$\begin{aligned}
A_s' =&2p'^i\left(\frac{-E_p^+}{E_{p'}(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)} + \frac{-E_p^-}{E_{p'}(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)}\right. \\
&+(1-\mu(E_{p'})^{-1})\frac{(p_0'+E_{p'}^-)(p_0-E_p^+)+(p_0'-E_{p'}^-)(p_0+E_p^+)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^-)^2)^2(p_0^2-(E_\Delta^+)^2)} \\
&\left.+(1+\mu(E_{p'})^{-1})\frac{(p_0'+E_{p'}^+)(p_0-E_p^-)+(p_0'-E_{p'}^+)(p_0+E_p^-)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^+)^2)^2(p_0^2-(E_\Delta^-)^2)}\right),
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
C_s' =&2p'^i\left(\frac{E_p^+}{E_{p'}(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)} + \frac{E_p^-}{E_{p'}(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)}\right. \\
&+(1-\mu(E_{p'})^{-1})\frac{(p_0'+E_{p'}^-)(p_0+E_p^-)+(p_0'-E_{p'}^-)(p_0-E_p^-)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^-)^2)^2(p_0^2-(E_\Delta^-)^2)} \\
&\left.+(1+\mu(E_{p'})^{-1})\frac{(p_0'+E_{p'}^+)(p_0+E_p^+)+(p_0'-E_{p'}^+)(p_0-E_p^+)-2|\Delta|^2}{(p_0'^2-(E_{\Delta'}^+)^2)^2(p_0^2-(E_\Delta^+)^2)}\right).
\end{aligned} \tag{3.64}$$

Of course, the first order derivative goes to zero by the integration over p^i because the each terms are an odd function of p^i. The second derivative is given as $A_s''B_s + 2A_s'B_s' + A_sB_s'' + C_s''D_s + 2C_s'D_s' + C_sD_s'' = 2A_s'' + (C_s'' - A_s'')\frac{2m^2}{E_p^2} + 2(A_s' - C_s')\frac{2m^2p^i}{E_p^4} + (A_s - C_s)B''$. We work out B_s'' and D_s'' as

$$\begin{aligned}
B_s'' =& -\frac{\delta^{ij}}{E_p^2} + \frac{p^ip^j}{E_p^4} + \frac{2m^2\delta^{ij}}{E_p^4} - \frac{6p^ip^jm^2}{E_p^6} \\
\overset{i=j}{=}& -\frac{5m^2}{E_p^4} + \frac{6m^4}{E_p^6},
\end{aligned} \tag{3.65}$$

$$D_s'' \overset{i=j}{=} \frac{5m^2}{E_p^4} - \frac{6m^4}{E_p^6} = -B_s''. \tag{3.66}$$

A''_s and C''_s are written as

$$\begin{aligned}
A''_s \overset{i=j}{=} & \frac{-6}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)} + \frac{3}{(p_0^2-(E_\Delta^-)^2)^2} + \frac{3}{(p_0^2-(E_\Delta^+)^2)^2} \\
& + \frac{10}{(p_0^2-(E_\Delta^-)^2)(p_0^2-(E_\Delta^+)^2)} + \frac{-3}{(p_0^2-(E_\Delta^-)^2)^2} + \frac{-3}{(p_0^2-(E_\Delta^+)^2)^2} \\
=& 2\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)} + \frac{1}{(p_0^2-(E_\Delta^-)^2)}\right), \qquad (3.67) \\
C''_s \overset{i=j}{=} & -2\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2} + \frac{1}{(p_0^2-(E_\Delta^+)^2)^2}\right) \\
& + 8\left(\frac{(E_p^-)^2}{(p_0^2-(E_\Delta^-)^2)^3} + \frac{(E_p^+)^2}{(p_0^2-(E_\Delta^+)^2)^3}\right). \qquad (3.68)
\end{aligned}$$

Hence the terms $(A'_s - C'_s)\frac{2m^2p^i}{E_p^4}$, $(A_s - C_s)B''_s$ and $(C''_s - A''_s)\frac{2m^2}{E_p^2}$ converge. The second order space derivative is written as

$$\partial_i\partial_j\Gamma_s(q)|_{q=0} = -\, g_0^2 I_0 \delta_{ij}. \qquad (3.69)$$

The result of the second order derivative of Γ_{ss} is

$$\partial_\mu\partial_\nu\Gamma_s(q)|_{q=0} = g_0^2 I_0 g_{\mu\nu}. \qquad (3.70)$$

The sigma meson part in the second order is given as

$$\mathcal{L}_s^{(2)} = \frac{1}{2}g_0^2 I_0 \partial^\mu s(x)\partial_\mu s(x) - \frac{1}{2}\left[M_s^2 - 2g_0^2(I_2 - 2\mu^2 I_0 - 2m^2 I_0)\right] s^2(x). \qquad (3.71)$$

The mass term can be identified

$$m_s^2 = M_s^2 - 2g_0^2 I_2 + 4g_0^2\mu^2 I_0 + 4g_0^2 m^2 I_0 = m_\pi^2 + 4g_0^2 m^2 I_0. \qquad (3.72)$$

3.3.3 Diquark-Baryon

The diquark-baryons are the complex scalar fields. The term $U^{(2)}$ generates the dd^* term and d^*d term separately. The difference of these terms are only the chemical potential dependency which will be derived from the first order derivative with temporal part in contrast with mesons. The mesons constructed from $\bar{q}q$ have no antiparticle (or self-conjugate) and have no chemical potential dependence. The diquark-baryons are, however, constructed from qq and their antiparticles are described by $\bar{q}\bar{q}$. The conserved baryon charge is written as

$$Q = \int d^3x J^0 = \int d^3x i(\Delta^*(x)\Pi_\Delta(x) - \Delta(x)\Pi_{\Delta^*}(x)), \qquad (3.73)$$

with the canonical conjugate momenta

$$\begin{aligned}
\Pi_\Delta(x) =& \partial_0\Delta(x) \\
\Pi_{\Delta^*}(x) =& \partial_0\Delta^*(x). \qquad (3.74)
\end{aligned}$$

The first order derivatives represent the conserved charge and the difference between dd^* and d^*d is opposites sign. We can identify the dd^* and d^*d without the first order derivative.

We consider dd^* term:

$$\begin{aligned}\Gamma_{dd^*}(q) =& \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}g_d^2(-G^+(p')\gamma_5t_2\tau_2G^-(p)\gamma_5t_2\tau_2)\\ =&\frac{i}{4}g_d^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\Bigg[\Bigg(\frac{(p_0'+E_{p'}^-)(p_0-E_p^-)}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)}\\ &+\frac{(p_0'-E_{p'}^+)(p_0+E_p^+)}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)}\Bigg)\left(1+\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\\ &+\Bigg(\frac{(p_0'+E_{p'}^-)(p_0+E_p^+)}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)}\\ &+\frac{(p_0'-E_{p'}^+)(p_0-E_p^-)}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)}\Bigg)\left(1-\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\Bigg]\\ \equiv&\frac{i}{4}g_d^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left(A_{dd^*}B_{dd^*}+C_{dd^*}D_{dd^*}\right).\end{aligned}\tag{3.75}$$

We work out the case $q=0$

$$\begin{aligned}\Gamma_{dd^*}(0) =&\frac{i}{2}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{p_0^2-(E_p^-)^2}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{p_0^2-(E_p^+)^2}{(p_0^2-(E_\Delta^+)^2)^2}\right)\\ =&\frac{i}{2}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\Bigg(\frac{1}{p_0^2-(E_\Delta^-)^2}+\frac{1}{p_0^2-(E_\Delta^+)^2}\\ &+\frac{|\Delta|^2}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{|\Delta|^2}{(p_0^2-(E_\Delta^-)^2)^2}\Bigg)\\ =&\frac{1}{2}g_d^2I_2-\frac{1}{2}g_d^2|\Delta|^2I_0.\end{aligned}\tag{3.76}$$

The first order time derivative is written as

$$\begin{aligned}\partial_0\Gamma_{dd^*}|_{q=0} =&\frac{i}{2}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\Bigg[\left(\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}-\frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2}\right)\\ &+\left(\frac{\mu}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{\mu}{(p_0^2-(E_\Delta^-)^2)^2}\right)\Bigg],\end{aligned}\tag{3.77}$$

where the first term is evaluated as

$$
\begin{aligned}
\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}-\frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2} =& \frac{E_p}{(p_0^2-(E_\Delta^-)^2-4\mu E_p)^2}-\frac{E_p}{(p_0^2-(E_\Delta^+)^2+4\mu E_p)^2}\\
=& \frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2}\left(1+2\frac{4\mu E_p}{p_0^2-(E_\Delta^-)^2}\right)\\
&-\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}\left(1-2\frac{4\mu E_p}{p_0^2-(E_\Delta^+)^2}\right)\\
2\left(\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}-\frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2}\right) =& 8\mu E_p^2\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^3}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^3}\right)\\
\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}-\frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2} =& -3\mu\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}\right).
\end{aligned}
\tag{3.78}
$$

Thus,

$$
\partial_0\Gamma_{dd^*}|_{q=0}=g_d^2\mu I_0. \tag{3.79}
$$

The second order derivative is written as

$$
\begin{aligned}
\partial_0\partial_0\Gamma_{dd^*}(q)|_{q=0} =& ig_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{-2p_0(p_0-(E_p^-))-(p_0^2-(E_p^-)^2)}{(p_0^2-(E_\Delta^-)^2)^3}\right.\\
&+\frac{-2p_0(p_0+(E_p^+))-(p_0^2-(E_p^+)^2)}{(p_0^2-(E_\Delta^+)^2)^3}\\
&\left.+\frac{4p_0^2(p_0^2-(E_p^-)^2)}{(p_0^2-(E_\Delta^-)^2)^4}+\frac{4p_0^2(p_0^2-(E_p^+)^2)}{(p_0^2-(E_\Delta^+)^2)^4}\right)\\
=& -\frac{i}{2}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}\right)\\
=& \frac{1}{2}g_d^2 I_0.
\end{aligned}
\tag{3.80}
$$

The first order space derivative is given by $2A'_{dd^*}$ and the second order derivative is $2A''_{dd^*}$.

We work out the space derivatives of A_{dd^*}:

$$\begin{aligned}A'_{dd^*} =&p'^i\left(\frac{p_0-E_p^-}{E_{p'}(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)}+\frac{-(p_0+E_p^+)}{E_{p'}(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)}\right.\\&\left.+\frac{2(1-\mu(E_{p'})^{-1})(p_0'+E_{p'}^-)(p_0-E_p^-)}{(p_0'^2-(E_{\Delta'}^-)^2)^2(p_0^2-(E_\Delta^-)^2)}+\frac{2(1+\mu(E_{p'})^{-1})(p_0'-E_{p'}^+)(p_0+E_p^+)}{(p_0'^2-(E_{\Delta'}^+)^2)^2(p_0^2-(E_\Delta^+)^2)}\right),\end{aligned}\tag{3.81}$$

$$\begin{aligned}A''_{dd^*} =&\delta^{ij}\left(\frac{p_0-E_p^-}{E_p(p_0^2-(E_\Delta^-)^2)^2}-\frac{p_0+E_p^-}{E_p(p_0^2-(E_\Delta^+)^2)^2}+\frac{2(p_0^2-(E_p^-)^2)}{(p_0^2-(E_\Delta^-)^2)^3}+\frac{2(p_0^2-(E_p^+)^2)}{(p_0^2-(E_\Delta^+)^2)^3}\right)\\&+p^ip^j\left(\frac{-(p_0-E_p^-)}{E_p^3(p_0^2-(E_\Delta^-)^2)^2}+\frac{p_0+E_p^+}{E_p^3(p_0^2-(E_\Delta^+)^2)^2}+\frac{4(p_0-E_p^-)}{E_p(p_0^2-(E_\Delta^-)^2)^3}\right.\\&\left.+\frac{4(p_0+E_p^+)}{E_p(p_0^2-(E_\Delta^+)^2)^3}+\frac{8(p_0^2-(E_p^-)^2)}{(p_0^2-(E_\Delta^-)^2)^4}+\frac{8(p_0^2-(E_p^+)^2)}{(p_0^2-(E_\Delta^+)^2)^4}\right)\\\overset{i=j}{=}&\frac{p_0-E_p^-}{E_p(p_0^2-(E_\Delta^-)^2)^2}-\frac{p_0+E_p^-}{E_p(p_0^2-(E_\Delta^+)^2)^2}+\frac{2}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{2}{(p_0^2-(E_\Delta^+)^2)^2}\\&-\frac{p_0-E_p^-}{E_p(p_0^2-(E_\Delta^-)^2)^2}+\frac{p_0+E_p^+}{E_p(p_0^2-(E_\Delta^+)^2)^2}-\frac{p_0-E_p^-}{E_p(p_0^2-(E_\Delta^-)^2)^2}\\&+\frac{p_0+E_p^+}{E_p(p_0^2-(E_\Delta^+)^2)^2}-\frac{2}{(p_0^2-(E_\Delta^-)^2)^2}-\frac{2}{(p_0^2-(E_\Delta^+)^2)^2}\\=&\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}.\end{aligned}\tag{3.82}$$

Since A'_{dd^*} is odd function of p, the first derivative vanishes. The second order derivative is written as

$$\partial_i\partial_j\Gamma_{dd^*}|_{q=0}=-\frac{1}{2}g_d^2I_0\delta_{ij}.\tag{3.83}$$

The result of the second order derivative of Γ_{dd^*} is

$$\partial_\mu\partial_\nu\Gamma_{dd^*}(q)|_{q=0}=\frac{1}{2}g_d^2I_0g_{\mu\nu}.\tag{3.84}$$

We consider d^*d term:

$$\begin{aligned}\Gamma_{d^*d}(q) =&\frac{i}{4}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}(-G^-(p')\gamma_5t_2\tau_2G^+(p)\gamma_5t_2\tau_2)\\=&\frac{i}{4}g_d^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{(p_0'-E_{p'}^-)(p_0+E_p^-)}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2+(E_\Delta^-)^2)}\right.\right.\\&\left.+\frac{(p_0'+E_{p'}^+)(p_0-E_p^+)}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)}\right)\left(1+\frac{\vec{p}'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\\&+\left(\frac{(p_0'-E_{p'}^-)(p_0-E_p^+)}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)}\right.\\&\left.\left.+\frac{(p_0'+E_{p'}^+)(p_0+E_p^-)}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)}\right)\left(1-\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\right],\end{aligned}\tag{3.85}$$

and its time derivative

$$\partial_0\Gamma_{d^*d}(q)|_{q=0} = \frac{i}{2}g_d^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{E_p}{(p_0^2-(E_\Delta^-)^2)^2}-\frac{E_p}{(p_0^2-(E_\Delta^+)^2)^2}\right)\right.$$
$$\left.-\left(\frac{\mu}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{\mu}{(p_0^2-(E_\Delta^+)^2)^2}\right)\right]. \tag{3.86}$$

Thus,

$$\partial_0\Gamma_{d^*d}(q)|_{q=0} = -g_d^2\mu I_0 \tag{3.87}$$

in the same way as dd^*. Note that $\Gamma_{d^*d}(q)$ gives the same result as $\Gamma_{dd^*}(q)$ except for $\partial_0\Gamma_{d^*d}(q)|_{q=0}$.

Since the diquarks are baryons, the diquark condensation break the baryon charge symmetry. The existence of the condensation arise the mixing terms dd and d^*d^*. We work out the diquark mixing terms dd and d^*d^*. The dd term is

$$\Gamma_{dd}(q) = \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}g_d^2H^+(p')\gamma_5t_2\tau_2H^+(p)\gamma_5t_2\tau_2$$
$$= \frac{i}{4}g_d^4(\Delta_0^*)^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{1}{(p_0'^2-(E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)}\right.\right.$$
$$\left.+\frac{1}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)}\right)\left(1+\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)+\left(\frac{1}{(p_0'^2-E_{\Delta'}^{+2})(p_0^2-(E_\Delta^-)^2)}\right.$$
$$\left.\left.+\frac{1}{(p_0'^2-(E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)}\right)\left(1-\frac{\vec{p}\,'\cdot\vec{p}+m^2}{E_{p'}E_p}\right)\right]. \tag{3.88}$$

We work out the $q=0$ case:

$$\Gamma_{dd}(0) = \frac{i}{2}g_d^4(\Delta_0^*)^2\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}\right)$$
$$= -\frac{1}{2}g_d^4(\Delta_0^*)^2I_0. \tag{3.89}$$

Similarly,

$$\Gamma_{d^*d^*}(q) = \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}g_d^2H^-(p')\gamma_5t_2\tau_2H^-(p)\gamma_5t_2\tau_2, \tag{3.90}$$

$$\Gamma_{d^*d^*}(0) = -\frac{1}{2}g_d^4(\Delta_0)^2I_0. \tag{3.91}$$

All the derivative terms are convergence. While we have defined a notation $g_d^2|\Delta_0|^2=|\Delta|^2$, the $g_d^2(\Delta_0)^2$ and $g_d^2(\Delta_0^*)^2$ are not expressed by $|\Delta|^2$.

The diquark-baryon part in the second order is given as

$$\mathcal{L}_{diq}^{(2)} = \frac{1}{2}g_d^2I_0\left(\partial^\mu d(x)\right)\left(\partial_\mu d^*(x)\right)-\frac{1}{2}(M_d^2-2g_d^2(I_2-|\Delta|^2I_0))d(x)d^*(x)$$
$$-ig_d^2I_0\mu(d^*(x)\partial_0d(x)-d(x)\partial_0d^*(x))-\frac{1}{2}g_d^4I_0(\Delta_0^*)^2(d(x))^2-\frac{1}{2}g_d^4I_0(\Delta_0)^2(d^*(x))^2, \tag{3.92}$$

with the mass

$$m_d^2 = M_d^2-2g_d^2(I_2-|\Delta|^2I_0). \tag{3.93}$$

3.3.4 Mixing Terms

Since the sigma meson and diquark-baryons are the scalar and due to the PG symmetry, they mix each other through the condensation. This property appears in the Lagrangian as mixing terms proportional to the condensation.

We work out the ds term:

$$\begin{aligned}
\Gamma_{ds}(q) =& \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(-g_0g_d\right)\left(G^+(p')\gamma_5t_2\tau_2H^+(p)+H^+(p')\gamma_5t_2\tau_2G^-(p)\right)\\
=&\frac{i}{4}g_0g_d^2\Delta_0^*\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{(p'_0+E^-_{p'})-(p_0-E^-_p)}{(p'^2_0-(E^-_{\Delta'})^2)(p_0^2-(E^-_{\Delta})^2)}\right.\right.\\
&\left.+\frac{(p_0+E^+_p)-(p'_0-E^+_{p'})}{(p'^2_0-(E^+_{\Delta'})^2)(p_0^2-(E^+_{\Delta})^2)}\right)\left(\frac{m}{E_{p'}}+\frac{m}{E_p}\right)\\
&+\left(\frac{(p'_0+E^-_{p'})-(p_0+E^+_p)}{(p'^2_0-(E^-_{\Delta'})^2)(p_0^2-(E^+_{\Delta})^2)}\right.\\
&\left.\left.+\frac{(p_0-E^-_p)-(p'_0-E^+_{p'})}{(p'^2_0-(E^+_{\Delta'})^2)(p_0^2-(E^-_{\Delta})^2)}\right)\left(\frac{m}{E_{p'}}-\frac{m}{E_p}\right)\right],
\end{aligned} \tag{3.94}$$

in the $q=0$ case

$$\begin{aligned}
\Gamma_{ds}(0) &= ig_0g_d^2\Delta_0^*\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{m}{E_p}\left(\frac{E^-_p}{(p_0^2-(E^-_\Delta)^2)^2}+\frac{E^+_p}{(p_0^2-(E^+_\Delta)^2)^2}\right)\\
&= ig_0g_d^2\Delta_0^*m\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-(E^-_\Delta)^2)^2}+\frac{1}{(p_0^2-(E^+_\Delta)^2)^2}\right)\\
&= -g_0g_d^2\Delta_0^*mI_0.
\end{aligned} \tag{3.95}$$

We work out the sd term:

$$\begin{aligned}
\Gamma_{sd}(q) =& \frac{i}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}(-g_0g_d)\left(H^+(p')G^+(p)\gamma_5t_2\tau_2+G^-(p')H^+(p)\gamma_5t_2\tau_2\right)\\
=&\frac{i}{4}g_0g_d^2\Delta_0^*\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left[\left(\frac{(p_0+E^-_p)-(p'_0-E^-_{p'})}{(p'^2_0-(E^-_{\Delta'})^2)(p_0^2-(E^-_{\Delta})^2)}\right.\right.\\
&\left.+\frac{(p'_0+E^+_{p'})-(p_0-E^+_p)}{(p'^2_0-(E^+_{\Delta'})^2)(p_0^2-(E^+_{\Delta})^2)}\right)\left(\frac{m}{E_{p'}}+\frac{m}{E_p}\right)\\
&+\left(\frac{(p_0-E^+_p)-(p'_0-E^-_{p'})}{(p'^2_0-(E^-_{\Delta'})^2)(p_0^2-(E^+_{\Delta})^2)}\right.\\
&\left.\left.+\frac{(p'_0+E^+_{p'})-(p_0+E^-_p)}{(p'^2_0-(E^+_{\Delta'})^2)(p_0^2-(E^-_{\Delta})^2)}\right)\left(\frac{m}{E_{p'}}-\frac{m}{E_p}\right)\right],
\end{aligned} \tag{3.96}$$

in the $q=0$ case

$$\begin{aligned}\Gamma_{sd}(0) =& i g_0 g_d^2 \Delta_0^* \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{m}{E_p} \left(\frac{E_p^-}{(p_0^2-(E_\Delta^-)^2)^2} + \frac{E_p^+}{(p_0^2-(E_\Delta^+)^2)^2} \right) \\ =& i g_0 g_d^2 \Delta_0^* m \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2} + \frac{1}{(p_0^2-(E_\Delta^+)^2)^2} \right) \\ =& - g_0 g_d^2 \Delta_0^* m I_0. \end{aligned} \tag{3.97}$$

We work out the d^*s term:

$$\begin{aligned}\Gamma_{d^*s}(q) =& \frac{i}{4} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} g_0 g_d \left(G^-(p')\gamma_5 t_2 \tau_2 H^-(p) + H^-(p')\gamma_5 t_2 \tau_2 G^+(p) \right) \\ =& \frac{i}{4} g_0 g_d^2 \Delta_0 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \Bigg[\Bigg(\frac{(p_0' + E_{p'}^+) - (p_0 - E_p^+)}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)} \\ & + \frac{(p_0 + E_p^-) - (p_0' - E_{p'}^-)}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)} \Bigg) \left(\frac{m}{E_{p'}} + \frac{m}{E_p} \right) \\ & + \Bigg(\frac{(p_0' + E_{p'}^+) - (p_0 + E_p^-)}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)} \\ & + \frac{(p_0 - E_p^+) - (p_0' - E_{p'}^-)}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)} \Bigg) \left(\frac{m}{E_{p'}} - \frac{m}{E_p} \right) \Bigg], \end{aligned} \tag{3.98}$$

in the $q=0$ case

$$\begin{aligned}\Gamma_{d^*s}(0) =& i g_0 g_d^2 \Delta_0 \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{m}{E_p} \left(\frac{E_p^+}{(p_0^2-(E_\Delta^+)^2)^2} + \frac{E_p^-}{(p_0^2-(E_\Delta^-)^2)^2} \right) \\ =& i g_0 g_d^2 \Delta_0 m \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2-(E_\Delta^+)^2)^2} + \frac{1}{(p_0^2-(E_\Delta^-)^2)^2} \right) \\ =& - g_0 g_d^2 \Delta_0 m I_0. \end{aligned} \tag{3.99}$$

We work out the sd^* term:

$$\begin{aligned}\Gamma_{sd^*}(q) =& \frac{i}{4} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} g_0 g_d \left(H^-(p') G^-(p)\gamma_5 t_2 \tau_2 + G^+(p') H^-(p)\gamma_5 t_2 \tau_2 \right) \\ =& \frac{i}{4} g_0 g_d^2 \Delta_0 \mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \Bigg[\Bigg(\frac{(p_0 + E_p^+) - (p_0' - E_{p'}^+)}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^+)^2)} \\ & + \frac{(p_0' + E_{p'}^-) - (p_0 - E_p^-)}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^-)^2)} \Bigg) \left(\frac{m}{E_{p'}} + \frac{m}{E_p} \right) \\ & + \Bigg(\frac{(p_0 - E_p^-) - (p_0' - E_{p'}^+)}{(p_0'^2 - (E_{\Delta'}^+)^2)(p_0^2-(E_\Delta^-)^2)} \\ & + \frac{(p_0' + E_{p'}^-) + (p_0 + E_p^+)}{(p_0'^2 - (E_{\Delta'}^-)^2)(p_0^2-(E_\Delta^+)^2)} \Bigg) \left(\frac{m}{E_{p'}} - \frac{m}{E_p} \right) \Bigg], \end{aligned} \tag{3.100}$$

in the $q = 0$ case

$$\begin{aligned}\Gamma_{sd^*}(0) =& ig_0 g_d^2 \Delta_0 \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{m}{E_p} \left(\frac{E_p^+}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{E_p^-}{(p_0^2 - (E_\Delta^-)^2)^2} \right) \\ =& ig_0 g_d^2 \Delta_0 m \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{(p_0^2 - (E_\Delta^+)^2)^2} + \frac{1}{(p_0^2 - (E_\Delta^-)^2)^2} \right) \\ =& - g_0 g_d^2 \Delta_0 m I_0. \end{aligned} \tag{3.101}$$

All the derivative terms converge.

The effective Lagrangian at order $k = 2$ can be written as

$$\begin{aligned}\mathcal{L}^{(2)} =& \frac{1}{2} g_0^2 I_0 (\partial_\mu \vec{\pi}(x))^2 - \frac{1}{2} m_\pi^2 \vec{\pi}^2(x) \\ &+ \frac{1}{2} g_0^2 I_0 (\partial_\mu s(x))^2 - \frac{1}{2} m_s^2 s^2(x) \\ &+ \frac{1}{2} g_d^2 I_0 (\partial^\mu d^*(x))(\partial_\mu d(x)) - \frac{1}{2} m_d^2 d^*(x) d(x) \\ &- i g_d^2 I_0 \mu (d^*(x) \partial_0 d(x) - d(x) \partial_0 d^*(x)) - \frac{1}{2} g_d^4 I_0 (\Delta_0^{*2} d^2(x) + \Delta_0^2 d^{*2}(x)) \\ &- 2 g_0 g_d^2 I_0 m\, s(x) (\Delta_0^* d(x) + \Delta_0 d^*(x)). \end{aligned} \tag{3.102}$$

The Lagrangian density has mass dimension 4, and I_0 and I_2 have mass dimensions 0 and 2, respectively. Hence when I_0 appears in the coefficients there are factors of mass dimension 2 such as squares of the derivative ∂_μ, the chemical potential μ, the quark mass m and the gap energy Δ_0 and Δ_0^*. The pion term has a simple form due to the pseudo-scalar nature of the pion field, while the scalar boson $s(x)$ couples to the diquark-baryon fields through the diquark condensates.

3.4 The Coupling Terms (k=3)

The coupling vertices involving combinations of three mesons or diquark fields are obtained from the third-order term:

$$\begin{aligned} U^{(3)} = -\frac{i}{6} \mathrm{tr}(\hat{S}\hat{K})^3 =& -\frac{i}{6} \mathrm{tr}(A^3 + ABC + BCA + CAB \\ &+ CBD + BDC + DCB + D^3). \end{aligned} \tag{3.103}$$

Non-local terms with derivatives of hadron fields converge, and $U^{(3)}$ is constructed with only local terms. This term generates leading combinations of $G^\pm$ and $H^\pm$ with γ_5. With the principle of taking only the divergent integrals, the possible combinations are GGG or GGH with γ_5. From dimensional analysis, the coupling constant should have mass dimension one. Thus for baryon-number conserving channels, the relevant factor is m. For baryon-number non-conserving channels the factors are Δ_0 or Δ_0^*. Hence, the terms for baryon-number conserving channels are generated from GGG and the terms for baryon-number non-conserving channels are generated from GGH. We drop the $\vec{\pi}^2$ therms since that terms will generate same coefficients as the s^2 and the first order $\vec{\pi}$ has to vanish due to the odd γ_5. Even they are dropped, we find 32 terms as:

$$\begin{aligned}(SK)^3 =& - g_0^3(G^+G^+G^+ + G^-G^-G^-)s^3 \\ & - g_0g_d^2(G^+G^+H^{'-}\gamma_5 + G^+H^{'-}\gamma_5G^+ + H^{'-}\gamma_5G^+G^+ \\ & + G^+H^{'-}G^-\gamma_5 + H^{'-}G^-\gamma_5G^+ + G^-\gamma_5G^+H^{'-} + G^-\gamma_5H^{'-}G^- \\ & + H^{'-}G^-G^-\gamma_5 + G^-G^-\gamma_5H^{'-})s^2d^* \\ & + g_0^2g_d(G^+G^+\gamma_5H^{'+} + G^+\gamma_5H^{'+}G^+ + H^{'+}G^+G^+\gamma_5 \\ & + H^{'+}G^+\gamma_5G^- + G^+\gamma_5G^-H^{'+} + G^-H^{'+}G^+\gamma_5 \\ & + H^{'+}\gamma_5G^-G^- + G^-H^{'+}\gamma_5G^- + G^-G^-H^{'+}\gamma_5)s^2d \\ & + g_0g_d^2(G^+G^+\gamma_5G^-\gamma_5 + G^-\gamma_5G^+G^+\gamma_5 + G^-\gamma_5G^+G^+\gamma_5 \\ & + G^-\gamma_5G^+\gamma_5G^- + G^+\gamma_5G^-G^-\gamma_5 + G^-G^-\gamma_5G^+\gamma_5)sdd^* \\ & + g_d^3(H^{'-}\gamma_5G^+\gamma_5G^-\gamma_5 + G^+\gamma_5G^-\gamma_5H^{'-}\gamma_5 + G^-\gamma_5H^{'-}\gamma_5G^+\gamma_5)dd^{*2} \\ & - g_d^3(G^+\gamma_5H^{'+}\gamma_5G^-\gamma_5 + H^{'+}\gamma_5G^-\gamma_5G^+\gamma_5 + G^-\gamma_5G^+\gamma_5H^{'+}\gamma_5)d^2d^*. \end{aligned} \tag{3.104}$$

where we have used

$$H^{'+} = -\frac{g_d\Delta_0^*}{p_0^2-(E_\Delta^\pm)^2}\gamma_5\tilde{\Lambda}_+ - \frac{g_d\Delta_0^*}{p_0^2-(E_\Delta^\mp)^2}\gamma_5\tilde{\Lambda}_- \ . \tag{3.105}$$

Taking trace

$$\begin{aligned}-\frac{i}{6}\mathrm{tr}(\hat{S}\hat{K})^3 =& -2g_0^3I_0ms^3 - g_0g_d^2I_0\Delta_0s^2d^* - g_0g_d^2I_0\Delta^*s^2d \\ & -2g_0g_d^2I_0msdd^* - g_d^4I_0\Delta_0dd^{*2} - g_d^4I_0\Delta_0^*d^2d^*, \end{aligned} \tag{3.106}$$

The final result for the $k = 3$ part of the effective Lagrangian is:

$$\begin{aligned}\mathcal{L}^{(3)} =& -2g_0I_0m\left[g_0^2s^3(x) + g_0^2s(x)\vec{\pi}^2(x) + g_d^2d^*(x)d(x)s(x)\right] \\ & - g_d^2I_0\left[g_0^2(s^2(x)+\vec{\pi}^2(x))(\Delta_0d^*(x)+\Delta_0^*d(x)) + g_d^2d^*(x)d(x)(\Delta_0d^*(x)+\Delta_0^*d(x))\right].\end{aligned} \tag{3.107}$$

Since the constituent quark mass is defined $m = m_0 + g_0\sigma_0$, the coupling constants for each terms have the condensations $g_0\sigma_0$, $g_d\Delta_0$ and $g_d\Delta_0^*$ and dimension one together with the dimensionless integral I_0. Obviously, the vacuum transition s^3 and sdd^* channels conserve the baryon number and the others do not conserve.

3.5 The Interaction Terms (k=4)

Four-point interaction terms involving mesons and diquarks derive from the fourth-order term,

$$U^{(4)} = \frac{i}{8}\mathrm{tr}(\hat{S}\hat{K})^4 \ . \tag{3.108}$$

The non-local pieces converge. By examination it turns out that only the possible combination is $GGGG$ with γ_5 and the coupling constants are dimensionless. The non-zero combinations are s^4, $\vec{\pi}^4$, $(dd^*)^2$, $s^2\vec{\pi}^2$, s^2dd^* and $\vec{\pi}dd^*$ due to the number of γ_5. Out of many terms that appear we give here a generic example:

$$U_{\pi}^{(4)} = \frac{i}{8}g_0^4\mathrm{tr}_{fcs}\int\frac{d^4p}{(2\pi)^4}(G^+\gamma_5G^+\gamma_5G^+\gamma_5G^+\gamma_5 + +G^-\gamma_5G^-\gamma_5G^-\gamma_5G^-\gamma_5)\vec{\pi}^4\,. \quad (3.109)$$

The first and second terms generate many terms. The point of calculation is to take trace for spinor. The products of gamma matrices and propagators are worked out by the trace formulas for instance as follows:

$$\begin{aligned}\mathrm{tr}_s\Lambda_-\gamma_0\gamma_5\Lambda_+\gamma_0\gamma_5\Lambda_-\gamma_0\gamma_5\Lambda_+\gamma_0\gamma_5 =&\mathrm{tr}_s\Lambda_- = 2,\\ \mathrm{tr}_s\Lambda_+\gamma_0\gamma_5\Lambda_-\gamma_0\gamma_5\Lambda_+\gamma_0\gamma_5\Lambda_-\gamma_0\gamma_5 =&\mathrm{tr}_s\Lambda_+ = 2,\end{aligned} \quad (3.110)$$

and all the other terms vanish. From the first term one finds:

$$\begin{aligned}\mathrm{tr}_sG^+\gamma_5G^+\gamma_5G^+\gamma_5G^+\gamma_5 =&2\frac{(p_0+E_p^-)^2(p_0-E_p^+)^2}{(p_0^2-(E_\Delta^-)^2)^2(p_0^2-(E_\Delta^+)^2)^2}\times 2\\ =&2\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}\right).\end{aligned} \quad (3.111)$$

The same result is obtained from the second term of Eq. (3.109), hence the result of the complete $\vec{\pi}^4$ term is

$$\begin{aligned}U_{\pi}^{(4)} =&\frac{i}{2}g_0^4\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}\right)\vec{\pi}^4\\ =&-\frac{1}{2}g_0^4I_0\vec{\pi}^4.\end{aligned} \quad (3.112)$$

Consider another example, the combination involving $\vec{\pi}^2dd^*$, which has 12 terms of the form $GGGG$ as

$$\begin{aligned}U_{\vec{\pi}^2dd^*}^{(4)} =&\frac{i}{8}g_0g_d^2\mathrm{tr}_{fcs}\int\frac{d^4p}{(2\pi)^4}\big[G^+\gamma_5G^+\gamma_5G^+\gamma_5G^+\gamma_5 + G^+\gamma_5G^-\gamma_5G^+\gamma_5G^+\gamma_5\\ &+G^+\gamma_5G^+\gamma_5G^-\gamma_5G^+\gamma_5 - G^+\gamma_5G^+\gamma_5G^-\gamma_5G^-\gamma_5 - G^+\gamma_5G^-\gamma_5G^-\gamma_5G^+\gamma_5\\ &+G^+\gamma_5G^-\gamma_5G^-\gamma_5G^-\gamma_5 + G^-\gamma_5G^+\gamma_5G^+\gamma_5G^+\gamma_5 - G^-\gamma_5G^+\gamma_5G^+\gamma_5G^-\gamma_5\\ &-G^-\gamma_5G^-\gamma_5G^+\gamma_5G^+\gamma_5 + G^-\gamma_5G^-\gamma_5G^+\gamma_5G^+\gamma_5 + G^-\gamma_5G^+\gamma_5G^-\gamma_5G^-\gamma_5\\ &+G^-\gamma_5G^-\gamma_5G^+\gamma_5G^+\gamma_5\big]\,\vec{\pi}^2dd^*.\end{aligned} \quad (3.113)$$

As we found, the non-zero terms come from only the two products as Eq. (3.110) and the result is Eq. (3.111). In the 12 terms, the 4 terms have minus sign and the 8 terms have plus sign, hence the 4 terms are remained. Thus we obtain

$$\begin{aligned}U_{\vec{\pi}^2dd^*}^{(4)} =&\frac{i}{8}g_0g_d^2\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}4\times 2\left(\frac{1}{(p_0^2-(E_\Delta^-)^2)^2}+\frac{1}{(p_0^2-(E_\Delta^+)^2)^2}\right)\vec{\pi}^2dd^*\\ =&-g_0^2g_d^2I_0\vec{\pi}^2dd^*.\end{aligned} \quad (3.114)$$

Thus, the 4th order of the fields $s^4, \vec{\pi}^4$ and $(dd^*)^2$ generate $-\frac{1}{2}I_0$ and $s^2\vec{\pi}^2, s^2dd^*$ and $\vec{\pi}dd^*$ generate $-I_0$. The final expression for the $k=4$ part of the effective Lagrangian is:

$$\begin{aligned}\mathcal{L}^{(4)} =& - g_0^4 I_0 s^2(x)\vec{\pi}^2(x) - g_0^2 g_d^2 I_0(s^2(x)+\vec{\pi}^2(x))d^*(x)d(x) \\ &- \frac{1}{2}g_0^4 I_0(s^4(x)+\vec{\pi}^4(x)) - \frac{1}{2}g_d^4 I_0 d^{*2}(x)d^2(x) \\ =& -\frac{1}{2}I_0\left[g_0^2(s^2(x)+\vec{\pi}^2(x)) + g_d^2 d^*(x)d(x)\right]^2 . \end{aligned} \tag{3.115}$$

It generates interactions between diquarks, scalar mesons and pions in all possible combinations.

3.6 Identification with a Generalized Linear Sigma Model

The limiting case of exact Pauli-Gürsey symmetry in color SU(2) is realized with $g_0 = g_d$ and $M_s = M_d$. The fourth order Lagrangian density is written as

$$\mathcal{L}^{(4)} = -\frac{1}{2}g_0^4 I_0(s^2(x)+\vec{\pi}^2(x)+d^*(x)d(x))^2 . \tag{3.116}$$

This form and the structure of the other terms suggest, not surprisingly, that the hadron Lagrangian is related to a generalized linear σ model. To demonstrate this, we introduce a fourth-order extended σ model Lagrangian density with σ, $\vec{\pi}$, Δ and Δ^* fields:

$$\mathcal{L}_\sigma^{(4)} = -\frac{1}{2}g_0^4 I_0(\sigma^2(x)+\vec{\pi}^2(x)+\Delta^*(x)\Delta(x))^2 . \tag{3.117}$$

Separating again mean fields and fluctuations by the relations $\sigma = \sigma_0 + s(x)$ and $\Delta = \Delta_0 + d(x)$ together with its complex conjugate, one finds

$$\begin{aligned}\sigma^2(x)+\vec{\pi}^2(x)+\Delta^*(x)\Delta(x) =& (\sigma_0+s(x))^2+\vec{\pi}^2(x)+(\Delta_0+d(x))(\Delta_0^*+d^*(x)) \\ =& \sigma_0^2+\Delta_0\Delta_0^*+2\sigma_0 s(x)+\Delta_0 d^*(x)+\Delta_0^* d(x) \\ &+ s^2(x)+\vec{\pi}^2(x)+d^*(x)d(x) , \end{aligned} \tag{3.118}$$

and the 4-th order Lagrangian (3.117) is rewritten as:

$$\begin{aligned}\mathcal{L}_\sigma^{(4)} =& -\frac{1}{2}g_0^4 I_0[v_0^4 + 2v_0^2(2\sigma_0 s(x)+\Delta_0 d^*(x)+\Delta_0^* d(x)) \\ &+ 2v_0^2(s^2(x)+\vec{\pi}^2(x)+d^*(x)d(x)) + 4\sigma_0^2 s^2(x) + 2|\Delta_0|^2 d^*(x)d(x) \\ &+ \Delta_0^2 d^{*2}(x) + \Delta_0^{*2}d^2(x) + 4\sigma_0\Delta_0 s(x)d^*(x) + 4\sigma_0\Delta_0^* s(x)d(x) \\ &+ 4\sigma_0(s^3(x)+s(x)d^*(x)d(x)+s(x)\vec{\pi}^2(x)) \\ &+ 2\Delta_0(d^{*2}(x)d(x)+s^2(x)d^*(x)+\vec{\pi}^2(x)d^*(x)) \\ &+ 2\Delta_0^*(d^*(x)d^2(x)+s^2(x)d(x)+\vec{\pi}^2(x)d(x)) \\ &+ (s^2(x)+\vec{\pi}^2(x)+d^*(x)d(x))^2], \end{aligned} \tag{3.119}$$

where we have defined $\sigma_0^2 + |\Delta_0|^2 = v_0^2$.

Hence one can write the hadron Lagrangian in the compact form of a generalized linear σ model with inclusion of diquark fields:

$$
\begin{aligned}
\mathcal{L}_{E\sigma} =& \frac{1}{2}g_0^2 I_0 \left[(\partial_\nu \vec{\pi}(x))^2 + (\partial_\nu \sigma(x))^2 + |(\partial_\nu - 2i\mu\delta_{\nu 0})\Delta(x)|^2\right] \\
&- \frac{1}{2}(M_s^2 - 2g_0^2 I_2 + 4g_0^2 I_0 \mu^2)(\sigma^2(x) + \vec{\pi}^2(x) + \Delta^*(x)\Delta(x)) \\
&- \frac{1}{2}g_0^4 I_0 \left[(\sigma^2(x) + \vec{\pi}^2(x) + \Delta^*(x)\Delta(x)) - v_0^2\right]^2 \ . \qquad (3.120)
\end{aligned}
$$

In order to write the Lagrangian in this form, we have added $-2g_0^2 I_0 \mu^2 \Delta^* \Delta$ to the mass term and combine with the $\sigma^2 + \vec{\pi}^2$ terms to arrive at the compact symmetric mass term shown in the second line. We then add the counter part, $+2g_0^2 I_0 \mu^2 \Delta^* \Delta$, to the kinetic energy term proportional to $(\partial_\nu \Delta^*)(\partial^\nu \Delta)$ and arrive at the compact form $\frac{1}{2}g_0^2 I_0 |(\partial_\nu - 2i\mu\delta_{\nu 0})\Delta|^2$. Since the diquark-baryon is constructed by two quarks (qq) and the antidiquark-baryon is by two antiquarks $(\bar{q}\bar{q})$, the baryon chemical potential can be written $\mu_B = 2\mu$ which appears in the kinetic term of the diquarks.

Consider next the explicit chiral symmetry breaking term. Obviously, we have a term linear in the scalar field, $2g_0 m_0 (I_2 - 2\mu^2 I_0)s$, which is proportional to the bare quark mass. This suggests the explicit chiral symmetry breaking term of the form

$$
\mathcal{L}_{SB} = 2g_0 m_0 (I_2 - 2\mu^2 I_0)\sigma(x). \qquad (3.121)
$$

This term provides a new mean-field mass equation including explicit chiral symmetry breaking,

$$
-M_s^2 \sigma_0 + 2g_0 (m_0 + g_0 \sigma_0)(I_2 - 2\mu^2 I_0) = 0 \ . \qquad (3.122)
$$

Self-consistent solution of this gap equation determines the dynamical quark mass, $m = m_0 + g_0 \sigma_0$, that appears also in the integrals I_2 and I_0.

3.7 Renormalization

The hadron Lagrangian (3.120) extracted from the NJL model involves the divergent quark loop integrals I_2 and I_0. The NJL approach is valid for quark momenta below a characteristic scale, $\Lambda \sim 0.6$ [GeV], at which the integrals are cut off in practice. Nonetheless, when written in the form (3.120) as a generalized linear sigma model, renormalization has to be performed. We follow the Eguchi method [42]. The kinetic term implies the wave function renormalization,

$$
\begin{aligned}
g_0^2 I_0 =& Z_M^{-1} \\
\sigma =& Z_M^{\frac{1}{2}} \sigma_R \quad \vec{\pi} = Z_M^{\frac{1}{2}} \vec{\pi}_R \\
\Delta =& Z_M^{\frac{1}{2}} \Delta_R \quad \Delta^* = Z_M^{\frac{1}{2}} \Delta_R^* \ . \qquad (3.123)
\end{aligned}
$$

The mass renormalization condition is

$$
M_s^2 - 2g_0^2 I_2 + 4g_0^2 I_0 \mu^2 = 0 \ . \qquad (3.124)
$$

The coupling constant renormalization is written as

$$\begin{aligned} 2g_0^4 I_0 =& Z_\lambda^{-1}\lambda_0 \ , \\ \lambda =& Z_M^2 Z_\lambda^{-1}\lambda_0 \ . \end{aligned} \tag{3.125}$$

The mean field is renormalized as

$$v = Z_M^{\frac{1}{2}} v_0 \ , \tag{3.126}$$

and the explicit chiral symmetry breaking term is renormalized as

$$\begin{aligned} 2g_0 m_0(I_2 - 2\mu^2 I_0) =& Z_{SB}^{-1}\varepsilon_0 \\ \varepsilon =& Z_M^{\frac{1}{2}} Z_{SB}^{-1}\varepsilon_0 \ . \end{aligned} \tag{3.127}$$

When expressed in terms of renormalized fields and couplings, the effective Lagrangian (3.120) reads:

$$\begin{aligned} \mathcal{L}_{E\sigma SB} =& \frac{1}{2}\left[(\partial_\nu \vec{\pi}_R(x))^2 + (\partial_\nu \sigma_R(x))^2 + |(\partial_\nu - i\mu_B\delta_{\nu 0})\Delta_R(x)|^2\right] \\ &- \frac{\lambda}{4}\left[\sigma_R^2(x) + \vec{\pi}_R^2(x) + \Delta_R^*(x)\Delta_R(x) - v^2\right]^2 + \varepsilon\sigma_R(x). \end{aligned} \tag{3.128}$$

Together with the quark term, we have now a consistent Lagrangian to derive the thermodynamical potential for the interacting quark-hadron system. The hadron dynamics itself is governed by the generalized linear σ model Lagrangian (3.128). The explicit chiral symmetry breaking term, $\varepsilon\sigma_R$, is related to the bare quark mass m_0. When the PG symmetry is not satisfied, the meson part and the diquark-baryon part are renormalized independently.

Chapter 4

Thermodynamics of Quark-Hadron Matter

We derived the quark mean field thermodynamical potential (the partition function) in Chapter 2 and the meson-diquark Lagrangian (extended linear sigma model) in Chapter 3. In this chapter we discuss properties of quark-hadron matter and compare with the lattice QCD result. First, we will discuss the phase structure at finite temperature and density in the mean field level by using the thermodynamical potential. At finite temperature, we include the Polyakov loop effect. Next, we discuss the hadron mass spectrum at finite temperature and density. The behavior of the mass at finite density can be found by analysis of the coupled mass matrix. We find the existence of the NG boson due to the spontaneous breaking of baryon number symmetry. Finally, the equation of state (EOS) for quark-hadron matter in various chemical potentials are discussed. The excitation of the NG boson called the Bogoliubov excitation may give the important contribution to matter property. The baryon density in the vacuum is also discussed. We will find our framework cannot reproduce the lattice calculation for the baryon density.

4.1 Thermodynamics of Mesons and Diquark-Baryons in the Gaussian Approximation

We now discuss the contributions of mesons and diquark-baryon fields to the thermodynamical potential, using the Gaussian approximation. The starting point is the extended linear σ model Lagrangian derived in the previous section. The whole Lagrangian is written as

$$\mathcal{L} = \mathcal{L}_{MF} + \mathcal{L}_{hadron} \,. \tag{4.1}$$

The mean field part has been worked out previously to provide the thermodynamical potential Ω_{MF}. The extended linear σ model includes hadron fields up to fourth order. The complete integrations over the hadron fields cannot be performed to this order. In the present work we restrict ourselves to the Gaussian approximation [77], taking only the second order (mass) terms into account and perform the hadron integrals for the partition function as

$$Z = Z_q \int \mathcal{D}s\mathcal{D}\vec{\pi}\mathcal{D}d\mathcal{D}d^* \exp\left(i\int d^4x \mathcal{L}^{(2)}_{hadron}\right) \,, \tag{4.2}$$

with

$$
\begin{aligned}
Z_q =& \exp\left(-\frac{V}{T}\Omega_{MF}\right) , \\
\mathcal{L}^{(2)}_{hadron} =& \frac{1}{2}(\partial_\mu \vec{\pi})^2 - \frac{1}{2}m_\pi^2 \vec{\pi}^2 + \frac{1}{2}(\partial_\mu s)^2 - \frac{1}{2}m_s^2 s^2 \\
& + \frac{1}{2}(\partial^\mu d^*)(\partial_\mu d) - \frac{1}{2}m_d^2 d^* d - i\mu(d^*\partial_0 d - d\partial_0 d^*) \\
& - \frac{1}{2}\Delta^{*2} d^2 - \frac{1}{2}\Delta^2 d^{*2} - 2m\Delta^* s d - 2m\Delta s d^* . \qquad (4.3)
\end{aligned}
$$

Here we have used the renormalized fields but omitted the index R for simplicity. For further convenience, we have introduced $\Delta = g_d \Delta_0$ and $\Delta^* = g_d \Delta_0^*$ in the above equation. The renormalized masses are defined as

$$
\begin{aligned}
m_\pi^2 &= \left[M_s^2 - 2g_0^2(I_2 - 2I_0\mu^2)\right]/(g_0^2 I_0) , \\
m_s^2 &= \left[M_s^2 - 2g_0^2(I_2 - 2I_0(\mu^2 + m^2))\right]/(g_0^2 I_0) , \\
m_d^2 &= \left[M_d^2 - 2g_d^2(I_2 - I_0|\Delta|^2)\right]/(g_0^2 I_0) . \qquad (4.4)
\end{aligned}
$$

The meson and diquark-baryon masses (4.4) found by bosonization and renormalization are identical to the masses obtained by solving the corresponding Bethe-Salpeter equations taking only the divergent integrals I_0 and I_2 in the quark-loop integrals. Of course, the use of the Gaussian approximation means ignoring important interaction terms generated by $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$.

The hadronic partition function is

$$
\begin{aligned}
Z_{hadron} =& \int \mathcal{D}\Phi \mathcal{D}\vec{\pi} \exp\left(-S^{(2)}\right) , \\
S^{(2)} =& \frac{1}{2}\sum_Q \int d^3x \left[\Phi^\dagger(x) M_n(Q) \Phi(x) + \vec{\pi}^\dagger(x) N_n(Q) \vec{\pi}(x)\right] , \qquad (4.5)
\end{aligned}
$$

where $Q = (i\omega_n, \vec{q})$, with $\omega_n = 2\pi n T$ being the boson Matsubara frequency and $\sum_Q = T\sum_n \int \frac{d^3q}{(2\pi)^3}$. The fields Φ are defined as $\Phi^T = (s, d, d^*)$ and $\vec{\pi}^T = (\pi^+, \pi^0, \pi^-)$. The matrix $M_n(Q)$ is the mass matrix in Euclidean space with $Q^2 = \vec{q}^2 + \omega_n^2$ and $Q_0 = i\omega_n$:

$$
M_n(Q) = \begin{pmatrix} Q^2 + m_s^2 & 2m\Delta^* & 2m\Delta \\ 2m\Delta & \frac{1}{2}(Q^2 + m_d^2) - 2\mu Q_0 & \Delta^2 \\ 2m\Delta^* & \Delta^{*2} & \frac{1}{2}(Q^2 + m_d^2) + 2\mu Q_0 \end{pmatrix} . \qquad (4.6)
$$

On the other hand, the matrix $N_n(Q)$ of the pion term is diagonal and proportional to the identity matrix as

$$
N_n^{ij}(Q) = \delta^{ij}(Q^2 + m_\pi^2) . \qquad (4.7)
$$

A related study of the thermodynamical potential beyond the mean field approximation, including hadron contributions in the two-color NJL model by using the Gaussian approximation, has been performed by He [67]. The effect of the pair mode in condensed matter has been investigated by Diener *et al.* [77]. This method is applied to the NJL model and the effect of the hadrons is estimated by a perturbation method for the quark density [67]. An expansion of the $N_c = 3$ two-flavor PNJL model beyond mean field approximation has been reported in ref. [25].

4.2 Parameters

Our NJL model has three parameters the current quark mass m_0, a quark loop momentum cut-off Λ and the coupling strength $G_0 = H_0$. The parameters are fixed at zero temperature and density. In the three color NJL model parameters are fixed from the experimental data which are the physical pion mass m_π and the pion decay constant f_π, and the chiral condensate $\langle\bar{\psi}_u\psi_u\rangle$ from the lattice QCD or QCD sum rule calculation. Following standard NJL relation [6–8, 64, 78]: The pion decay constant

$$f_\pi^2 = m^2 I_0^\Lambda \tag{4.8}$$

and the chiral condensate

$$\langle\bar{\psi}_u\psi_u\rangle = -mI_2^\Lambda, \tag{4.9}$$

where $\langle\bar{\psi}_u\psi_u\rangle^{1/3} \sim \langle\bar{\psi}_d\psi_d\rangle^{1/3}$. The current quark mass is fixed from the Gell-Mann, Oakes, Renner relation

$$m_\pi^2 = -\frac{m_0\langle\bar{\psi}\psi\rangle}{f_\pi^2}. \tag{4.10}$$

We have used the regularized divergent integrals I_2^Λ and I_0^Λ as the defined divergent integrals I_2 and I_0 at zero temperature and density with the cut-off Λ as

$$\begin{aligned} I_2^\Lambda =& i\mathrm{tr}_{fc}\int\frac{d^4p}{(2\pi)^4}\theta(\Lambda^2-\vec{p}^2)\left(\frac{1}{p^2-m^2+i\epsilon}+\frac{1}{p^2-m^2+i\epsilon}\right)\\ =&2N_fN_ci\int\frac{d^4p}{(2\pi)^4}\frac{\theta(\Lambda^2-\vec{p}^2)}{p^2-m^2+i\epsilon}, \end{aligned} \tag{4.11}$$

$$I_0^\Lambda =-2N_fN_ci\int\frac{d^4p}{(2\pi)^4}\frac{\theta(\Lambda^2-\vec{p}^2)}{(p^2-m^2+i\epsilon)^2}. \tag{4.12}$$

They depend on the color N_c, hence the pion decay constant and the chiral condensate depend on the color N_c and the pion mass does not depend on N_c. We use the parameters following Ref. [65], which shown in Table 4.1.

inputs	f_π 75.4 [MeV]	$-\langle\bar{\psi}\psi\rangle^{1/3}$ 218 [MeV]	m_π 140 [MeV]
parameters	m_0 5.4 [MeV]	Λ 657 [MeV]	$G_0 = H_0$ 7.23 [GeV^{-2}]

Table 4.1: The first line is physical quantities as inputs and the second line is fitted parameters of the model.

4.3 Polyakov Loop and Its Effective Poitential

For the discussion of the equation of state of quark-hadron matter, it is important to take into account the quark confinement effect. Some but not all aspects of confinement are treated by introducing the Polyakov loop as the order parameter of the deconfinement transition. The coupling of this Polyakov loop to the quark sector suppresses color non-singlet degree of freedom in the thermodynamics of the hadronic phase. However, this minimally necessary conditions does not yet account for the dynamical generation of localized color-singlet clusters as compact wave packets. This is a limitation that (so far) excludes a proper description of the hadron-to-quarks crossover at high density.

We adopt here the Fukushima method and add the Polyakov loop potential $U(\Phi[A], \Phi^*[A]; T)$ to the two-color NJL Lagrangian [22]. The derivative ∂^μ acting on the quark field is replaced by the covariant derivative $D^\mu = \partial^\mu - iA^\mu$. The temporal background color gauge field is introduced $A_4 = iA^0$ with $A^0 = gA^0_a \frac{t_a}{2}$ and the SU(2) Pauli matrices $t_a(a = 1, 2, 3)$ in color space. In the Polyakov gauge this temporal gauge field is diagonal in the color space. For the SU(2) color group it is represented as $t_3\theta$ where θ is real.

The Polyakov loop potential U is written as [65]

$$U(\Phi, T) = -bT(24\Phi^2 e^{-\beta a} + \ln(1 - \Phi^2)) \ , \tag{4.13}$$

in terms of the trace of the Polyakov loop

$$\Phi = \frac{1}{N_c} \mathrm{tr} e^{i\beta A_4} = \cos(\beta\theta) \ . \tag{4.14}$$

The logarithmic term comes from the Haar measure in the SU(2) color space. The two parameters are taken from the discussion of Brauner *et al.* [65]. The critical temperature of the Polykov loop is set at $T_0(N_c = 3) = 270$ [MeV] for the pure gauge case without coupling to quarks. The parameter a is related to the critical temperature as $a = T_c \ln 24 = 858.1$ [MeV]. The parameter b is estimated by strong-coupling expansion of lattice QCD $b = (\sigma_s/a)^{1/3}$ where $\sigma_s = (425\text{MeV})^2$ is the physical string tension, then we adopt $b^{1/3}$ =210.5[MeV]. According to Ref. [79], the N_c-dependence of T_0 goes approximately as $T_0/\sqrt{\sigma} \simeq 0.6 + 0.45/N_c^2$, suggesting roughly a 10% difference between $T_0(N_c = 2)$ and $T_0(N_c = 3)$ that we can ignore.

The effect of the Polyakov loop implies the replacement

$$\ln(1 + e^{-\beta E^+_\Delta}) \to \frac{1}{2} \ln(1 + 2\Phi e^{-\beta E^+_\Delta} + e^{-2\beta E^+_\Delta}) \ , \tag{4.15}$$

in the thermodynamical potential. Hence, the Fermi function $n_F(E)$ in the mean field equation is replaced by

$$\tilde{n}_F(E) = \frac{1 + \Phi e^{\beta E}}{1 + 2\Phi e^{\beta E} + e^{2\beta E}} \ . \tag{4.16}$$

When the Polyakov loop $\Phi \to 0$, the function $\tilde{n}_F$ obviously corresponds to the distribution function of two-particles, which means the two quarks survive as a composite colorless object and a single quark (colored particle) degree of freedom is suppressed. Hence, the $\Phi \to 0$ phase can be identified with the hadronic phase. On the other hand, when $\Phi \to 1$, a single quark degree of freedom is active, which can be considered as the deconfined phase. We note that, by construction, the effect of the Polyakov loop does not influence the physics at zero temperature but has a major impact on the order parameters for the chiral and deconfinement transitions, as demonstrated in the next section.

4.4 Phase Diagram and Order Parameters

The thermodynamical potential in the mean field approximation has the information of the phase structure and phase transition. The thermodynamical potential including the Polyakov loop effect is

$$\begin{aligned}\Omega_{MF} =& -4\int \frac{d^3p}{(2\pi)^3}\left[E_\Delta^+ + E_\Delta^- + T\ln(1+2\Phi e^{-\beta E_\Delta^+} + e^{-2\beta E_\Delta^+}) + T\ln(1+2\Phi e^{-\beta E_\Delta^-} + e^{-2\beta E_\Delta^-})\right] \\ &+ \frac{(m-m_0)^2}{2G_0} + \frac{|\Delta|^2}{2H_0} - bT(24\Phi^2 e^{-\beta a} + \ln(1-\Phi^2)),\end{aligned} \tag{4.17}$$

where the dycamical quark mass $m = m_0 + g_0\sigma_0$ and the diquark condensate $g_d|\Delta_0| = |\Delta|$ have been introduced. The dynamical quark mass m is determined selfconsistently

$$\frac{\partial \Omega_{MF}}{\partial m} = 0. \tag{4.18}$$

Since the current quark mass is small, we can identify the chiral condensate as quark mass $g_0\sigma_0 \simeq m$. The order parameters are determind by solving the gap equations

$$\frac{\partial \Omega_{MF}}{\partial m} = \frac{\partial \Omega_{MF}}{\partial |\Delta|} = \frac{\partial \Omega_{MF}}{\partial \Phi} = 0, \tag{4.19}$$

namaly for chiral condensate:

$$4\int \frac{d^3p}{(2\pi)^3}\frac{m}{E_p}\left[\frac{E_\Delta^+}{E_p^+}(1-2\tilde{n}_F(E_\Delta^+)) + \frac{E_\Delta^-}{E_p^-}(1-2\tilde{n}_F(E_\Delta^-))\right] = \frac{m-m_0}{G_0}, \tag{4.20}$$

for diquark condensate:

$$4|\Delta|\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{E_\Delta^+}(1-2\tilde{n}_F(E_\Delta^+)) + \frac{1}{E_\Delta^-}(1-2\tilde{n}_F(E_\Delta^-))\right] = \frac{|\Delta|}{H_0}, \tag{4.21}$$

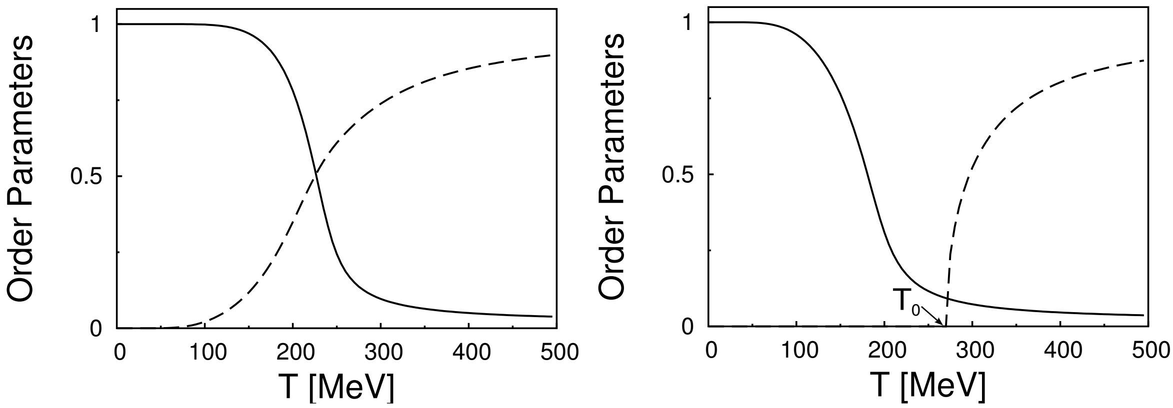
and for Polyakov loop:

$$2\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{\Phi + \cosh(E_\Delta^+)} + \frac{1}{\Phi + \cosh(E_\Delta^-)}\right] = b\Phi\left(\frac{1}{1-\Phi^2} - 24e^{-\beta a}\right). \tag{4.22}$$

The behavior of their order parameters at several situations are investigated.

Finite temperature ($T \neq 0$) **and zero chemical potential** ($\mu = 0$)

It is known that the chiral and deconfinement transition are strongly correlated [22, 39]. To see this behavior in two color system, we show the numerical result of the chiral condensate and the Polyakov loop at finite temperature and zero chemical potential ($\mu = 0$) in comparison with the PNJL model and the NJL model in Fig. 4.1. The quark-Polyakov loop coupling is introduced in the PNJL model (Fig. 4.1(a)), while the NJL model, the quark and Polyakov loop behave independently (Fig. 4.1(b)). The Polyakov loop reproduces the pure gauge case in the NJL model case. Since the chiral condensate σ_0 does not go to zero exactly due to the small current quark mass m_0, the chiral transition temperature is defined at $\sigma_c = \frac{1}{2}\sigma_0$ in the NJL model case. The critical temperature of the Polyakov loop is set at $T_0 = 270$ [MeV] for the pure gauge case without coupling to quarks as we mentioned. In the PNJL model case, the crossover temperatures of both order parameters defined at half of their full values are now $T_\chi \simeq T_{dec} \simeq 225$ [MeV].



(a) PNJL model case (b) NJL model case

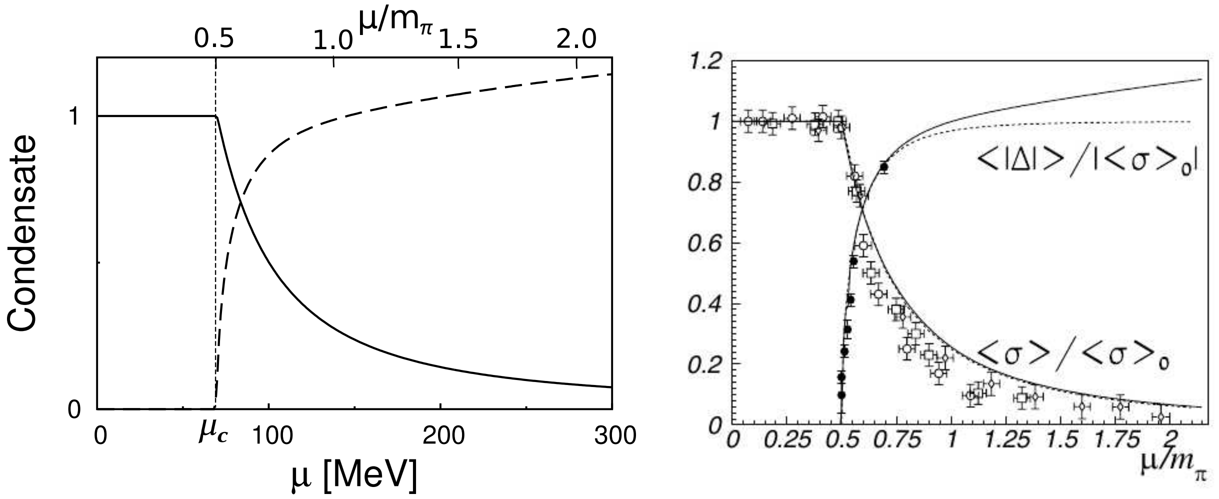
Figure 4.1: The behaviors of the chiral condensate (solid line) and the Polyakov loop (dashed line) as functions of temperature at zero chemical potential, for the cases of the PNJL model and the NJL model in unit of $\sigma_0(T=0)$.
(a): The PNJL model includes the quark-Polyakov loop coupling, in which the chiral and de-confinement transition are strongly correlated.
(b): The NJL model dose not have correlation, in which the behavior of the Polyakov loop corresponds the pure gauge case.

Zero temperature $(T = 0)$ **and finite chemical potential** $(\mu \neq 0)$

The order parameters of our model are the chiral condensate σ_0, the Polyakov loop Φ and the diquark condensate $|\Delta_0|$. The diquark condensate arises from the critical chemical potential μ_c (diquark transition). From the discussion of the symmetry [55] (or we will see in Sec. 4.7), the diquark transition occurs at the half pion mass, $\mu_c = m_\pi/2$, and the chiral transition occurs at the same time due to the PG symmetry. These condensations are calculated by lattice QCD [48, 49]. Our numerical result of the chiral and diquark condensations as functions of the chemical potential at the zero temperature $(T = 0)$ are shown in Fig. 4.2(a) and comparison with the lattice QCD result is made in Fig. 4.2(b) [64].

Chiral and diquark condensation

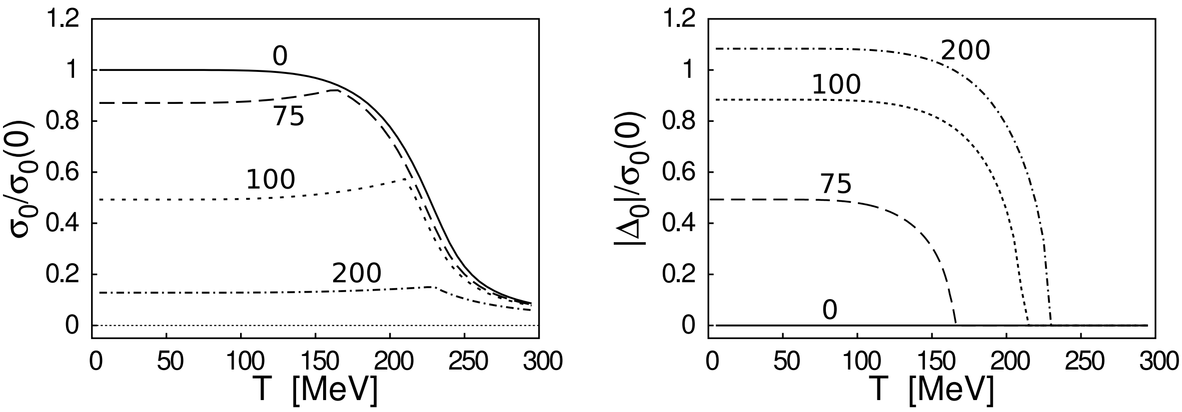
The phase boundary of the de-confinement transition ($\Phi \sim 0.5$) is insensitive to the chemical potential as discussed by Brauner *et al.* [65]. Hence, we plot in Fig. 4.3 only σ_0 and $|\Delta_0|$ for various chemical potentials as functions of temperature. The chiral condensate σ_0 shown by the top smooth curve ($\mu = 0$) in Fig. 4.3(a) stay unchanged until diquark condensate sets in at $\mu_c = m_\pi/2 = 70$ [MeV]. The chiral condensate σ_0 at $\mu = 75$ [MeV] is depleted in the small temperature region, where the diquark condensate $|\Delta_0|$ is finite as shown in the right hand figure 4.3(b). This behavior continues as the chemical potential increases as shown for σ_0 and $|\Delta_0|$ at $\mu = 100$ and 200 [MeV] [55, 64]. These behaviors agree with the results shown in Fig. 3 of Ref. [65].



(a) Our calculation

(b) Lattice calculation

Figure 4.2: The chiral (solid line) and diquark (dashed line) condensation as a function of chemical potential in unit of $\sigma_0(T = 0)$. The left figure (a) is our calculation solving the gap equations. The right figure (b) quoted from Ref. [64] is lattice simulation by Ref. [49].



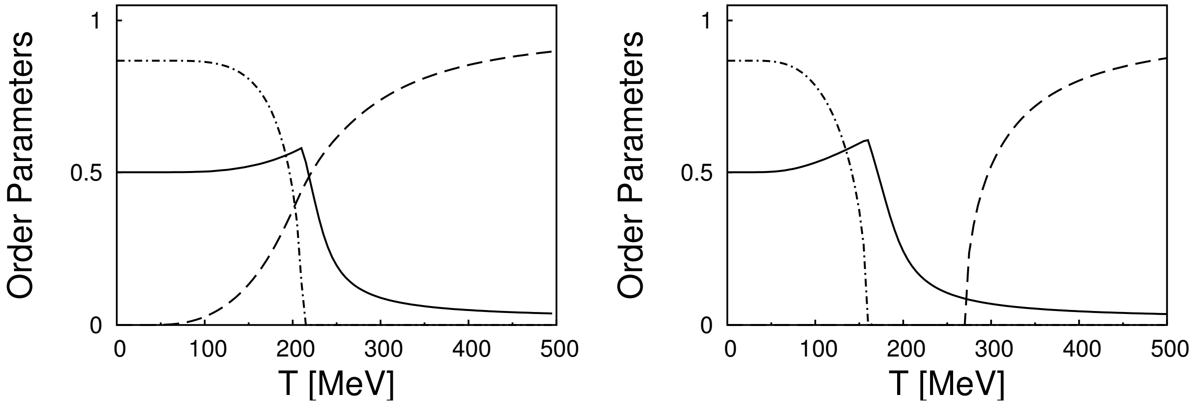
(a) The chiral condensate

(b) The diquark condensate

Figure 4.3: The chiral and diquark condensate for various chemical potentials as functions of temperature in units of $\sigma_0(T = 0)$. The solid line corresponds to $\mu = 0$ [MeV], the dashed line to $\mu = 75$ [MeV], the dotted line to $\mu = 100$ [MeV] and the dash-dotted line to $\mu = 200$ [MeV] in both. The chiral condensate decreases with the chemical potential μ increasing in contrast to the diquark condensate increases with μ.

Phase structure at finite chemical potential $\mu = 100$ [MeV]

We then explore the behavior of all order parameters as a function of temperature at finite chemical potential $\mu = 100$ [MeV] in Fig. 4.4. We observe the Polyakov loop is rather insensitive to the chemical potential by comparing the result in Fig. 4.4 ($\mu \neq 0$) with those in Fig. 4.1 ($\mu = 0$). The correlation of the diquark condensation and the Polyakov loop arise through the chiral condensation. Since in the NJL model the chiral condensation and the Polyakov loop are independent, the diquark condensation and the Polyakov loop are also independent.



(a) PNJL model case (b) NJL model case

Figure 4.4: The behavior of the chiral condensation (solid line), the Polyakov loop (dashed line) and the diquark condensation (dash-dotted line) as a function of temperature and finite chemical potential $\mu = 100$ [MeV] in comparison with PNJL model and the NJL model in unit of $\sigma_0(T = 0)$. The diquark condensation correlates the Polyakov loop through the chiral condensation in the PNJL model case.

4.5 BEC-BCS Crossover

Since the diquark-baryons are bosonic particles, and the diquark condensation, Bose-Einstein condensation (BEC), arise above the critical chemical potential $\mu_c = m_\pi/2$ as we mentioned. Further, the diquark-baryons are constructed from two-quarks (fermions) and hence they can form a Cooper pair for higher chemical potential ($\mu \gg \mu_c$). The transition from BEC state to BCS state is called BEC-BCS crossover.

It is known from the condensed matter physics that in the non-relativistic system the BEC and BCS states are distinguished by $\mu < 0$ and $\mu > 0$, respectively, where μ is the fermion chemical potential. In relativistic system (our system) their condensates are characterized by the fermionic excitation gap Δ_{ex}, defined as the minimum of the fermionic excitation energy $\Delta_{ex} = \min\{E_\Delta^-, E_\Delta^+\}$. In the BCS state, the fermionic excitation energy has the gap $|\Delta|$ which is the solution of the diquark gap equation (4.21) and thus $\Delta_{ex} = |\Delta|$, which means the minimum of the quasi-particle excitation energy is located at finite momentum $|\vec{p}| \neq 0$. The Coorper paired fermions have a finite momentum $|\vec{p}| \neq 0$ with the opposite direction. On the other hand, in the BEC state, all the condensed particles (bosons) are located at the ground state. In this point of view, the fermionic excitation gap Δ_{ex} may be located at $|\vec{p}| = 0$.

In our framework, the dynamical quark mass $m = m_0 + g_d\sigma_0$ and the diquark gap $|\Delta| = g_d|\Delta_0|$ in the fermionic excitation spectra $E^{\pm}_{\Delta} = \sqrt{(E_p \pm \mu)^2 + |\Delta|^2}$ ($E_p = \sqrt{\vec{p}^2 + m^2}$) are functions of the chemical potential. As μ increases from $\mu_c = m_\pi/2$, the diquark gap $|\Delta|$ increases, and conversely the dynamical quark mass m decreases as shown in Fig. 4.2. Hence the location of the minimum of the quark excitation E^-_Δ can be determined by a comparison between the dynamical quark mass and the chemical potential. On the other hand, the minimum of the antiquark excitation E^+_Δ may be always larger than the quark excitation E^-_Δ and can be neglected. Thus, The fermionic exitation gap Δ_{ex} can be defined by the minimum of the quark excitation $\min\{E^-_\Delta\}$. The fermionic excitation energy gap can be evaluated as

$$\Delta_{ex} = \min\{E^-_\Delta, E^+_\Delta\} = \begin{cases} \sqrt{(m-\mu)^2 + |\Delta|^2} & \mu < \mu_0 \\ |\Delta| & \mu > \mu_0 \end{cases}, \tag{4.23}$$

where μ_0 is so-called the crossover point. When the chemical potential near the diquark transition point μ_c, the minimum of the quark excitation spectrum E^-_Δ is located at $|\vec{p}| = 0$. For very large chemical potential $\mu > \mu_0$ the minimum of E^-_Δ occurs at a finite $|\vec{p}| \simeq \mu$ since $m \to m_0$. This behavior is demonstrated in Fig. 4.5 showing the quasi-particle energy of quarks, $E^-_\Delta(\vec{p})$ at $\mu = 100$ [MeV] and at $\mu = 200$ [MeV]. At the chemical potential $\mu = 100$ [MeV], the minimum of the quasi-particle energy is at zero momentum. The quasi-particle energy E^-_Δ at $\mu = 200$ [MeV] has a minimum at finite momentum. The crossover point is insensitive to the temperature (see Fig. 4.3) as long as the diquark condensation is finite.

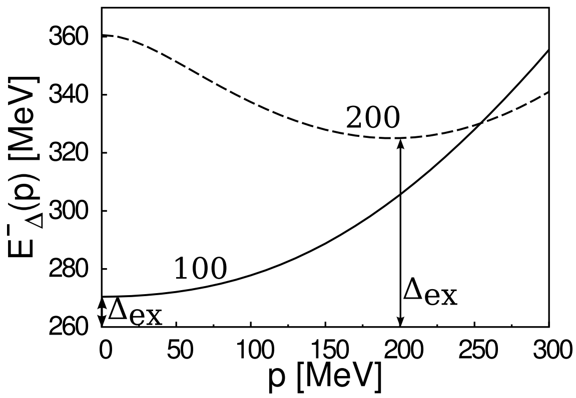


Figure 4.5: The quasi-particle energies of quark as functions of momentum $|\vec{p}|$ in unit MeV. The quasi-particle energy of quarks at $\mu = 100$ [MeV] (solid line) and at $\mu = 200$ [MeV] (dashed line) as functions of momentum p. The minimum is located at $|\vec{p}| = 0$ in BEC state ($\mu = 100$[MeV]) and at $|\vec{p}| \simeq \mu$ in BCS state ($\mu = 200$[MeV]).

We then show the comparison of the dynamical quark mass and the chemical potential in Fig. 4.6. The quark mass m starts to decrease from $\mu_c = m_\pi/2$, at this point the diquark condensation occurs, with $m > \mu$. Then the chemical potential exceeds the quark mass from $\mu \simeq 120$ [MeV], which is the crossover point μ_0. Once the chemical potential μ exceeds the dynamical quark mass, $\mu \geq m$, the system undergoes the BEC-BCS crossover and turns into the BCS phase as μ increases further.

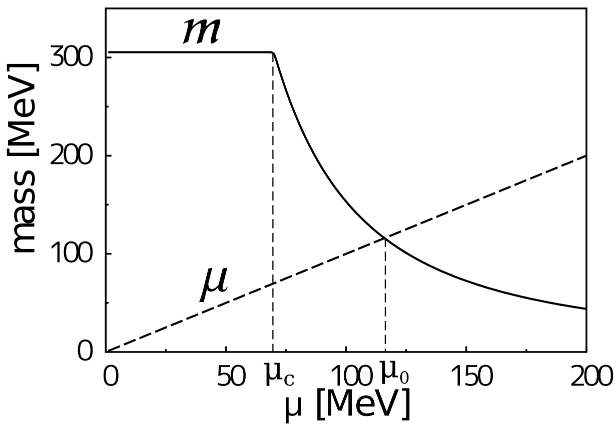


Figure 4.6: The dynamical quark mass m (solid line) vs. the chemical potential μ (dashed line). The diquark transition point is $\mu_c = m_\pi/2$ and the BEC-BCS crossover point corresponds to the crossing point, $\mu_0 \simeq 120$ [MeV].

4.6 The Behavior of Hadron Masses at Finite Temperature

The hadron mass spectrum can be described by analyzing the hadronic partition function Z_{hadron} in Eq. (4.5). We start with the mesons of the sigma and pion at finite temperature and zero chemical potential ($\mu = 0$). In this case, the diquark condensate is zero and there is no sigma-diquark mixing. The Lagrangian for each fields at this state can be written as

$$\mathcal{L}_s^{(2)} = \frac{1}{2}\left(\partial^\mu s(x)\right) - \frac{1}{2}m_s^2 s^2(x), \tag{4.24}$$

$$\mathcal{L}_\pi^{(2)} = \frac{1}{2}(\partial_\mu \vec{\pi}(x))^2 - \frac{1}{2}m_\pi^2 \vec{\pi}^2(x), \tag{4.25}$$

$$\mathcal{L}_{diq}^{(2)} = \frac{1}{2}(\partial^\mu d^*(x))(\partial_\mu d(x)) - \frac{1}{2}m_d^2 d^* d, \tag{4.26}$$

with the masses

$$m_s^2 = \left[M_s^2 - 2g_0^2 I_2 + 4I_0 m^2\right]/(g_0^2 I_0), \tag{4.27}$$

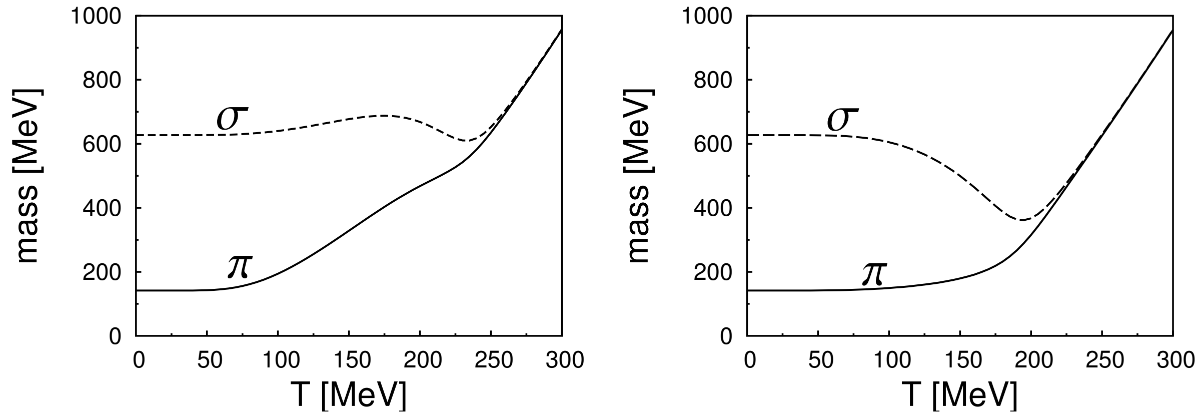
$$m_\pi^2 = \left[M_s^2 - 2g_0^2 I_2\right]/(g_0^2 I_0), \tag{4.28}$$

$$m_d^2 = \left[M_d^2 - 2g_d^2 I_2\right]/(g_0^2 I_0). \tag{4.29}$$

Obviously, the pion mass and the diquark mass coincide due to the PG symmetry, so we discuss the dispersion relation for the sigma and pion. The behavior of these masses is similar to that found in the SU(3) color NJL model. The only difference is the role of the Polyakov loop for two colors here as compared to the standard color SU(3). The effect of the Polyakov loop is included through the integrals I_2 and I_0.

We show the sigma and pion masses as functions of temperature with and without the quark-Polyakov-loop coupling at zero chemical potential ($\mu = 0$) in Fig. 4.7. The explicit chiral symmetry breaking by the small quark mass, $m_0 = 5.4$ [MeV], gives the Nambu-Goldstone pion a small mass, $m_\pi = 140$ [MeV] at zero temperature, while the sigma mass, $m_s = 610$ [MeV] stays around twice the dynamical quark mass $m = m_0 + g_0\sigma_0$. As the temperature increases, the pion mass starts to increase, while the sigma mass decreases until the temperature approaches the crossover temperature of about 180 [MeV]. The two masses meet at the crossover temperature and increase jointly as shown in Fig. 4.7(b).

When the quark-Polyakov-loop coupling is introduced, the pion mass increases earlier with temperature as shown in Fig. 4.7(a). The reason is that the chiral condensate σ_0 (or the quark mass m) drops more slowly with increasing temperature as shown in Fig. 4.1(a). The sigma mass stays almost constant up to the crossover temperature and then both the pion and sigma masses increase rapidly with increasing temperature.



(a) With the Polyakov loop (b) Without the Polyakov loop

Figure 4.7: The sigma (dashed line) and pion masses (solid line) in MeV as functions of temperature in MeV at zero chemical potential (μ=0).
(a): For the case of the PNJL model with coupling of the quarks and Polyakov loop.
(b): For the case of the NJL model with decoupled Polyakov loop.

4.7 The Behavior of Hadron Masses at Finite Chemical Potential

We analyze the behavior of the masses at finite chemical potential and zero temperature ($T = 0$). In the diquark condensed phase $\mu > m_\pi/2$, the mass matrix for the scalar fields has non-diagonal components. On the other hand, the pion does not depend on the other fields due to the its pseudo-scalar nature.

4.7.1 Pion

We first discuss the behavior of the pion mass at finite chemical potential. The renormalized pion mass is written as

$$m_\pi^2 = (M_s^2 - 2g_0^2 I_2 + 4g_0^2 I_0 \mu^2)/(g_0^2 I_0) = \frac{\varepsilon}{\sigma_0}\ . \tag{4.30}$$

Consider the case of $|\Delta| = 0$ and define divergent integrals for $\mu = 0$ as:

$$I_2^0 = 2i\mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p_0^2 - E_p^2} \quad I_0^0 = -2i\mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p_0^2 - E_p^2)^2}. \tag{4.31}$$

I_2 for finite μ with $|\Delta| = 0$ is written in terms of I_2^0 and I_0^0

$$\begin{aligned} I_2 =& i\mathrm{tr}_{fc} \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{p_0^2 - (E_p^+)^2} + \frac{1}{p_0^2 - (E_p^-)^2} \right) \\ =& I_2^0 + 2\mu^2 I_0^0. \end{aligned} \tag{4.32}$$

In the region of interest, $I_0 = I_0^0$ and m_π^2 becomes

$$m_\pi^2 = (M_s^2 - 2g_0^2 I_2^0)/(g_0^2 I_0^0) \equiv m_{\pi 0}^2 \ . \tag{4.33}$$

Hence, the pion mass does not depend on μ for $|\Delta_0| = 0$. The pion mass stays constant at 140 [MeV] and proportional to the square root of the bare quark mass until the onset of diquark condensation.

The gap equation for the diquark with the Pauli-Gürsey symmetry $G_0 = H_0$ provides $g_0 = g_d$ and $M_s = M_d$ and therefore $M_s^2 - 2g_0^2 I_2 = 0$ for the finite $|\Delta_0|$. The pion mass becomes

$$m_\pi^2 = (M_s^2 - 2g_0^2 I_2 + 4g_0^2 I_0 \mu^2)/(g_0^2 I_0) = (2\mu)^2. \tag{4.34}$$

This shows that the pion mass in the diquark condensed phase increases linearly with μ.

4.7.2 Diagonalization of Sigma and Diquark-Baryon Mass Matrix

The sigma meson mixes with diquark-baryons, and furthermore diquarks mix with each other, since the sigma and diquark Lagrangian density with kinetic and mass terms is written in a matrix form:

$$\mathcal{L}_{sd}^M = -\frac{1}{2}\Phi^\dagger \begin{pmatrix} \partial^2 + m_s^2 & 2\Delta^* m & 2\Delta m \\ 2\Delta m & \frac{1}{2}(\partial^2 + m_d^2) + 2i\mu\partial^0 & \Delta^2 \\ 2\Delta^* m & \Delta^{*2} & \frac{1}{2}(\partial^2 + m_d^2) - 2i\mu\partial^0 \end{pmatrix} \Phi \ , \tag{4.35}$$

with the scalar fields representation $\Phi^t = (s, d, d^*)$. The (bare) diquark-baryon mass is given by:

$$\begin{aligned} m_d^2 =& (M_d^2 - 2g_d^2 I_2 + 2g_d^2 I_0 |\Delta|^2)/(g_0^2 I_0) \\ =& \begin{cases} m_\pi^2 - 4\mu^2 & |\Delta| = 0 \\ 2|\Delta|^2 & |\Delta| \neq 0 \end{cases} , \end{aligned} \tag{4.36}$$

with the diquark gap equation $M_d^2 - 2g_d^2 I_2 = 0$ and $g_d = g_0$ for $|\Delta| \neq 0$. The (bare) sigma mass is larger than the pion mass, $m_s^2 = m_\pi^2 + 4m^2$, with m the dynamical quark mass:

$$m_s^2 = (M_s^2 - 2g_0^2 I_2 + 4g_0^2 I_0 \mu^2 + 4g_0^2 I_0 m^2)/(g_0^2 I_0) = m_\pi^2 + 4m^2 \ . \tag{4.37}$$

We write the mass matrix in the momentum representation (Minkowski space) as

$$M(\omega, \vec{q}) = \begin{pmatrix} q^2 - m_s^2 & -2\Delta^* m & -2\Delta m \\ -2\Delta m & \frac{1}{2}(q^2 - m_d^2) + 2\mu\omega & -\Delta^2 \\ -2\Delta^* m & -\Delta^{*2} & \frac{1}{2}(q^2 - m_d^2) - 2\mu\omega \end{pmatrix} , \tag{4.38}$$

where $q^2 = \omega^2 - \vec{q}^2$. The physical mass spectra for sigma, diquark and antidiquark are obtained solving the dispersion relation $\det M(\omega, 0) = 0$ with respect to ω^2.

Low chemical potential $\mu < \mu_c$ ($|\Delta| = 0$)

Before sloving the complete solution of Eq. (4.38), we first analyze the case of $|\Delta| = 0$, in which there are no mixing terms. Hence, the sigma meson and the diquark (antidiquark) masses are determined independently. The sigma mass $m_s^2 = m_\pi^2 + 4m^2$ is obviously constant since the both of the pion mass and the dynamical quark mass are constant in this region. The physical diquark and anti-diquark dispersion in the case of $|\Delta| = 0$ are

$$\omega^2 - m_d^2 \pm 4\mu\omega = 0, \tag{4.39}$$

where the bare mass is $m_d^2 = m_\pi^2 - 4\mu^2$ and upper plus sign is for the diquark and lower minus sign for the anti-diquark. The solutions of Eq. (4.39) are

$$\omega = m_\pi \pm 2\mu, \tag{4.40}$$

where the diquark-baryon corresponds to $m_\pi - 2\mu$ and the antidiquark to $m_\pi + 2\mu$ since the baryon number is 1 and -1, respectively. As anticipated the diquark masses equal to the pion mass m_π at $\mu = 0$ due to the Pauli-Gürsey symmetry. The diquark and antidiquark masses vary linearly as functions of the baryon chemical potential $\mu_B = 2\mu$. This behavior can be understood by a simple picture. When $\mu_B = 0$, the bare diquark and antidiquark masses may be located at ± 140 [MeV] as measured from the vacuum as shown in Fig. 4.8. When $\mu_B \neq 0$, their excitation energies (ω_Δ and ω_{Δ^*}) change linearly by μ_B for the diquark and by $-\mu_B$ for the antidiquark. When the baryon chemical potential μ_B reaches the bare diquark mass m_d, the physical diquark mass ω_Δ becomes zero (NG boson) and the diquark condensation appears (BEC).

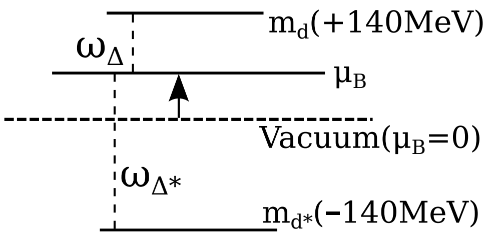


Figure 4.8: A schematic picture of the dispersions for the diquark and antidiquark. The bare diquark and antidiquark masses exist at ± 140 [MeV]. Their dispersions are measured from the fermi surface μ_B.

Without the mixing term

Now we consider the case of finite $|\Delta|$. In the absence of the sigma-diquark mixing term, $-2m\Delta^{(*)}$, the sigma meson indipendent with the diquarks. The sigma mass approaches the pion mass as the dynamical quark mass tends to zero due to the chiral symmetry restoration. In this case the diquark-antidiquark mass matrix become

$$D(\omega, \vec{q}) = \begin{pmatrix} \frac{1}{2}(q^2 - m_d^2) + 2\mu\omega & -\Delta^2 \\ -\Delta^{*2} & \frac{1}{2}(q^2 - m_d^2) - 2\mu\omega \end{pmatrix}. \tag{4.41}$$

The dispersion relation $\det D(\omega,0)=0$ generates

$$\omega_{\Delta^*} = \sqrt{16\mu^2+4|\Delta|^2}, \tag{4.42}$$
$$\omega_{\Delta} = 0\ . \tag{4.43}$$

The physical diquark-baryon mass ω_Δ is zero. These behaviors are shown in Fig. 4.9(b).

With the mixing term

Finally, we investigate the dispersion for all scalar particles with the full mass matrix M in the diquark condensate phase. The sigma and diquarks masses are given by solving the dispersion relation $\det M(\omega,0)=0$ as

$$\omega^2\left[\omega^4-2\omega^2(2|\Delta|^2+10\mu^2+2m^2)+16\mu^2(|\Delta|^2+4\mu^2+4m^2)\right]=0, \tag{4.44}$$

with substituting $m_d^2=2|\Delta|^2$ and $m_s^2=m_\pi+4m^2=4\mu^2+4m^2$ in the diquark condensate phase. One trivial solution $\omega^2=0$ is the NG boson and the other solutions are

$$\omega^2=2|\Delta|^2+10\mu^2+2m^2\pm\sqrt{(6\mu^2+2|\Delta|^2+2m^2)^2-48\mu^2m^2}. \tag{4.45}$$

In the limit of $|\Delta|\to 0$ at the onset of diquark condensation ($\mu=m_\pi/2$) the dispersion Eq. (4.45) becomes

$$\omega^2=\begin{cases}4\mu^2+4m^2=m_s^2\\ 16\mu^2=4m_\pi^2\end{cases}. \tag{4.46}$$

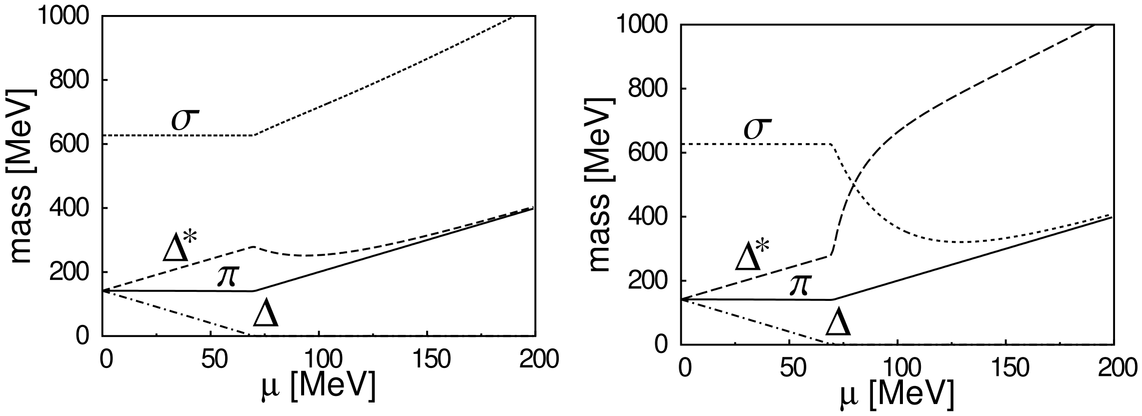
The upper solution corresponds to the squared sigma mass m_s^2 (without mixing effect) and the lower one is the squared antidiquark mass at $\mu=m_\pi/2$. In the limit $m\to 0$ for large μ the dispersion Eq. (4.45) leads to the solutions

$$\omega^2=\begin{cases}4\mu^2=m_\pi^2\\ 16\mu^2+4|\Delta|^2\end{cases}. \tag{4.47}$$

One solution represents the linearly increasing pion mass. The other one agrees with the squared antidiquark-baryon mass Eq. (4.42). Thus, the effect of the mixing terms disappears at very large μ.

4.7.3 Numerical Plot

We show in Fig. 4.9(a) the results of the diagonalized masses. The effect of the coupling of the sigma meson and the diquark-baryon is very large. When the coupling is neglected, the sigma mass drops slightly as μ approaches the crossover chemical potential $\mu\sim 150$ [MeV] and then increases together with the pion mass as shown in Fig. 4.9(b). At the same time, the antidiquark mass increases rapidly with the chemical potential after diquark condensation. On the other hand, in the case of strong scalar-diquark coupling, the quantum-mechanical "non-crossing" rule is at work. The sigma mass increases continuously and the antidiquark mass joins the slowly increasing pion mass.



(a) Hadron spectrum (b) Hadron spectrum without mixing term

Figure 4.9: The sigma (dotted line), pion (solid line), diquark (dash-dotted line) and antidiquark (dashed line) masses in MeV as functions of the chemical potential in MeV at zero temperature ($T = 0$).
(a): For the case of full dispersion.
(b): For the case of dispersion neglecting mixing term.

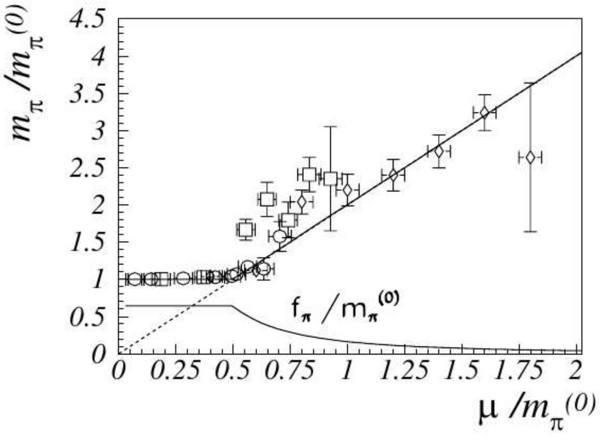
We show the lattice calculation result in Fig. 4.10 to compare with our result. Our diquark and antidiquark correspond to 0^+. The authors [50] calculate the hadron mass spectrum in baryon-rich medium with the diquark source term. The pion mass behaves constant in the phase where the diquark condensate vanishes and linearly increasing in diquark condensed phase. The diquark and antidiquark behave linearly increasing and decreasing respectively with the chemical potential at low density. The antidiquark becomes massless NG boson in the "physical" limit at high density.

Thus, our analysis of the mass spectra agree with the lattice simulation and further we worked out the sigma meson and the diquark mass in the diquark condensed phase. These sigma and diquark mass behavior is interesting to check the lattice simulation because the "non-crossing" rule may arise in this region.

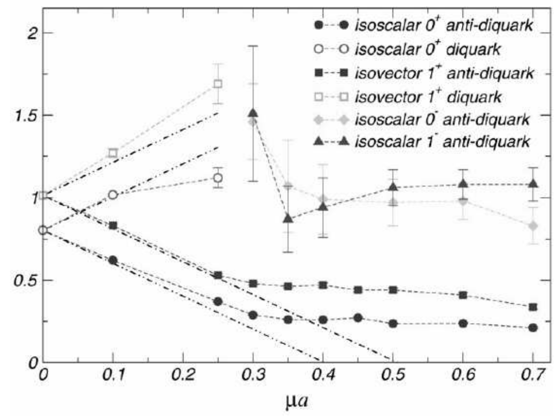
4.8 Equation of State of Quark-Hadron Matter

Next, consider the equation of state (EOS) of quark-hadron matter at various chemical potentials μ as functions of the temperature T. The pressure is given by the thermodynamical potential, $p = -\Omega$. For the case of only quarks, $\Omega = \Omega_{MF}$ (4.17), the pressure is essentially zero as shown by the thin solid line in Fig. 4.11(a) in the confined region (small temperature), and gradually increases with temperature as deconfinement sets in featuring the entanglement of chiral symmetry and Polyakov loop effects. Finally the pressure becomes large at high temperature above the crossover transition.

In the confined region, the essential active degrees of freedom are hadrons. Their dynamics is governed by the hadron Lagrangian derived previously, in which mesons and diquark-baryons interact. We calculate thermodynamical potential, taking only the mass terms and integrating



(a) The pion mass in lattice



(b) The diquark mass in lattice

Figure 4.10: (a): The lattice simulation of the pion mass quoted from Ref. [64] in lattice simulation by Ref. [49].
(b): The lattice simulation of the diquark for several J^P from Ref. [50]. Our diquark and anti-diquark correspond to 0^+.

out the hadron fields by the Gaussian approximation as explained in Sect. 4.1 as

$$\Omega_{hadron} = \frac{1}{2}\sum_Q (\ln\det M_n(Q) + \ln\det N_n(Q)), \tag{4.48}$$

where

$$M_n(Q) = \begin{pmatrix} Q^2 + m_s^2 & 2m\Delta^* & 2m\Delta \\ 2m\Delta & \frac{1}{2}(Q^2 + m_d^2) - 2\mu Q_0 & \Delta^2 \\ 2m\Delta^* & \Delta^{*2} & \frac{1}{2}(Q^2 + m_d^2) + 2\mu Q_0 \end{pmatrix}, \tag{4.49}$$

$$N_n^{ij}(Q) = \delta^{ij}(Q^2 + m_\pi^2), \tag{4.50}$$

in Euclidean space $Q^2 = \vec{q}^2 + \omega_n^2$ with the Matsubara frequencies for boson $Q_0 = i\omega_n$ and $\sum_Q = T\sum_n \int \frac{d^3q}{(2\pi)^3}$. The pion thermodynamical potential can be worked out simply

$$\Omega_\pi = 3\int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{2}\omega_\pi(\vec{q}) + T\ln(1 - e^{-\beta\omega_\pi(\vec{q})})\right], \tag{4.51}$$

where the zero point energy is $\omega_\pi(\vec{q}) = \sqrt{\vec{q}^2 + m_\pi^2}$. The behavior of the pion mass m_π at finite chemical potential and zero temperature has been discussed. The behavior of thermodynamical potential for the scalar particles are different between $|\Delta| = 0$ and $|\Delta| \neq 0$. In the case of $|\Delta| = 0$ where the non-diagonal components in the matrix $M_n(Q)$ disappear, the thermodynamical potential for the sigma meson is

$$\Omega_s = \int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{2}\omega_s(\vec{q}) + T\ln(1 - e^{-\beta\omega_s(\vec{q})})\right], \tag{4.52}$$

with the zero point energy $\omega_s(\vec{q}) = \sqrt{\vec{q}^2 + m_s^2} = \sqrt{\vec{q}^2 + m_\pi^2 + 4m^2}$ and for the diquarks is

$$\begin{aligned}\Omega_\Delta &= \int \frac{d^3q}{(2\pi)^3}\left[\frac{1}{2}\omega_\Delta(\vec{q}) + \frac{1}{2}\omega_{\Delta^*}(\vec{q}) + T\ln(1-e^{-\beta\omega_\Delta(\vec{q})}) + T\ln(1-e^{-\beta\omega_{\Delta^*}(\vec{q})})\right]\\ &= \int \frac{d^3q}{(2\pi)^3}\left[\omega_\pi(\vec{q}) + T\ln(1-e^{-\beta(\omega_\pi(\vec{q})-2\mu)}) + T\ln(1-e^{-\beta(\omega_\pi(\vec{q})+2\mu)})\right], \end{aligned} \tag{4.53}$$

where the zero point energy is the solution of the dispersion relation

$$\omega^2 - \vec{q}^2 - m_d^2 \pm 4\mu\omega = 0 \tag{4.54}$$

as

$$\omega_\Delta(\vec{q}) = \sqrt{\vec{q}^2 + m_\pi^2} - 2\mu = \omega_\pi(\vec{q}) - 2\mu, \tag{4.55}$$

$$\omega_{\Delta^*}(\vec{q}) = \sqrt{\vec{q}^2 + m_\pi^2} + 2\mu = \omega_\pi(\vec{q}) + 2\mu. \tag{4.56}$$

In the diquark condensed phase $|\Delta| \neq 0$, all the scalar excitation energies are described by the solution of the dispersion relation as

$$\begin{aligned}\det M(\omega,\vec{q}) =&\omega^6 - [(\vec{q}^2+m_s^2) + 2(\vec{q}^2+m_d^2+8\mu^2)]\omega^4\\ &+ [\vec{q}^2(\vec{q}^2+2m_d^2) + 2(\vec{q}^2+m_s^2)(\vec{q}^2+m_d^2+8\mu^2) - 16m^2|\Delta|^2]\omega^2\\ &- \vec{q}^2[(\vec{q}^2+m_s^2)(\vec{q}^2+2m_d^2) + 16m^2|\Delta|^2] = 0,\end{aligned} \tag{4.57}$$

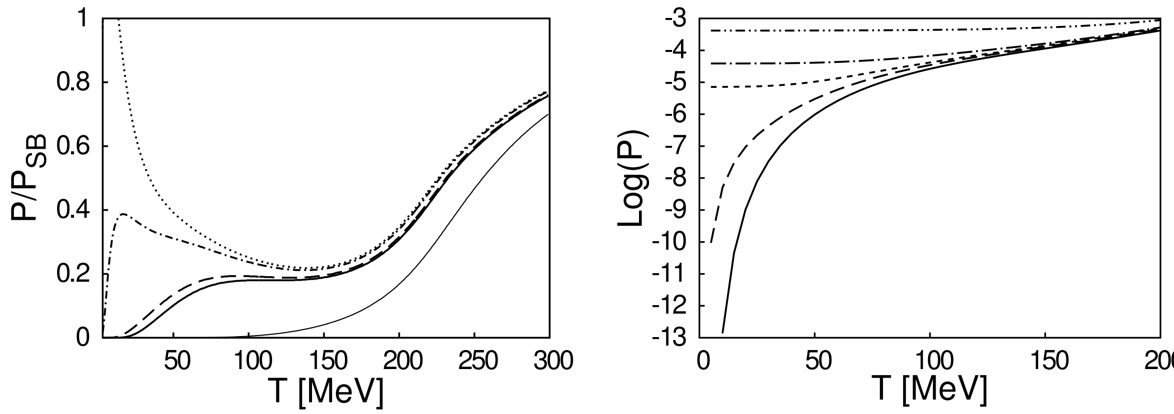
with respect to ω. We write the solutions as $\omega_i(\vec{q})$ $(i = s, d, d^*)$. Thermodynamical potential is written as

$$\Omega_{scalar} = \int \frac{d^3q}{(2\pi)^3}\sum_i\left[\omega_i(\vec{q}) + T\ln(1-e^{-\beta\omega_i(\vec{q})})\right]. \tag{4.58}$$

The momentum integrals in zero point energies of these bosons are divergent. As the standard treatment we introduce the momentum cut-off to regularize the integrals. However, this cut-off is a free parameter since the bosonic momenta are external variables. We employ the NJL cut-off $\Lambda = 657$ [MeV] because our starting point NJL model is an effective model below the cut-off Λ. The zero point energy at $T = 0$ and $\mu = 0$ is then subtracted from the thermodynamical potential. For the case $\Delta = 0$, this procedure makes the contribution of the zero-point energy terms vanish. Hence, the whole contribution to the pressure from the hadrons comes from the temperature dependent terms for $\Delta = 0$.

Shown in Fig. 4.11(a) is the pressure at $\mu = 0, 30, 60$ and 66 [MeV]. Given the small pion and diquark-baryon masses, the contributions of these degrees of freedom make the pressure significantly different from that with quark degrees of freedom only. This result is qualitatively similar to the one of the color SU(3) PNJL model [25], but here, due to the additional pressure of the diquark-baryon fields, the effect of the hadron contributions is much larger. The pressure increases rapidly as the chemical potential approaches the critical chemical potential $\mu_c = m_\pi/2$. This rapid increase of the pressure is caused by the diquark-baryon mode whose energy drops as $\omega_\Delta = \omega_\pi - 2\mu$ with increasing μ. When the temperature becomes small ($T \sim 10$ [MeV]), we can expand the logarithm and the pressure can be written approximately as

$$p \simeq T\int\frac{d^3q}{(2\pi)^3}e^{-(\omega_\pi(\vec{q})-2\mu)/T} \sim T\int_0^{q_{max}}\frac{q^2dq}{2\pi^2}e^{-(\omega_\pi(\vec{q})-2\mu)/T}, \tag{4.59}$$



(a) EOS contribution of hadrons compare with the only quarks

(b) The logarithmic scale of the pressure

Figure 4.11: (a): The pressure divided by the Stefan-Boltzmann pressure for various chemical potentials $\mu = 0$ (solid line), 30 (dashed line), 60 (dashed-dotted line) and 66 (dotted line) [MeV] below diquark condensation as functions of temperature. Shown also is the one with only the quark degree of freedom (thin solid line).
(b): The logarithm of the pressure $\ln(P)$ for various chemical potentials for $\mu = 0$ (solid line), 60 (dashed line) [MeV] below $\mu_c = 70$ [MeV] and for $\mu = 80$ (dotted line), 100 (dashed-dotted line) and 200 (dashed double-dotted line) [MeV] above μ_c as functions of temperature. The unit of pressure is GeV^4.

where q_{max} is appreciable only for the diquark-baryon mode (would be zero mode) as $\mu \to \mu_c$. Since the pressure is divided by the Stefan-Boltzmann pressure, the pressure ratio close to the critical temperature shows rapid growth when μ approaches the critical chemical potential.

Consider now the pressure over a wide range of chemical potentials μ, in particular above the critical μ_c for diquark condensation. The pressure is then dominated by quarks through their zero point motion renormalized by the vacuum value, with additional contributions from the zero point motion of mesons and diquark-baryons. Given that these zero point motion effects do not vanish at zero temperature, it is more appropriate to present the pressure as such, not divided by the Stefan-Boltzmann pressure, on a logarithmic scale. Results are shown in Fig. 4.11(b). The zero point motion is largely influenced by the presence of the Bogoliubov spectrum of the diquark zero mode which will be discussed in Sec. 4.9 [19, 67, 77, 80, 81]. The pressure curves in Fig. 4.11(b) are displayed in the whole μ range as functions of temperature. Shown in Fig. 4.11(b) are the pressures at $\mu = 0, 60$ [MeV] below $\mu_c = m_\pi/2$ and $\mu = 80, 100$ and 200 [MeV] above μ_c. The pressure below the critical chemical potential drops to zero at $T = 0$, while the pressure at $\mu > \mu_c$ stays finite at zero temperature. The pressure at low temperature increases rapidly across the critical chemical potential. The pressure at high temperature is dominated by the de-confined quark contribution and insensitive to the chemical potential.

4.9 The Bogoliubov Excitation

The hadronic contributions to the pressure and the quark density involve spectra of hadrons. It is important to know how the zero mode behaves at finite momentum in the diquark-condensed phase. The Bogoliubov excitation [19,67,77,80,81] for the zero mode results from the solution $\omega(\vec{q})$ of the dispersion Eq. (4.57). The gapless Bogoliubov mode is a linear dispersion in the low energy limit. In our system, the Bogoliubov mode is realized not only the diquark-antidiquark pairing but also the sigma-diquark mixing. The solution of the dispersion Eq. (4.57) is computed numerically in Fig. 4.12 at $\mu = 80$ [MeV]. To see the existence of the Bogoliubov mode, we drop the sigma-diquark mixing terms in the full mass matrix (4.38) and arrive at an analytic expression for $\omega(\vec{q})$ of the Bogoliubov mode. Consider the reduced mass matrix

$$D(\omega, \vec{q}) = \begin{pmatrix} \omega^2 - \vec{q}^2 - m_d^2 + 4\mu\omega & -2\Delta^2 \\ -2\Delta^{*2} & \omega^2 - \vec{q}^2 - m_d^2 - 4\mu\omega \end{pmatrix}, \tag{4.60}$$

and solve the dispersion relation $\det D(\omega, \vec{q}) = 0$ with respect to $\omega(\vec{q})$ for the lowest mode:

$$\begin{aligned} \omega^2 =& \vec{q}^2 + 2|\Delta|^2 + 8\mu^2 \\ & - \sqrt{(\vec{q}^2 + 2|\Delta|^2 + 8\mu^2)^2 - \vec{q}^2(\vec{q}^2 + 4|\Delta|^2)} \\ \sim& \frac{\vec{q}^2(\vec{q}^2 + 4|\Delta|^2)}{2(\vec{q}^2 + 2|\Delta|^2 + 8\mu^2)} . \end{aligned} \tag{4.61}$$

In the BEC limit, the pion mass $m_\pi = 2\mu$ is much larger than $|\Delta|$. The Bogoliubov excitation is then written as

$$\omega(\vec{q}) \sim \sqrt{\frac{\vec{q}^2}{2m_\pi}\left(\frac{\vec{q}^2}{2m_\pi} + \frac{2|\Delta|^2}{m_\pi}\right)} . \tag{4.62}$$

The zero mode varies linearly with $|\vec{q}|$ at small momentum. With inclusion of the sigma-diquark coupling, the dispersion equation for $\omega(\vec{q})$ with the full mass matrix is solved numerically. For $m_\pi \gg |\Delta|$ one confirms that the zero mode has the Bogoliubov excitation spectrum as expressed in Eq. (4.62). As we show in Fig. 4.12, the Bogoliubov excitation in our system is linearly increasing.

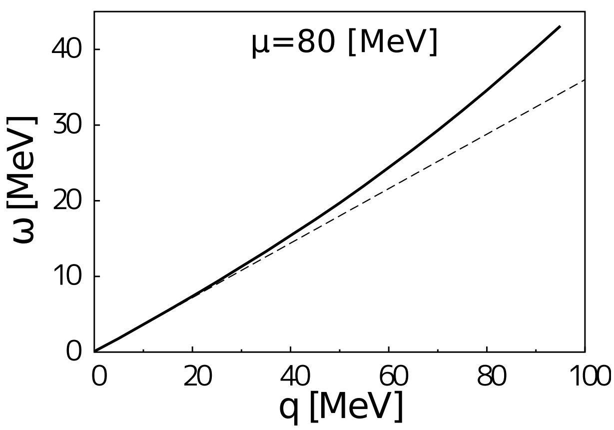


Figure 4.12: The Bogoliubov excitation at $\mu = 80$ [MeV] (solid line) as a function of momentum. The dashed line denotes a line in momentum. In the low momentum region, the excitation energy increases linearly with momentum.

The Bogoliubov excitation gives large contribution to thermodynamics in the BEC state $70 < \mu \lesssim 120$ [MeV]. On the other hand, in the BCS state $\mu \gtrsim 120$ [MeV] the excitation contribution might be small as discussed in Ref. [77]. As an example, this property will be shown in the discussion of the baryon density in the next section 4.10.

4.10 Baryon Density

An interesting point of comparison with color SU(2) lattice simulations concerns the quark density as a function of chemical potential μ at zero temperature. When the baryon number symmetry is broken, the diquark condensate becomes finite and the diquark-baryon becomes a Nambu-Goldstone boson. From the onset of diquark condensation, the quark number becomes finite. The quark density derived from the thermodynamical potential at mean field level is:

$$\rho_{MF} = -\frac{\partial \Omega_{MF}}{\partial \mu} = \mathrm{tr} \int \frac{d^3 p}{(2\pi)^3} \left[\frac{E_p^+}{E_\Delta^+} - \frac{E_p^-}{E_\Delta^-} \right]. \tag{4.63}$$

Here, we write only the zero point oscillation terms, dropping the temperature dependent terms.

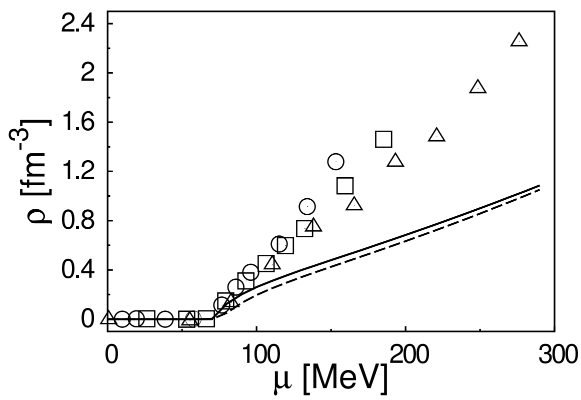


Figure 4.13: The quark density in unit of fm^{-3} as a function of the chemical potential μ in MeV. Shown by various points are the results of SU(2) lattice simulations [49]. The dashed curve denotes the result of the mean field approximation and the solid curve the results of the mean field and Gaussian approximation of the hadron contributions.

In Fig. 4.13 the quark density is presented as a function of the chemical potential μ. As compared with lattice simulations [49] the mean field quark density comes out to be smaller than the lattice results by about a factor of two. The quark density becomes non-zero when the diquark condensate develops, starting at $\mu = m_\pi/2$. The quark density increases with the chemical potential.

Going beyond mean field level implies adding hadronic contributions to the quark density. In a first step we use the Gaussian approximation, dropping higher order terms of the hadron Lagrangian, and integrate out the meson and diquark-baryon fields. The hadronic part of the density is found by taking the derivative of Ω_{hadron}, Eqs. (4.51) and (4.58), with respect to the chemical potential μ:

$$\rho_{hadron} = -\frac{\partial \Omega_{hadron}}{\partial \mu} = -\frac{\partial}{\partial \mu} \int \frac{d^3 q}{(2\pi)^3} \left[\sum_{i}^{3} \frac{\omega_i(\mu)}{2} + \frac{3}{2} \omega_\pi(\mu) \right] , \tag{4.64}$$

where ω_i $(i = \sigma, d, d^*)$ and ω_π are again the sigma meson, diquark, antidiquark and pion energies, respectively. Only the temperature independent terms are written. Note that ρ_{hadron} vanishes for $\Delta = 0$, while it becomes finite in the diquark condensed phase. The zero point energies are calculated numerically and the momentum integrals are performed introducing the NJL cut-off $\Lambda = 657$ [MeV]. The result is shown in Fig. 4.13 by the solid curve. As we mentioned, the main contribution of the correction of the baryon density ρ_{hadron} comes from the Bogoliubov excitation. The contribution is appreciable in the BEC state near the diquark transition. When the chemical potential is increasing and going in the BCS state, it does not grow much. Evidently, using the Gaussian approximation, the effect of the hadron fields on the quark density is very small. This was anticipated by He [67] in their analysis of the hadron contributions.

Chapter 5

Non-perturbative Treatment of Hadron Interaction

In this chapter we review the Gaussian variational approach and the Gaussian effective action. This approach is expected to be important to explore high density hadron matter. Although we do not apply the method to our hadron Lagrangian yet, we would like to discuss the framework for the future extension of the present work.

5.1 Introduction to a Non-perturbative Method

The thermodynamics in the Gaussian approximation cannot reproduce the baryon density of the lattice QCD calculation. Obviously, the Gaussian approximation misses important hadronic interaction terms generated by the higher order pieces, $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$, of the hadron effective Lagrangian. These interactions include, for example, scalar boson exchange between diquarks and mesons in various possible combinations. The strength of these couplings is controlled by the constant λ in Eq. (3.125). We find $\lambda = 33$ in the present parameter set. Altogether, the net attraction provided by such mechanisms is expected to decreace the vacuum energy and increase the density significantly as a function of μ.

An interesting idea for a systematic treatment of non-perturbative interaction terms were introduced by Kuti who, however, did not publish the idea and his idea was reviewed by Cornwall *et al.* in the appendix of Ref. [82]. The idea is to adopt the Schrödinger representation in the quantum field theory and solve the Schrödinger equation. The systematic discussion was done by Barnes and Ghandour [83]. They introduced a trial Gaussian wave function based on the variational problem and described the renormalized energy expectation value in the ϕ^4 theory with non-perturbative interaction. This approach is called the Gaussian variational approach or Gaussian functional approximation. The ground state energy (density) is obtained by solving the Schrödinger equation with the trial Gaussian wave functional in an variational method. The obtained ground state energy is equivalent to that obtained by Gaussian effective potential [84, 85].

The ground state at the mean filed approximation level, it seems that there is no Nambu-Goldstone (NG) particle. The massless NG particle will appear as a collective state [86]. Hence, the Gaussian functional method requires so-called "mean field approximation (MFA) + random phase approximation (RPA)" in the many-body literature. This framework is unsatisfactory since we require the ground state energy which is equivalent to the thermodyanamical potential at zero temperature and finite density with NG particle (Bogoliubov excitation). The effective action with the optimized expansion [87] is one of powerful non-perturbative approach which fully respects the NG theorem [88]. The effective action with the optimazed expansion is related to the Gaussian effective potential and the Gaussian functional method. The effective action is based on the path integral quantization approach and the Gaussian functional approximation is the Schrödinger quantization. Hence it is not surprising that the optimized expansion method and the Gaussian functional are related.

5.2 Functional Formulation in the Canonical Quantization

The canonical quantization is one of standard quantization prescription. The classical canonical variables q and p are replaced by operators which satisfy the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$. The classical field $\phi(t, \vec{x})$ is quantized by replacing it as the operator $\hat{\phi}(t, \vec{x})$ and taking the canonical commutation relation $[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\hbar\delta(\vec{x} - \vec{y})$ with its canonical conjugate momentum operator $\hat{\pi}(t, \vec{y})$.

In Schrödinger wave dynamical treatment, a quantum mechanical state is realized by a wave function $\psi(t, \vec{x})$. The quantization can be done by considering the correspondence relation $\hat{x}\psi(t, \vec{x}) \to x\psi(t, \vec{x})$ and $\hat{p}\psi(t, \vec{x}) \to -i\hbar\frac{d}{dx}\psi(t, \vec{x})$. The Gaussian functional approach is constructed based on this correspondence. For a field theory involving the field operator $\hat{\phi}(x)$, the wave function is a functional of a c-number $\phi(\vec{x})$:

$$|\psi\rangle \to \Psi[\phi]. \tag{5.1}$$

The action of the operator $\hat{\phi}(x)$ on $|\psi\rangle$ is realized by multiplying $\Psi[\phi]$ by $\phi(\vec{x})$:

$$\hat{\phi}|\psi\rangle \to \phi(\vec{x})\Psi[\phi]. \tag{5.2}$$

The action of the canonical conjugate momentum operator $\hat{\pi}(x)$ on $|\psi\rangle$ is realized by functional differentation

$$\hat{\pi}(x)|\psi\rangle \to -i\hbar\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi]. \tag{5.3}$$

Note that we have suppressed the time variable t.

Introducing eigenstates at a fixed time of the field $\phi(\vec{x})$, denoted as $|\phi\rangle$, the wave function $\Psi[\phi]$ is expressed

$$\langle\phi|\psi\rangle = \Psi[\phi]. \tag{5.4}$$

The inner product is defined by the functional integration:

$$\langle\psi_1|\psi_2\rangle \to \int \mathcal{D}\phi\Psi_1^*[\phi]\Psi_2[\phi]. \tag{5.5}$$

The analogy with ordinary quantum mechanics is clear.

Energy eigenstates statisfy the Schrödinger equation

$$\int d\vec{x}\mathcal{H}\left[-i\hbar\frac{\delta}{\delta\phi(\vec{x})}, \phi(\vec{x})\right]\Psi[\phi] = E\Psi[\phi], \tag{5.6}$$

where $\mathcal{H}[\pi(\vec{x}), \phi(\vec{x})]$ is the Hamiltonian density. The time development of an energy eigenstate can be written as

$$\Psi[\phi; t] = e^{-iEt/\hbar}\Psi[\phi]. \tag{5.7}$$

The Gaussian functional approach is based on this functional treatment of the quantum field theory.

5.3 Gaussian Trial Wave Function in Quantum Mechanics

In this section we illustrate the Gaussian variational approach by considering a simple quantum mechanical model following [83]. We consider one dimensional harmonic oscillator with a quartic term in the potential (taking $\hbar = 1$). The model Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\mu}{2}x^2 + \lambda_0 x^4 \tag{5.8}$$

gives the Schrödinger equation

$$H|\psi(x)\rangle = E|\psi(x)\rangle. \tag{5.9}$$

In the case of $\lambda_0 = 0$, the ground state wave function and the ground state energy can be written as

$$|\psi_0(x)\rangle = \left(\frac{\mu}{\pi}\right)^{1/4} e^{-\mu x^2/2} \tag{5.10}$$

$$E_0 = \frac{1}{2}\mu. \tag{5.11}$$

For $\lambda_0 > 0$, we introduce a simple trial vacuum wave fucion as

$$|\psi(x)\rangle = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}, \tag{5.12}$$

which gives the expectation value of the energy

$$E(\alpha) = \frac{1}{4}\alpha + \frac{\mu^2}{4\alpha} + \frac{3\lambda_0}{4\alpha^2}. \tag{5.13}$$

The parameter α satisfies the minimization condition for the energy, $\frac{dE(\alpha)}{d\alpha} = 0$, to give the vacuum energy

$$\alpha^2 = \mu^2 + \frac{6\lambda_0}{\alpha}, \tag{5.14}$$

which approaches μ^2 as $\lambda_0 \to 0$.

Now we consider the effective potential for this model. The effective potential $V(x_0)$ is defined as the minimum value of the energy in the set of all normalized state vectors in which x has the expectation value x_0:

$$V(x_0) = \min\{\langle x_0|H|x_0\rangle\}, \tag{5.15}$$

with

$$\begin{aligned}\langle x_0|x|x_0\rangle =& x_0, & (5.16)\\ \langle x_0|x_0\rangle =& 1. & (5.17)\end{aligned}$$

We extend the Gaussian trian wave function to evaluate the effective potential for our model as

$$|\alpha, x_0\rangle = \mathcal{N} e^{-\alpha(x-x_0)^2/2}, \tag{5.18}$$

where $\mathcal{N}$ is the normalization constant satisfying normalization condition (5.17). The expectation value depending on x_0 and α is evaluated as (see Appendix E):

$$\begin{aligned}V(\alpha, x_0) =& \langle \alpha, x_0|H|\alpha, x_0\rangle\\ =& \mathcal{N}^2 \int dx H e^{-\alpha(x-x_0)^2}\\ =& \frac{1}{4}\alpha + \frac{1}{4}\frac{\mu^2}{\alpha} + \frac{3\lambda_0}{4\alpha^2} + \frac{1}{2}\mu^2 x_0^2 + \lambda_0 x_0^4 + \frac{3\lambda_0}{\alpha} x_0^2. \end{aligned}\tag{5.19}$$

The parameters satisfy the minimum condition

$$\begin{aligned}\frac{dV}{d\alpha} =& \frac{1}{4} - \frac{\mu^2}{4\alpha^2} - \frac{6\lambda_0}{4\alpha^3} - \frac{3\lambda_0}{\alpha^2} x_0^2 = 0, & (5.20)\\ \frac{dV}{dx_0} =& x_0 \left(\mu^2 + 4\lambda_0 x_0^2 - \frac{6\lambda_0}{\alpha^2}\right) = 0, & (5.21)\end{aligned}$$

which lead to

$$x_0 = 0, \quad \alpha^2 = \mu^2 + 12\lambda_0 x_0^2 + \frac{6\lambda_0}{\alpha}. \tag{5.22}$$

Note that the second order derivative d^2V/dx_0^2 and $d^2V/d\alpha^2$ are positive values. We obtain the effective potential $V(x_0)$ by substituting the minimum conditions (5.22) in (5.19). The renormalized mass m and coupling λ are obtained from the effective potential,

$$m^2 = \left.\frac{d^2V}{dx_0^2}\right|_{\min}, \quad \lambda = \frac{1}{4!}\left.\frac{d^4V}{dx_0^4}\right|_{\min}. \tag{5.23}$$

The parameter α also is a function of x_0, so we calculate the derivative of α with respect to x_0 to evaluate these values:

$$\begin{aligned}2\alpha\frac{d\alpha}{dx_0} =& 24\lambda_0 x_0 - \frac{6\lambda_0}{\alpha^2}\frac{d\alpha}{dx_0}\\ \frac{d\alpha}{dx_0} =& \frac{12\lambda_0 x_0 \alpha^2}{\alpha^3 + 3\lambda_0}\end{aligned}\tag{5.24}$$

We evaluate the minimum of the second order derivative

$$\left.\frac{d^2V}{dx_0^2}\right|_{\min} = \left.\left(\mu^2 + 12\lambda_0 x_0^2 + \frac{6\lambda_0}{\alpha} - \frac{6\lambda_0 x_0}{\alpha^2}\frac{d\alpha}{dx_0}\right)\right|_{x_0=0} = \mu^2 + \frac{6\lambda_0}{\alpha} = m^2. \tag{5.25}$$

We further work out

$$\begin{aligned}\left.\frac{d^3V}{dx_0^3}\right|_{\min} &= \left.\left(24\lambda_0 x_0 - \frac{72\lambda_0^2}{\alpha^3+3\lambda_0}x_0 - \frac{144\lambda_0^2}{\alpha^3+3\lambda_0}x_0 + \frac{216\lambda_0^2\alpha^2x_0^2}{(\alpha^3+3\lambda_0)^2}\frac{d\alpha}{dx_0}\right)\right|_{x_0=0} \\ &=0. \end{aligned} \tag{5.26}$$

Hence the coupling is

$$\begin{aligned}\frac{1}{4!}\left.\frac{d^4V}{dx_0^4}\right|_{\min} &= \frac{1}{4!}\left(24\lambda_0 - \frac{72\lambda_0^2}{\alpha^3+3\lambda_0} - \frac{144\lambda_0^2}{\alpha^3+3\lambda_0}\right) \\ &=\lambda_0 - \frac{9\lambda_0}{\alpha^3+3\lambda_0} = \lambda_0\left(\frac{\alpha^3-6\lambda_0}{\alpha^3+3\lambda_0}\right) = \lambda. \end{aligned} \tag{5.27}$$

We would like to skip the disccution of the interpretation of the renormalized values m and λ because our interest is focused on the Gaussian trial function formulation. In the next section, we will discuss the application of the Gaussian trial function to the field theory and the spontaneous symmetry breaking of the system.

5.4 The Gaussian Functional Method for the O(4) Symmetric Linear Sigma Model

We attempt to apply the Gaussian trial function to the field theory. Our main interest is the treatment of the spontaneously symmetry breaking in the field theory. As an example, we consider the O(4) symmetric linear sigma model and verify the method includes the existence of NG boson as the two particle bound state [86, 89, 90].

5.4.1 The Gaussian Functional

The O(4) symmetric linear sigma model Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \boldsymbol{\phi})^2 - V(\boldsymbol{\phi}^2), \tag{5.28}$$

with the potential

$$V(\boldsymbol{\phi}^2) = -\frac{1}{2}\mu_0^2\boldsymbol{\phi}^2 + \frac{\lambda_0}{4}(\boldsymbol{\phi}^2)^2 \tag{5.29}$$

and a column vector

$$\boldsymbol{\phi} = (\phi_0, \phi_1, \phi_2, \phi_3) = (\sigma, \vec{\pi}). \tag{5.30}$$

This Lagrangian can be derived from the NJL model by the standard bosonization technique [42]. Hence we assume that the spontaneously chiral symmetry breaking is realized by the vacuum expectation value of the σ field as $\langle\sigma\rangle$ in the mean field approximation and the $\vec{\pi}$ field can be interpreted as NG boson. Actually, the chiral symmetry breaking parttern in two flavor SU(2) $\times$ SU(2) $\rightarrow$ SU(2) ($\dim \mathrm{SU}(N) = N^2 - 1$) generates 3 NG particles and our model

Lagrangian breaking pattern $\mathrm{O}(4) \to \mathrm{O}(3)$ ($\dim \mathrm{O}(N) = N(N-1)/2$) generates also 3 NG particles. The explicit chiral symmetry breaking term is introduced

$$\mathcal{L}_{SB} = -\mathcal{H}_{SB} = \varepsilon\sigma, \tag{5.31}$$

which generates the finite pion mass, as suggested by the NJL model .

The canonical conjugate momentum is defined by $\pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_i)} = \partial_0\phi_i$ ($i = 0, 1, 2, 3$) and the Hamiltonian density is obtained by Legendre transformation as

$$\mathcal{H} = \int d\vec{y}\delta(\vec{y}-\vec{x})\sum_i \left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi_i(\vec{x})\phi_i(\vec{y})} + \frac{1}{2}\nabla_x\phi_i(\vec{x})\nabla_y\phi_i(\vec{y}) + V(\boldsymbol{\phi}^2) - \mathcal{H}_{SB}\right), \tag{5.32}$$

where

$$V\left(\boldsymbol{\phi}^2\right) = -\frac{1}{2}\mu_0^2\phi_i(\vec{x})\phi_i(\vec{y}) + \frac{\lambda_0}{4}\left(\phi_i(\vec{x})\phi_i(\vec{y})\right)^2. \tag{5.33}$$

We introduce the Gaussian ground state functional Ansatz with the vacuum expectation values $\langle\phi_i\rangle$

$$\Psi_0[\boldsymbol{\phi}] = \mathcal{N}\exp\left(-\frac{1}{4\hbar}\int d\vec{x}d\vec{y}[\phi_i(\vec{x}) - \langle\phi_i(\vec{x})\rangle]G_{ij}^{-1}(\vec{x},\vec{y})[\phi_j(\vec{y}) - \langle\phi_j(\vec{y})\rangle]\right), \tag{5.34}$$

where $\mathcal{N}$ is the normalization constant. While the vacuum expectation value for pion will be chosen as the vanishing value, which is treated as finite in the evalutation of the ground state energy expectation value. The form G_{ij}^{-1} have been defined as

$$G_{ij}(\vec{x},\vec{y}) = \frac{1}{2}\delta_{ij}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2 + m_i^2}}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{5.35}$$

where m_i is called the "dressed mass" which is determined self-consistently [83, 90]. Since the inverse satisfies

$$\int d\vec{z}G_{ij}(\vec{x},\vec{z})G_{ij}^{-1}(\vec{z},\vec{y}) = \delta(\vec{x}-\vec{y})\delta_{ij}, \tag{5.36}$$

it can be written as

$$G_{ij}^{-1}(\vec{x},\vec{y}) = 2\delta_{ij}\int \frac{d\vec{k}}{(2\pi)^3}\sqrt{\vec{k}^2 + m_i^2}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}. \tag{5.37}$$

We have explicitly kept $\hbar$ to keep track of quantum corrections and count the number of loops in our calculation. The quantum corrections may come from the functional integrals in the calculation of the energy density. We work out the energy (density) expectation value $\langle\Phi_0|\mathcal{H}|\Phi_0\rangle$

therm by term below (see Appendix E). The kinetic term:

$$
\begin{aligned}
(kinetic) =& -\frac{\hbar^2}{2}\langle\Psi_0|\frac{\delta^2}{\delta\phi_i(\vec{x})\phi_i(\vec{y})}|\Psi_0\rangle \\
=& -\frac{\hbar^2}{2}\langle\Psi_0|\int d\vec{y}\delta(\vec{x}-\vec{y})\sum_i\int d\vec{x}_1[-\frac{1}{2\hbar}\delta(\vec{x}_1-\vec{x})G^{-1}_{ii}(\vec{x}_1,\vec{y})]|\Psi_0\rangle \\
& -\frac{\hbar^2}{2}\langle\Psi_0|\int d\vec{y}\delta(\vec{x}-\vec{y})\sum_i\left[\int d\vec{x}_1\left(-\frac{1}{2\hbar}[\phi_i(\vec{x}_1)-\langle\phi_i(\vec{x})\rangle]G^{-1}_{ii}(\vec{x}_1,\vec{y})\right)\right] \\
& \qquad\qquad \times\left[\int d\vec{x}_2\left(-\frac{1}{2\hbar}[\phi_i(\vec{x}_2)-\langle\phi_i(\vec{x}_2)\rangle]G^{-1}(\vec{x},\vec{x}_2)\right)\right]|\Psi_0\rangle \\
=&\frac{\hbar}{4}\sum_i G^{-1}_{ii}(\vec{x},\vec{y})-\frac{\hbar}{8}\langle\Psi_0|\int d\vec{y}\delta(\vec{x}-\vec{y})\sum_i\int d\vec{x}_1d\vec{x}_2G^{-1}_{ii}(\vec{x}_1,\vec{y})G^{-1}(\vec{x},\vec{x}_2) \\
& \times[\phi_i(\vec{x}_1)\phi_i(\vec{x}_2)-\langle\phi_i(\vec{x}_1)\rangle\phi_i(\vec{x}_2)-\phi_i(\vec{x}_2)\langle\phi_i(\vec{x}_1)\rangle+\langle\phi_i(\vec{x}_1)\rangle\langle\phi_i(\vec{x}_2)\rangle]|\Psi_0\rangle \\
=&\frac{\hbar}{4}\sum_i G^{-1}_{ii}(\vec{x},\vec{x})-\frac{\hbar}{8}\int d\vec{y}d\vec{x}_1d\vec{x}_2\sum_i\delta(\vec{x}-\vec{y})G^{-1}_{ii}(\vec{x}_1,\vec{y})G^{-1}_{ii}(\vec{x},\vec{x}_2)G_{ii}(\vec{x}_1,\vec{x}_2) \\
=&\frac{\hbar}{8}\sum_i G^{-1}_{ii}(\vec{x},\vec{x})=\frac{\hbar}{4}\sum_i\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}.
\end{aligned} \tag{5.38}
$$

The shift term:

$$
\begin{aligned}
(shift) =&\frac{1}{2}\langle\Psi_0|\int d\vec{y}\sum_i\delta(\vec{x}-\vec{y})\nabla_x\phi_i(\vec{x})\nabla_y\phi_i(\vec{y})|\Psi_0\rangle \\
=& -\frac{1}{2}\sum_i\nabla_x^2\left(\langle\phi_i\rangle^2+\hbar\,G_{ii}(\vec{x},\vec{y})|_{\vec{x}=\vec{y}}\right) \\
=& -\frac{\hbar}{4}\sum_i\int\frac{d\vec{k}}{(2\pi)^3}\frac{-\vec{k}^2}{\sqrt{\vec{k}^2+m_i^2}}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\Bigg|_{\vec{x}=\vec{y}} \\
=&\frac{\hbar}{4}\sum_i\int\frac{d\vec{k}}{(2\pi)^3}\frac{\vec{k}^2+m_i^2-m_i^2}{\sqrt{\vec{k}^2+m_i^2}} \\
=&\frac{\hbar}{4}\sum_i\int\frac{d\vec{k}}{(2\pi)^3}\sqrt{\vec{k}^2+m_i^2}-\frac{\hbar}{4}\sum_i m_i^2\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}},
\end{aligned} \tag{5.39}
$$

where the vacuum expectation value have been assumed to be translationally invariant $\langle\phi_i(\vec{x})\rangle = \langle\phi_i(0)\rangle = \langle\phi_i\rangle$. The second order term in the potential:

$$
\begin{aligned}
(second) =& -\frac{1}{2}\mu_0^2\langle\Psi_0|\int d\vec{y}\delta(\vec{x}-\vec{y})\sum_i\phi_i(\vec{x})\phi_i(\vec{y})|\Psi_0\rangle \\
=& -\frac{1}{2}\mu_0^2\sum_i\left(\langle\phi_i\rangle^2+\hbar G_{ii}(\vec{x},\vec{x})\right) \\
=& -\frac{1}{2}\mu_0^2\sum_i\langle\phi_i\rangle^2-\frac{\hbar}{2}\mu_0^2\sum_i\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}.
\end{aligned} \tag{5.40}
$$

We write the fourth order of the fields as

$$(\boldsymbol{\phi}^2)^2 = (\sigma^2+\vec{\pi}^2)^2 = \sigma^4+\vec{\pi}^4+2\sigma^2\vec{\pi}^2 = \sum_i \phi_i^4 + 2\phi_i^2\phi_j^2. \tag{5.41}$$

The fourth order term in the potential:

$$\begin{aligned}
(fourth) =&\frac{\lambda_0}{4}\langle\Psi_0|\int d\vec{y}\delta(\vec{x}-\vec{y})\sum_i \phi_i^4+2\phi_i^2\phi_j^2|\Psi_0\rangle\\
=&\frac{\lambda_0}{4}\sum_i\big(3(\hbar G_{ii}(\vec{x},\vec{x}))^2+6\hbar\langle\phi_i\rangle^2G_{ii}(\vec{x})+\langle\phi_i\rangle^4\\
&+2(\hbar G_{ii}(\vec{x},\vec{x})+\langle\phi_i\rangle^2)(\hbar G_{jj}(\vec{x},\vec{x})+\langle\phi_j\rangle^2)\big)\\
=&\frac{\lambda_0}{4}(\langle\boldsymbol{\phi}\rangle^2)^2+\frac{3\hbar^2}{4}\lambda_0\sum_i\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}\right)^2\\
&+\frac{3\hbar}{2}\lambda_0\sum_i\langle\phi_i\rangle^2\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}\right)\\
&+\frac{\lambda_0}{2}\hbar^2\sum_{i<j}\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}\right)\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_j^2}}\right)\\
&+\frac{\lambda_0}{2}\hbar\sum_{i\neq j}\langle\phi_i\rangle^2\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_j^2}}\right),
\end{aligned} \tag{5.42}$$

where we have written as

$$\begin{aligned}
2G_{ii}(\vec{x},\vec{x})G_{jj}(\vec{x},\vec{x}) =&2\sum_{i<j}\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}}\right)\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_j^2}}\right),\\
2[\langle\phi_i\rangle^2G_{jj}(\vec{x},\vec{x})+\langle\phi_j\rangle^2G_{ii}(\vec{x},\vec{x})] =& 2\sum_{i\neq j}\langle\phi_i\rangle^2\left(\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_j^2}}\right),
\end{aligned} \tag{5.43}$$

which are understood by the representation of $(\boldsymbol{\phi}^2)^2$. The explicit symmetry breaking term:

$$-\langle\Psi_0|\mathcal{H}_{SB}|\Psi_0\rangle = -\varepsilon\langle\phi_0\rangle. \tag{5.44}$$

We obtain the energy density as

$$\begin{aligned}
\mathcal{E}(m_i,\langle\phi_i\rangle) =& -\varepsilon\langle\phi_0\rangle-\frac{1}{2}\mu_0^2\langle\boldsymbol{\phi}\rangle^2+\frac{\lambda_0}{4}\left[\langle\boldsymbol{\phi}\rangle^2\right]^2\\
&+\sum_i\hbar I_1(m_i)-\frac{1}{2}\mu_0^2\hbar\sum_i I_0(m_i)-\frac{1}{2}\hbar\sum_i m_i^2I_0(m_i)\\
&+\frac{3\lambda_0}{2}\hbar\sum_i\langle\phi_i\rangle^2I_0(m_i)+\frac{\lambda_0}{2}\hbar\sum_{i\neq j}\langle\phi_i\rangle^2I_0(m_j)\\
&+\frac{3\lambda_0}{4}\hbar^2\sum_i I_0^2(m_i)+\frac{\lambda_0}{2}\hbar^2\sum_{i<j}I_0(m_i)I_0(m_j),
\end{aligned} \tag{5.45}$$

where we have defined

$$I_0(m_i) = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{\vec{k}^2+m_i^2}} = G_{ii}(\vec{x},\vec{x}) = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m_i^2+i\epsilon}, \tag{5.46}$$

$$I_1(m_i) = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\sqrt{\vec{k}^2+m_i^2} = \frac{1}{4}G_{ii}^{-1}(\vec{x},\vec{x}) = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2-m_i^2+i\epsilon). \tag{5.47}$$

We identigy $\hbar I_1(m_i)$ with the familiar "zero-point" energy of a free scalar filed of mass m_i. The coupling λ_0 dependence terms are non-perturbative correction terms including one-loop $\mathcal{O}(\hbar)$ and two-loop $\mathcal{O}(\hbar^2)$.

5.4.2 The Gap Equations

The fields are difined as $\phi(\vec{x}) = (\sigma(\vec{x}), \vec{\pi}(\vec{x}))$, and hence the variational parameters are rewritten as $\langle\phi_i\rangle = (\langle\sigma\rangle, \langle\vec{\pi}\rangle)$ and $m_i = (M, \mu)$ to fit the field representation, where M and μ are the masses of σ and $\vec{\pi}$, respectively. We also rewrite the energy density in this notation as

$$\begin{aligned}\mathcal{E}(M,\mu,\langle\sigma\rangle,\langle\vec{\pi}\rangle) = &-\varepsilon\langle\sigma\rangle - \frac{1}{2}\mu_0^2[\langle\sigma\rangle^2+\langle\vec{\pi}\rangle^2] + \frac{\lambda_0}{4}[\langle\sigma\rangle^2+\langle\vec{\pi}\rangle^2]^2 + \hbar I_1(M) + 3\hbar I_1(\mu)\\ &- \frac{1}{2}\hbar[\mu_0^2+M^2]I_0(M) - \frac{3}{2}\hbar[\mu_0^2+\mu^2]I_0(\mu) + \frac{3}{2}\hbar\lambda_0\langle\sigma\rangle^2[I_0(M)+I_0(\mu)]\\ &+ \frac{1}{2}\hbar\lambda_0\langle\vec{\pi}\rangle^2[I_0(M)+5I_0(\mu)] + \frac{3}{4}\hbar^2\lambda_0 I_0^2(M) + \frac{15}{4}\hbar^2\lambda_0 I_0^2(\mu) + \frac{3}{2}\hbar^2\lambda_0 I_0(M)I_0(\mu).\end{aligned} \tag{5.48}$$

This value satisfies the minimum condition

$$\frac{\partial\mathcal{E}}{\partial\langle\phi_i\rangle} = 0, \quad \frac{\partial\mathcal{E}}{\partial m_i} = 0, \tag{5.49}$$

with $i = 0, 1, 2, 3$. The derivatives with respect to $\langle\phi_i\rangle$ are

$$\frac{\partial\mathcal{E}}{\partial\langle\sigma\rangle} = -\varepsilon + \langle\sigma\rangle\left[-\mu_0^2+\lambda_0\langle\sigma\rangle^2+\lambda_0\langle\vec{\pi}\rangle^2+3\hbar\lambda_0 I_0(M)+3\hbar\lambda_0 I_0(\mu)\right] = 0, \tag{5.50}$$

$$\frac{\partial\mathcal{E}}{\partial\langle\vec{\pi}\rangle} = \langle\vec{\pi}\rangle\left[-\mu_0^2+\lambda_0\langle\sigma\rangle^2+\lambda_0\langle\vec{\pi}\rangle^2+\hbar\lambda_0 I_0(M)+5\hbar\lambda_0 I_0(\mu)\right] = 0. \tag{5.51}$$

We choose $\langle\vec{\pi}\rangle = 0$ from the second equation and $\langle\sigma\rangle = v$. From the first equation, we obtain a relation

$$\mu_0^2 = -\varepsilon/v + \lambda_0 v^2 + 3\hbar\lambda_0 I_0(M) + 3\hbar\lambda_0 I_0(\mu). \tag{5.52}$$

The divergent integrals I_0 and I_1 are functions of M and μ. On the other hand, the derivative of $I_1(m_i)$ with respect to m_i can be represented as

$$\frac{\partial I_1(m_i)}{\partial m_i} = m_i I_0(m_i). \tag{5.53}$$

The derivative with respect to M is

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial M} =& \hbar M I_0(M) - \frac{1}{2}\hbar\mu_0^2\frac{\partial I_0(M)}{\partial M} - \hbar M I_0(M) - \frac{1}{2}\hbar M^2\frac{\partial I_0(M)}{\partial M} + \frac{3}{2}\hbar\lambda_0 v^2\frac{\partial I_0(M)}{\partial M}\\ &+ \frac{3}{2}\hbar^2\lambda_0 I_0(M)\frac{\partial I_0(M)}{\partial M} + \frac{3}{2}\hbar^2\lambda_0 M I_0(\mu)\frac{\partial I_0(M)}{\partial M} = 0,\end{aligned} \tag{5.54}$$

and hence the sigma mass is

$$M^2 = -\mu_0^2 + 3\lambda_0 v^2 + 3\hbar\lambda_0 I_0(M) + 3\hbar\lambda_0 I_0(\mu). \tag{5.55}$$

The derivative with μ is

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial \mu} =& 3\hbar\mu I_0(\mu) - \frac{3}{2}\hbar\mu_0^2\frac{\partial I_0(\mu)}{\partial \mu} - 3\hbar\mu I_0(\mu) - \frac{3}{2}\hbar\mu^2\frac{\partial I_0(\mu)}{\partial \mu} + \frac{3}{2}\hbar\lambda_0 v^2\frac{\partial I_0(\mu)}{\partial \mu}\\ &+ \frac{15}{2}\hbar^2\lambda_0 I_0(\mu)\frac{\partial I_0(\mu)}{\partial \mu} + \frac{3}{2}\hbar^2\lambda_0 I_0(M)\frac{\partial I_0(\mu)}{\partial \mu} = 0,\end{aligned} \tag{5.56}$$

and hence the pion mass is

$$\mu^2 = -\mu_0^2 + \lambda_0 v^2 + \hbar\lambda_0 I_0(M) + 5\hbar\lambda_0 I_0(\mu). \tag{5.57}$$

Inserting Eq. (5.52), we obtain two coupled equation

$$M^2 = \varepsilon/v + 2\lambda_0 v^2, \tag{5.58}$$
$$\mu^2 = \varepsilon/v + 2\hbar\lambda_0 I_0(\mu) - 2\hbar\lambda_0 I_0(M). \tag{5.59}$$

We evaluate the "dressed masses" M and μ at this stage. From the discussion of the chiral symmetry, we find $\varepsilon = f_\pi m_\pi^2$ and $v \sim f_\pi$. Hence the coupling constant is identfied with $2\lambda_0 = (M^2 - m_\pi^2)/f_\pi^2$, the gap equation can be written as

$$f_\pi^2 = \left(\frac{M^2 - m_\pi^2}{\mu^2 - m_\pi^2}\right)\hbar\left[I_0(\mu) - I_0(M)\right]. \tag{5.60}$$

We fix the parameters $f_\pi = 93$ [MeV] and $m_\pi = 140$ [MeV]. The integrals I_0 and I_1 are understood to be regularized via a UV momentu cut-off Λ. This new parameter Λ, however, is a free parameter as well as we have seen in the previous chapter. One choice of the value is the NJL model cut-off because the sigma model can be derived from the NJL model with the bosonization technique [42] as we derived in two color case. We investigate the behavior of the gap equation (5.60) using the several cut-off as shown in Fig. 5.1. It is known that the pion mass $m_\pi = 140$ [MeV] and the "sigma mass" $m_\sigma \sim 600$ [MeV] in the NJL model result. Note that the "sigma meson mass" is not well-defined since its decay width is too wide. We adopt the result of the NJL model calculation for the sigma meson mass. According to Fig. 5.1, if we identify $\mu = 140$ [MeV] with the pion mass, $M = 0$ [MeV] cannot be identified with sigma mass. On the other hand, if we identify $M \sim 600$ [MeV] with the sigma mass, μ cannot be identified with pion mass even if we choose any cut-off.

In the chiral limit ($\varepsilon = 0$), the pion mass should vanish due to the Nambu-Goldstone (NG) theorem. However, the two mass equations (5.55) and (5.57) admit only massive solutions $M > \mu > 0$ for positive values of λ_0 and μ_0^2 and any real ultraviolet cut-off Λ in the momentum integrals $I_0(m_i)$ and $I_1(m_i)$ as shown in Fig. 5.2. The pion mass equation (5.57) is massless in the tree level $\mathcal{O}(\hbar^0)$, but the non-perturbative one-loop corrections $\mathcal{O}(\hbar)$ give it a finite mass. The pion (ϕ_1, ϕ_2, ϕ_3) excitations $\hbar I_1(\mu)$ in the energy density (5.45) are massive, with $\mu \neq 0$ in the mean field approximation even in the chiral limit. While the Gaussian functional approach can contain the information of non-perturbative interaction term, it apparently does not satisfy the NG theorem [91].

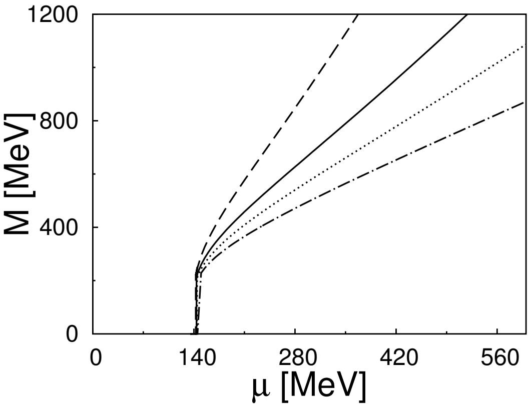


Figure 5.1: The behavior of the gap equation (5.60) using the several cut-off in (M, μ) plane. The cut-off Λ is chosen as 400 [MeV] (dashed line), 657 [MeV] (solid line), 1.0 [GeV] (dotted line) and 2.0 [GeV] (dashed-dotted line). $\Lambda = 657$ [MeV] is the NJL cut-off used previous chapter.

5.4.3 The Bethe-Salpeter Equation

It is known from the quantum many body literature that the mean field approximation (MFA) does not respect internal symmetries. Actually, Nambu's proof of the existence of massless particle is realized by considering the collective mode [1]. The NG particles apper as poles in the two-particle propagator, hence they are considered as a bound states of the two massive elementary excitations [86]. In the quantum many-body thoery language, it corresponds to considering the random pahse approximation (RPA). The MFA is one of framework to define the single particle state in the many-body system. Implementing the RPA into MFA state, it corresponds to consider the scattering sate. The state can be obtained by solving the Bethe-Salpeter equation (or equivalently, four-point Green function Schwinger-Dyson equation).

The $\sigma - \pi$ scattering

We specify the two body dynamics in terms of the Bethe-Salpeter (BS) equation, or equivalently four-point Schwinger-Dyson equation. We focus on the s-channel part of the total scattering

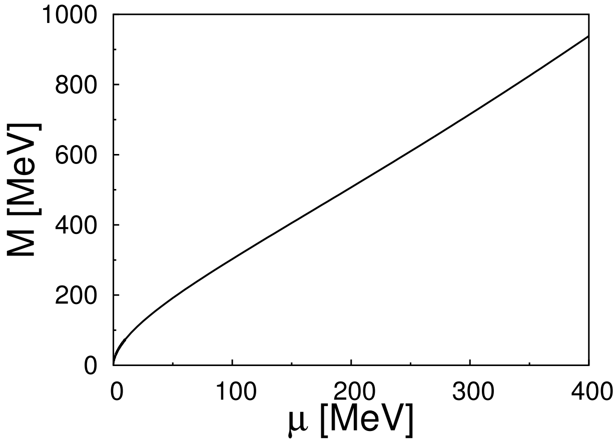


Figure 5.2: The solution to the gap equation in the chiral limit ($\varepsilon = 0$) using the NJL cut-off $\Lambda = 657$ [MeV] in (M, μ) plane. Obviously, it is not satisfy the NG therorem: $M > \mu > 0$.

amplitude. The Feynmann diagram Fig. 5.3 gives the BS equation as

$$D_\pi(s) = V_\pi(s) + V_\pi(s)\Pi_\pi(s)D_\pi(s), \tag{5.61}$$

$$\Pi_\pi(s) = i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M^2 + i\epsilon][(k-P)^2 - \mu^2 + i\epsilon]} = I_{M\mu}(s), \tag{5.62}$$

$$V_\pi(s) = 2\lambda_0 \left[1 + \left(\frac{2\lambda_0 v^2}{s - \mu^2}\right)\right] = 2\lambda_0 \left[1 + \frac{M^2 - \varepsilon/v}{s - \mu^2}\right], \tag{5.63}$$

where $s = (p_1 + p_2)^2 \equiv P^2$ is the center-of-mass (CM) energy. The solution is written as

$$D_\pi(s) = \frac{V_\pi(s)}{1 - V_\pi(s)\Pi_\pi(s)}. \tag{5.64}$$

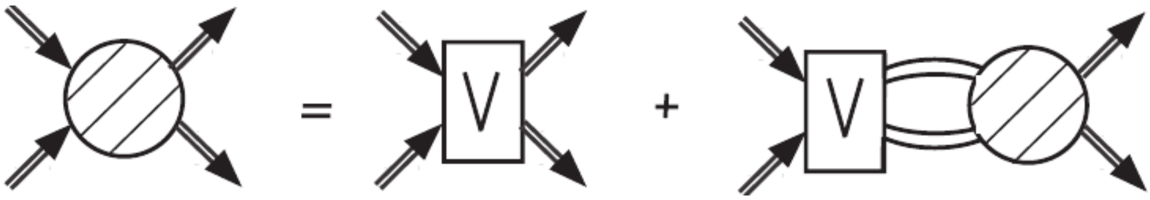
We consider the system of zero CM energy $\sqrt{s} = 0$ for the study of pion property. The polarization function $V_\pi(0)\Pi_\pi(0)$ is calculated using Eqs. (5.58) and (5.59) as

$$\begin{aligned} V_\pi(0)\Pi_\pi(0) =& \frac{2\lambda_0\hbar}{M^2 - \mu^2}[I_0(M) - I_0(\mu)] \left[1 - \frac{M^2 - \varepsilon/v}{\mu^2}\right] \\ =& \frac{\varepsilon/v - \mu^2}{M^2 - \mu^2} \left[1 - \frac{M^2 - \varepsilon/v}{\mu^2}\right] \\ =& 1 + \frac{\varepsilon/v - M^2}{M^2 - \mu^2} - \frac{\varepsilon/v(M^2 - \varepsilon/v)}{\mu^2(M^2 - \mu^2)} - \frac{\varepsilon/v - M^2}{M^2 - \mu^2} \\ =& 1 - \frac{\varepsilon}{v}\frac{M^2}{\mu^2(M^2 - \mu^2)} + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.65}$$

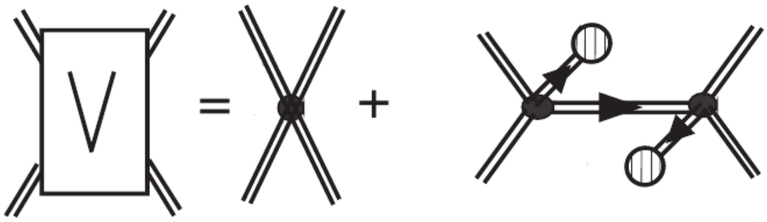
where we have used the integral I_0 defined in Eq. (5.46) in the four-dimensonal integral from

$$I_0(m_i) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_i^2 + i\epsilon}. \tag{5.66}$$

Thus, we verify the existence of the massless particle in the chiral limit and certainly the explicit chiral symmetry breaking term $\mathcal{H}_{SB} = -\varepsilon\sigma$ might give the finite mass for the pion.



(a) The square box represents the potential and the round blob is the BS amplitude itself. The double solid line denotes the dreesed meson.



(b) The potential entering the BS equation. The shaded blob together with the bouble line leading to it (the tadpole) denotes the vacuum expectation value of the field and the solid dot in the intersection of the four lines denotes the bare four-point coupling.

Figure 5.3: The diagramatic representation of the Bethe-Salpeter equation referred from Ref. [89].

The $\pi - \pi$ scattering

We found the existence of the massless particle as a pole of the BS equation in $\sigma - \pi$ channel. On the other hand the sigma mass would appear as a pole in $\pi - \pi$ channel. This channel has two distinct intermediate states, one with two "elementary" sigma field ϕ_0 and the other with two "elementary" pion fields $\phi_1,\ \phi_2,\ \phi_3$. The SD equations couple these two channels:

$$\begin{aligned}
D_{MM}(s) =& V_{MM}(s) + \frac{1}{2}V_{MM}(s)I_{MM}(s)D_{MM}(s) + \frac{3}{2}V_{M\mu}(s)I_{\mu\mu}(s)D_{\mu M}(s), && (5.67)\\
D_{M\mu}(s) =& V_{M\mu}(s) + \frac{1}{2}V_{M\mu}(s)I_{\mu\mu}(s)D_{\mu\mu}(s) + \frac{1}{2}V_{MM}(s)I_{MM}(s)D_{M\mu}(s), && (5.68)\\
D_{\mu M}(s) =& V_{\mu M}(s) + \frac{1}{2}V_{\mu M}(s)I_{MM}(s)D_{MM}(s) + \frac{1}{2}V_{\mu\mu}(s)I_{\mu\mu}(s)D_{\mu M}(s), && (5.69)\\
D_{\mu\mu}(s) =& V_{\mu\mu}(s) + \frac{1}{2}V_{\mu\mu}(s)I_{\mu\mu}(s)D_{\mu\mu}(s) + \frac{3}{2}V_{\mu M}(s)I_{MM}(s)D_{M\mu}(s). && (5.70)
\end{aligned}$$

This can be cast into matrix form as

$$\boldsymbol{D}_\sigma = \boldsymbol{V} + \frac{1}{2}\boldsymbol{V}\Pi\boldsymbol{D}_\sigma, \qquad (5.71)$$

where

$$\boldsymbol{D}_\sigma = \begin{pmatrix} D_{MM} & D_{M\mu} \\ D_{\mu M} & \frac{1}{3}D_{\mu\mu} \end{pmatrix}, \tag{5.72}$$

$$\begin{aligned} \boldsymbol{V} =& \begin{pmatrix} V_{MM} & V_{M\mu} \\ V_{\mu M} & \frac{1}{3}V_{\mu\mu} \end{pmatrix} \\ =& 2\lambda_0 \begin{pmatrix} 3\left[1+3\frac{M^2-\varepsilon/v}{s-M^2}\right] & \left[1+3\frac{M^2-\varepsilon/v}{s-M^2}\right] \\ & \\ \left[1+3\frac{M^2-\varepsilon/v}{s-M^2}\right] & \left[5+3\frac{M^2-\varepsilon/v}{s-M^2}\right] \end{pmatrix}, \end{aligned} \tag{5.73}$$

and

$$\boldsymbol{\Pi} = \begin{pmatrix} I_{MM} & 0 \\ 0 & 3I_{\mu\mu} \end{pmatrix}. \tag{5.74}$$

The invariant function $I_{ii}(s)$ is given by

$$I_{ii}(s) = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-m_i^2+i\varepsilon][(k-P)^2-m_i^2+i\varepsilon]}, \tag{5.75}$$

where $s = P^2$. We define

$$\chi =1+3\frac{M^2-\varepsilon/v}{s-M^2}, \tag{5.76}$$

$$V_{M\mu}(s) =V_{\mu M}(s) = 2\lambda_0\chi, \tag{5.77}$$

$$V_{MM}(s) =2\lambda_0\cdot 3\chi, \tag{5.78}$$

$$V_{\mu\mu}(s) =2\lambda_0(4+\chi), \tag{5.79}$$

for simply. We find that the Eq. (5.67) and Eq. (5.69) are coupled and Eq. (5.70) and Eq. (5.68) are coupled. It turns out that the coupled equations are splited into two systems with two unknown. The solution can be written as

$$\boldsymbol{D}_\sigma = \left(1-\frac{1}{2}\boldsymbol{V\Pi}\right)^{-1}\boldsymbol{V}. \tag{5.80}$$

The discriminant of the Eq. (5.80) is

$$\begin{aligned}
\mathcal{D}(s) =& \det\left[1-\frac{1}{2}\boldsymbol{V}\boldsymbol{\Pi}\right] \\
=& \left(1-\frac{1}{2}V_{MM}(s)I_{MM}(s)\right)\left(1-\frac{1}{2}V_{\mu\mu}(s)I_{\mu\mu}(s)\right) \\
&- \left(\frac{1}{2}V_{\mu M}(s)I_{MM}(s)\right)\left(\frac{3}{2}V_{M\mu}(s)I_{\mu\mu}(s)\right) \\
=& 1-\frac{1}{2}(V_{MM}(s)I_{MM}(s)+V_{\mu\mu}(s)I_{\mu\mu}(s)) \\
&+ \frac{1}{4}V_{MM}(s)I_{MM}(s)V_{\mu\mu}(s)I_{\mu\mu}(s)-\frac{3}{4}V_{\mu M}(s)I_{MM}(s)V_{M\mu}(s)I_{\mu\mu}(s) \\
=& 1-\frac{1}{2}(V_{MM}(s)I_{MM}(s)+V_{\mu\mu}(s)I_{\mu\mu}(s)) \\
& \lambda_0^2\left[3\chi(4+\chi)-3\chi^2\right]I_{MM}(s)I_{\mu\mu}(s) \\
=& 1-\frac{1}{2}(V_{MM}(s)I_{MM}(s)+V_{\mu\mu}(s)I_{\mu\mu}(s))+6\lambda_0 I_{MM}(s)I_{\mu\mu}(s)V_{M\mu}(s),
\end{aligned} \tag{5.81}$$

which is requaired non-zero to have its inverse matrix. The inverse matrix is

$$\left(1-\frac{1}{2}\boldsymbol{V}\boldsymbol{\Pi}\right)^{-1}=\frac{1}{\mathcal{D}(s)}\begin{pmatrix}1-\frac{1}{2}V_{\mu\mu}(s)I_{\mu\mu}(s) & \frac{3}{2}V_{M\mu}(s)I_{\mu\mu}(s)\\ \frac{1}{2}V_{\mu M}(s)I_{MM}(s) & 1-\frac{1}{2}V_{MM}(s)I_{MM}(s)\end{pmatrix}. \tag{5.82}$$

Thus, the solution is represented with the discriminant $\mathcal{D}$

$$D_{MM}(s) = \frac{1}{\mathcal{D}(s)}(V_{MM}(s)-12\lambda_0 I_{\mu\mu}(s)V_{M\mu}(s)), \tag{5.83}$$

$$D_{\mu\mu}(s) = \frac{1}{\mathcal{D}(s)}(V_{\mu\mu}(s)-12\lambda_0 I_{MM}(s)V_{M\mu}(s)), \tag{5.84}$$

$$D_{\mu M}(s) = D_{M\mu} = \frac{1}{\mathcal{D}(s)}V_{M\mu} = \frac{1}{\mathcal{D}(s)}V_{M\mu}. \tag{5.85}$$

Elementary and composite states in the s-channel manifest themselves as poles in the $\boldsymbol{D}_\sigma(s)$ matrix, or equivalently as roots of

$$(s-M^2)\mathcal{D}(s)=0. \tag{5.86}$$

In the chiral limit ($\varepsilon=0$), we work out

$$(s-M^2)V_{MM} = 6\lambda_0[s+2M^2], \tag{5.87}$$

$$(s-M^2)V_{\mu\mu} = 2\lambda_0(5s-2M^2), \tag{5.88}$$

$$(s-M^2)V_{M\mu} = 2\lambda_0(s+2M^2). \tag{5.89}$$

The physical sigma mass can be obtained by substituting above relations into (5.86) and replaceing $s=m_\sigma^2$ (CM system) as

$$m_\sigma^2 = M^2\left(\frac{1+2\lambda_0[3I_{MM}(m_\sigma)-I_{\mu\mu}(m_\sigma)]-24\lambda_0^2 I_{MM}(m_\sigma)I_{\mu\mu}(m_\sigma)}{1-\lambda_0[3I_{MM}(m_\sigma)+5I_{\mu\mu}(m_\sigma)]+12\lambda_0^2 I_{MM}(m_\sigma)I_{\mu\mu}(m_\sigma)}\right). \tag{5.90}$$

We show the numerical plot Eq. (5.90) in Fig. 5.4 with the NJL cut-off. We can only identify the variational parameter M with the physical sigma meson mass m_σ at the beginning.

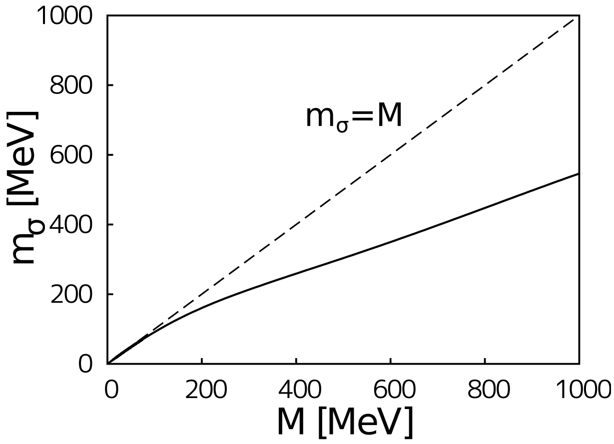


Figure 5.4: The solution m_σ to the Bethe-Salpeter equation (solid line) as a function of the variational parameter M with a cut-off $\Lambda = 657$ [MeV] in the chiral limit in comparision with $m_\sigma = M$ (dashed line).

5.5 The Gaussian Effective Action with the Optimized Expansion

We find the "masses" M and μ have to be considered only as variational parameters; they do not correspond to physical masses. The physical masses have to be determined as poles of the full propagator in the discussed approximation. The optimaized expansion (OE) method [87, 92] based on the effective action approach (see Appendix G) is one of other approach as inspired by the spontaneously symmetry breaking and the NG theorem [88]. An advantage of the OE is the physical mass can be found as poles of the full propagator in the discussed approximation. We would like to review the OE method and see the proof of NG theorem. We also see the effective potential defined from the effective action is equivalent to the Gaussian functional approach [93].

5.5.1 The Optimized Expansion

We discuss the scalar ϕ^4 theory. The classical action in (4-dimensional) Euclidean space is given as

$$S[\Phi] = \int d^4x \left[\frac{1}{2}\Phi(x)(-\partial^2 + m^2)\Phi(x) + \lambda\Phi^4(x)\right]. \tag{5.91}$$

In this section we write the four dimensional notation d^4x as dx for simply. The generating functional is defined by

$$Z[J] = \int \mathcal{D}\Phi \exp\left[-S[\Phi] + \int dx J(x)\Phi(x)\right], \tag{5.92}$$

and the effective action is

$$\Gamma[\phi] = W[J] - \int dx J(x)\phi(x) \tag{5.93}$$

with one-particle-irreducibe (1PI) Green function $W[J] = \ln Z[J]$. The background field

$$\begin{aligned}\phi(x) =& \frac{\delta W}{\delta J} \\ =& \int \mathcal{D}\Phi\Phi \exp\left[-S[\Phi] + \int dx J(x)\Phi(x)\right]\end{aligned} \tag{5.94}$$

is the vacuum expactation value of the scalor field. The effective action satisfies

$$\frac{\delta \Gamma}{\delta \phi(x)} = -J(x), \tag{5.95}$$

and is stationary in the physical theory when the sources are absent. When the background fields are constant, the true vacuum is given. The true vacuum expectation value (VEV) can be found as the stationary point of the effective potential as

$$V(\phi) = -\frac{\Gamma[\phi]|_{\phi=const.}}{\int dx}. \tag{5.96}$$

We write the classical action as

$$\begin{aligned}S_\epsilon[\Phi] =& S^{(0)} + \epsilon S^{(1)} \\ =& \frac{1}{2}\Phi(-\partial^2 + \Omega^2)\Phi + \epsilon\left[\frac{1}{2}\Phi(m^2 - \Omega^2)\Phi + \lambda\Phi^4\right],\end{aligned} \tag{5.97}$$

where the parameter ϵ has been introduced to identify the order of the perturbation and is set equal to one at the end. The coordinate index x and the integrations over it have been suppressed for notational simplicity. We choose ϕ_0 to satisfy the classical equation of motion for the modified action

$$\frac{\delta S_\epsilon}{\delta \phi_0} = -J. \tag{5.98}$$

The auxiliary field $\Omega(x)$ has been introduced in such a way that $Z[J]$ dose not depend on them. However, in the truncated series the dependence on $\Omega(x)$ apprears. We require the kth-order approximant of the physical quantity $W[J]$ to be as insensitive as possible to the small variation of Ω, by choosing Ω to satisfy

$$\frac{\delta W_k}{\delta \Omega} = 0 \tag{5.99}$$

and the effective action, as a Legendre transform, satisfies

$$\frac{\delta \Gamma_k}{\delta \Omega} = 0. \tag{5.100}$$

We expand S_ϵ around ϕ_0:

$$\begin{aligned}S_\epsilon[\Phi] =& S_\epsilon[\phi_0] + \left.\frac{\delta S_\epsilon}{\delta \Phi}\right|_{\phi_0}(\Phi - \phi_0) + \frac{1}{2}\left.\frac{\delta^2 S_\epsilon}{\delta \Phi^2}\right|_{\phi_0}(\Phi - \phi_0)^2 \\ &+ \frac{1}{6}\left.\frac{\delta^3 S_\epsilon}{\delta \Phi^3}\right|_{\phi_0}(\Phi - \phi_0)^3 + \frac{1}{24}\left.\frac{\delta^4 S_\epsilon}{\delta \Phi^4}\right|_{\phi_0}(\Phi - \phi_0)^4 + \cdots .\end{aligned} \tag{5.101}$$

The derivatives are written as

$$\begin{aligned}\left.\frac{\delta^2 S_\epsilon}{\delta\Phi^2}\right|_{\phi_0} &= -\,\partial^2 - \Omega^2 + \epsilon\left.\frac{\delta^2 S^{(1)}}{\delta\Phi^2}\right|_{\phi_0}, \\ \left.\frac{\delta^{3,4} S_\epsilon}{\delta\Phi^{3,4}}\right|_{\phi_0} &= \epsilon\left.\frac{\delta^{3,4} S^{(1)}}{\delta\Phi^{3,4}}\right|_{\phi_0}.\end{aligned} \tag{5.102}$$

Thus, the expansion becomes

$$\begin{aligned}S_\epsilon[\Phi] =& S_\epsilon[\phi_0] + J(\Phi-\phi_0) + \frac{1}{2}\left[-\partial^2 - \Omega^2 + \epsilon\left.\frac{\delta^2 S^{(1)}}{\delta\Phi^2}\right|_{\phi_0}\right](\Phi-\phi_0)^2 \\ &+ \frac{1}{6}\epsilon\left.\frac{\delta^3 S^{(1)}}{\delta\Phi^3}\right|_{\phi_0}(\Phi-\phi_0)^3 + \frac{1}{24}\epsilon\left.\frac{\delta^4 S^{(1)}}{\delta\Phi^4}\right|_{\phi_0}(\Phi-\phi_0)^4 + \cdots .\end{aligned} \tag{5.103}$$

Relabeling $\Phi - \phi_0 \to \Phi' \to \Phi$ then the partition function can be written as

$$\begin{aligned}Z[J] =& \exp\left(-\frac{1}{2}\phi_0(-\partial^2+\Omega^2)\phi_0 - \epsilon\left[\frac{1}{2}(m^2-\Omega^2)\phi_0^2 + \lambda\phi_0^4\right] + J\phi_0\right) \\ &\times\int\mathcal{D}\Phi\exp\left(\left(-\frac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi\right)\right) \\ &\quad\times\exp\left(-\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}\Phi^2 + \frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\phi_0^3}\Phi^3 + \frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}\Phi^4\right]\right),\end{aligned} \tag{5.104}$$

up to fourth order of the field Φ. The notation of the derivatives have been replaced such as $\left.\frac{\delta^2 S^{(1)}}{\delta\Phi^2}\right|_{\phi_0} \to \frac{\delta^2 S^{(1)}}{\delta\phi_0^2}$ for simply. The integration is evaluated as

$$\begin{aligned}&\int\mathcal{D}\Phi\exp\left(-\frac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi\right)\exp\left(-\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}\Phi^2 + \frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\phi_0^3}\Phi^3 + \frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}\Phi^4\right]\right) \\ &= \int\mathcal{D}\Phi\exp\left(-\frac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi\right) \\ &\quad\times\left(1-\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}\Phi^2 + \frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\phi_0^3}\Phi^3 + \frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}\Phi^4\right] + \mathcal{O}(\epsilon^2)\right) \\ &\sim \int\mathcal{D}\Phi e^{-\frac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi} \\ &\quad -\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}e^{-\frac{1}{2}(-\partial^2+\Omega^2)\Phi^2}\Phi^2 + \frac{1}{6}\frac{\delta^3 S^{(1)}}{\delta\phi_0^3}e^{-\frac{1}{2}(-\partial^2+\Omega^2)\Phi^2}\Phi^3 + \frac{1}{24}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}e^{-\frac{1}{2}(-\partial^2+\Omega^2)\Phi^2}\Phi^4\right] \\ &= \left(\det[-\partial^2+\Omega^2]\right)^{-1/2}\left(1-\epsilon\left[\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}(-\partial^2+\Omega^2)^{-1} + \frac{1}{8}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}(-\partial^2+\Omega^2)^{-2}\right]\right) \\ &= \exp\left[-\frac{1}{2}\mathrm{tr}\ln(-\partial^2+\Omega^2) + \ln\left[1-\epsilon\left(\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}(-\partial^2+\Omega^2)^{-1} + \frac{1}{8}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}(-\partial^2+\Omega^2)^{-2}\right)\right]\right] \\ &= \exp\left[-\frac{1}{2}\mathrm{tr}\ln(-\partial^2+\Omega^2) - \epsilon\left(\frac{1}{2}\frac{\delta^2 S^{(1)}}{\delta\phi_0^2}(-\partial^2+\Omega^2)^{-1} + \frac{1}{8}\frac{\delta^4 S^{(1)}}{\delta\phi_0^4}(-\partial^2+\Omega^2)^{-2}\right) + \mathcal{O}(\epsilon^2)\right],\end{aligned} \tag{5.105}$$

in the first order of ϵ. We calculate the derivatives:

$$\frac{\delta^2 S^{(1)}}{\delta\phi_0^2} = m^2 - \Omega^2 + 12\lambda\phi_0^2, \tag{5.106}$$

$$\frac{\delta^4 S^{(1)}}{\delta\phi_0^4} = 24\lambda. \tag{5.107}$$

After taking $\epsilon = 1$, the effective action is obtain as

$$\Gamma[\phi_0] = \int dx \left(-\frac{1}{2}\phi_0(x)(-\partial^2 + m^2)\phi_0(x) - \lambda\phi_0^4(x) \right) - \frac{1}{2}\mathrm{tr}\ln(-\partial^2 + \Omega^2) + \frac{1}{2}\int dx(\Omega^2 - m^2 - 12\lambda\phi_0^2(x))(-\partial^2 + \Omega^2)^{-1} - 3\lambda\int dx(-\partial^2 + \Omega^2)^{-2}, \tag{5.108}$$

with recovering the space arguments and the integrations over them and the trace is understood including the integration over the space.

If we limit ourselves to the constant Ω, the effective potential $V(\phi)$ (5.96) can be expressed as

$$V(\phi_0) = \frac{1}{2}m^2\phi_0^2 + \lambda\phi_0^4 + I_1(\Omega) - \frac{1}{2}\Omega^2 I_0(\Omega) + \frac{1}{2}m^2 I_0(\Omega) + 6\lambda\phi_0^2 I_0(\Omega) + 3\lambda I_0^2(\Omega), \tag{5.109}$$

with denoting the constant classical field $\phi_0(x) = \phi_0$. We have introduced the momentum integrals

$$I_1(\Omega) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ln(p^2 + \Omega^2), \tag{5.110}$$

$$I_0(\Omega) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 + \Omega^2}. \tag{5.111}$$

It is notable that the effective potential is similar to the energy density (5.45). The auxiliary field Ω can be found as a root of the euqation $\frac{\partial V(\phi)}{\partial\Omega} = 0$ as

$$\Omega^2 = m^2 + 12\lambda_0\phi_0^2 + 12\lambda_0 I_0(\Omega). \tag{5.112}$$

5.5.2 Nambu-Goldstone Boson

We would like to see the existence of the NG boson in the OE method. We consider N component scalar field $\mathbf{\Phi} = (\Phi_1, \cdots, \Phi_N)$ theory following the previous subseciton. The classical action in OE method is given by

$$S_\epsilon[\mathbf{\Phi}, G] = \int dxdy \frac{1}{2}\mathbf{\Phi}(x)G^{-1}(x,y)\mathbf{\Phi}(y) + \epsilon\left[\int dxdy\frac{1}{2}\mathbf{\Phi}(x)[(-\partial^2 + m^2)\delta(x-y) - G^{-1}(x,y)]\mathbf{\Phi}(y) + \int dx\lambda(\mathbf{\Phi}^2(x))^2\right], \tag{5.113}$$

with an arbitrary free propagator $G(x,y)$. The effective action, as a series in an artificial parameter ϵ, can be obtained as a sum of vacuum 1PI diagram with Feynman rules of the modified theory. The given order expression for the effective action is optimized, choosing $G(x, y)$ which fulfills the gap equation

$$\frac{\delta \Gamma_n}{\delta G^{-1}(x,y)} = 0, \tag{5.114}$$

to make the dependence on the unphysical field as weak as possible. The action has the O(N) symmetry and we assume the symmetry is sponteneously broken to O($N-1$) with $N-1$ NG boson. The trial propagators for massive field is difined as

$$G^{-1}(x,y) =(-\partial^2 + \Omega^2(x))\delta(x-y), \tag{5.115}$$

and for massless fields are

$$g^{-1}(x,y) =(-\partial^2 + \omega^2(x))\delta(x-y). \tag{5.116}$$

The effective action in the first order of the OE is obtained

$$\begin{aligned}\Gamma[\varphi] =& -\int dx \left[\frac{1}{2}\varphi(x)(-\partial^2+m^2)\varphi(x) + \lambda(\varphi^2(x))^2\right] - \frac{1}{2}\mathrm{tr}\ln G^{-1} - \frac{N-1}{2}\mathrm{tr}\ln g^{-1} \\ &+ \frac{1}{2}\int dx(\Omega^2(x) - m^2 - 12\lambda\varphi^2(x))G(x,x) + \frac{N-1}{2}\int dx(\omega^2(x) - m^2 - 4\lambda\varphi^2(x))g(x,x) \\ &- 3\lambda\int dx G^2(x,x) - (N^2-1)\lambda\int dx g^2(x,x) - 2(N-1)\lambda\int dx G(x,x)g(x,x),\end{aligned} \tag{5.117}$$

with the classcal field $\varphi(x)$ The effective action is required to satisfy the stability with respect to small changes of variational parameters

$$\frac{\delta\Gamma}{\delta\Omega^2} = \frac{\delta\Gamma}{\delta\omega^2} = 0, \tag{5.118}$$

which lead to the gap equations

$$\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 12\lambda G(x,x) - 4(N-1)\lambda g(x,x) =0, \tag{5.119}$$
$$\omega^2(x) - m^2 - 4\lambda\varphi^2(x) - 4\lambda G(x,x) - 4(N+1)\lambda g(x,x) =0, \tag{5.120}$$

which determine the functionals $\Omega[\varphi]$ and $\omega[\varphi]$.

Considering a constant background $\phi = (\phi_1, \cdots, \phi_N)$, the action gives the effective potential

$$\begin{aligned}V(\phi) =& \frac{1}{2}m^2\phi^2 + \lambda(\phi^2)^2 + I_1(\Omega) + (N-1)I_1(\omega) \\ & \frac{1}{2}(\Omega^2 - m^2 - 12\lambda\phi^2)I_0(\Omega) + \frac{N-1}{2}(\omega^2 - m^2 - 4\lambda\phi^2)I_0(\omega) \\ & - 3\lambda I_0^2(\Omega) - (N^2-1)\lambda I_0^2(\omega) - 2(N-1)\lambda I_0(\Omega)I_0(\omega),\end{aligned} \tag{5.121}$$

with the functions $\Omega(\phi)$ and $\omega(\phi)$ determined by

$$\Omega^2 - m^2 - 12\lambda\phi^2 - 12\lambda I_0(\Omega) - 4(N-1)\lambda I_0(\omega) =0, \tag{5.122}$$
$$\omega^2 - m^2 - 4\lambda\phi^2 - 4\lambda I_0(\Omega) - 4(N+1)\lambda I_0(\omega) =0, \tag{5.123}$$

where

$$I_1(\Omega) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ln(p^2+\Omega^2), \tag{5.124}$$

$$I_0(\Omega) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+\Omega^2}. \tag{5.125}$$

The effective potential agrees with the energy density in the Gaussian functional approach taking $N = 4$. In the OE approach, by generalising the Gaussian effective potential (GEP) to space-time dependent fields, the Gaussian effective action (GEA) can be obtained. It enables us to derive not only the effective potential, but also 1PI Green function with arbitrary external momenta in the Gaussian approximation.

The minimum of the GEP is at stationary point ϕ_0 fulfilling

$$\frac{\partial V}{\partial \phi_i} = (m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega))\phi_i = 0, \tag{5.126}$$

where the index i runs over 1 to N. In the unsymmetric minimum we have

$$B = m^2 + 4\lambda\phi^2 + 12\lambda I_0(\Omega) + 4(N-1)\lambda I_0(\omega) = 0. \tag{5.127}$$

The parameters Ω and ω are only variational parameters in the free propagator, and do not correspond to physcal masses of scalar particles. The physical masses have to be determined as poles of the full propagator in the discussed approximation. The inverse of that propagator can be obtained as a second derivative of the GEA. Performing the Fourier tranform, the two-point vertex is calculated as

$$\begin{aligned}\Gamma_i(p) &= \left.\frac{\delta^2\Gamma}{\delta\varphi_i^2}\right|_{\varphi(x)=\phi} \\ &= \left.\left[p^2 + B + 8\lambda\varphi_i^2 + 12\lambda\varphi_i\Omega I_{-1}(p,\Omega)\frac{\delta\Omega}{\delta\varphi_i} + 4(N-1)\lambda\varphi_i\omega I_{-1}(p,\omega)\frac{\delta\omega}{\delta\varphi_i}\right]\right|_{\varphi_i(x)=\phi},\end{aligned} \tag{5.128}$$

with

$$I_{-1}(p,\Omega) = 2\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+\Omega^2)((p+q)^2+\Omega^2)}. \tag{5.129}$$

The variationals with respect to Ω and ω are evaluated from (5.119) and (5.120):

$$\begin{aligned}2\Omega\frac{\delta\Omega}{\delta\varphi_i} - 24\lambda\varphi_i - 12\lambda\Omega I_{-1}(p,\Omega)\frac{\delta\Omega}{\delta\varphi_i} - 4(N-1)\lambda\omega I_{-1}(p,\omega)\frac{\delta\omega}{\delta\varphi_i} &= 0 \\ \frac{\delta\Omega}{\delta\varphi_i} = \frac{12\lambda\varphi_i + 2(N-1)\lambda\omega I_{-1}(p,\omega)\frac{\delta\omega}{\delta\varphi_i}}{\Omega(1-6\lambda I_{-1}(p,\Omega))}&,\end{aligned} \tag{5.130}$$

and

$$\begin{aligned}2\omega\frac{\delta\omega}{\delta\varphi_i} - 8\lambda\varphi_i - 4\lambda\Omega I_{-1}(p,\Omega)\frac{\delta\Omega}{\delta\varphi_i} - 4(N+1)\lambda\omega I_{-1}(p,\omega)\frac{\delta\omega}{\delta\varphi_i} &= 0 \\ \frac{\delta\omega}{\delta\varphi_i} = \frac{4\lambda\varphi_i + 2\lambda\Omega I_{-1}(p,\Omega)\frac{\delta\Omega}{\delta\varphi_i}}{\omega(1-2(N+1)\lambda I_{-1}(p,\omega))}&.\end{aligned} \tag{5.131}$$

These two coupled equation are solved as

$$
\begin{aligned}
\frac{\delta\Omega}{\delta\varphi_i} =& \frac{12\lambda\varphi_i - 16(N+2)\lambda^2\varphi_i I_{-1}(p,\omega)}{\Omega(1-6\lambda I_{-1}(p,\Omega) - 2(N+1)\lambda I_{-1}(p,\omega) + 8(N+2)\lambda^2 I_{-1}(p,\Omega)I_{-1}(p,\omega))}, & (5.132)\\
\frac{\delta\omega}{\delta\varphi_i} =& \frac{4\lambda\varphi_i}{\omega(1-6\lambda I_{-1}(p,\Omega) - 2(N+1)\lambda I_{-1}(p,\omega) + 8(N+2)\lambda^2 I_{-1}(p,\Omega)I_{-1}(p,\omega))}. & (5.133)
\end{aligned}
$$

Then the two-point vertex is given as

$$
\Gamma_i = p^2 + B + 8\lambda\phi^2 A, \tag{5.134}
$$

where

$$
A = 1 + \frac{18\lambda I_{-1}(p,\Omega) + 2(N-1)\lambda I_{-1}(p,\omega) - 24(N+2)\lambda^2 I_{-1}(p,\Omega)I_{-1}(p,\omega)}{1-6\lambda I_{-1}(p,\Omega) - 2(N+1)\lambda I_{-1}(p,\omega) + 8(N+2)\lambda^2 I_{-1}(p,\Omega)I_{-1}(p,\omega)}. \tag{5.135}
$$

In the asymmetric state $\phi_1 = \phi_0$ and $\phi_i = 0$ $(i = 2, \cdots, N)$ with $B = 0$, an inverse propagator $\Gamma_1(p) = p^2 + 8\lambda A(p)\phi_0^2$ corresponds to a massive particle and $N-1$ inverse propagator $\Gamma_i(p) = p^2$ of NG bosons. The mass for massive particle is given as $\Gamma_1(m) = 0$, which agrees with the sigma mass Eq. (5.90) in $N = 4$ case.

5.6 Discussions

We have found that the Gaussian functional or Gaussian effective action approaches fully respect to the NG theorem and correspond to MPA + RPA framework. We need the thermodynamical potential including the effect of non-perturbative interaction. The thermodynamical potintial is generated from the partition function with Euclidean action. In this sense, it may be better to accept the Gaussian effective action approach. The fields in the Euclidean action should be integrated out to generate the thermodynamical potential. Hence we desire a Gaussian form action or higher order terms can be treated perturbatively.

The physical mass and coupling constant are derived from two- and four-point vertex in the effective action. We have discussed the physical masses at unrenormalized level. Some of masses in effective action correspond to NG boson and others include the interaction effect. If the physical coupling constant would be small, the partition function for new classical fields (which are φ in previous section) up to second order can be integrated out and higher order terms are treated perturbatively. We expect that our mesons and diquark-baryons are replaced by the new classical fields and eventually reproduce the lattice QCD result.

Chapter 6

Summary

6.1 Summary

We have investigated two-color quark-hadron matter at finite temperature and chemical potential using a Nambu-Jona-Lasinio model supplemented by Polyakov loop dynamics. Ultimately all the quark-hadron matter properties will have to be derived from the QCD Lagrangian. As a first step, we treat the chiral properties of the strong interaction using the NJL model accompanied by Polyakov loop terms for confinement effects in the sense that color non-singlets are suppressed in the hadron phase. Our primary aim of the present study is to investigate the emergence and dynamics of baryonic degrees of freedom in addition to mesonic modes. Baryons are realized as diquarks in $N_c = 2$ QCD. Their role as bosons is fundamentally different from the $N_c = 3$ case in which baryons are fermions subject to the Pauli exclusion principle. Nonetheless, the $N_c = 2$ case is of conceptual interest because corresponding lattice simulations are not limited by the sign problem at non-zero, real chemical potential μ.

A full bosonization leads to an extended linear sigma model incorporating diquark degrees of freedom in the effective Lagrangian which establishes a connection between the hadron Lagrangian and an underlying quark dynamical approach including diquark degrees of freedom. Connecting the NJL approach to standard linear sigma model is not a new issue. However, we find it useful to draw such a connection incorporating in addition the diquark (baryon) sector in a consistent way. It turns out that even at this stage, the derived meson-diquark Lagrangian must be treated non-perturbatively in order to reproduce the two-color lattice result.

The thermodynamical properties have been discussed and compared with the lattice QCD calculation. The derived effective Lagrangian include the both of quark and hadron properties. The phase structure is examined in the mean field approximation approach. The behavior of chiral and diquark condensates and their intertwining at finite μ, reflecting the underlying Pauli-Gürsey symmetry, are discussed. The Polyakov loop plays an important role at finite temperature. The deconfinement transition is now correlated with the chiral condensate. The characteristic temperatures for the chiral (T_χ) and deconfinement (T_{dec}) cross-overs become about equal ($T_\chi \simeq T_{dec} \simeq 225$ [MeV]) once the quarks are coupled to the Polyakov loop. At finite chemical potential and low temperature, $T < T_{dec}$, the diquark condensate plays an important role. A non-zero diquark condensate emerges at $\mu = \frac{1}{2}m_\pi$, the onset of Bose-Einstein condensation in this model. As μ increases the diquark condensate grows quickly and stays finite at large chemical potential. With increasing diquark condensate, the chiral condensate decreases subject to the condition $\sigma_0^2 + |\Delta_0|^2 \simeq const.$

The meson and diquark masses behave naturally in accordance with the symmetry breaking pattern associated with chiral and diquark condensation. The pion mass first stays constant and then grows linearly with the chemical potential once diquark condensation starts. The sigma meson mass is about twice the dynamical quark mass until the onset of diquark condensation. Due to the strong coupling of the sigma with diquark-baryons, the mixed sigma-diquark modes behave quite differently from the case without coupling.

The equation-of-state (EOS) of the quark-hadron system has interesting properties as a function of increasing chemical potential. When μ approaches its critical value $\mu_c = m_\pi/2$ from below, the pressure divided by its Stefan-Boltzmann limit increase rapidly at low temperature. This reflects the fact that the energy of the lowest diquark mode drops as $m_\pi - 2\mu$ and hence contributes prominently to the pressure. It becomes a zero mode at $\mu = \mu_c$, the onset of diquark condensation, at which point the system develops a Bose-Einstein condensation (BEC) phase. As the chemical potential increases beyond $\mu \simeq m_\pi$, the dynamical mass of quark quasiparticles drops below μ and the system undergoes a BEC-BCS crossover. With μ further increasing, the quark quasiparticle energy develops a minimum at finite momentum p. Above the deconfinement transition, the EOS is governed by quark degrees of freedom, but the hadronic modes still have a significant influence up to the crossover temperature range.

The quark density ρ as function of the chemical potential is a quantity of interest that is readily accessible in two-color lattice QCD. It turns out that the mean field approximation of the (P)NJL model underestimates this density by about a factor of two. Corrections treated in Gaussian approximation do not change this result significantly. This is not surprising since the Gaussian approximation misses important correlations between diquarks and scalar bosons generated by the higher order terms ($\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$) of the hadronic effective Lagrangian. The net attractive interactions produced by strong couplings in $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$ are expected to raise the density considerably.

We have reviewed the Gaussian functional approach as a non-perturbative treatment. The Schrödinger quantization prescription is constructed in the field theory. The ground state energy including a non-perturbative interaction have been worked out in a simple model. The sponteneous symmetry breaking and the Nambu-Goldstone theorem is fully supported in the Gaussian functional approximation. This framework is expected to apply our effective hadron Lagrangian and reproduce the lattice QCD result.

6.2 Outlook

We have derived an effective hadron Lagrangian from two color NJL type Lagrangian and many thermodynamical properties especially the emergence and dynamics of baryonic degree of freedom have been investigated. We would like to see the outlook briefly.

6.2.1 Non-perturbative Treatment in Hadron Lagrangian

We have investigated the thermodynamical property in the Gaussian approximation in addition to the mean field approximation (MPA). The approximation gives the hadronic contribution in quark-hadron matter. We find that the baryonic contribution is important in medium. However, the Gaussian approximation misses the hadron-hadron correlation. We treat only the non-interactive hadron Lagrangian based on the MPA approach quark model. In this sense, it can be identified as the mean field approximation in the hadron level. To include the correlation between hadrons, we have to treat the non-perturbative interaction term, which corresponds the random phase approximation (RPA) in the many body theory language. The Gaussian functional approximation can be expected to correspond the RPA. The Nambu-Goldstone theorem is guaranteed in the Gaussian functional or Gaussian effective action approximation. Hence, the Gaussian functional or Gaussian effective action approximation can be expected to a non-perturbative treatment in our hadron Lagrangian. As we discussed in previous chapter, the new physical mass and coupling constant are described by the original mass and coupling constant which we derived from the NJL model by a full bosonization. We expect that a new effective action is described by these new physical mass and coupling constant and the interaction can be treated perturbatively.

6.2.2 Confinement Effect from the QCD Lagrangian

Ultimately all the quark-hadron matter properties will have to be derived from the QCD Lagrangian. To describe hadron, the quark confinement is a most important property of QCD as well as chiral symmetry. The NJL model is lack of confinement and the Polyakov loop terms play a role of confinement kinematics in the sense of suppressing a color non-singlet degree of freedom in hadron phase at finite temperature. Recently, remarkable studies have been developed. The first one is so-called non-local PNJL model [94, 95]. The four-fermion interaction is described by non-local distribution instead of the local delta function. The original local NJL model interaction is strong at quark momenta $|\vec{p}| < \Lambda$ and turned off at $|\vec{p}| > \Lambda$ in terms of the NJL cut-off Λ. The non-local interaction is expected to be closer to QCD and its running coupling strength. It implies that the gluon fields provide a Wilson line between the non-local fermionic bilinears. The second is a reformulation of QCD in terms of the Cho-Faddeev-Niemi-Shabanov (CFNS) decomposition of the Yang-Mills field [96, 97]. The non-local NJL type interaction is derived from the reformulation of QCD by CFNS separation. Hence, it is expected that a full treatment of the confinement effect based on the gluon fields and chiral symmetry from first principle. We will attempt to using the non-local PNJL model as our new starting Lagrangian and manipulate bosonization.

6.2.3 Construction of Real Baryons

Our interest is aimed at investigating the property of baryonic matter. In $N_c = 2$ QCD, baryons are realized as diquark and its role is fundamentally different form the real fermionic baryon in $N_c = 3$ case. We have successfully applied the bosonization technique for baryons in two color system. However, for the three color system we have not worked out the hadronisation program yet. We have to expand our bosonization (hadronisation) technique to include three body fermion system. We just mention several studies for describing baryons from quark level model based on auxiliary field method. Baryons as quark-diquark model picture is discussed in the auxiliary fields framework [73, 98–102]. However, the diquark field in this picture is treated as a fundamental field and has a finite size. Another study of baryon using auxiliary fields is discussed in strong coupling limit [103].

Appendix A

Notations and Conventions

A.1 System of Measurement

We accept the so-called ‘Natural unit’ as $c = \hbar = k_B = 1$, where c is the speed of light, $\hbar$ is the reduced Planck constant (or simply called Planck constant) and k_B is the Boltzmann constant. We note the comparison:

$$\begin{aligned} 1[\text{eV}] =& 1.16 \times 10^4[\text{K}] \quad &(A.1)\\ 1[\text{MeV}] =& 1.78 \times 10^{-30}[\text{kg}] \quad &(A.2)\\ 197[\text{MeV}] =& 1[\text{fm}^{-1}]. \quad &(A.3) \end{aligned}$$

A useful relation used above is $\hbar c = 197$ [MeV· fm].

A.2 Relativistic Notation

Greek indices μ, ν, etc. generally run over the four space-time coordinate labeling $0, 1, 2, 3$ and Latin indices i, j, k, etc. run over the three spatial coordinate labeling $1, 2, 3$. We define

$$x^\mu = (ct, \vec{x}), \quad x_\mu = (ct, -\vec{x}). \tag{A.4}$$

We adopt the summation convention (Einstein notation):

$$\sum_{\mu=0}^{3} a^\mu b_\mu = a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3. \tag{A.5}$$

The Minkowski space-time metric is $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In the Euclidean space, Greek indices run over $1, 2, 3, 4$ and $x^4 = ix^0$ and the metric is $g_{\mu\nu} = -\delta_{\mu\nu}$. Using the metric tensor we define

$$x^\mu = g^{\mu\nu} x_\nu. \tag{A.6}$$

Derivatives with respect to coordinates are written as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right). \tag{A.7}$$

The energy-momentum vector

$$p^\mu = (E/c, \vec{p}), \quad p_\mu = (E/c, -\vec{p}) \tag{A.8}$$

satisfies

$$p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = m^2. \tag{A.9}$$

We shall also use

$$px = p \cdot x = p^\mu x_\mu = Et - \vec{p} \cdot \vec{x}. \tag{A.10}$$

We sometimes write $ct = x_0 = x^0$ and $E/c = p_0 = p^0$.

A.3 The Pauli matrix

The Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.11}$$

have the property

$$\mathrm{tr}[\sigma^i] = 0, \quad \det[\sigma] = -1, \tag{A.12}$$
$$(\sigma^i)^2 = I, \tag{A.13}$$
$$[\sigma^i, \sigma^j] = 2\varepsilon_{ijk}\sigma_k, \tag{A.14}$$
$$\sigma_i\sigma_j = \delta_{ij} I + i\varepsilon_{ijk}\sigma_k. \tag{A.15}$$

We sometimes write

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \tag{A.16}$$

and

$$\sigma^\mu = (\sigma_0, \sigma_i), \quad \bar{\sigma}^\mu = (\sigma_0, -\sigma_i), \tag{A.17}$$
$$\sigma_\mu = (\sigma_0, -\sigma_i), \quad \bar{\sigma}_\mu = (\sigma_0, \sigma_i). \tag{A.18}$$

The symmetric matrices are σ_0, σ_1 and σ_3 and the antisymmetric one is σ_2. In the text, we write the Pauli matrices for two color space as t_i and for two flavor space as τ_i.

Appendix B

Dirac Algebra

B.1 The Gamma Matrix

We note some useful properties of Dirac algebra following some textbooks [104–107].

The Dirac gamma matrices satisfy the anti-commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{B.1}$$

and

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \tag{B.2}$$

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{B.3}$$

The notation

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{B.4}$$

satisfies

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = 1. \tag{B.5}$$

The gamma matrices are expressed in the chiral representation :

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{B.6}$$

with the 2×2 identity matrix I. It can be further written as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_\mu = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{B.7}$$

Products of gamma matrices:

$$\gamma_\mu\gamma^\mu = 4, \tag{B.8}$$

$$\gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha, \tag{B.9}$$

$$\gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu = 4g_{\alpha\beta}. \tag{B.10}$$

Trace formula of gamma matrices:

$$\begin{aligned}
\mathrm{tr}[1] =&4, && \text{(B.11)}\\
\mathrm{tr}[\gamma^\mu] =&0, && \text{(B.12)}\\
\mathrm{tr}[\gamma^\mu\gamma^\nu] =&4g^{\mu\nu}, && \text{(B.13)}\\
\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] =&4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}), && \text{(B.14)}\\
\mathrm{tr}[\gamma_5] =&0, && \text{(B.15)}\\
\mathrm{tr}[\gamma_5 \not{a}\not{b}] =&0, && \text{(B.16)}\\
\mathrm{tr}[\not{a}_1\not{a}_2\cdots\not{a}_n] =&0, \quad \text{for } n \text{ odd} && \text{(B.17)}
\end{aligned}$$

with the slash notation $a^\mu\gamma_\mu = \not{a}$.

B.2 Dirac Bilinear Form

The Lorentz spinor bilinear expressions like $\bar{\psi}\psi, \bar{\psi}\gamma^\mu\psi$, etc. are represented as $\bar{\psi}\Gamma^\alpha\psi$ with α denoting the Lorentz transformation property. The 16 independent 4×4 Dirac matrices Γ^α are

$$\Gamma^S = \mathbf{1}, \quad \Gamma^V_\mu = \gamma_\mu, \quad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu], \quad \Gamma^A_\mu = \gamma_\mu\gamma_5, \quad \Gamma^P = i\gamma_5,. \tag{B.18}$$

They satisfy $(\Gamma^\alpha)^2 = \Gamma^\alpha\Gamma_\alpha = \pm 1, \Gamma_\alpha = (\Gamma^\alpha)^{-1}$ and $\gamma^0(\Gamma^\alpha)^\dagger\gamma^0 = \Gamma^\alpha$, hence the forms $\bar{\psi}\Gamma^\alpha\psi$ are hermitian.

B.3 Charge Conjugation

The charge conjugation matrix interchanges particle and anti-particle:

$$\begin{aligned}
C = i\gamma_2\gamma_0 = -C^{-1} = -C^\dagger = -C^T&, && \text{(B.19)}\\
\psi^c =&C\bar{\psi}^T, && \text{(B.20)}\\
[C,\gamma_5] =&0, && \text{(B.21)}\\
C^{-1}\Gamma^\alpha C =&\eta_\alpha\Gamma^{\alpha T}, && \text{(B.22)}
\end{aligned}$$

with the values η_α are summarized in the following table:

α	S	V	T	A	P
η_α	1	-1	-1	1	1

B.4 Euclidean Dirac Matrices

The Euclidean time space coordinate, instead of the Minkowski space, is introduced as

$$x^\mu = (x_0, \vec{x}) \to x^\mu = x_\mu = (\vec{x}, x_4), \tag{B.23}$$

with $x_4 = ix_0$ and the index μ runs over 1 to 4.

In thermal field theory, a system is described in Euclidean space with minus sign. Since the Dirac field is applied to thermodynamics, we have to define the Euclidean Dirac matrices:

$$\gamma_{E\mu} = \gamma_E^\mu = (\vec{\gamma}, \gamma_4), \quad \text{with} \quad \gamma_4 = i\gamma_0. \tag{B.24}$$

The commutation relation in Euclidean space is defined as

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}. \tag{B.25}$$

The scalar product of two Euclidean four-vectors a_μ, b_ν is described as

$$a_\mu b^\nu = a_\mu b_\nu \delta_{\mu\nu} = \sum_{i=1}^{4} a_i b_i. \tag{B.26}$$

We introduce the slash notation

$$\not{a} = a_\mu \gamma_E^\mu = a_4\gamma_4 + \vec{a} \cdot \vec{\gamma}. \tag{B.27}$$

In particular, we use

$$\not{\partial} = \gamma_4 \partial_4 + \vec{\gamma} \cdot \nabla. \tag{B.28}$$

B.5 The Scalar Diquark

The Dirac bilinear form in the diquark form will be written as $\psi^T \Gamma \psi$. Our diquark should satisfy the Lorentz scalar, parity plus, flavor singlet and color singlet in the two flavor and two color space. The infinitesimal Lorentz transformation is given as

$$\psi^T \Gamma \psi \to \psi^{T'} \Gamma \psi' = \psi^T S^T \Gamma S \psi, \tag{B.29}$$

$$S = 1 - \frac{i}{4}\varepsilon_{\mu\nu}\sigma^{\mu\nu}. \tag{B.30}$$

Using the charge conjugation C

$$(\sigma^{\mu\nu})^T = -C^{-1}\sigma^{\mu\nu}C, \tag{B.31}$$

thus we obtain

$$S^T \Gamma S = \Gamma - \frac{i}{4}\Gamma\varepsilon_{\mu\nu}\sigma^{\mu\nu} + \frac{i}{4}C^{-1}\varepsilon_{\mu\nu}\sigma^{\mu\nu}C\Gamma. \tag{B.32}$$

The Lorentz scalar bilinear satisfies

$$\Gamma\varepsilon_{\mu\nu}\sigma^{\mu\nu} = C^{-1}\varepsilon_{\mu\nu}\sigma^{\mu\nu}C\Gamma, \tag{B.33}$$

which leads

$$[\sigma^{\mu\nu}, C\Gamma] = 0. \tag{B.34}$$

The γ_5 satisfies $[\sigma^{\mu\nu}, \gamma_5] = 0$, hence the Lorentz scalar matrix is $\Gamma = C^{-1}\gamma_5$ and the Lorentz scalar bilinear is $\psi^T C^{-1} i\gamma_5 \psi$. The parity transformation is:

$$\psi \to \psi' = \gamma_0 \psi, \tag{B.35}$$

$$\psi^T C^{-1} i\gamma_5 \to \psi^{T'} C^{-1} i\gamma_5 = \psi^T C^{-1} i\gamma_5 \gamma_0. \tag{B.36}$$

Hence, $\psi^T C^{-1} i\gamma_5 \psi$ is parity plus. From the condition of the flavor and color singlet, we introduce the anti-symmetric operators $(i\tau_2)_{ij} = \varepsilon_{ij}$ (flavor space) and $(it_2)_{ij} = \varepsilon_{ij}$ (color space). Finally, our desired diquark is obtained using Eq. (B.19) as

$$\psi^T C i\gamma_5 t_2 \tau_2 \psi. \tag{B.37}$$

Appendix C

Fierz Transformations

The NJL-type interaction is derived from a local coupling between color currents $J^a_\mu = \bar{\psi}\gamma_\mu t^a \psi$ involving the quark fields ψ and the generators for the color group t^a. Taking four spinor fields $\psi_{1,2,3,4}$. Its four-point interaction Lagrangian can be written as $G(\bar{\psi}_1\Gamma^\alpha\psi_2)(\bar{\psi}_3\Gamma_\alpha\psi_4)$ with the product of the Dirac bilinears using a specific Dirac matrix Γ^α and the coupling constant G. There exists the equivalent form $G'(\bar{\psi}_1\Gamma^\beta\psi_4)(\bar{\psi}_3\Gamma_\beta\psi_2)$, if there are any relation between the coefficients G and G' which is known as the Fierz identities [108].

The Fierz identities can be discussed using general mathematics theory. We do not discuss general Fierz identities (see [109–112]) but see the case of the Lorentz group (Dirac algebra) and SU(N) algebra to derive two color and two flavor NJL model interaction [78].

C.1 Dirac Algebra

C.1.1 Fierz Identity

The set of the Dirac matrices

$$\{\Gamma^\alpha\} = \{\mathbf{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma_5, i\gamma_5\} \quad (\mu, \nu = 0, 1, 2, 3) \tag{C.1}$$

gives the basis for the Dirac matrix space where the Greek index α is taken from 1 to 5 corresponding scalar, vector, tensor and so on. The orthogonality relation is

$$\mathrm{tr}(\Gamma^\alpha\Gamma_\beta) = 4\delta^\alpha_\beta. \tag{C.2}$$

Any 4×4 matrix X is expanded in terms of the basis as

$$X = x_\alpha\Gamma^\alpha = \frac{1}{4}\mathrm{tr}(X\Gamma_\alpha)\Gamma^\alpha = \frac{1}{4}\mathrm{tr}(X\Gamma^\alpha)\Gamma_\alpha. \tag{C.3}$$

The summation over the Greek indices are taken implicitly likewise the Einstein notation. From these relation we find the identity in writing the component explicitly as

$$\delta_{ij}\delta_{kl} = \frac{1}{4}\Gamma^\alpha_{il}\Gamma_{\alpha,kj}, \tag{C.4}$$

where the Latin indices represent the components and run from 1 to 4. This is the basic Fierz identity. Multiplying matrices Γ^α and Γ^β, this leads the usual Fierz identities

$$\begin{aligned}\Gamma^\alpha_{ij}\Gamma^\beta_{kl} =&\frac{1}{4}(\Gamma^\alpha\Gamma_\rho)_{il}(\Gamma^\beta\Gamma_\sigma)_{kj}\\ =&\frac{1}{16}\mathrm{tr}(\Gamma^\alpha\Gamma_\rho\Gamma^\beta\Gamma_\sigma)\Gamma^\rho_{il}\Gamma^\sigma_{kj}.\end{aligned} \tag{C.5}$$

When we consider a interaction Lagrangian, the Greek indices are chosen to form Lorentz scalar quantities such as $\gamma^\mu\gamma_\mu$. The resulting Fierz identities are summarized below:

$$\begin{pmatrix}(\mathbf{1})(\mathbf{1})\\ (\gamma_\mu)(\gamma^\mu)\\ (\sigma_{\mu\nu})(\sigma^{\mu\nu})\\ (\gamma_\mu\gamma_5)(\gamma^\mu\gamma_5)\\ (i\gamma_5)(i\gamma_5)\end{pmatrix}(ij:kl) =\frac{1}{4}\begin{pmatrix}1 & 1 & 1 & -1 & -1\\ 4 & -2 & 0 & -2 & 4\\ 6 & 0 & -2 & 0 & -6\\ -4 & -2 & 0 & -2 & -4\\ -1 & 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}(\mathbf{1})(\mathbf{1})\\ (\gamma_\mu)(\gamma^\mu)\\ (\sigma_{\mu\nu})(\sigma^{\mu\nu})\\ (\gamma_\mu\gamma_5)(\gamma^\mu\gamma_5)\\ (i\gamma_5)(i\gamma_5)\end{pmatrix}(il:kj) \tag{C.6}$$

with corresponding to Eq. (C.5).

C.1.2 Diquark Channel

The Fierz rearrangement is considered as the interchange $(ij : kl) \to (il : kj)$ usually. In other words, a particle ψ remains a particle and the same applies for antiparticle $\bar{\psi}$. In the text, however, we discuss another transformation such as $(ij : kl) \to (ik : lj)$ which corresponds particle-particle (diquark) channel. This means that we have to change a particle to antiparticle and vice verse. In the Dirac algebra, the transformation for a particle to antiparticle is the charge conjugation matrix C (see Appendix B.3). By using the charge conjugation property, we find

$$\bar{\psi}_1\Gamma^\alpha\psi_2 = \eta_\alpha\bar{\psi}^c_2\Gamma^\alpha\psi^c_1. \tag{C.7}$$

If we denote the usual Fierz transformation as $(1234) \to (1432)$, the desired form can be written as

$$(1234) \to(124^c3^c) \to (13^c4^c2). \tag{C.8}$$

Hence, we find the Fierz matrix for diquark channel in contrast with Eq. (C.6) as

$$\begin{pmatrix}(\mathbf{1})(\mathbf{1})\\ (\gamma_\mu)(\gamma^\mu)\\ (\sigma_{\mu\nu})(\sigma^{\mu\nu})\\ (\gamma_\mu\gamma_5)(\gamma^\mu\gamma_5)\\ (i\gamma_5)(i\gamma_5)\end{pmatrix}(ij:kl) =\frac{1}{4}\begin{pmatrix}-1 & -1 & -1 & 1 & 1\\ 4 & -2 & 0 & -2 & 4\\ 6 & 0 & -2 & 0 & -6\\ 4 & 2 & 0 & 2 & 4\\ 1 & -1 & 1 & 1 & -1\end{pmatrix}\begin{pmatrix}(C^*)(C)\\ (\gamma_\mu C^*)(C\gamma^\mu)\\ (\sigma_{\mu\nu}C^*)(C\sigma^{\mu\nu})\\ (\gamma_\mu\gamma_5C^*)(C\gamma^\mu\gamma_5)\\ (i\gamma_5C^*)(Ci\gamma_5)\end{pmatrix}(ik:lj). \tag{C.9}$$

Before closing this section, we discuss the Fierz transformation in particle-particle channel in terms of the previous section. To obtain the Fierz transformation in particle-particle channel, we need new basis, denoting Λ^α, for the matrix space. In the Dirac algebra, it is given for particle-particle pair $\Lambda_\alpha = \Gamma_\alpha C$ and for antiparticle-antiparticle pair $\bar{\Lambda}_\alpha = \bar{C}\Gamma_\alpha$. They obviously satisfy the orthogonality relation

$$\mathrm{tr}[\Lambda^\alpha\bar{\Lambda}_\beta] \to 4\delta^\alpha_\beta. \tag{C.10}$$

Repeating same discussion, we may find

$$\begin{aligned}\sum_{\alpha=1}^{5}\Gamma^{\alpha}_{ij}\Gamma^{\alpha}_{kl} =&\eta_\alpha\frac{1}{16}\sum_{\alpha,\beta}\mathrm{tr}(\Gamma^\alpha\Lambda^\alpha(\Gamma^\alpha)^T\bar{\Lambda}^\alpha)\Lambda^\alpha_{ik}\bar{\Lambda}^\alpha_{lj}\\ =&\eta_\alpha\frac{1}{16}\sum_{\alpha,\beta}\mathrm{tr}(\Gamma^\alpha\Gamma^\beta CC^{-1}\Gamma^\alpha C\bar{C}\Gamma^\beta)\Lambda^\beta_{ik}\bar{\Lambda}^\beta_{lj}\\ =&-\eta_\alpha\frac{1}{16}\sum_{\alpha,\beta}\mathrm{tr}(\Gamma^\alpha\Gamma^\beta\Gamma^\alpha\Gamma^\beta)\Lambda^\beta_{ik}\bar{\Lambda}^\beta_{lj},\end{aligned} \tag{C.11}$$

which is Eq. (C.9).

C.2 SU(N) Algebra

C.2.1 Fierz Identity

We consider SU(N) algebra in the fundamental representation. The SU(N) is constructed by the $N\times N$ special unitary matrix U which is described as $U = \exp(iX)$ where X is a Hermitian matrix and satisfies $\mathrm{tr}X = 0$. The Hermitian matrix X is expanded by the N^2-1 generators of SU(N), especially the SU(2) generators are $\sigma_a/2$ (σ_a are the Pauli matrices with $a = 1,2,3$) and the SU(3) are $\lambda_a/2$ (λ_a are the Gell-Mann matrices with $a = 1,\cdots,8$). They form matrix spaces and satisfy the orthogonality relations

$$\mathrm{tr}[\sigma_a\sigma_b] =2\delta_{ab}, \tag{C.12}$$
$$\mathrm{tr}[\lambda_a\lambda_b] =2\delta_{ab}. \tag{C.13}$$

In the case of any fundamental representation of SU(N) algebra $\{T_a\}$ ($a = 1,\cdots,N^2-1$) satisfies $\mathrm{tr}[T_aT_b] = C\delta_{ab}$. Since these matrices are traceless, the basis need identity matrix to expand any $N\times N$ matrix like Eq. (C.14). Then the set $\{\mathbf{1},T_a\}$ gives the basis for SU(N) and any $N\times N$ complex matrix X can be expanded in terms of

$$X = X_0\mathbf{1} + \sum_{a=1}^{N^2-1}X_aT_a, \tag{C.14}$$

with

$$X_0 = \frac{1}{N}\mathrm{tr}(X), \quad X_a = \frac{1}{C}\mathrm{tr}(XT_a). \tag{C.15}$$

We substitute Eq. (C.15) into Eq. (C.14) and take the general elements

$$(X)_{ij} = \frac{1}{N}(X)_{kk}\delta_{ij} + \frac{1}{C}\sum_{a=1}^{N^2-1}(X)_{lk}(T_a)_{kl}(T_a)_{ij}, \tag{C.16}$$

and obtain from the coefficients of $(X)_{lk}$ the completeness relation

$$\delta_{il}\delta_{kj} = \frac{1}{N}\delta_{ij}\delta_{kl} + \frac{1}{C}\sum_{a=1}^{N^2-1}(T_a)_{ij}(T_a)_{kl}. \tag{C.17}$$

From this relation, we find the square of the generators is proportional to identity matrix as

$$\sum_a T_a T_a = C_2(R)\mathbf{1}, \tag{C.18}$$

where $C_2(R)$ is the conventional quadratic Casimir invariant depending on the representation. In the fundamental representation denoting f

$$C_2(f) = C\frac{N^2-1}{N^2}. \tag{C.19}$$

From Eq. (C.17) and Eq. (C.18), we obtain the Fierz identity for SU(N)

$$\delta_{ij}\delta_{kl} = \frac{1}{N}\delta_{il}\delta_{kj} + \frac{1}{C}\sum_{a=1}^{N^2-1}(T_a)_{il}(T_a)_{kj}, \tag{C.20}$$

$$\sum_{a=1}^{N^2-1}(T_a)_{ij}(T_a)_{kl} = C\frac{N^2-1}{N^2}\delta_{il}\delta_{kj} - \frac{1}{N}\sum_{b=1}^{N^2-1}(T_b)_{il}(T_b)_{kj}. \tag{C.21}$$

We note the SU(2) case

$$\delta_{ij}\delta_{kl} = \frac{1}{2}\delta_{il}\delta_{kj} + \frac{1}{2}\sum_{a=1}^{3}(\sigma_a)_{il}(\sigma_a)_{kj}, \tag{C.22}$$

$$\sum_{a=1}^{3}(\sigma_a)_{ij}(\sigma_a)_{kl} = \frac{3}{2}\delta_{il}\delta_{kj} - \frac{1}{2}\sum_{b=1}^{3}(\sigma_b)_{il}(\sigma_b)_{kj}. \tag{C.23}$$

C.2.2 Diquark Channel

In general, the Fierz transformation in particle-particle channel is constructed by new basis Λ_a which may be related with the original basis by using transpose operator like discussed in C.1.2. We denote the original basis as Γ^α_a with the representation α. The basis matrices Λ^α_a with the representation α is assumed to be normalized as

$$\mathrm{tr}(\Lambda^\alpha_a \bar{\Lambda}^\beta_b) = \delta^{\alpha\beta} g_{ab}, \tag{C.24}$$

with the metric g_{ab}. The Fierz transformation is defined similarly with Eq. (C.11) as

$$\sum_a (\Gamma^\alpha_a)_{ij}(\Gamma^\alpha_a)_{kl} = \sum_\beta C_{\alpha\beta}\sum_b (\Lambda^\beta_b)_{ik}(\bar{\Lambda}^\beta_b)_{lj}, \tag{C.25}$$

with the coefficient

$$C_{\alpha\beta} = \sum_a \mathrm{tr}(\Gamma^\alpha_a \Lambda^\beta_b (\Gamma^\alpha_a)^T \bar{\Lambda}^\beta_b). \tag{C.26}$$

The matrix space in particle-particle channel of SU(N) in the fundamental representation decomposes into two irreducible representation, the symmetric and antisymmetric tensors. We denote the corresponding basis matrices respectively as S_a and A_a. The symmetric matrices S_a

are simply the symmetric matrices T_a with the identity matrix, while the antisymmetric A_a are the antisymmetric T_a to have the same norm C. We identify $\Gamma_a^\alpha = T_a$ with the fundamental representation, the symmetric case $\Lambda_a^\alpha = S_a$ and the antisymmetric case $\Lambda_a^\alpha = A_a$ and denoting the representation index α as $\mathcal{T}$ as the fundamental representation, $\mathcal{S}$ as the symmetric case and $\mathcal{A}$ as the antisymmetric case in the above discussion. We find

$$C_{\mathcal{TS}} = \frac{N-1}{N}, \quad C_{\mathcal{TA}} = -\frac{N+1}{N}. \tag{C.27}$$

Then we obtain the identity

$$\delta_{ij}\delta_{kl} = \frac{1}{C}\sum_a (S_a)_{ik}(S_a)_{lj} + \frac{1}{C}\sum_a (A_a)_{ik}(A_a)_{lj}, \tag{C.28}$$

$$\sum_a (T_a)_{ij}(T_a)_{kl} = \frac{N-1}{N}\sum_a (S_a)_{ik}(S_a)_{lj} - \frac{N+1}{N}\sum_a (A_a)_{ik}(A_a)_{lj}. \tag{C.29}$$

We note the SU(2) case

$$\delta_{ij}\delta_{kl} = \frac{1}{2}\left(\delta_{ik}\delta_{lj} + (\sigma_1)_{ik}(\sigma_1)_{lj} + (\sigma_3)_{ik}(\sigma_3)_{lj} + (\sigma_2)_{ik}(\sigma_2)_{lj}\right), \tag{C.30}$$

$$\sum_a (\sigma_a)_{ij}(\sigma_a)_{kl} = \frac{1}{2}\left(\delta_{ik}\delta_{lj} + (\sigma_1)_{ik}(\sigma_1)_{lj} + (\sigma_3)_{ik}(\sigma_3)_{lj}\right) - \frac{3}{2}(\sigma_2)_{ik}(\sigma_2)_{lj}. \tag{C.31}$$

C.3 NJL model interaction

We consider two color and two flavor NJL model interaction following [64]. We start from the color current interaction (2.14) and show that performing a Fierz transformation we obtain the interaction term (2.21) with the coupling coefficients related by (2.22). We first consider mesonic channel $\bar{q}q$ and next diquark channel qq.

We rewrite the interaction term and keep track explicitly of all color, flavor and Dirac indices:

$$\begin{aligned}\mathcal{L}_{int} =& - G_c \sum_{a=1}^{3} (\bar{\psi}\gamma^\mu t_a \psi)(\bar{\psi}\gamma_\mu t^a \psi) \\ =& - G_c \sum_{a=1}^{3} \left[\bar{\psi}_{i,p,\mu}\psi_{j,q,\nu}\bar{\psi}_{k,r,\rho}\psi_{l,s,\sigma}(\gamma_\alpha)_{\mu\nu}(\gamma^\alpha)_{\rho\sigma}(t_a)_{ij}(t_a)_{kl}\delta_{pq}\delta_{rs}\right],\end{aligned} \tag{C.32}$$

with color indices i, j, k, l, flavor indices p, q, r, s and Dirac indes $\mu, \nu.\rho, \sigma$.

Fierz transformation for flavor indices in $\bar{q}q$ channel is performed using the relatoin:

$$\delta_{pq}\delta_{rs} = \frac{1}{2}\sum_{b=0}^{3}(\tau_b)_{ps}(\tau_b)_{rq}, \tag{C.33}$$

where τ_b are Pauli matrices with $\tau_0 = \mathbf{1}$. For color indices:

$$\sum_{a=1}^{3}(t_a)_{ij}(t_a)_{kl} = \frac{3}{2}\delta_{il}\delta_{kj} - \frac{1}{2}\sum_{c=1}^{3}(t_c)_{il}(t_c)_{kj}. \tag{C.34}$$

For Dirac indices:

$$(\gamma_\alpha)_{\mu\nu}(\gamma^\alpha)_{\rho\sigma} = \delta_{\mu\sigma}\delta_{\rho\nu} - \frac{1}{2}(\gamma_\alpha)_{\mu\sigma}(\gamma^\alpha)_{\rho\nu} - \frac{1}{2}(\gamma_\alpha\gamma_5)_{\mu\sigma}(\gamma^\alpha\gamma_5)_{\rho\nu} + (i\gamma_5)_{\mu\sigma}(i\gamma_5)_{\rho\nu}. \tag{C.35}$$

Fierz transformation for $\bar{q}q$ channel is performed by substituting above relations as

$$\begin{aligned}\mathcal{L}_{int} =& \frac{3}{4}G_c \sum_{b=0}^{3} \left[(\bar{\psi}\tau_b\psi)^2 + (\bar{\psi}i\gamma_5\tau_b\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma_\alpha\tau_b\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma_\alpha\gamma_5\tau_5\psi)^2\right] \\ & - \frac{1}{4}G_c \sum_{b=0}^{3}\sum_{a=1}^{3} \left[(\bar{\psi}t_a\tau_b\psi)^2 + (\bar{\psi}i\gamma_5 t_a\tau_b\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma_\alpha t_a\tau_b\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma_\alpha\gamma_5 t_a\tau_b\psi)^2\right].\end{aligned} \tag{C.36}$$

The first term is color singlet and second term is color triplet. The color singlet scalar and pseudoscalar part is written as

$$\mathcal{L}_{\bar{q}q} = \frac{G_0}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right], \tag{C.37}$$

from which we can easily read

$$G_0 = \frac{3}{2}G_c. \tag{C.38}$$

We next perform diquark channel qq as below. For flavor indices:

$$\delta_{pq}\delta_{rs} = \frac{1}{2}\sum_{b=0}^{3}(\tau_b)_{pr}(\tau_b)_{sq}. \tag{C.39}$$

For color indices:

$$\sum_{a=1}^{3}(t_a)_{ij}(t_a)_{kl} = \frac{1}{2}\left[\delta_{ik}\delta_{lj} + (t_1)_{ik}(t_a)_{lj} + (t_3)_{ik}(t_a)_{lj}\right] - \frac{3}{2}(t_2)_{ik}(t_2)_{lj}. \tag{C.40}$$

For Dirac indices:

$$(\gamma_\alpha)_{\mu\nu}(\gamma^\alpha)_{\rho\sigma} = (C^*)_{\mu\rho}(C)_{\sigma\nu} - \frac{1}{2}(\gamma^\alpha C^*)_{\mu\sigma}(C\gamma^\alpha)_{\sigma\nu} - \frac{1}{2}(\gamma_\alpha\gamma_5 C^*)_{\mu\rho}(C\gamma^\alpha\gamma_5)_{\sigma\nu} + (i\gamma_5 C^*)_{\mu\rho}(i\gamma_5 C)_{\sigma\rho}. \tag{C.41}$$

We find

$$\begin{aligned}\mathcal{L}_{int} =& \frac{3}{4}G_c \sum_{b=0}^{3}\left[(\bar{\psi}\tau_b t_2 C\bar{\psi}^T)(\psi^T C\tau_b t_2\psi) + (\bar{\psi}i\gamma_5\tau_b t_2 C\bar{\psi}^T)(\psi^T Ci\gamma_5\tau_b t_2\psi)\right. \\ & \left. - \frac{1}{2}(\bar{\psi}\gamma_\alpha\tau_b t_2 C\bar{\psi}^T)(\psi^T C\gamma^\alpha\tau_b t_2 C\psi) - \frac{1}{2}(\bar{\psi}\gamma_\alpha\gamma_5\tau_b t_2 C\bar{\psi}^T)(\psi^T C\gamma^\alpha\gamma_5\tau_b t_2\psi)\right] \\ & - \frac{1}{4}G_c \sum_{b=0}^{3}\sum_{S=0,1,3}\left[(\bar{\psi}\tau_b t_S C\bar{\psi})(\psi^T C\tau_b t_S\psi) + (\bar{\psi}i\gamma_5\tau_b t_S\bar{\psi}^T)(\psi^T Ci\gamma_5\tau_b t_S\psi)\right. \\ & \left. - \frac{1}{2}(\bar{\psi}\gamma_\alpha\tau_b t_S C\bar{\psi}^T)(\psi^T C\gamma^\alpha\tau_b t_S\psi) - \frac{1}{2}(\bar{\psi}\Gamma_\alpha\gamma_5\tau_b t_S C\bar{\psi}^T)(\psi^T C\Gamma^\alpha\gamma_5\tau_b t_S\psi)\right].\end{aligned} \tag{C.42}$$

The color singlet scalar diquark part is written as

$$\mathcal{L}_{qq} = \frac{H_0}{2} G_c (\bar{\psi} i\gamma_5 \tau_2 t_2 C \bar{\psi}^T)(\psi^T C i\gamma_5 \tau_2 t_2 \psi), \tag{C.43}$$

from which we can easily read

$$H_0 = \frac{3}{2} G_c. \tag{C.44}$$

Appendix D

Thermal Field Theory

In this appendix, we review the formulation for thermal field theory following [113–115]. Especially, we formulate to construction of Bose and Fermi statistics. First, we discuss the construction of a partition function described by fields using path integral formalism. To introduce the temperature, we go to the Euclidean metric with minus sign instead of the Minkowski:

$$(t^2 - \vec{x}^2) \to -(\tau^2 + \vec{x}^2), \tag{D.1}$$

taking $it \to \tau$, which is called the imaginary time formalism. Due to the periodicity or anti-periodicity of the thermal field, we introduce so-called the Matsubara frequencies in the momentum representation. We work out the partition function and the thermodynamical potential density for the complex scalar field and the Dirac field using the Matsubara formalism. Finally, we discuss a technique of the summation of the frequencies.

D.1 The Partition Function

We take a Schrödinger picture field $\hat{\phi}(\vec{x}, 0)$ and the conjugate momentum $\hat{\pi}(\vec{x}, 0)$ at $t = 0$. The field operator eigenstate $|\phi\rangle$ satisfy

$$\hat{\phi}(\vec{x}, 0)|\phi\rangle = \phi(\vec{x})|\phi\rangle, \tag{D.2}$$

where $\phi(\vec{x})$ is the eigenvalue. We have the usual completeness and orthogonality conditions

$$\int d\phi(\vec{x})|\phi\rangle\langle\phi| = 1, \tag{D.3}$$

$$\langle\phi_a|\phi_b\rangle = \delta(\phi_a(\vec{x}) - \phi_b(\vec{x})). \tag{D.4}$$

Similarly, the conjugate momentum operator eigenstate $|\pi\rangle$ satisfy

$$\hat{\pi}(\vec{x}, 0)|\pi\rangle = \pi(\vec{x})|\pi\rangle, \tag{D.5}$$

with the eigenvalue $\pi(\vec{x})$. The completeness and orthogonality conditions are

$$\int \frac{d\pi(\vec{x})}{2\pi}|\pi\rangle\langle\pi| = 1, \tag{D.6}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\pi_a(\vec{x}) - \pi_b(\vec{x})). \tag{D.7}$$

We take a eigenstate for this system $|\phi_a\rangle$ at $t = 0$, then the time evolution of the system is described $e^{-iHt}|\phi_a\rangle$ where H is Hamiltonian of the system. Taking the state $|\phi_b\rangle$ realized after the time development, the transition amplitude between these states is $\langle\phi_b|e^{-iHt}|\phi_a\rangle$. By the path integral method, the transition amplitude is expressed

$$\langle\phi_b|e^{-iHt}|\phi_a\rangle = \int \mathcal{D}\pi \int_{\phi(\vec{x},0)=\phi_a(\vec{x})}^{\phi(\vec{x},t)=\phi_b(\vec{x})} \mathcal{D}\phi \exp\left[i\int_0^t dt' \int d^3x(\pi\dot{\phi} - \mathcal{H}(\phi,\pi))\right] \tag{D.8}$$

where $\mathcal{H}$ is Hamiltonian density.

On the other hand, a partition function for statistical mechanics is assigned as

$$Z(\beta) = \sum_n e^{-\beta E_n} = \mathrm{tr} e^{-\beta\hat{H}} = \int d\phi_a \langle\phi_a|e^{-\beta\hat{H}}|\phi_a\rangle, \tag{D.9}$$

where β is the inverse of temperature and the Boltzmann constant is taken $k_B = 1$. The trace is evaluated by using a complete set of eigenvectors of H as

$$H|n\rangle = E_n|n\rangle. \tag{D.10}$$

From the last line in Eq. (D.9), the partition function could be interpreted as expression for time transition amplitude from 0 to β, if we adopt imaginary time formulation $\tau = it$. Thus, the partition function is expressed

$$Z(\beta) = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp\left[\int_0^\beta d\tau \int d^3x(\pi\frac{\partial\phi}{\partial\tau} - \mathcal{H}(\phi,\pi))\right] \tag{D.11}$$

by applying the path integral method. Where "periodic" means periodicity of $\phi_a(\vec{x}) = \phi(\vec{x},0) = \phi(\vec{x},\beta)$ and this notation is omitted afterward. If we suppose the system is described by Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \tag{D.12}$$

the conjugate momentum is defined as $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$. The momentum is integrated out immediately and the numerical factor independent of β is suppressed. Then the partition function can be written as

$$Z(\beta) = \int \mathcal{D}\phi \exp\left(-S_E(\beta)\right), \tag{D.13}$$

with the Euclidean action

$$S_E(\beta) = \int_0^\beta d\tau \int d^3x \frac{1}{2}\left((\partial_\tau\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right). \tag{D.14}$$

D.2 The Matsubara Propagator

We consider Klein-Gordon Lagrangian as an example and introduce the symbol

$$\int_0^\beta d^4x = \int_0^\beta d\tau \int d^3x, \tag{D.15}$$

and $x = (\vec{x}, \tau)$. Klein-Gordon Lagrangian is a Gaussian type for the fields, hence the functional integral is evaluated as

$$\begin{aligned} Z(\beta) =& \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int_0^\beta d^4x\phi(x)G^{-1}\phi(x)\right) \\ =&(\det G)^{1/2} = \exp\left(\frac{1}{2}\ln\det G\right) = \exp\left(\frac{1}{2}\mathrm{tr}\ln G\right), \end{aligned} \tag{D.16}$$

where

$$G^{-1} = -\partial_\tau^2 - \nabla^2 + m^2 \tag{D.17}$$

is the inverse of the Green function.

In the field theory, a generation functional $Z(\beta; j)$ with $Z(\beta) = Z(\beta; j = 0)$ is given as

$$\begin{aligned} Z(\beta; j) =& \int \mathcal{D}\phi \exp\left[-S_E(\beta) + \int_0^\beta d^4x j(\vec{x}, \tau)\phi(\vec{x}, \tau)\right] \\ =&Z(\beta) \exp\left[\frac{1}{2}\int_0^\beta d^4x d^4x' j(x)G(x, x')j(x')\right], \end{aligned} \tag{D.18}$$

with $x = (\vec{x}, \tau), x' = (\vec{x}', \tau')$. The Green function $G(x, x')$ satisfies

$$(-\partial_\tau^2 - \nabla^2 + m^2)G(x, x') = \delta(x - x'). \tag{D.19}$$

The functional differentiation gives the propagator with imaginary time as

$$\frac{1}{Z(\beta)}\frac{\delta^2 Z(\beta; j)}{\delta j(\vec{x}, \tau_1)\delta j(\vec{x}, \tau_2)}\bigg|_{j=0} = \frac{1}{Z(\beta)}\int \mathcal{D}\phi\ \phi(\vec{x}, \tau_1)\phi(\vec{x}, \tau_2)\exp\left(-S_E(\beta)\right). \tag{D.20}$$

From the discussion of appendix E, we obtain the propagator as

$$\langle\phi|\phi(x')\phi(x)|\phi\rangle = G(x, x') = \Delta(x - x'). \tag{D.21}$$

The right hand side of Eq. (D.20) may be regarded as the thermal average of T-product of field operator. We omit the space index $\vec{x}$ henceforth. We recall that the thermal average of an operator A is defined as

$$\langle A\rangle_\beta = \frac{1}{Z(\beta)}\mathrm{tr}(Ae^{-\beta H}) \tag{D.22}$$

and T-product is

$$T(\phi(-i\tau_1)\phi(-i\tau_2)) = \begin{cases} \phi(-i\tau_1)\phi(-i\tau_2) & \text{if} \quad \tau_1 > \tau_2 \\ \phi(-i\tau_2)\phi(-i\tau_1) & \text{if} \quad \tau_2 > \tau_1 \end{cases}. \tag{D.23}$$

Hence, the thermal average of T-product of field operator is described as

$$\begin{aligned} \langle T(\phi(-i\tau_1)\phi(-i\tau_2))\rangle_\beta =& \frac{1}{Z(\beta)}\mathrm{tr}\left[e^{-\beta H}T(\phi(-i\tau_1)\phi(-i\tau_2))\right] \\ =& \frac{1}{Z(\beta)}\int d\phi\langle\phi|e^{-(\beta-\tau_1)H}\phi e^{-(\tau_1-\tau_2)H}\phi e^{-\tau_2 H}|\phi\rangle, \end{aligned} \tag{D.24}$$

where $\tau_1 > \tau_2$ have been assumed for simply. The field operator is defined as $\phi(-i\tau) = e^{H\tau}\phi e^{-H\tau}$ in Heisenberg picture have been used. Then one inserts a complete set of states of the field operator at 'times' τ_1 and τ_2 and repeats the procedure leading to the path integral formalism. It is also that $Z(\beta; j)$ can be written as

$$Z(\beta; j) = \mathrm{tr}\left[e^{-\beta H} T\left(e^{\int_0^\beta d^4x j(\vec{x},\tau)\phi(-i\tau)}\right)\right]. \tag{D.25}$$

From the periodicity of the paths in path integral formalism, the thermal average has a property

$$\langle T(\phi(-i\beta)\phi(-i\tau))\rangle_\beta = \langle T(\phi(0)\phi(-i\tau))\rangle_\beta. \tag{D.26}$$

Because of periodicity of the trace, the imaginary time propagator $\Delta(\tau)$ has a property

$$\Delta(\tau) = \langle T(\phi(-i\tau)\phi(0))\rangle_\beta = \Delta(\tau - \beta) \tag{D.27}$$

in any time interval $[0, \beta]$. The Fourier transform of this propagator is given as

$$\Delta(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}\Delta(\tau), \tag{D.28}$$

and its inverse is

$$\Delta(\tau) = T\sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau}\Delta(i\omega_n). \tag{D.29}$$

Since $\Delta(\tau)$ is periodic, the Fourier transform is taken over a finite interval $[0, \beta]$, so that frequencies ω_n is taken discrete values as

$$\omega_n = 2\pi n T, \tag{D.30}$$

which are called Matsubara frequencies. Form above discussion, the way of treating field theory as statistical mechanics is

$$\begin{aligned} it &\to \tau \\ \int d^4x &\to \int_0^\beta d\tau \int d^3x \qquad \text{(D.31)} \\ i\int \frac{d^4p}{(2\pi)^4} &\to -T\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3} \end{aligned}$$

$$\text{(D.32)}$$

are replaced, then

$$p_0 \to i\omega_n = i2\pi n T \tag{D.33}$$

is used. Thus, Eq. (D.16) is now represented as

$$Z(\beta) = \exp\left(-\frac{1}{2}\int d^3x \frac{d^3p}{(2\pi)^3}\sum_n \ln(\omega_n^2 + \omega^2)\right), \tag{D.34}$$

where we have used the definition of the Klein-Gordon propagator in the momentum space (with Minkowski metric) as

$$G = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p_0^2 - \omega^2}, \tag{D.35}$$

with $\omega^2 = \vec{p}^2 + m^2$.

The discussion up to here is obtained supposing that a symmetry for particle exchange is that of boson. We have another quantum statistics, namely fermion. The fermion has anti-symmetric property for particle exchange as

$$\langle\psi|\psi(x')\psi(x)|\psi\rangle = -\langle\psi|\psi(x)\psi(x')|\psi\rangle. \tag{D.36}$$

This means that

$$\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta), \tag{D.37}$$

hence,

$$\Delta(0) = -\Delta(\beta). \tag{D.38}$$

Therefore, The Matsubara frequencies for fermion is assigned as

$$\omega_n = \frac{(2n-1)\pi}{\beta} \quad \text{or} \quad \omega_n = \frac{(2n+1)\pi}{\beta}. \tag{D.39}$$

The remaining problem is how to take the summation over the Matsubara frequencies, which will be discussed later.

Before closing this section, we discuss a grand canonical ensemble. The grand canonical partition function is defined as

$$Z(\beta, \mu) = \mathrm{tr} e^{-\beta(H - \mu\cdot N)} = \int \mathcal{D}\pi\mathcal{D}\phi \exp\left[\int_0^\beta d^4x \left(\pi\frac{\partial\phi}{\partial\tau} - \mathcal{H} + \mu\mathcal{N}\right)\right], \tag{D.40}$$

where $\mu \cdot N = \sum_\alpha \mu_\alpha N_\alpha$, N_α is a particle number operator and μ_α is the chemical potential corresponding the particle. $\mathcal{N}$ is the conserved particle number density which is the 0-th component of the conserved current. Namely, it is the quantity relating density ρ:

$$N = Q = \int d^3x\rho = \int d^3xJ^0 = \int d^3x\mathcal{N}. \tag{D.41}$$

The conserved particle number $\mathcal{N}$ in case of fermion corresponds the baryon number and/or the lepton number, which depends on the field, while the conjugate momentum does not. Hence, the partition function would be described as the form of (D.13). In boson case, however, it also depends on momentum so we have to go through the Hamiltonian form of (D.40).

D.3 The Complex Scalar Field

We discuss several example in this and next section. First, we consider the complex scalar field. The complex scalar field Lagrangian is

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi. \tag{D.42}$$

The conserved particle number is obtained from the Noether current as

$$\mathcal{N} = i(\Phi^* \partial_0 \Phi - \Phi \partial_0 \Phi^*) = i(\Phi^* \Pi - \Phi \Pi^*), \tag{D.43}$$

where $\Pi = \partial_0 \Phi$ is the canonical conjugate momentum. The complex field is decomposed into real and imaginary part as

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{D.44}$$

$$\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{D.45}$$

By using these real scalar field, Hamiltonian and particle number are written as

$$\mathcal{H} = \frac{1}{2}[(\pi_1)^2 + (\pi_2)^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] \tag{D.46}$$

$$\mathcal{N} = \phi_2 \pi_1 - \phi_1 \pi_2. \tag{D.47}$$

Thus, the partition function of scalar field is given as

$$\begin{aligned} Z &= \int \mathcal{D}\pi_1 \mathcal{D}\pi_2 \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\left[\int_0^\beta d^4x \left(i\pi_1\dot{\phi}_1 + i\pi_2\dot{\phi}_2 \right.\right. \\ &\quad \left.\left. - \frac{1}{2}[(\pi_1)^2 + (\pi_2)^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] + \mu[\phi_2\pi_1 - \phi_1\pi_2]\right)\right] \\ &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\left[-\frac{1}{2}\int_0^\beta d^4x \left(\left(\frac{\partial\phi_1}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_2}{\partial\tau}\right)^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 \right.\right. \\ &\quad \left.\left. + m^2\phi_1^2 + m^2\phi_2^2 + 2i\mu\left(\phi_1\frac{\partial\phi_2}{\partial\tau} - \phi_2\frac{\partial\phi_1}{\partial\tau}\right) - \mu^2\phi_1^2 - \mu^2\phi_2^2\right)\right] \end{aligned} \tag{D.48}$$

from Eq. (D.40). The second line means performing completing square with conjugate momentum, it is integrated out, then the numerical constant is dropped. The derivative terms of the fields ϕ_1 and ϕ_2 are performed by partial integration. Its Fourier transformation is replaced by (D.31) and (D.33). The Euclidean action is described in the matrix form:

$$S_E = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{n=-\infty}^{\infty} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{D.49}$$

where $\omega^2 = \vec{p}^2 + m^2$. The functional integration is done by Eq. (E.13) and described as

$$Z = \exp\left(-\int d^3x \int \frac{d^3p}{(2\pi)^3} \sum_n \frac{1}{2} \ln[(\omega_n^2 + \omega^2 - \mu^2)^2 + 4\mu^2\omega_n^2]\right). \tag{D.50}$$

We compute further

$$\begin{aligned}
\ln[(\omega_n^2+\omega^2-\mu^2)^2+4\mu^2\omega_n^2] &= \ln[(\omega_n^2+\omega^2-\mu^2+2i\mu\omega_n)(\omega_n^2+\omega^2-\mu^2-2i\mu\omega_n)] \\
&= \ln[((\omega_n+i\mu)^2+\omega^2)((\omega_n-i\mu)^2+\omega^2)] \\
&= \ln[(\omega_n+i(\omega+\mu))(\omega_n-i(\omega-\mu))(\omega_n+i(\omega-\mu))(\omega_n-i(\omega+\mu))] \\
&= \ln[(\omega_n^2+(\omega+\mu)^2)(\omega_n^2+(\omega-\mu)^2)] \\
&= \ln(\omega_n^2+(\omega+\mu)^2)+\ln(\omega_n^2+(\omega-\mu)^2). \qquad \text{(D.51)}
\end{aligned}$$

We consider the summation over ω_n. First of all, we take derivative of $\ln$ term with $\alpha = \omega \pm \mu$

$$\frac{\partial}{\partial\alpha}\sum_n \ln(\omega_n^2+\alpha^2) = \sum_n \frac{2\alpha}{\omega_n^2+\alpha^2} = \sum_n \frac{2\alpha/(2\pi T)^2}{n^2+\alpha^2/(2\pi T)^2} = \frac{1}{T}\coth\left(\frac{\alpha}{2T}\right), \qquad \text{(D.52)}$$

the sum formula:

$$\sum_{n=-\infty}^{\infty}\frac{1}{x^2+n^2} = \frac{\pi}{x}\coth(\pi x), \qquad \text{(D.53)}$$

has been used. Then, we integrate the result with α:

$$\int d\alpha \coth\left(\frac{\alpha}{2T}\right) = 2T\ln|\sinh(\alpha/2T)| = 2T\left[\frac{\alpha}{2T}+\ln(1-e^{-\alpha/T})+\ln\frac{1}{2}\right]. \qquad \text{(D.54)}$$

Ignoring the constant term $\ln(1/2)$ independent of T and ω, which does not affect the thermodynamics, the sum of the frequency of logarithmic function for bosons is described as

$$\sum_n \frac{1}{2}\ln(\omega_n^2+\alpha^2) = \frac{\alpha}{2T}+\ln(1-e^{-\alpha/T}). \qquad \text{(D.55)}$$

Thus, the partition function for bosons is given as

$$Z = \exp\left(-\int d^3x\int\frac{d^3p}{(2\pi)^3}[\beta\omega+\ln(1-e^{-\beta(\omega+\mu)})+\ln(1-e^{-\beta(\omega-\mu)})]\right). \qquad \text{(D.56)}$$

Thermodynamical potential (density) is obtained from this partition function:

$$\begin{aligned}
\Omega &= \frac{F}{V} = -\frac{1}{V}T\ln Z \\
&= \int\frac{d^3p}{(2\pi)^3}\left(\omega+T\ln(1-e^{-(\omega+\mu)/T})+T\ln(1-e^{-(\omega-\mu)/T})\right). \qquad \text{(D.57)}
\end{aligned}$$

D.4 The Dirac Field

Lagrangian of Dirac field is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \qquad \text{(D.58)}$$

and the conserved particle number is

$$\mathcal{N} = \bar{\psi}\gamma_0\psi, \qquad \text{(D.59)}$$

which does not depend on the conjugate momentum. Thus, the exponential part of the partition function (D.40) can be described only field by taking Legendre transformation. Then, momentum integral is performed soon and the numerical constant is dropped. The partition function for fermions is written as

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int_0^\beta d^4x \bar{\psi}(i(\partial_\tau + \mu)\gamma_4 + i\nabla \cdot \vec{\gamma} + m)\psi\right), \tag{D.60}$$

with the Euclidean Dirac matrices discussed in Appendix B.4 as $t \to -i\tau$, $\gamma_0 \to -i\gamma_4$ and $p_0 \to -ip_4 = i\omega_n$ with the frequency for fermion. The fields $\psi, \bar{\psi}$ are integrated out as Grassmann variable and the partition function in momentum representation is

$$Z = \exp\left(\ln \det[(\omega_n + i\mu)\gamma_4 - \vec{p} \cdot \vec{\gamma} + m]\right), \tag{D.61}$$

where determinant is taken over (Euclidean) Dirac matrices, coordinate space, momentum space and the Matsubara frequencies. We use the identity

$$\ln \det A = \operatorname{tr} \ln A, \tag{D.62}$$

except the Dirac matrices. We consider the determinant of Dirac matrices in the chiral representation (see Appendix B.4) as

$$\det \begin{pmatrix} mI & i(\omega_n + i\mu)I - \vec{\sigma} \cdot \vec{p} \\ i(\omega_n + i\mu)I + \vec{\sigma} \cdot \vec{p} & mI \end{pmatrix} = \left((\omega_n + i\mu)^2 + \omega^2\right)^2, \tag{D.63}$$

with $\omega^2 = \vec{p}^2 + m^2$ and I being 2×2 unit matrix. Since the summation is taken over both positive and negative frequencies, we can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \ln\left((\omega_n + i\mu)^2 + \omega^2\right)^2 =& 2\sum_{n=1}^{\infty}\left[\ln((\omega_n + i\mu)^2 + \omega^2) + \ln((-\omega_n + i\mu)^2 + \omega^2)\right] \\ =& 2\sum_{n=1}^{\infty} \ln[((\omega_n + i\mu)^2 + \omega^2)((\omega_n - i\mu)^2 + \omega^2)] \\ =& 2\sum_{n=1}^{\infty}\left[\ln(\omega_n^2 + (\omega + \mu)^2) + \ln(\omega_n^2 + (\omega - \mu)^2)\right],\end{aligned} \tag{D.64}$$

from Eq. (D.51). The partition function is now

$$Z = \exp\left(\int d^3x \int \frac{d^3p}{(2\pi)^3} \sum_{n=1}^{\infty} 2\left[\ln(\omega_n^2 + (\omega + \mu)^2) + \ln(\omega_n^2 + (\omega - \mu)^2)\right]\right). \tag{D.65}$$

The factor 2 corresponds to the degree of spin degeneracy and the summation is taken over only positive integer dealing with particle and anti-particle, which are the difference between the scalar field and the Dirac field.

The summation of frequencies for fermions $\omega_n = (2n-1)\pi T$ is performed as below. The derivative of $\ln$ term is

$$\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty} \ln(\omega_n^2 + \alpha^2) = \sum_{n=1}^{\infty} \frac{2\alpha}{\omega_n^2 + \alpha^2} = \sum_{n=1}^{\infty} \frac{2\alpha/(\pi T)^2}{(2n-1)^2 + \alpha^2/(\pi T)^2} = \frac{1}{2T}\tanh\left(\frac{\alpha}{2T}\right), \tag{D.66}$$

where the sum formula has been used following:

$$\sum_{n=1}^{\infty} \frac{1}{x^2 + (2n-1)^2} = \frac{\pi}{4x} \tanh\left(\frac{\pi x}{2}\right). \tag{D.67}$$

The integration over α is written as

$$\int d\alpha \tanh\left(\frac{\alpha}{2T}\right) = 2T \ln|\cosh(\alpha/2T)| = 2T[\frac{\alpha}{2T} + \ln(1 + e^{-\alpha/T}) + \ln\frac{1}{2}]. \tag{D.68}$$

Hence, the sum of frequency of logarithmic function for fermions is

$$\sum_n \ln(\omega_n^2 + \alpha^2) = \frac{\alpha}{2T} + \ln(1 + e^{-\alpha/T}). \tag{D.69}$$

The partition function is given as

$$Z = \exp\left[2\int d^3x \int \frac{d^3p}{(2\pi)^3} \left\{\frac{\omega}{T} + \ln(1 + e^{-(\omega-\mu)/T}) + \ln(1 + e^{-(\omega+\mu)/T})\right\}\right]. \tag{D.70}$$

The thermodynamical potential is

$$\Omega = -2\int \frac{d^3p}{(2\pi)^3} \left(\omega + T\ln(1 + e^{-(\omega-\mu)/T}) + T\ln(1 + e^{-(\omega+\mu)/T})\right). \tag{D.71}$$

D.5 Frequency Sum

We discuss a technique of the summation of the frequencies. We take a summation of the Matsubara frequency as follows

$$S = \sum_n h(\omega_n), \tag{D.72}$$

where $h(\omega_n)$ is a single variable function and frequencies ω_n is $2n\pi T$ in boson case or $(2n+1)\pi T$ in fermion case. The idea for taking the summation of frequencies is to replace summation to complex integration by introducing a virtual function of complex variable $g(z)$ which has simple pole at $z = i\omega_n$. The product of function gh is integrated along a suitable path in the complex plane, and the whole summation S is treated as summation of residue. The typical choice of g are

$$g(z) = \begin{cases} \frac{\beta}{\exp(\beta z)-1} \\ \frac{\beta}{\exp(\beta z)+1} \end{cases} \quad \text{or} \quad g(z) = \begin{cases} \frac{\beta}{2}\coth(\beta z/2) \\ \frac{\beta}{2}\tanh(\beta z/2). \end{cases} \tag{D.73}$$

The upper one is used for bosons and the lower one for fermion in both choices. We adopt the first one. Note that the first choices are similar to the Bose or Fermi distribution. The integration is manipulated along the imaginary axis as

$$-\frac{\zeta}{2\pi i}\oint dz g(z) h(-iz) = -\zeta \sum_n \mathrm{Res}(g(z)h(-iz))|_{z=i\omega_n} = \sum_n h(\omega_n) = S. \tag{D.74}$$

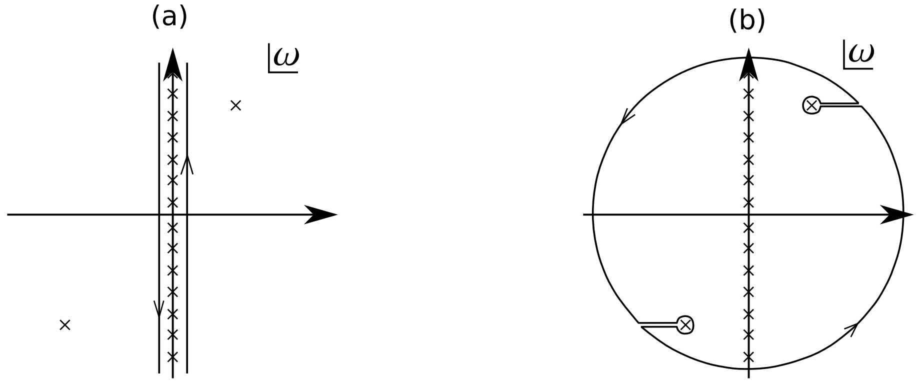


Figure D.1: The integration path is deformed from (a) to (b). The cross marks along the vertical axis are singularities for the function g and the isolate cross marks are for the function h.

For proof of the second equality, the residues of "the counting function" g are chosen as ζ and assuming integration path is closed at $z = \pm i\infty$[1]. The integration path can be modified to satisfy the condition that the singularity for g and $h(-iz)$ are avoided. In the case of the product gh is damped in the limit of $|z| \to \infty$. The original integration path can be modified to infinitely large circle as shown in Fig. D.1 (b). Then, the contribution for integration along outmost circle vanishes and the integration around singularities for h are calculated. If we assume that $h(-iz)$ has some isolated singularity $\{z_k\}$, we obtain

$$S = -\frac{\zeta}{2\pi i}\oint h(-iz)g(z) = -\zeta\sum_k \mathrm{Res}h(-iz)g(z)|_{z=z_k}. \tag{D.75}$$

Thus, the infinite summation S is simplified to a feasible operation to evaluate a finite number residue.

Examples

We calculate the frequency sum in some simple cases as examples. For simplicity, we consider fermion a case which is $\omega_n = (2n+1)\pi T$. Same discussion holds in boson case. In general, the function h needs the factor, which does not contribute thermodynamics, to vanish the contribution from the large outmost circle which appears in the modified integration path in the limit of $|z| \to \infty$. This non-essential factor is omitted usually, since the function h decays faster than z^{-1} in many physical case. We shall discuss the case where the introduction of such a factor is necessary.

Linear Fractional Function

We take a function $h(\omega_n)$ as

$$h(\omega_n) = -\frac{T}{i\omega_n e^{-i\omega_n\delta} - \xi}, \tag{D.76}$$

[1]The difference between the first and the second choices in Eq. (D.73) is the residue. The residue of the first functions are $\zeta = \pm 1$, $+1$ for boson and -1 for fermion and that of the second functions is $\zeta = 1$ for both cases.

where $\xi = \omega - \mu$ and δ is a positive infinitesimal value which is taken $\delta \to 0$ finally. The integrand $h(-iz)g(z)$ converges at $z \to -\infty$ by the factor $e^{z\delta}$ and $z \to \infty$ by the Fermi factor $(e^{\beta z}+1)^{-1}$. We work it out as

$$\begin{aligned} S &= \sum_n h(\omega_n) \\ &= -\sum_n \text{Res} \frac{\beta}{e^{z\beta}+1} \frac{-T}{ze^{z\delta}-\xi}\bigg|_{z=i\omega_n} \\ &= \lim_{z\to\xi}(z-\xi)\frac{1}{e^{z\beta}+1}\frac{1}{z-\xi} \\ &= \frac{1}{e^{\xi\beta}+1} = f(\omega-\mu). \end{aligned} \tag{D.77}$$

We get the well known formula

$$-T\sum_n \frac{1}{i\omega_n - (\omega-\mu)} = f(\omega-\mu), \tag{D.78}$$

where $f(\omega) = (\exp(\beta\omega)+1)^{-1}$ is well-known Fermi distribution.

Second Order Fractional Function

We take a function as

$$h(\omega_n) = -\frac{T}{(i\omega_n - \xi)^2}. \tag{D.79}$$

We omit the convergence factor here and next. This function has pole of order two at $z = \xi$. The frequency sum is

$$\begin{aligned} S &= \sum_n \text{Res}\ \frac{1}{e^{z\beta}+1}\frac{1}{(z-\xi)^2}\bigg|_{z=i\omega_n} \\ &= \lim_{z\to\xi}\frac{d}{dz}\left[(z-\xi)^2\frac{1}{e^{z\beta}+1}\frac{1}{(z-\xi)^2}\right] \\ &= \lim_{z\to\xi}\left[2(z-\xi)\frac{1}{e^{z\beta}+1}\frac{1}{(z-\xi)^2} - (z-\xi)^2\frac{\beta e^{z\beta}}{(e^{z\beta}+1)^2}\frac{1}{(z-\xi)^2} - (z-\xi)^2\frac{1}{e^{z\beta}+1}\frac{2}{(z-\xi)^3}\right] \\ &= -\frac{\beta e^{\xi\beta}}{(e^{\xi\beta}+1)^2} = \frac{d}{d\xi}f(\xi). \end{aligned} \tag{D.80}$$

Logarithmic Function

We take

$$h(\omega_n) = -T\ln(\beta(-i\omega_n + \xi)). \tag{D.81}$$

In generally, ξ is a function such as $\omega(\vec{p}) - \mu$, we fix it a certain value ξ_a firstly. The function $h(z)$ has branch cut along the real axis $z \in (-\infty, \xi_a]$. We choose the integration path as Fig.

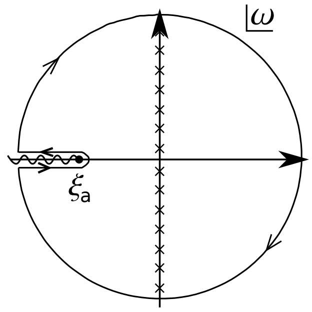


Figure D.2: Choice of the integration path for the logarithmic function Eq. (D.81).

D.2. The path around the branch cut on the real axis is enclosed by $\epsilon^{\pm} = \epsilon \pm i\eta$ with positive infinitesimal value η. Since the contribution of the integration is only around $\epsilon^{\pm}$, the complex integration is manipulated as

$$\begin{aligned} S =& \oint \frac{dz}{2\pi i} T \ln(\beta(z-\xi_a)) \frac{\beta}{e^{\beta z}+1} \\ =& \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left[\ln(\epsilon^+ - \xi_a) - \ln(\epsilon^- - \xi_a)\right] \frac{\beta}{e^{\epsilon\beta}+1}, \end{aligned} \tag{D.82}$$

where we have used $(e^{\epsilon^{\pm}}+1)^{-1} \sim (e^{\epsilon}+1)^{-1}$ and the path have been expanded as $(-\infty, \infty)$. We have

$$g(\epsilon) = \frac{\beta}{e^{\epsilon\beta}+1} = -\frac{\partial}{\partial\epsilon} \ln(1+e^{-\epsilon\beta}), \tag{D.83}$$

and the integrated function goes zero at the limit of $\epsilon \to \pm\infty$, then we compute the partial integration and obtain

$$\begin{aligned} S =& \frac{T}{2\pi i} \int_{-\infty}^{\infty} d\epsilon \left(\frac{1}{\epsilon^+ - \xi_a} - \frac{1}{\epsilon^- - \xi_a}\right) \ln(1+e^{\epsilon\beta}) \\ =& -T \int_{-\infty}^{\infty} d\epsilon \delta(\epsilon - \xi_a) \ln(1+e^{-\epsilon\beta}) \\ =& -T \ln(1+e^{-\xi_a\beta}) = -T\ln(1+e^{-\xi\beta}). \end{aligned} \tag{D.84}$$

In the second line, we have used the identity

$$\lim_{\eta\to 0} \frac{1}{x+i\eta} = -i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right), \tag{D.85}$$

where $\mathcal{P}$ means the principal value and in the last equal means the fixed value ξ_a is replaced by a function ξ.

The convergence factor

We see a case where the summation over the Matsubara frequencies diverges. This situation appears for example in the work of Diener et. al. [77]. Taking the function

$$h(\omega_n) = -T\frac{i\omega_n + \xi}{(i\omega_n)^2 - E^2}, \tag{D.86}$$

where $\xi = \sqrt{\vec{p}^2 + m^2}$ and $E = \sqrt{\xi^2 + |\Delta_0|^2}$ with BCS gap Δ_0 following [77]. The Matsubara sum is formally divergent and we explicitly write the convergence factor

$$h(\omega_n) = -T\frac{i\omega_n + \xi}{(i\omega_n)^2 - E^2}e^{i\omega_n\delta}, \tag{D.87}$$

with a positive infinitesimal value δ which is taken $\delta \to 0$ finally. Hence,

$$\begin{aligned}
S = &- \oint_c \frac{dz}{2\pi i}\frac{z+\xi}{z^2 - E^2}\frac{1}{e^{\beta z}+1}e^{z\delta} \\
&= \oint_{c'} \frac{dz}{2\pi i}\frac{z+\xi}{(z-E)(z+E)}\frac{1}{e^{\beta z}+1}e^{z\delta} \\
&= \lim_{z\to -E}(z+E)\frac{z+\xi}{(z-E)(z+E)}\frac{1}{e^{\beta z}+1}e^{z\delta} \\
&=\frac{E-\xi}{2E}\frac{1}{e^{\beta z}+1}.
\end{aligned} \tag{D.88}$$

The integral path c and c' is shown in Fig. D.3.

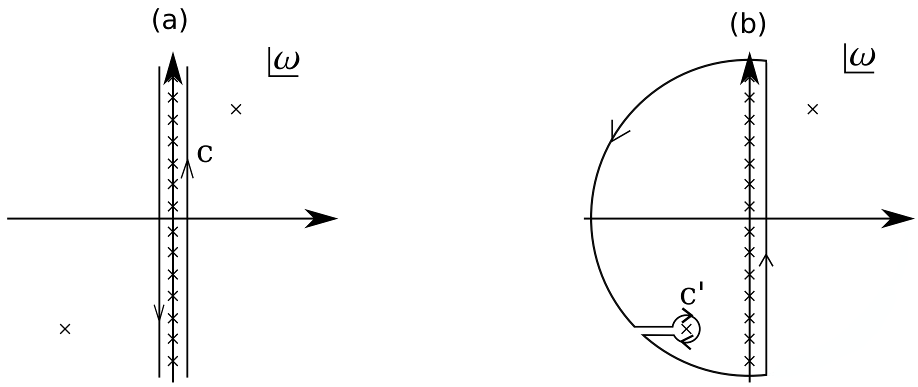


Figure D.3: The integration path is deformed from (a) to (b). The path c' is counterclockwise for the path c.

Appendix E

The Gaussian Integral

We write several Gaussian integral formulas ($\alpha > 0$)

$$\int dx e^{-\alpha x^2} = \left(\frac{\pi}{\alpha}\right)^{1/2}, \tag{E.1}$$

$$\int dx e^{-\alpha x^2+\beta x+\gamma} = \exp\left(\frac{\beta^2}{4\alpha}+\gamma\right)\left(\frac{\pi}{\alpha}\right)^{1/2}, \tag{E.2}$$

$$\int dx x^n e^{-\alpha x^2} = \begin{cases} \frac{(n-1)!!}{(2\alpha)^{n/2}}\left(\frac{\pi}{\alpha}\right)^{1/2} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd,} \end{cases} \tag{E.3}$$

$$\int dx x e^{-\alpha(x-F)^2} = \int dx'(x'+F)e^{-\alpha x'^2} = F\left(\frac{\pi}{\alpha}\right)^{1/2}, \tag{E.4}$$

$$\int dx x^2 e^{-\alpha(x-F)^2} = \int dx(x+F)^2 e^{-\alpha x^2} = \left(\frac{1}{2\alpha}+F^2\right)\left(\frac{\pi}{\alpha}\right)^{1/2}, \tag{E.5}$$

$$\int dx x^4 e^{-\alpha(x-F)^2} = \int dx(x^4+6F^2x^2+F^4)e^{-\alpha x^2} = \left(\frac{3}{4\alpha^2}+\frac{3}{\alpha}F^2+F^4\right)\left(\frac{\pi}{\alpha}\right)^{1/2}. \tag{E.6}$$

We take the product of n Gaussian integrals with $a_i > 0$ following [107],

$$\int dx_1 \dots dx_n \exp\left(-\frac{1}{2}\sum_n a_n x_n^2\right) = \frac{(2\pi)^{n/2}}{\prod_{i=1}^n a_i^{1/2}}. \tag{E.7}$$

Let A be a diagonal matrix with elements $a_1, \dots, a_n$ and x be an n-vector $(x_1, \dots, x_n)$, the integral becomes

$$\int d^n x e^{-(x,Ax)/2} = (2\pi)^{n/2}(\det A)^{-1/2}, \tag{E.8}$$

with the inner product

$$\sum_n a_n x_n^2 = (x, Ax). \tag{E.9}$$

Since this holds for any diagonal matrix, it also holds for any real symmetric, positive, non-singular matrix. Defining the measure $(dx) = d^n x(2\pi)^{-n/2}$, we obtain

$$\int (dx)e^{-(x,Ax)/2} = (\det A)^{-1/2}. \tag{E.10}$$

In the same way, we obtain for Hermitian matrix A and the measure $(dz) = d^n z(2\pi i)^{-n/2}$,

$$\int (dz^*)(dz)e^{-(z^*,Az)} = (\det A)^{-1}. \tag{E.11}$$

We now assume that we can generalise the above formulas to an infinite-dimensional function space without mathematical proof. The inner product for a function $\phi(x)$ is

$$(\phi, \phi) = \int dx[\phi(x)]^2. \tag{E.12}$$

The Gaussian integral is generalized in the functional form as

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int dx\phi(x)A\phi(x)\right) =(\det A)^{-1/2}. \tag{E.13}$$

If ϕ is a complex field, we obtain

$$\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left(-\int dx\phi^*(x)A\phi(x)\right) = (\det A)^{-1}. \tag{E.14}$$

Finally, we write the Gaussian integral with the Grassmann number $\eta, \bar{\eta}$,

$$\int d\bar{\eta}d\eta e^{-\bar{\eta}A\eta} = \det A. \tag{E.15}$$

In the functional form it is

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-\int dx\bar{\psi}(x)A\psi(x)\right) = \det A. \tag{E.16}$$

Suppose the Gaussian functional [83, 90]

$$\psi[\phi] = N\exp\left(-\frac{1}{2}\int dxdy[\phi(x) - f(x)]G(x,y)[\phi(y) - f(y)]\right), \tag{E.17}$$

with the weight function $G(x, y)$ and the fluctuation function $f(x)$. The scalar product

$$\langle\psi|\psi\rangle = N^2\int \mathcal{D}\phi \exp\left(-\int dxdy[\phi(x) - f(x)]G(x,y)[\phi(y) - f(y)]\right) \tag{E.18}$$

is defined as 1 by the normalization constant N which is obviously $(\det G^{-1})^{1/4}$. We want to calculate $\langle\psi|\phi(x)|\psi\rangle$, $\langle\psi|\phi(x)\phi(y)|\psi\rangle$ and $\langle\psi|(\phi(x)\phi(y))^2)|\psi\rangle$. To calcualte these values, we introduce a source term in the Gaussian functional as

$$\psi_J[\phi] = N\exp\left(-\frac{1}{2}\int dxdy[\phi(x) - f(x)]G(x,y)[\phi(y) - f(y)] + \int dx\phi(x)J(x)\right). \tag{E.19}$$

With a new variable $\phi' = \phi - f$, the scalar product is

$$\begin{aligned}
\langle\psi_J|\psi_J\rangle =& N^2 \int \mathcal{D}\phi' \exp\left(-\int dxdy\phi'(x)G(x,y)\phi'(y) + \int dx[\phi'(x) + f(x)]J(x)\right) \\
=& N^2 \int \mathcal{D}\phi' \exp\left(-\int dxdy[\phi'(x) - \frac{1}{2}\int dx_1 J(x_1)G^{-1}(x,x_1)]G(x,y)\right. \\
&\left.\times[\phi'(y) - \frac{1}{2}\int dx_2 J(x_2)G^{-1}(x_2,y)] + \int dx J(x)f(x) + \frac{1}{4}\int dxdy J(x)G^{-1}(x,y)J(y)\right) \\
=& \exp\left(\int dx J(x)f(x) + \frac{1}{4}\int dxdy J(x)G^{-1}(x,y)J(y)\right), \qquad \text{(E.20)}
\end{aligned}$$

where we have used the relation

$$\int dz G(x,z)G^{-1}(z,y) = \delta(x-y). \qquad \text{(E.21)}$$

Then we calculate

$$\begin{aligned}
\langle\psi|\phi(x)|\psi\rangle =& \frac{\delta}{\delta J(x)}\ \langle\psi_J|\psi_J\rangle|_{J=0} \\
=& \left(f(x) + \frac{1}{2}\int dx_1 J(x_1)G^{-1}(x,x_1)\right)\exp(\dots)\Big|_{J=0} \\
=& f(x), \qquad \text{(E.22)} \\
\langle\psi|\phi(x)\phi(y)|\psi\rangle =& \frac{\delta^2}{\delta J(x)\delta J(y)}\ \langle\psi_J|\psi_J\rangle|_{J=0} \\
=& \frac{1}{2}G^{-1}(x,y) + f(x)f(y). \qquad \text{(E.23)}
\end{aligned}$$

Further,

$$\begin{aligned}
\langle\psi|(\phi(x)\phi(y))^2|\psi\rangle =& \langle\psi'|(\phi'(x)+f(x))^2(\phi'(y)+f(y))^2|\psi'\rangle \\
=& \frac{3}{4}G^{-1}(x,y)G^{-1}(x,y) + \frac{1}{2}f^2(x)G^{-1}(x,x) \\
&+ 2f(x)f(y)G^{-1}(x,y) + \frac{1}{2}f^2(y)G^{-1}(y,y) + (f(x)f(y))^2, \qquad \text{(E.24)}
\end{aligned}$$

with

$$\psi' = N'\exp\left(-\frac{1}{2}\int dxdy\phi'(x)G(x,y)\phi'(y)\right), \qquad \text{(E.25)}$$

and defining normalization $\langle\psi'|\psi'\rangle = 1$. If we take $x = y$,

$$\langle\psi|\phi^4(x)|\psi\rangle = \frac{3}{4}G^{-1}(x,x)G^{-1}(x,x) + 3f^2(x)G^{-1}(x,x) + f^4(x). \qquad \text{(E.26)}$$

Hence, the Gaussian integrals in functional form (E.22), (E.23), (E.26) correspond usual integrals (E.4), (E.5), (E.6).

Appendix F

Mott Transition

The NJL model or PNJL model lack of the confinement at zero temperature. The hadrons (mesons and diquark-baryons) can decay to the two quark state $\bar{q}q$ and qq at finite momentum $\vec{q}$. This can happen at the point where the meson or diquark energy becomes larger than the two-particle continuum $\omega_{\bar{q}q}$ or ω_{qq} for the decay process at zero momentum. This transition is called Mott transition [116]. This property in two color NJL model is discussed in Ref. [67]. According to Ref. [67] the two-particle continua $\omega_{\bar{q}q}$ and ω_{qq} are different at the BEC and the BCS sides as

$$\omega_{\bar{q}q} = \begin{cases} \sqrt{(m-\mu)^2 + |\Delta|^2} + \sqrt{(m+\mu)^2 + |\Delta|^2} & \mu_B < \mu_0 \\ |\Delta| + \sqrt{(m+\mu)^2 + |\Delta|^2} & \mu_B > \mu_0 \end{cases}, \tag{F.1}$$

$$\omega_{qq} = \begin{cases} 2\sqrt{(m-\mu)^2 + |\Delta|^2} & \mu_B < \mu_0 \\ 2|\Delta| & \mu_B > \mu_0 \end{cases}, \tag{F.2}$$

where the baryon chemical potential is defined as $\mu_B = 2\mu$ with the quark chemical potential μ and the BEC-BCS crossover point is $\mu_0 \sim 120$ [MeV]. We refer the left upper figure in Fig.9 in Ref. [67] as shown in Fig. F.1. According to the figure, the effective range of the baryon chemical potential is less than five times of the pion mass $\mu_B/m_\pi < 5$. All of our calculations should be restricted in this range.

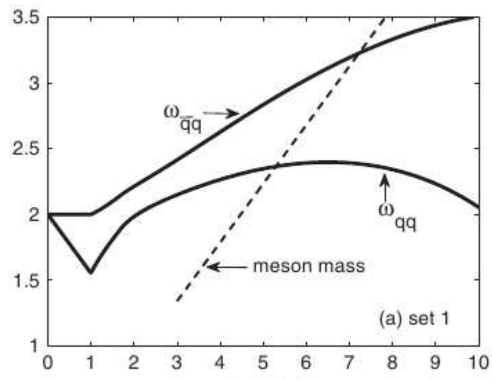


Figure F.1: The two-particle continua in units of m as functions of the baryon chemical potential μ_B in units of m_π [67]. Note that the author uses different parameter set from ours.

Appendix G

The Loop Expansion of the Effective Action and the Effective Potential

We discuss the derivation of the effective action following the text book [107]. We consider the real scalar field

$$
\begin{aligned}
S[\phi] &= \int dx \mathcal{L}(\phi(x)) \\
\mathcal{L}(\phi(x)) &= \frac{1}{2} \left(\partial\phi(x)\right)^2 - V(\phi), \\
V(\phi) &= \frac{1}{2} m^2 \phi^2(x) - \lambda\phi^4(x),
\end{aligned}
\tag{G.1}
$$

where dx stands for d^4x for simply. The Lagrangian is invariant under $\phi \to -\phi$, but this symmetry is not shared by the solution

$$
\left.\frac{dV}{d\phi}\right|_{\phi=\phi_0} = 0 \tag{G.2}
$$

where $\phi_0 \neq 0$. The symmetry has been spontaneously broken. These statements are classical. Quantum consideration enter with loops. The generating functional is written as

$$
Z[J] = \int \phi \exp\left(\frac{i}{\hbar} S[\phi] + i\hbar \int dx J(x)\phi(x)\right), \tag{G.3}
$$

which is assumed normalized. We explicitly write the Plank constant. The connected Green's function is

$$
W[J] = -i \ln Z[J], \tag{G.4}
$$

or

$$
e^{\frac{i}{\hbar} W[J]} = Z[J]. \tag{G.5}
$$

Since the generation functional is related to the vacuume expectation value with the source J

$$
Z[J] \propto \langle 0, \infty | 0, -\infty \rangle = \langle 0^+ | 0^- \rangle_J, \tag{G.6}
$$

the connected generation functional can be written as

$$e^{iW[J]} =\langle 0^+|0^-\rangle_J. \tag{G.7}$$

The classical field $\phi_c(x)$ is defined by

$$\phi_c(x) =\frac{\delta W[J]}{\delta J(x)} = \frac{\delta Z[J]}{\delta J(x)}\frac{\delta W[J]}{\delta Z} = \frac{\langle 0^+|\phi|0^-\rangle_J}{\langle 0^+|0^-\rangle_J}, \tag{G.8}$$

and is seen to depend on the source. The vacuum expectation value is defined as

$$\phi_0(x) = \lim_{J\to 0} \phi_c(x). \tag{G.9}$$

The effective action is defined by the Legendre transformation on $W[J]$ as

$$\Gamma[\phi_c] =W[J] - \int dx J(x)\phi(x), \tag{G.10}$$

and obeys

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} =- \hbar J(x). \tag{G.11}$$

Hence, we obtain the gap equation in the limit of J to zero as

$$\left.\frac{d\Gamma[\phi_c]}{d\phi_c}\right|_{\phi_c=\phi_0} = 0. \tag{G.12}$$

We would like to calculate $W[J]$ by the saddle-point method of the path integral. Taking an itegral

$$I = \int dx e^{-f(x)} \tag{G.13}$$

and $f(x)$ is stationary at some point x_0 and then $f(x)$ is expaned about x_0

$$f(x) = f(x_0) + \frac{1}{2}\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \cdots, \tag{G.14}$$

thus the integral is seen to be

$$I \sim e^{-f(x_0)}\int dx e^{-\frac{1}{2}f''(x_0)(x-x_0)^2}. \tag{G.15}$$

The integral becomes a Gaussian. We expand the action around ϕ_0

$$
\begin{aligned}
S[\phi, J] =& S[\phi] + \hbar \int dx J(x)\phi(x) \\
=& S[\phi_0, J] + \int dx \left.\frac{\delta S[\phi, J]}{\delta\phi(x)}\right|_{\phi_0} [\phi(x) - \phi_0] \\
&+ \frac{1}{2}\int dx_1 dx_2 [\phi(x_1) - \phi_0] \left.\frac{\delta^2 S[\phi, J]}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi_0} [\phi(x_2) - \phi_0] + \cdots \\
=& S[\phi_0, J] + \hbar \int dx J(x)[\phi(x) - \phi_0] \\
&+ \frac{1}{2}\int dx_1 dx_2 [\phi(x_1) - \phi_0] \left(- \left[\partial^2 + \left.\frac{d^2 V(\phi)}{d\phi^2}\right|_{\phi_0}\right] \delta(x_1 - x_2)\right) [\phi(x_2) - \phi_0] + \cdots \\
=& S[\phi_0, J] + \hbar \int dx J(x)\phi'(x) + \frac{1}{2}\int dx \phi'(x) \left(- \left[\partial^2 + V''(\phi_0)\right]\right) \phi'(x) + \cdots ,
\end{aligned}
\tag{G.16}
$$

where we have put $\phi' = \phi - \phi_0$ in the last line. Substituting this into Eq. (G.5) and applying to saddle-point integration Eq. (G.15) gives (relabelling ϕ' as ϕ)

$$
\begin{aligned}
\exp\left(\frac{i}{\hbar}W[J]\right) =& \exp\left(\frac{i}{\hbar}S[\phi_0, J]\right) \int \mathcal{D}\phi \exp\left(-\frac{i}{\hbar}\frac{1}{2}\int dx \phi[\partial^2 + V''(\phi_0)]\phi\right) \\
=& \exp\left(\frac{i}{\hbar}S[\phi_0, J]\right) \left(\det(\partial^2 + V''(\phi_0))\right)^{-1/2} ,
\end{aligned}
\tag{G.17}
$$

where we have rescaled the field ϕ into $\hbar^{1/2}\phi$ and gone to Euclidean space to eliminate the imaginary unit i. Hence, we obtain

$$
W[J] = S[\phi_0] + \hbar \int dx \phi_0(x) J(x) + i\frac{\hbar}{2}\mathrm{tr}\ln[\partial^2 + V''(\phi_0)], \tag{G.18}
$$

which is the loop expansion of $W[J]$ ignoring the higer order of $\hbar$.

To find the loop expansion of $\Gamma[\phi_c]$ we need $S[\phi_c]$. Putting $\phi_0 = \phi_c - \phi(\phi' = \phi_c)$

$$
\begin{aligned}
S[\phi_0] =& S[\phi_c - \phi] = S[\phi_c] - \int dx \left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi_0} \phi_0(x) + \mathcal{O}(\hbar^2) \\
=& S[\phi_c] - \hbar \int dx J(x)\phi_0(x) + \mathcal{O}(\hbar^2).
\end{aligned}
\tag{G.19}
$$

Then

$$
\Gamma[\phi_c] = S[\phi_c] + i\frac{\hbar}{2}\mathrm{tr}\ln[\partial^2 + V''(\phi_0)]. \tag{G.20}
$$

The trace tr is taken over the coordinate space. We define the n-point vertex function as

$$
\Gamma^{(n)}(x_1, \cdots, x_n) = \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)} \tag{G.21}
$$

in the coodinatespace, and in the momentum space

$$\Gamma^{(n)}(p_1, \cdots, p_n) = \frac{\delta^n \Gamma[\tilde{\phi}]}{\delta\tilde{\phi}(p_1)\ldots\delta\tilde{\phi}(p_n)}, \tag{G.22}$$

where $\tilde{\phi}$ is the Fourier transformed field.

We next expand the effective action $\Gamma[\phi_c]$ in powers of ϕ_c as

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma^{(n)}(x_1, \cdots, x_n)\phi_c(x_1)\cdots\phi_c(x_n), \tag{G.23}$$

and in the momentum representation as

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dp_1 \cdots dp_n \delta(p_1 + \cdots + p_n)\Gamma^{(n)}(p_1, \cdots, p_n)\tilde{\phi}_c(x_1)\cdots\tilde{\phi}_c(x_n). \tag{G.24}$$

We further have another expansion which is in terms of ϕ_c and its derivatives as

$$\Gamma[\phi_c] = \int dx \left[-U(\phi_c(x)) + \frac{1}{2}(\partial\phi_c(x))^2 + \ldots\right]. \tag{G.25}$$

The function $U(\phi_c)$ is called the effective potential. In the case of $\phi_c = \langle\phi\rangle = a$, a constant, all derivative term in the above expansion vanish:

$$\Gamma[a] = -\Omega U(a), \tag{G.26}$$

where Ω is the total volume of space-time. Comparing the expansion of $\Gamma[\phi]$ in momentum space and the above one, we have

$$U(a) = -\sum_{n=0}^{\infty} \frac{1}{n!}\Gamma^{(n)} a^n (p_i = 0). \tag{G.27}$$

Renormalisation conditions are stated in terms of U as

$$\begin{aligned} \Gamma^{(2)}(p_i = 0) &= \left.\frac{d^2 U(\phi_c)}{d\phi_c^2}\right|_{\langle\phi\rangle} = m^2, \\ \Gamma^{(4)}(p_i = 0) &= \left.\frac{d^4 U(\phi_c)}{d\phi_c^4}\right|_{\langle\phi\rangle} = \lambda. \end{aligned} \tag{G.28}$$

In addition, the vacuum expectation value can be written as

$$\left.\frac{dU(\phi_c)}{d\phi_c}\right|_{\langle\phi\rangle} = 0. \tag{G.29}$$

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