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PRESENTATIONS OF PERIODIC MAPS ON ORIENTED CLOSED SURFACES OF GENERA UP TO 4

Dedicated to Professor Akio Kawauchi for his 60th birthday

SUSUMU HIROSE

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Abstract

For oriented closed surfaces of genera up to 4, we list presentations of periodic maps by Dehn twists. As an application of these presentations, we provide examples of non-holomorphic Lefschetz fibrations.

1. Introduction

J. Nielsen [29] classified periodic maps on orientable surfaces by using data describing the homomorphisms from the orbifold fundamental groups of orbit spaces to the cyclic groups. On the other hand, M. Dehn [9] showed that any orientation preserving homeomorphism is isotopic to a product of Dehn twists. Since these results are classical, finding presentations of periodic maps by Dehn twists from the data of Nielsen's classification is a natural problem. We call these presentations *Dehn twist presentations*. J. Birman and H. Hilden [6] obtained a presentation of the hyperelliptic involution. Y. Matsumoto [27] obtained a presentation of a certain involution on Σ_2, an oriented closed surface of genus 2. Using a method similar to [27], T. Ito [22] reobtained the presentation of the hyperelliptic involution obtained by Birman and Hilden. M. Korkmaz [23] generalized Matsumoto's presentation to the higher genus. Y. Gurtas made further generalization [17, 18] and obtained a presentation of certain fixed point free periodic maps [19]. For hyperelliptic periodic maps, M. Ishizaka [21] obtained presentations by the investigation on hyperelliptic degenerations and their splittings. S. Takamura [34] will give Dehn twist decompositions of some automorphisms of Riemann surfaces. In this paper, we list Dehn twist presentations of periodic maps on orientable closed surfaces of genera up to 4.

This paper is organized as follows. In Section 2, we review Nielsen's classification of periodic maps. In Section 3, we list periodic maps such that any periodic maps on orientable closed surfaces of genera up to 4 are power of these maps, and list the Dehn twist presentation of these periodic maps. In Section 4, we introduce methods

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to find presentations listed in Section 3. In Section 5, as an application of the list of Dehn twist presentations, we show examples of non-holomorphic Lefschetz fibrations. In Section 6, we verify the presentations given in Section 3 and list the powers of periodic maps whose presentations are given there.

In order to find and verify presentations, we had to check the action of Dehn twists on simple closed curves very often. The author checked this sometimes by hand and sometimes by "Teruaki for Mathematica", implemented by K. Ahara and T. Sakasai. Many of results in this paper can not be found without this program.

2. Nielsen's classification of periodic maps

An orientation preserving homeomorphism f from a surface Σ_g to itself is *periodic map* if there is a positive integers n such that $f^n = id_{\Sigma_g}$. The *period* of f is the smallest positive integer satisfying the above condition. Two periodic maps f and f' on Σ_g are *conjugate* if there is an orientation preserving homeomorphism g from Σ_g to itself such that $f' = g \circ f \circ g^{-1}$. In this section, we will review the classification of conjugacy classes of periodic maps on surfaces by Nielsen [29]. We follow a description by Smith [30] and Yokoyama [36].

Let f be a periodic map on Σ_g, whose period is n. A point p on Σ_g is a *multiple point* of f if there is a positive integer k less than n so that $f^k(p) = p$. Let M_f be the set of multiple points of f. The orbit space Σ_g/f of f is defined by identifying x in Σ_g with $f(x)$. Let $\pi_f\colon \Sigma_g \to \Sigma_g/f$ be the quotient map, then π_f is an n-fold branched covering ramified at $\pi_f(M_f)$. The set $\pi_f(M_f)$ is denoted by B_f, and each element of B_f is called a *branch point* of f. We choose a point x in $\Sigma_g/f - B_f$, and a point $\tilde{x}$ in $\pi_f^{-1}(x)$. We define a homomorphism $\Omega_f : \pi_1(\Sigma_g/f - B_f, x) \to \mathbb{Z}_n$ as follows: Let l be loop in $\Sigma_g/f - B_f$, whose base point is x, and $[l]$ be an element of $\pi_1(\Sigma_g/f - B_f, x)$ represented by l. Let $\tilde{l}$ be a lift of l over Σ_g which begins from $\tilde{x}$. There is a positive integer r less than or equal to n so that the terminal point of $\tilde{l}$ is $f^r(\tilde{x})$. We define $\Omega_f([l]) = r$. Since $\mathbb{Z}_n$ is an Abelian group, we naturally define a homomorphism $\omega_f : H_1(\Sigma_g/f - B_f) \to \mathbb{Z}_n$ induced from Ω_f. For each point of $B_f = \{Q_1, \ldots, Q_b\}$, let D_i be a disk in Σ_g/f, which contains Q_i in its interior and is sufficiently small so that no other points of B_f are in D_i. Let S_{Q_i} be the boundary of D_i with clockwise orientation.

Theorem 2.1 ([29]). *Two periodic maps f and f' on Σ_g are conjugate if and only if the following three conditions are satisfied*:
(1) *the period of f = the period of f',*
(2) *the number of elements in B_f = the number of elements in $B_{f'}$,*
(3) *under a proper change of numbering on elements in $B_{f'}$, $\omega_f(S_{Q_i}) = \omega_{f'}(S_{Q_i})$ for each i.*

Let $\theta_i = \omega_f(S_{Q_i})$. By the above theorem, the data $[g, n; \theta_1, \dots, \theta_b]$ determines a periodic map up to conjugacy. The following proposition shows a necessary and sufficient condition for the data $[g, n; \theta_1, \dots, \theta_b]$ corresponding to a periodic map.

Proposition 2.2. *There is a periodic map whose data is* $[g, n; \theta_1, \dots, \theta_b]$ *if and only if the following conditions are satisfied*:
(1) $\theta_1 + \cdots + \theta_b = 0 \bmod n$,
(2) *if* Σ_g/f *is a sphere, then* $\gcd\{\theta_1, \dots, \theta_b\} = 1 \bmod n$,
(3) *let* g' *be the genus of* Σ_g/f *and* $q_i = \gcd\{\theta_i, n\}$, *then*

$$2g - 2 = n(2g' - 2) + \sum_{i\colon \text{branch points}} (n - q_i).$$

In the above proposition, (1) means that ω_f should be a homomorphism and $S_{Q_1} + \cdots + S_{Q_b}$ should be null-homologous, (2) means that ω_f should be a surjection, and (3) is the Riemann–Hurwitz formula.

In the following, we will use the expression $(n, \theta_1/n + \cdots + \theta_b/n)$ in place of $[g, n; \theta_1, \dots, \theta_b]$. This data $(n, \theta_1/n + \cdots + \theta_b/n)$ is called the *total valency*, which is introduced by Ashikaga and Ishizaka [5]. In the above data, we call θ_i/n the *valency* of Q_i, and often express it as an irreducible fraction.

3. Presentation of periodic maps by Dehn twists

For a simple closed curve a in Σ_g, we define the *right Dehn twist* t_a about a as illustrated in Fig. 1. We call each of t_a and t_a^{-1} the *Dehn twist* about a. The aim of this paper is to obtain presentations of periodic maps by Dehn twists, up to isotopy and conjugacy, from total valencies. Once if we obtain a Dehn twist presentation of a periodic map f, then we obtain a Dehn twist presentation of f^k automatically. Therefore, we make a list of periodic maps on Σ_g ($g \leq 4$) so that any periodic map on Σ_g is a power of maps in this list. This list is made by a kind of "Sieve of Eratosthenes". The list for the genus 3 case is referred from Ishizaka's paper [20, Lemma 1.1].

Proposition 3.1. *For* $g = 1, 2, 3, 4$, *any periodic map on* Σ_g (*if* $g = 1$, *with multiple points*) *is a power of a periodic map in the following list*:
$g = 1$,

$$f_{1,1} = \left(6, \frac{1}{6} + \frac{1}{3} + \frac{1}{2}\right), \quad f_{1,2} = \left(4, \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right),$$

$g = 2$,

$$f_{2,1} = \left(10, \frac{1}{10} + \frac{2}{5} + \frac{1}{2}\right), \quad f_{2,2} = \left(8, \frac{1}{8} + \frac{3}{8} + \frac{1}{2}\right),$$
$$f_{2,3} = \left(6, \frac{1}{6} + \frac{1}{6} + \frac{2}{3}\right), \quad f_{2,4} = \left(6, \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3}\right),$$

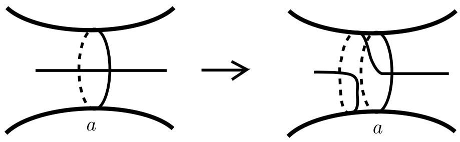


Fig. 1.

$g = 3$,

$$f_{3,1} = \left(14,\ \frac{1}{14} + \frac{3}{7} + \frac{1}{2}\right), \quad f_{3,2} = \left(12,\ \frac{1}{12} + \frac{5}{12} + \frac{1}{2}\right),$$

$$f_{3,3} = \left(8,\ \frac{1}{8} + \frac{1}{8} + \frac{3}{4}\right), \quad f_{3,4} = \left(4,\ \frac{1}{2} + \frac{1}{2}\right),$$

$$f_{3,5} = (2,\ ), \quad f_{3,6} = \left(12,\ \frac{1}{12} + \frac{1}{4} + \frac{2}{3}\right),$$

$$f_{3,7} = \left(8,\ \frac{1}{8} + \frac{1}{4} + \frac{5}{8}\right), \quad f_{3,8} = \left(9,\ \frac{1}{9} + \frac{1}{3} + \frac{5}{9}\right),$$

$$f_{3,9} = \left(7,\ \frac{1}{7} + \frac{2}{7} + \frac{4}{7}\right),$$

$g = 4$,

$$f_{4,1} = \left(18,\ \frac{1}{18} + \frac{4}{9} + \frac{1}{2}\right), \quad f_{4,2} = \left(16,\ \frac{1}{16} + \frac{7}{16} + \frac{1}{2}\right),$$

$$f_{4,3} = \left(10,\ \frac{1}{10} + \frac{1}{10} + \frac{4}{5}\right), \quad f_{4,4} = \left(10,\ \frac{2}{5} + \frac{1}{2} + \frac{1}{2} + \frac{3}{5}\right),$$

$$f_{4,5} = \left(15,\ \frac{1}{15} + \frac{1}{3} + \frac{3}{5}\right), \quad f_{4,6} = \left(12,\ \frac{1}{12} + \frac{1}{3} + \frac{7}{12}\right),$$

$$f_{4,7} = \left(10,\ \frac{1}{10} + \frac{3}{10} + \frac{3}{5}\right), \quad f_{4,8} = \left(12,\ \frac{1}{12} + \frac{1}{6} + \frac{3}{4}\right),$$

$$f_{4,9} = \left(6,\ \frac{1}{6} + \frac{1}{3} + \frac{2}{3} + \frac{5}{6}\right), \quad f_{4,10} = \left(6,\ \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2}\right),$$

$$f_{4,11} = \left(6,\ \frac{1}{2} + \frac{1}{2}\right), \quad f_{4,12} = \left(5,\ \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right).$$

The powers of these periodic maps are listed in §6.2.

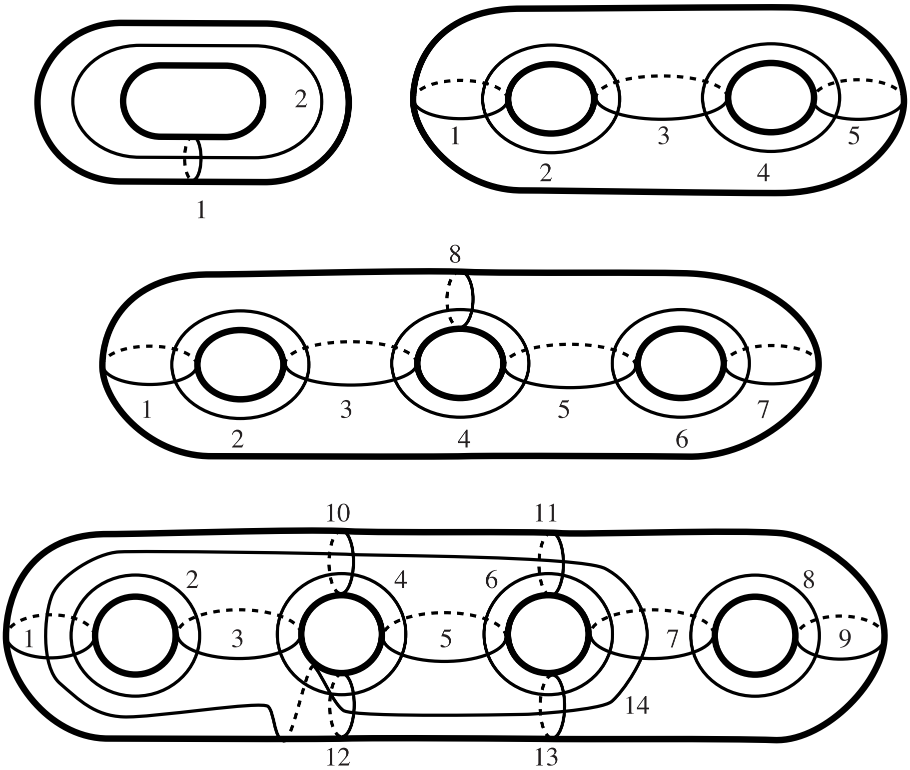


Fig. 2.

Theorem 3.2. *Periodic maps $f_{i,j}$ have Dehn twist presentation expressed as follows, where k means a right Dehn twist about a simple closed curve indicated by the letter k, and the order of the twists is from left to right, e.g., for $4 \cdot 3 \cdot 2 \cdot 1$, 4 is applied first*:
$g = 1$,

$$f_{1,1} = 1 \cdot 2, \quad f_{1,2} = 1 \cdot 2 \cdot 1,$$

$g = 2$,

$$f_{2,1} = 4 \cdot 3 \cdot 2 \cdot 1, \quad f_{2,2} = 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \quad f_{2,3} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$
$$f_{2,3} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^3,$$

$g=3$,

$$f_{3,1}=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\quad f_{3,2}=6\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,$$
$$f_{3,3}=7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\quad f_{3,4}=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^3,$$
$$f_{3,5}=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^5,\quad f_{3,6}=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 8,$$
$$f_{3,7}=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 5\cdot 4\cdot 3\cdot 8,\quad f_{3,8}=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 8,$$
$$f_{3,9}=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 8,$$

$g=4$,

$$f_{4,1}=8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,\quad f_{4,2}=8\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,$$
$$f_{4,3}=9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1,$$
$$f_{4,4}=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot(9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^5,$$
$$f_{4,5}=8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 11,\quad f_{4,6}=6\cdot 5\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 11,$$
$$f_{4,7}=8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 7\cdot 6\cdot 5\cdot 4\cdot 11,\quad f_{4,8}=3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 6\cdot 11\cdot 14,$$
$$f_{4,9}=1\cdot 2\cdot 3\cdot 4\cdot 10\cdot 9^{-1}\cdot 8^{-1}\cdot 7^{-1}\cdot 6^{-1}\cdot 11^{-1},$$
$$f_{4,10}=1\cdot 2\cdot 12^{-1}\cdot 4^{-1}\cdot 5^{-1}\cdot 6^{-1}\cdot 11^{-1}\cdot 8\cdot 9,\quad f_{4,11}=(2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 13\cdot 7)^2\cdot 9^{-1},$$
$$f_{4,12}=2\cdot 3\cdot 4\cdot 12\cdot 3\cdot 4\cdot 10\cdot 3\cdot 8^{-1}\cdot 7^{-1}\cdot 6^{-1}\cdot 13^{-1}\cdot 7^{-1}\cdot 6^{-1}\cdot 11^{-1}\cdot 7^{-1}.$$

We will verify these presentations in §4.1 and in §6.1.

4. Methods to find presentations in Theorem 3.2

An involution (periodic map of period 2) I of Σ_g is called a *hyperelliptic involution* if there are $2g+1$ fixed points of I, and the isotopy class of I is also denoted by I. An orientation-preserving homeomorphism ϕ from Σ_g to itself is said to be *hyperelliptic* if ϕ commutes with I. For hyperelliptic periodic maps, there is a method to find their presentation investigated by Ishizaka [21].

4.1. Hyperelliptic case. By the investigation on hyperelliptic degenerations, Ishizaka showed:

Theorem 4.1 ([21]). *Let* $\phi_1=(4g+2,1/(4g+2)+2g/(2g+1)+1/2)$, $\phi_2=(4g,1/4g+(2g-1)/4g+1/2)$ *and* $\phi_3=(2g+2,1/(2g+2)+1/(2g+2)+g/(g+1))$. *Any hyperelliptic periodic map is equal to the one of following*: (A) ϕ_1^k, (B) ϕ_2^k, (C) ϕ_3^k, (D) $I\phi_3^k$.

By the maps in this theorem, hyperelliptic periodic maps listed in Theorem 3.2 are rewritten as follows. When genus $=1$, $f_{1,1}=\phi_1$, $f_{1,2}=\phi_2$. When genus $=2$,

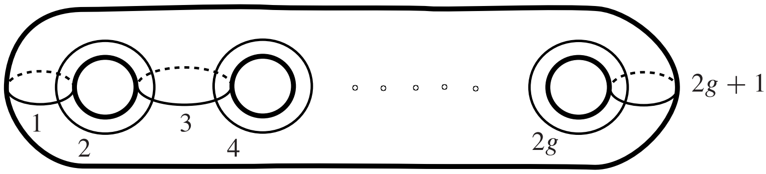


Fig. 3.

$f_{2,1} = \phi_1$, $f_{2,2} = \phi_2$, $f_{2,3} = \phi_3$, $f_{2,4} = I\phi_3^2$. When genus = 3, $f_{3,1} = \phi_1$, $f_{3,2} = \phi_2$, $f_{3,3} = \phi_3$, $f_{3,4} = I\phi_3^2$, $f_{3,5} = I\phi_3^4$. When genus = 4, $f_{4,1} = \phi_1$, $f_{4,2} = \phi_2$, $f_{4,3} = \phi_3$, $f_{4,4} = I\phi_3^4$.

By using splitting families, Ishizaka showed:

Theorem 4.2 ([21]). *Let* $1, \ldots, 2g+1$ *be simple closed curves in* Σ_g *as shown in* Fig. 3. *Then,*

(1) $\phi_1 = 2g \cdots 2 \cdot 1$,

(2) $\phi_2 = 2g \cdot 2g \cdots 2 \cdot 1$,

(3) $\phi_3 = 2g+1 \cdot 2g \cdots 2 \cdot 1$.

By the above theorem and the equation $I = 1 \cdot 2 \cdot 2g \cdot 2g+1 \cdot 2g+1 \cdot 2g \cdots 2 \cdot 1$ shown by Birman and Hilden [6], we obtain presentations for hyperelliptic periodic maps.

In what follows, we explain methods to find presentations for non-hyperelliptic periodic maps in Theorem 3.2.

4.2. Method to find Dehn twist presentations of $f_{3,6}$, $f_{3,8}$, $f_{4,5}$. Let f be a periodic map on Σ_g whose valency data is $(n, \theta_1/n + \theta_2/n + \cdots + \theta_b/n)$. Let D^2 be the unit disk in $\mathbb{C}$ whose center is 0. The homeomorphism on $\Sigma_g \times D^2$ defined by $(x, t) \to (f^{-1}(x), \exp(2\pi i/n) \cdot t)$ generates the action of $\mathbb{Z}_n$ on $\Sigma_g \times D^2$. The quotient space $(\Sigma_g \times D^2)/\mathbb{Z}_n$ has b quotient singular points [branch points of f, 0]. Applying the Hirzebruch–Jung resolution, we obtain a resolution map $r \colon M \to \Sigma_g \times D^2/\mathbb{Z}_n$. Now we take a holomorphic map $\psi \colon \Sigma_g \times D^2 \to D^2$ given by $\psi(x, t) = t^n$. Since ψ is $\mathbb{Z}_n$-equivalent, ψ determines a holomorphic map $\bar{\psi} \colon (\Sigma_g \times D^2)/\mathbb{Z}_n \to D^2$. We then consider the composite map $r' = \bar{\psi} \circ r \colon M \to D^2$. The preimage $(r')^{-1}(0)$ is a closed surface and 2-spheres transversely intersect each other. S. Takamura, in the second of his series of works [31, 32, 33, 34, 34] on degenerations of complex curves, explains the multiplicities of the components of $(r')^{-1}(0)$ associated to this map r'. We review his explanation.

Let Q be a branch point of f whose valency is θ/n. First, we define a sequence of positive integers $m_0 > m_1 > \cdots > m_\delta$ by setting $m_0 = n$, $m_1 = \theta$, and $m_2, m_3, \ldots, m_\delta = \gcd(m_0, m_1)$ by the division algorithm:

$$m_{i-1} = r_i m_i - m_{i+1} \quad (0 \le m_{i+1} < m_i), \quad i = 1, 2, \ldots, \delta - 1.$$

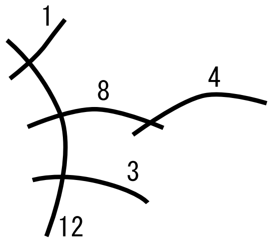


Fig. 4. The line with multiplicity 12 originates from the orbit space $\Sigma_3/f_{3,6}$, and other lines from branch points of $f_{3,6}$.

Next, we take δ copies of $\Theta_1,\dots,\Theta_\delta$ of 2-spheres ($=\mathbb{P}^1$). For $i=1,\dots,\delta$, we take a D^2-bundle N_i over the 2-sphere Θ_i such that the Chern number of N_i is $-r_i$, and take a D^2-bundle N_0 over a 2-disk. We patch N_i and N_{i+1} ($i=0,1,\dots,\delta-1$) by plumbing. Then we get a 4-manifold with a boundary which is homeomorphic to the boundary of the regular neighborhood of $[Q,0]$ in $(\Sigma_g\times D^2)/\mathbb{Z}_n$. We remove a neighborhood of $[Q,0]$ in $(\Sigma_g\times D^2)/\mathbb{Z}_n$ and glue the above 4-manifold. We apply this process for each singular points in $(\Sigma_g\times D^2)/\mathbb{Z}_n$, then we get the smooth 4-manifold M. Let Θ_0 be the surface in M, which originates from $\Sigma_g/f\times\{0\}$. Then, the regular neighborhood of Θ_0 is a D^2-bundle whose Euler number is $-(\theta_1+\theta_2+\cdots+\theta_b)/n$, and

$$(r')^{-1}(0)=m_0\Theta_0+\sum_{Q:\text{ branch point of } f} m_1\Theta_1+\cdots+m_\delta\Theta_\delta.$$

For example, we apply the above process to $f=f_{3,6}=(12,1/12+2/3+1/4)$, then $(\Sigma_3/f_{3,6})\times 0$ is replaced by spheres intersecting transversely as is shown in Fig. 4. We blow-down 2-spheres whose self-intersection numbers are -1, and continue to blow-down until there is no 2-sphere with self-intersection -1. For our example, Fig. 5 illustrate how this blow-down process is going on. At the end of this process, there remains a plane curve $x^4=y^3$ with an isolated singular point $(0,0)$.

Let Ψ be a map from $\mathbb{C}^2$ to $\mathbb{C}$ defined by $\Psi(x,y)=x^4-y^3$. If ϵ is a sufficiently small positive number and $S^3_\epsilon=\{(x,y)\in\mathbb{C}^2\mid |x|^2+|y|^2=\epsilon^2\}$, then $L_\epsilon=S^3_\epsilon\cap\Psi^{-1}(0)$ is a knot (in general, link) in S^3_ϵ. Let ψ be a map from $S^3_\epsilon-L_\epsilon$ to S^1 defined by $\psi(x,y)=\Psi(x,y)/|\Psi(x,y)|$, then ψ is a surface bundle map over S^1 whose fiber is a Seifert surface of L_ϵ. This fibration $\psi\colon S^3_\epsilon-L_\epsilon\to S^1$ is called a *Milnor-fibration* [28] of $x^4-y^3=0$. The monodromy of ψ is an orientation preserving diffeomorphism on the fiber surface with a boundary. If we cap the boundary of this surface with a disk and extend this monodromy by the identity map on this disk, then the conjugacy class of this diffeomorphism is $f_{3,6}$. Therefore, what we should do is finding a Dehn twist presentation of the topological monodromy of the Milnor-fibration of $x^4-y^3=0$. It is well-known that the knot around the singular point of $x^4-y^3=0$ is the $(3,4)$-torus knot [28] and it is easy to find the Dehn twist presentation of the monodromy from

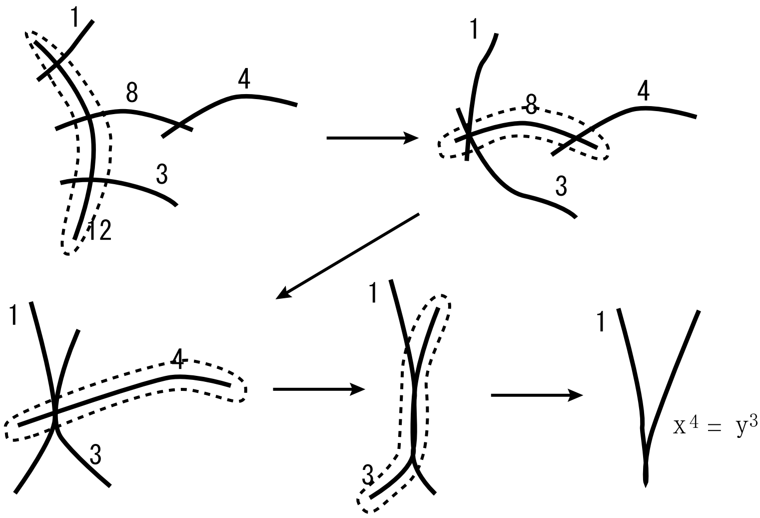


Fig. 5. 2-spheres with self-intersection -1 are surrounded by dotted circles.

this fact. Still, as a practical method to find presentations of periodic maps not only $f_{3,6}$ but also other cases which will appear later, we explain the theory of *divide link* invented by A'Campo [2, 3, 4]. This type of singularities and their monodromies are also investigated in the context of *real morsification*, for example, by Gusein–Zade [1].

First, we perturb $x^4 - y^3 = 0$. By a method introduced by H. Goda, M. Hirasawa, and Y. Yamada [15], we draw the picture of perturbed $x^4 - y^3 = 0$ in a plane. We draw a rectangle whose horizontal length is equal to the power for x (in our example, this is 4) and vertical length is equal to the power for y (in our example, this is 3), and divide this rectangle into unit squares. We draw a line from the right upper corner along the diagonals of squares. If we arrive at an edge of the rectangle, we continue to draw this line along the trajectory of light which is going on the line drawn before and reflected by the mirror on the edge (see the left hand of Fig. 6). If we arrive at a corner of the rectangle, we stop to draw. We smoothen corners of the line picture already drawn, then we get a curve which is the perturbed $x^4 - y^3 = 0$ (see the right hand of Fig. 6). We can embed this curve properly in the unit disk. This is an example of a *divide*.

The divide P is a relative, generic immersion of a compact 1-manifold in a unit disk D in $\mathbb{R}^2$. The link $L(P)$ of divide P is defined by

$$L(P) = \{(u, v) \in TD \mid u \in P,\ v \in T_u P,\ |u|^2 + |v|^2 = 1\} \subset S^3.$$

Let P be a divide shown in the right hand of Fig. 6, then $L(P)$ is isotopic to L_ϵ. In Fig. 6, we regard the horizontal coordinate as height. This divide P can be deformed

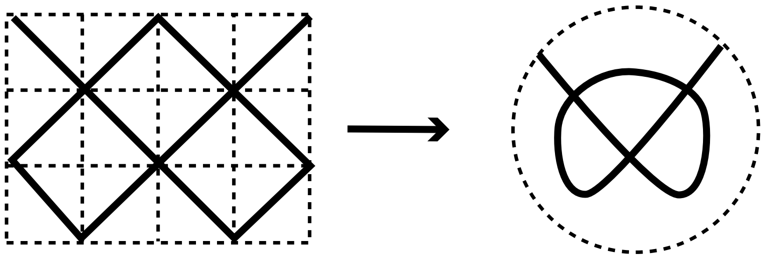


Fig. 6. The divide of $x^4 - y^3 = 0$.

into a position whose height function has only one local maximal value and only one local minimal value. A divide which satisfies this condition is an *ordered Morse* divide. For an ordered Morse divide, there is a method, introduced by O. Couture, B. Perron [8], H. Goda, M. Hirasawa and Y. Yamada [15], to visualize its link $L(P)$ and a fiber surface of $L(P)$. We deform the divide superposed on parallel lines as shown on the left hand of the top of Fig. 7, where dotted lines mean a position of the parallel lines. As shown on the right hand of the top of Fig. 7, we put a disk in place of each dotted line, we attach a pair of twisted bands to the boundary of the disks in place of each crossing, one of these bands is on the front and the other is on the back, and we attach a twisted band to the boundary of disks in place of each maximal or minimal arc. Let $F(P)$ be the surface obtained from the above process. The boundary of the surface $F(P)$ is isotopic to $L(P)$, and $F(P)$ is a fiber surface of $L(P)$. In order to make the later explanation simpler, we deform the surface as illustrated in the bottom of Fig. 7. This surface is a genus 3 surface with one boundary. We cap this boundary by a disk. We fix the identification of this closed surface with the surface illustrated as genus 3 case in Fig. 2, which identifies each curve with the curve of the same number.

A surface R in S^3 is a *Murasugi sum* of two surfaces R_1 and R_2 in S^3 if the following conditions are satisfied (see [13]):

(1) $R = R_1 \cup_\Delta R_2$, where Δ is a 2-disk such that $\partial\Delta = \mu_1 \cup \nu_1 \cup \cdots \cup \mu_n \cup \nu_n$, where μ_i (resp. ν_i) is a proper arc in R_1 (resp. R_2).

(2) There exist 3-balls B_1, B_2 in S^3 such that

- $B_1 \cup B_2 = S^3$, $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S^2$, and
- $R_1 \subset B_1$, $R_2 \subset B_2$, and $R_1 \cap \partial B_1 = R_2 \cap \partial B_2 = \Delta$.

If $(S^3, \partial R_i)$ is a fibered link whose fiber is R_i and monodromy is ϕ_i, then for the Murasugi sum R of R_1 and R_2, the link $(S^3, \partial R)$ is a fibered link whose fiber is R and whose monodromy is $\phi_1 \cdot \phi_2$ (see [13, 14]). A *positive Hopf band* is an annulus embedded in S^3 as in Fig. 8. In this paper, we treat only positive Hopf bands, so we call these *Hopf bands* for short. The boundary of a Hopf band is called a *Hopf link*. The Hopf link is a fibered link whose fiber is the Hopf band and whose monodromy is a right handed Dehn twist about the core circle of the Hopf band. In the bottom of Fig. 7, let B_i be a Hopf band whose core is the circle i. Then $F(P)$ is a Murasugi

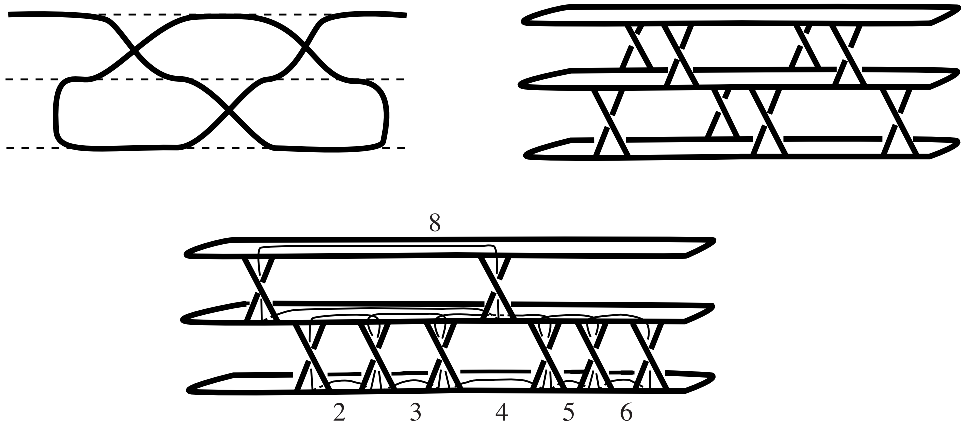


Fig. 7.

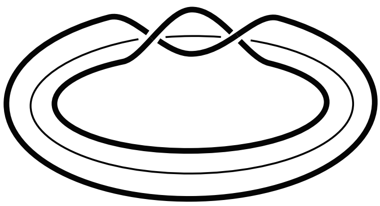


Fig. 8.

sum of B_i's. Therefore, the monodromy for $L(P)$ is the product of monodromies of Hopf bands B_i, that is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8$.

We apply the above method, then we see that $f_{3,8}$ is the monodromy of the Milnor-fibration of $x^3y - y^3 = 0$ capped by disks and $f_{4,5}$ is that of $x^5 - y^3 = 0$ capped by a disk. The divide for $x^3y - y^3 = 0$ and the fiber of the divide link for this divide is illustrated in the upper part of Fig. 9, and those for $x^5 - y^3 = 0$ is shown in the lower part of the same figure. From this figure, we obtain presentations of these periodic maps.

4.3. Digression: A family of periodic maps and their presentations by right Dehn twists. In this subsection, we introduce Dehn twist presentations of a family of periodic maps listed in the following. These presentations are obtained by the method explained in 4.2.

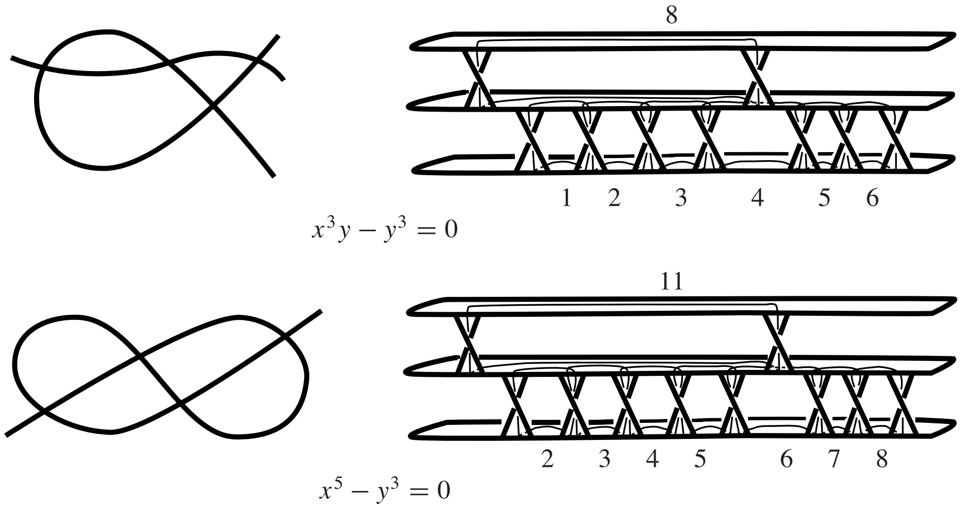


Fig. 9.

Phase 0:

$$g = 1, \quad \left(3, \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right), \quad x^3 - y^3 = 0, \quad \text{for} \quad k = 0,$$

$$g = 3k + 1, \quad \left(3k + 3, \frac{k}{k+1} + \frac{1}{3k+3} + \frac{1}{3k+3} + \frac{1}{3k+3}\right), \quad x^{3k+3} - y^3 = 0, \quad \text{for} \quad k \geq 1.$$

Phase I:

$$g = 3k + 3, \quad \left(3(3k + 4), \frac{2}{3} + \frac{k+1}{3k+4} + \frac{1}{3(3k+4)}\right), \quad x^{3k+4} - y^3 = 0.$$

Phase II:

$$g = 3k + 3, \quad \left(3(2k + 3), \frac{3k+5}{3(2k+3)} + \frac{k+1}{2k+3} + \frac{1}{3(2k+3)}\right), \quad x^{2k+3}y - y^3 = 0.$$

Phase III:

$$g = 3k + 4, \quad \left(3(3k + 5), \frac{2k+3}{3k+5} + \frac{1}{3} + \frac{1}{3(3k+5)}\right), \quad x^{3k+5} - y^3 = 0.$$

On each line of the above list, g is a genus, the second one is a total valency data, and the third equation defines a plane curve with a singularity whose monodromy capped by disk(s) is the periodic map defined by the total valency data. An integer $k \geq 0$ in the above list is called a *level*.

In Fig. 11, we illustrate Milnor fibers for plane curve singularities listed above. In this figure, a long horizontal line indicates a 2-disk, and a short vertical line indi-

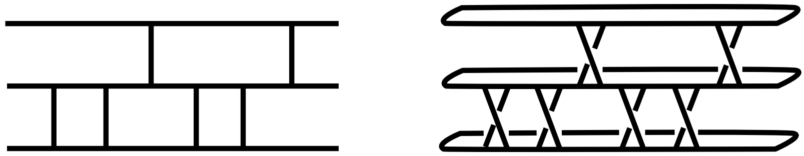


Fig. 10.

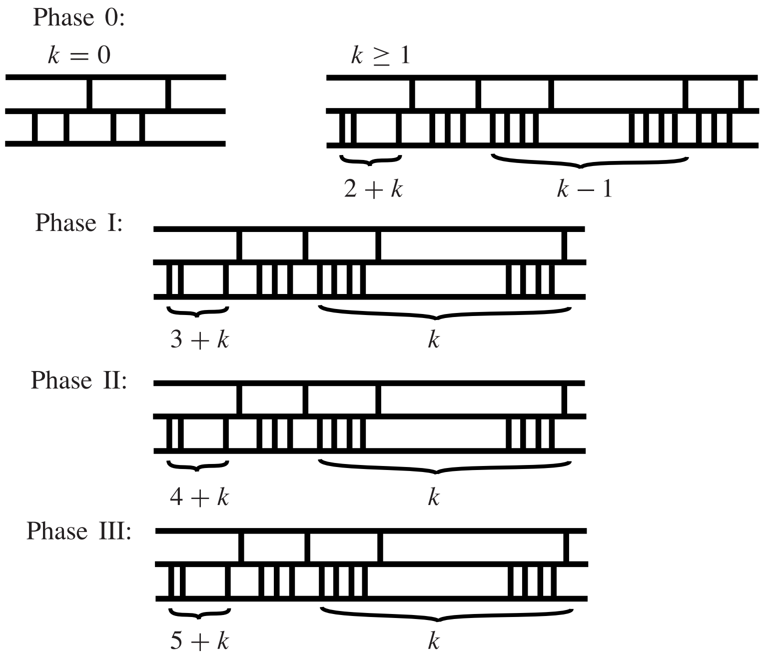


Fig. 11.

cates a twisted band connecting two 2-disks (see, for example, Fig. 10). We decompose these fibers into Murasugi sum of Hopf bands, then we obtain Dehn twist presentations of the periodic maps. In this subsection, we express Dehn twist presentations by figures. In each figure, there are two types of simple closed curves drawn on surfaces, thicker curves and thinner curves. We take a product of the right Dehn twists about thinner simple closed curves from right to left and take a product of the right Dehn twists about thicker simple closed curves from right to left. The product of Dehn twists obtained as a result is a Dehn twist presentation of the periodic map. In Fig. 12, the presentations of the level 0 case is given. We remark that these are monodromies of well-known singularities (see for example [1, Chapter 4]). Namely, I is the monodromy of the E_6 singularity: $x^4 - y^3 = 0$, II is the monodromy of the E_7 singularity: $x^3y - y^3 = 0$, and III is the monodromy of the E_8 singularity: $x^5 - y^3 = 0$.

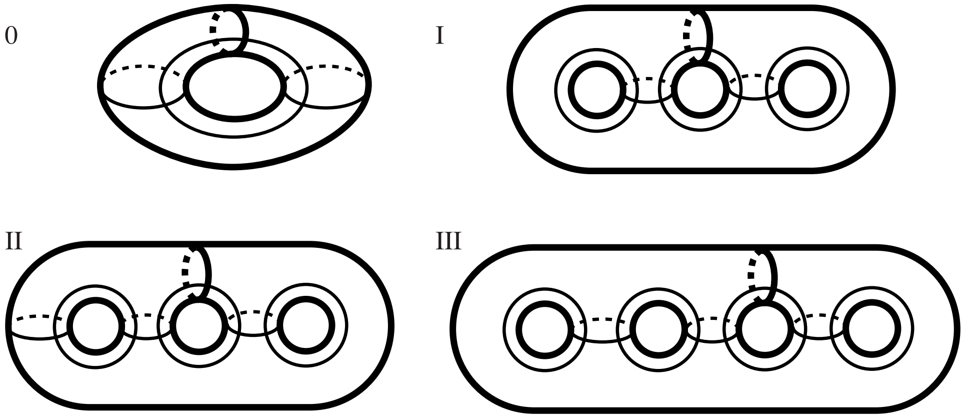


Fig. 12.

As we can observe in the presentations of level 0 case, there is the following relationships among four phases. For a presentation of phase 0, we attach handles to both sides and append one thinner circle in each handle, then we get a presentation of phase I of the same level. For a presentation of phase I, we append one more thinner circle to the left handle, then we get a presentation of phase II of the same level. For a presentation of phase II, we attach a handle to left side and append one thinner circle in the attached handle, then we get a presentation of phase III of the same level. Therefore, if we obtain presentations of all periodic maps of phase 0, then we obtain Dehn twist presentations of all listed periodic maps. For example, we illustrate the presentation of periodic maps of phase 0 of level 1 to 4, in Fig. 13.

For the level $k \geq 2$, we draw figures expressing Dehn twist presentations of periodic maps of phase 0 as follows. First of all, we prepare several pieces listed in Fig. 14. When the level k is even, we prepare one piece of type 0, $(k-2)/2$ pieces of type A, $(k-2)/2$ pieces of type B, and one piece of type T. When the level k is odd, we prepare one piece of type 0, $(k-3)/2$ pieces of type A, $(k-1)/2$ pieces of type B, and one piece of type T. Next, we put these pieces on a line from the left to the right. When the level k is even, the piece of type 0 comes the first, and type A, type B, $\dots$, type B, and the piece of type T comes the last. When the level k is even, the piece of type 0 comes the first, and type B, type A, $\dots$, type B, and the piece of type T comes the last. Finally, we glue these pieces along the parts indicated in Fig. 15, such that the arcs drawn on these parts are glued properly. Then we obtain a figure expressing the Dehn twist presentation.

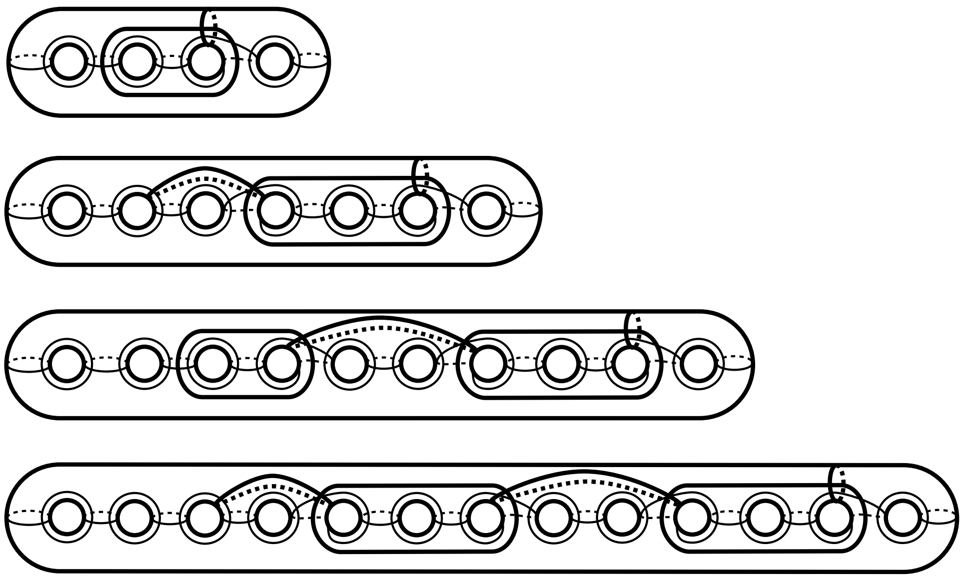
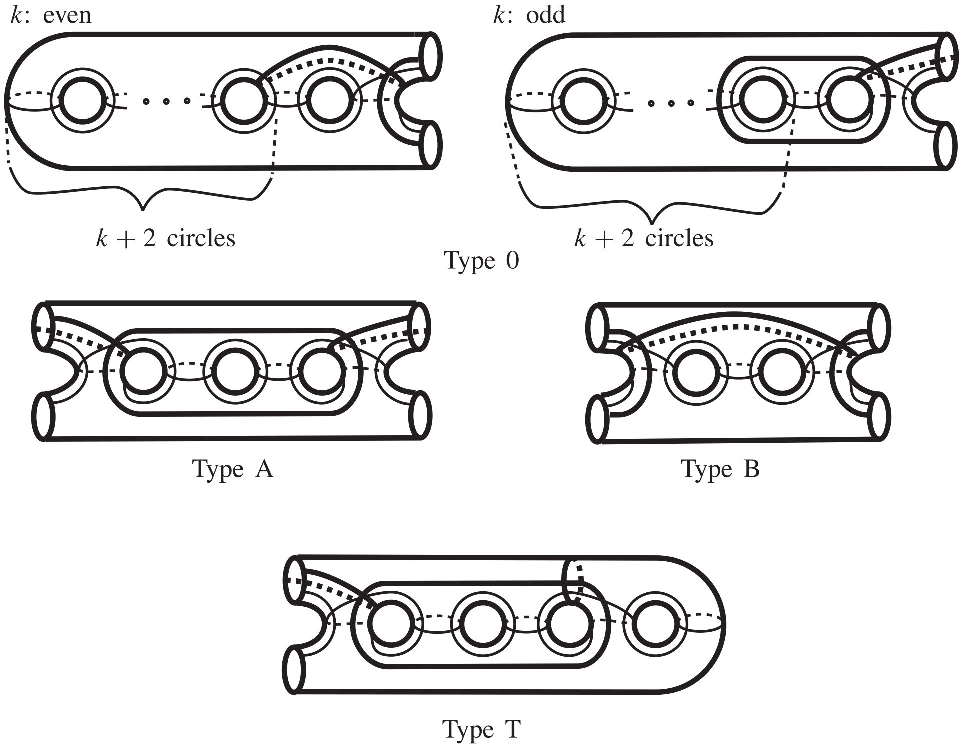
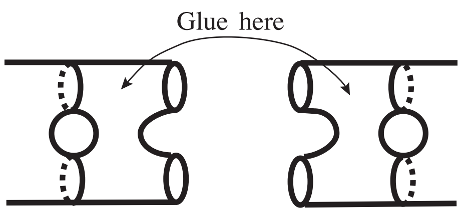


Fig. 13.



k: even
$k+2$ circles
k: odd
$k+2$ circles
Type 0
Type A
Type B
Type T

Fig. 14.



Glue here

Fig. 15.

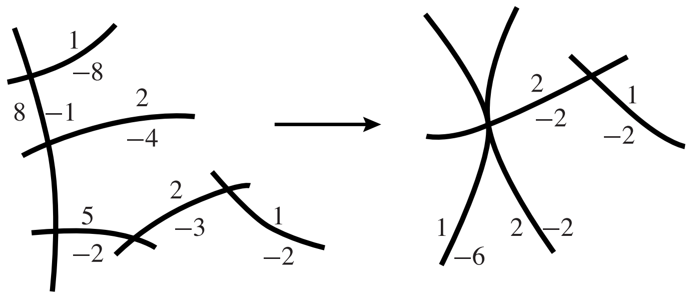


Fig. 16.

4.4. Method to find Dehn twist presentations of $f_{3,7}$, $f_{3,9}$, $f_{4,6}$, $f_{4,7}$. Here we explain a method to find a presentation for $f_{3,7}$. We find presentation for other three periodic maps by the same method.

Following the explanation in 4.2, we apply Hirzebruch–Jung resolution to $\Sigma_3 \times D^2/\mathbb{Z}_8$ of $f_{3,7} = (8, 1/8 + 1/4 + 5/8)$, then we get a configuration of lines on the left hand of Fig. 16. After we blow-down -1-curves for this diagram then we get the right hand of Fig. 16. Although the curve illustrated in this picture has a non-isolated singularity, we can not blow-down anymore. Therefore, we can not apply the method introduced in 4.2 to this case. If the singular curve is splitted into Lefschetz type singular fibers, then the number of singular fibers should be $\chi(\text{a singular curve}) - \chi(\Sigma_3) = 9$. Hence, we tried to find a presentation of $f_{3,7}$ by 9 right handed Dehn twists. Fortunately, we can find a product of right Dehn twists about 1 to 8 written in genus 3 case of Theorem 3.2, whose length is 9 and whose action on $H_1(\Sigma_3; \mathbb{Z})$ is period 8. We check the action of this product to simple closed curves on Σ_3 by using "Teruaki for Mathematica", implemented by K. Ahara and T. Sakasai.

4.5. Method to find Dehn twist presentations of $f_{4,9}$, $f_{4,10}$, $f_{4,11}$, $f_{4,12}$. An *essential* 1-*submanifold* of Σ_g is a disjoint union of simple closed curves in Σ_g such that (1) each component does not bound a disk in Σ_g and (2) no two components are homotopic. An orientation preserving self-homeomorphism of Σ_g is *reducible* if it leaves some essential 1-submanifold of Σ_g invariant. Four homeomorphisms $f_{4,9}$, $f_{4,10}$, $f_{4,11}$, $f_{4,12}$ of Σ_4 are reducible. Let f be a reducible periodic map of Σ_g, and l be a simple closed curve l on Σ_g/f so that $\pi_f^{-1}(l)$ is an essential 1-submanifold of Σ_g. Let N be a thin regular neighborhood of l in Σ_g/f, and let $\coprod N_i = \pi_f^{-1}(N)$ be the decomposition into connected components. We denote $F^c = \Sigma_g - \coprod N_i$, then $f|_{F^c}$ is a periodic map. The restriction of f_{F^c} to ∂f_{F^c} bounds a rotation g on disks D_i, D_i', so that $\partial D_i \amalg \partial D_i' = \partial N_i$. Let $\tilde{F} = F^c \bigcup_{\partial F^c} \big((\coprod D_i) \amalg (\coprod D_i') \big)$, then we can define a periodic map $\tilde{f}$ on $\tilde{F}$ so that $\tilde{f}|_{F^c} = f|_{F^c}$, $\tilde{f}|_{(\coprod D_i) \amalg (\coprod D_i')} = g$. We say this map $\tilde{f}$ is obtained from f by an *equivariant* 2-*surgery* along l. The genus of each components of $\tilde{F}$ is smaller than g.

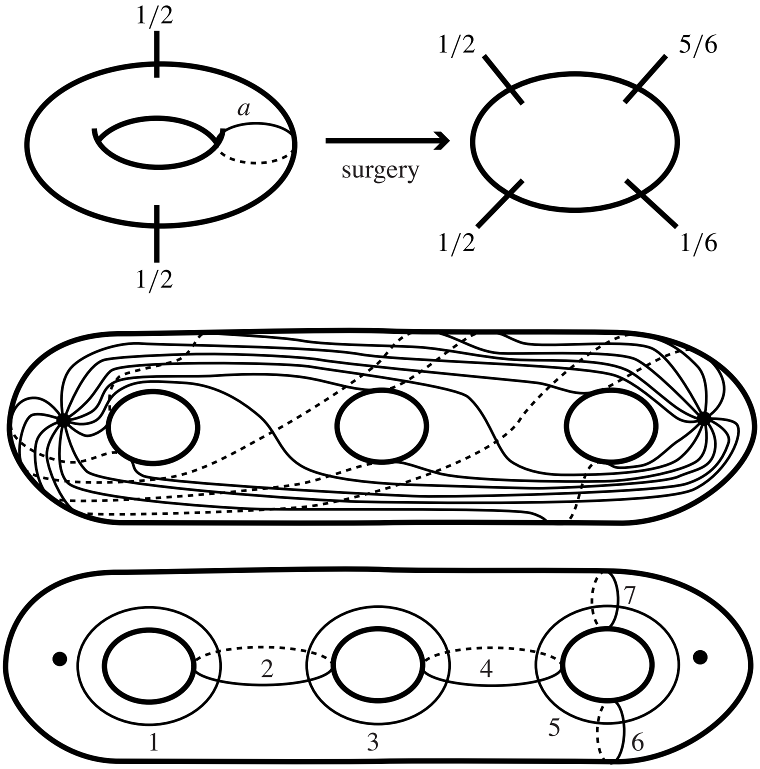


Fig. 17.

For example, we explain how we find the presentation of $f_{4,11} = (6, 1/2 + 1/2)$. We make an equivariant 2-surgery along the circle a as in the top of Fig. 17. Then the resulting periodic map $\tilde{f}_{4,11}$ is equal to $(6, 1/6 + 5/6 + 1/2 + 1/2) = f_{3,2}^2 = (12, 1/12 + 5/12 + 1/2)^2$ on Σ_3. The middle of Fig. 17 illustrate the graph in Σ_3 which is the inverse image of the arc in $\Sigma_3/f_{3,2}$ connecting the branch points whose valencies are $1/12$ and $5/12$. In this picture, the left (resp. right) vertex is the preimage of the branch point whose valency is $1/12$ (resp. $5/12$). Up to isotopy fixing these vertices, the product of twists $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, where each number means a right handed Dehn twist about the circle shown in the bottom of Fig. 17, fixes the graph set-wisely, and transforms the each edge to the edge clock-wisely adjacent at the left vertex. We dig small holes around vertices and glue these boundaries, then we get a surface F homeomorphic to Σ_4. Namely, there is a homeomorphism from F to the surface illustrated in genus 4 case in Fig. 2, which maps 1 to 2, 2 to 3, 3 to 4, 4 to 5, 5 to 6, 6 to 13, 7 to 7 and the circle obtained from the boundary to 9. Therefore, $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 13 \cdot 7)^{-2} \cdot f_{4,11}$ is isotopic to some powers of Dehn twist about 9. By checking the action of the above map into simple closed curves on Σ_4, we find the Dehn twist presentation for $f_{4,11}$.

4.6. Method to find a Dehn twist presentation of $f_{4,8}$. For this periodic map, we can not use the above methods to find a presentation. We write a picture of the cell decomposition of Σ_4 as in the top of Fig. 26, from this figure we observe the action of this map on the simple closed curves, and apply Lickorish's method [26] to find the presentation expressed in Theorem 3.2.

5. Application: examples of non-holomorphic Lefschetz fibrations

The remarkable works by Donaldson [10] and Gompf [16] show that Lefschetz fibrations are one of the most interesting objects for the study on 4-dimensional topology. We recall the definition:

DEFINITION 5.1. A *Lefschetz fibration* on an oriented compact smooth 4-manifold M over an oriented smooth surface S is a smooth map $f\colon M \to S$ which is a submersion everywhere except at finitely many non-degenerate critical points $p_1, \dots, p_r$, near which f identifies in orientation-preserving complex coordinates with the model map $(z_1, z_2) \mapsto z_1^2 + z_2^2$.

The smooth fibers of f are compact oriented surfaces and diffeomorphic each other. If the genus of the fiber is g, we call M a *genus g* Lefschetz fibration. In this paper, we always assume that the images $q_1, \dots, q_k$ of critical points $p_1, \dots, p_k$ are distinct. Under this assumption, each singular fiber is obtained by collapsing a simple closed curve (which we call a *vanishing cycle*) in the smooth fiber. The monodromy of a Lefschetz fibration f over S^2 is characterized by the homomorphism from the fundamental group of $S^2 \setminus \{q_1, \dots, q_k\}$ to the mapping class group $\mathcal{M}_g$. We choose a base point t_0 of $S^2 \setminus \{q_1, \dots, q_k\}$, and k embedded arcs A_i $(i = 1, \dots, k)$ beginning at t_0 and ending at q_i so that $\operatorname{int} A_i \cap \operatorname{int} A_j = \emptyset$ if $i \neq j$ and the arcs sit on the order $A_1, A_2, \dots, A_k$ counter-clockwisely around t_0. Let l_i be the simple loop beginning at t_0, going beside A_i, going around q_i counter-clockwisely, and going beside A_i back to t_0. Then $l_1 \cdot l_2 \cdots l_k$ is homotopic to the trivial loop. The image of l_i by the monodromy is the right Dehn twist t_i about an essential simple closed curve on Σ_g. Then the monodromy is characterized by the relation $t_1 \cdot t_2 \cdots t_k = id_{\Sigma_g}$ of the right Dehn twists, which we call a *positive relation*. On the other hand, any positive relation defines a Lefschetz fibration over S^2. From here, we write ϕ_ψ in place of $\psi^{-1} \cdot \phi \cdot \psi$, and $\bar{\phi}$ in place of ϕ^{-1}, for abbreviation.

We explain a sort of "fiber sum" of Lefschetz fibrations. A Lefschetz fibration over S^2 defined by the positive relation $W = id_{\Sigma_g}$ is denoted by $Lf(W)$. A Lefschetz fibration over D^2 defined by the product of right Dehn twists (*positive word*) W is denoted by $Lf_{D^2}(W)$. Let $id_{\Sigma_g} = V = V_1 \cdot V_2$ and $id_{\Sigma_g} = W = W_1 \cdot W_2$ be the positive relations so that V_1 is conjugate to W_1 in $\mathcal{M}_g$, i.e. there is an element ϕ of $\mathcal{M}_g$ so that $\phi^{-1} \cdot V_1 \cdot \phi = W_1$. Then $id_{\Sigma_g} = V_{1\phi} \cdot W_1$, where $V_{1\phi}$ is a positive word whose letters are conjugates of each letter of V_1 by ϕ, i.e. if $V_1 = 1 \cdot 2 \cdot 3 \cdot 4$ then $V_{1\phi} = 1_\phi \cdot 2_\phi \cdot 3_\phi \cdot 4_\phi$. We define

$V_1 \sharp W_2 = V_{1\phi} \cdot W_2$. The positive relation $V_1 \sharp W_2 = id_{\Sigma_g}$ defines a Lefschetz fibration $Lf(V_1 \sharp W_2)$. We can construct this Lefschetz fibration from $Lf_{D^2}(V_{1\phi})$ and $Lf_{D^2}(W_2)$ by glueing along their boundaries. By the Novikov additivity, the signature of $Lf(V_1 \sharp W_2)$ is $\sigma(Lf(V_1 \sharp W_2)) = \sigma(Lf_{D^2}(V_{1\phi})) + \sigma(Lf_{D^2}(W_2)) = \sigma(Lf_{D^2}(V_1)) + \sigma(Lf_{D^2}(W_2))$. The last equation follows from the equivalence of $Lf_{D^2}(V_{1\phi})$ and $Lf_{D^2}(V_1)$. For the Lefschetz fibration $Lf(W)$ (resp. $Lf_{D^2}(W)$), the number of singular fiber is denoted by $n(Lf(W))$ (resp. $n(Lf_{D^2}(W))$), which is equal to the word length of W. The *slope* $\lambda(Lf(W))$ is defined by

$$\lambda(Lf(W)) = 12 - \frac{4}{1 + \sigma(Lf(W))/n(Lf(W))}.$$

By using a sort of "fiber sum" explained above, we construct examples of non-holomorphic Lefschetz fibrations from our presentation of periodic maps. Let f_1 and f_2 be periodic maps on Σ_g with periods n_1 and n_2 respectively, and X_i be a presentation of f_i by right hand Dehn twists. We assume that there are integers k_1 and k_2 so that $f_1^{k_1}$ is conjugate to $f_2^{k_2}$ and $0 < k_i < n_i$ $(i = 1, 2)$. Then $X_1^{k_1} \sharp X_2^{n_2-k_2} = id_{\Sigma_g}$ is a positive relation and we define a Lefschetz fibration $Lf(X_1^{k_1} \sharp X_2^{n_2-k_2})$. We show that the following examples are not holomorphic by the method introduced by Endo and Nagami [12].

EXAMPLE 1. On Σ_3, $f_{3,3}^4 = (2, 1/2 + 1/2 + 1/2 + 1/2) = f_{3,6}^6$ up to conjugacy. Let X_1 be a Dehn twist presentation of $f_{3,3}$ shown in Theorem 3.2, and X_2 be that of $f_{3,6}$. From the above construction, we get a genus 3 Lefschetz fibration $Lf(X_1^4 \sharp X_2^6)$ over S^2. By calculating the Meyer's signature cocycles, we get $\sigma(Lf(X_1^4 \sharp X_2^6)) = \sigma(Lf_{D^2}(X_1^4)) + \sigma(Lf_{D^2}(X_2^6)) = (-16) + (-20) = -36$. Easily, we see $n(Lf(X_1^4 \sharp X_2^6)) = 64$. We assume that $Lf(X_1^4 \sharp X_2^6)$ is hyperelliptic. Since the vanishing cycle of each singular fiber $Lf(X_1^4 \sharp X_2^6)$ is non-separating, by the result of Endo [11], we obtain the equation $\sigma(Lf(X_1^4 \sharp X_2^6)) = n(Lf(X_1^4 \sharp X_2^6)) \times -(g + 1)/(2g + 1)$. But this contradicts $\sigma(Lf(X_1^4 \sharp X_2^6)) = -36$. Therefore $Lf(X_1^4 \sharp X_2^6)$ is not hyperelliptic. K. Konno [24] showed that if a non-hyperelliptic Lefschetz fibration $Lf(W)$ is isotopic to a holomorphic fibration of genus 3 then $\lambda(Lf(W)) \geq 3$. But in our case, $\lambda(Lf(X_1^4 \sharp X_2^6)) = 20/7 < 3$, hence $Lf(X_1^4 \sharp X_2^6)$ is non-holomorphic.

REMARK 5.2. Although we do not need to find ϕ such that $\phi^{-1} X_1^4 \phi = X_2^6$ in order to calculate the signature of $Lf(X_1^4 \sharp X_2^6)$, we find the explicit form of ϕ for its own interest.

The product of Dehn twists $X_1^4 = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4$ transforms the simple closed curves $1, \dots, 8$, which are as shown in genus 3 case of Fig. 2, to the simple closed curves shown in Fig. 18. The product of Dehn twists $X_2^6 = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2)^6$ transforms the same simple closed curves to the simple closed curves shown in Fig. 19. Since $\bar{7} \cdot \bar{6} \cdot \bar{5} \cdot 1 \cdot 2 \cdot 3$ transforms Fig. 19 to Fig. 18, $(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)^4 = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2)^6 \bar{7} \cdot \bar{6} \cdot \bar{5} \cdot 1 \cdot 2 \cdot 3$,

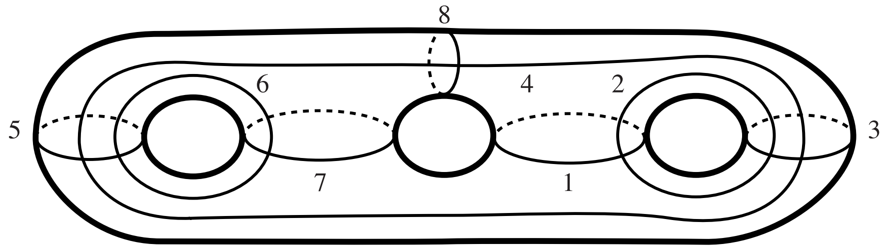


Fig. 18.

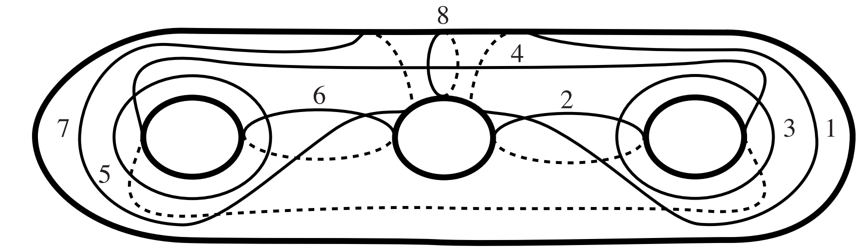


Fig. 19.

where $\bar{\phi}$ denotes ϕ^{-1}. On the other hand, $\overline{(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^4}$ transforms 1 to 5, 2 to 6, and 3 to 7. Therefore, $(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)^4 = 5\cdot 6\cdot 7(8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2)^6\bar{7}\cdot\bar{6}\cdot\bar{5} = (8_{\bar7\bar6\bar5}\cdot 7_{\bar7\bar6\bar5}\cdot 6_{\bar7\bar6\bar5}\cdot 5_{\bar7\bar6\bar5}\cdot 4_{\bar7\bar6\bar5}\cdot 3_{\bar7\bar6\bar5}\cdot 2_{\bar7\bar6\bar5})^6$.

EXAMPLE 2. On Σ_4, $f^4_{4,8} = (3, 1/3 + 1/3 + 1/3) = f^6_{4,1}$ up to conjugacy. Let X_1 be a Dehn twist presentation of $f_{4,8}$ shown in Theorem 3.2, and X_2 be that of $f_{4,1}$. Then $\sigma(Lf(X_1^4 \sharp X_2^{12})) = \sigma(Lf_{D^2}(X_1^4)) + \sigma(Lf_{D^2}(X_2^{12})) = (-18) + (-54) = -72$, and $n(Lf(X_1^4 \sharp X_2^{12})) = 132$. By the same argument as we gave in the above example, we see $Lf(X_1^4 \sharp X_2^{12})$ is not hyperelliptic. Z.J. Chen [7] and K. Konno [25] showed that if a non-hyperelliptic Lefschetz fibration $Lf(W)$ is isotopic to a holomorphic fibration of genus 4, then $\lambda(Lf(W)) \geq 24/7$. But in our case, $\lambda(Lf(X_1^4 \sharp X_2^{12})) = 16/5$, hence $Lf(X_1^4 \sharp X_2^{12})$ is non-holomorphic.

REMARK 5.3. We see $\lambda(Lf(W_3^8 \sharp W_4^6)) = 24/7$, so we can not determine whether $Lf(W_3^8 \sharp W_4^6)$ is holomorphic or not by our method.

EXAMPLE 3. On Σ_4, $f^6_{4,6} = (2, 1/2 + 1/2) = f^5_{4,3}$ up to conjugacy. Let X_1 be a Dehn twist presentation of $f_{4,6}$ shown in Theorem 3.2, and X_2 be that of $f_{4,3}$. Then $\sigma(Lf(X_1^6 \sharp X_2^5)) = \sigma(Lf_{D^2}(X_1^6)) + \sigma(Lf_{D^2}(X_2^5)) = (-32) + (-25) = -57$, $n(Lf(X_1^6 \sharp X_2^5)) = 105$ and $\lambda(Lf(X_1^6 \sharp X_2^5)) = 13/4$. By the same argument as we explained in the above example, we see that $Lf(X_1^6 \sharp X_2^5)$ is non-holomorphic.

6. Verification of presentations and powers of periodic maps

6.1. Verification of presentations in Theorem 3.2. As we explained before, the presentations for the hyperelliptic periodic maps are obtained by Ishizaka [21]. The periodic maps $f_{3,6}$, $f_{3,7}$, $f_{3,8}$, $f_{3,9}$, $f_{4,5}$, $f_{4,6}$, $f_{4,7}$, $f_{4,8}$, $f_{4,9}$, $f_{4,10}$, $f_{4,11}$, $f_{4,12}$ are non-hyperelliptic periodic maps. For these periodic maps, we will verify the presentation by investigating the action of Dehn twist on simple closed curves and arcs in Σ_g. These are three cases to consider

(1) not reducible with at least two fixed points ($f_{3,7}$, $f_{3,8}$, $f_{3,9}$, $f_{4,6}$, $f_{4,7}$),
(2) not reducible with only one fixed point ($f_{3,6}$, $f_{4,5}$, $f_{4,8}$), and
(3) reducible ($f_{4,9}$, $f_{4,10}$, $f_{4,11}$, $f_{4,12}$).

We verify the presentations case by case.

(1) not reducible with at least two fixed points:

$f_{3,7} = (8, 1/8 + 1/4 + 5/8)$: The orbit space $\Sigma_3/f_{3,7}$ is a 2-sphere with three branch points, whose valencies are 1/8, 1/4 and 5/8. In the top of Fig. 20, the vertex p_1 is the preimage of the branch point with valency 1/8 by $\pi_{f_{3,7}}$, the vertex p_2 is the preimage of the branch point with valency 5/8, and edges are the preimages of arc connecting these branch points. Up to isotopy fixing these vertices, a product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5' \cdot 4 \cdot 3 \cdot 8$ (these loops are shown in the bottom of Fig. 20) fixes the graph set-wisely, and transforms each edge to the edge clock-wisely adjacent at the vertex p_1. If we forget p_2, then $5 = 5'$, so we conclude $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 8 = f_{3,7}$.

$f_{3,8} = (9, 1/9 + 1/3 + 5/9)$: In the top of Fig. 21, the vertex p_1 is the preimage of the branch point with valency 1/9 by $\pi_{f_{3,7}}$, the vertex p_2 is the preimage of the branch point with valency 5/9, and edges are the preimages of arc connecting these branch points. We number the thick edge by 0, and other edges by I, II, …, VIII clockwise around p_1. A product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 8$ (these loops are shown in the bottom Fig. 21) transforms 0 to I, I to II, …, VIII to 0, so we conclude $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 8 = f_{3,8}$.

$f_{3,9} = (7, 1/7 + 2/7 + 4/7)$: We delete edges I and VI from Fig. 21, then we get a graph on Σ_3 which is the preimage of a edge connecting branch points with valencies 1/7 and 4/7 by $\pi_{f_{3,9}}$. A product of Dehn twists $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8$ (these loops are shown in the bottom of Fig. 21) transforms edges clockwise at p_1, so we conclude $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 8 = f_{3,9}$.

$f_{4,6} = (12, 1/12 + 1/3 + 7/12)$: In the top of Fig. 22, the vertex p_1 is the preimage of the branch point with valency 1/12 by $\pi_{f_{4,6}}$, the other vertex is the preimage of the branch point with valency 7/12, and edges are preimages of arc connecting these branch points. We observe the action of the product of Dehn twists $6 \cdot 5' \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11$ (these loops are shown in the bottom of Fig. 22). If we forget the vertex which is not p, then $5 = 5'$, so we conclude $6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11 = f_{4,6}$.

$f_{4,7} = (10, 1/10 + 3/10 + 3/5)$: In the top of Fig. 23, the vertex p_1 is the preimage of the branch point with valency 1/10 by $\pi_{f_{4,7}}$, the other vertex is the preimage of the branch point with valency 3/10, and edges are the preimages of arc connecting these branch points. We observe the action of a product of Dehn twists $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 7' \cdot 6 \cdot 5 \cdot 4 \cdot 11$

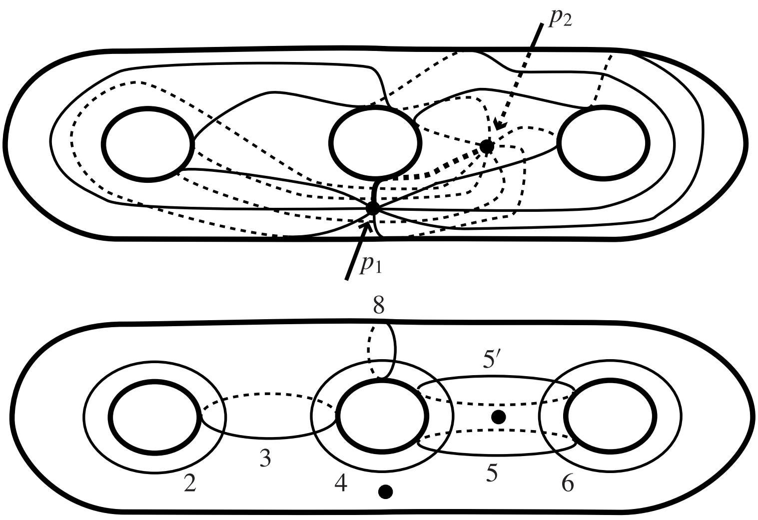


Fig. 20.

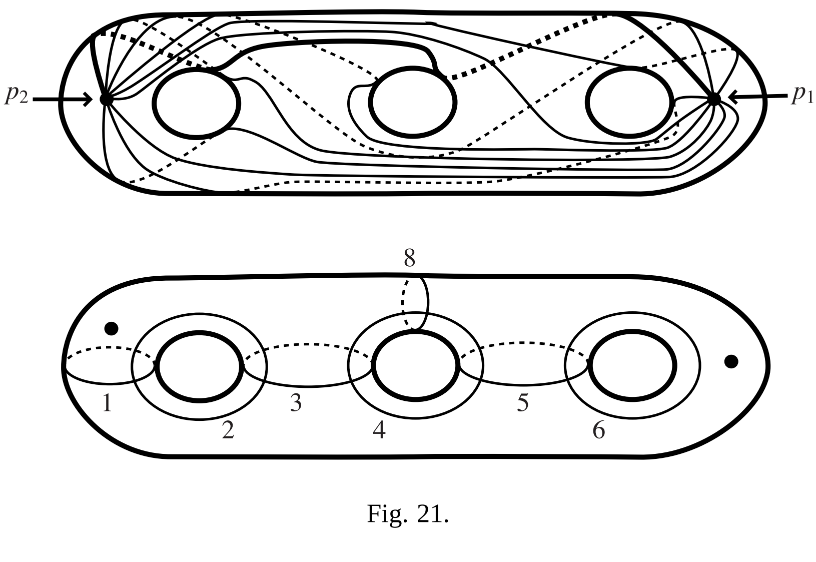


Fig. 21.

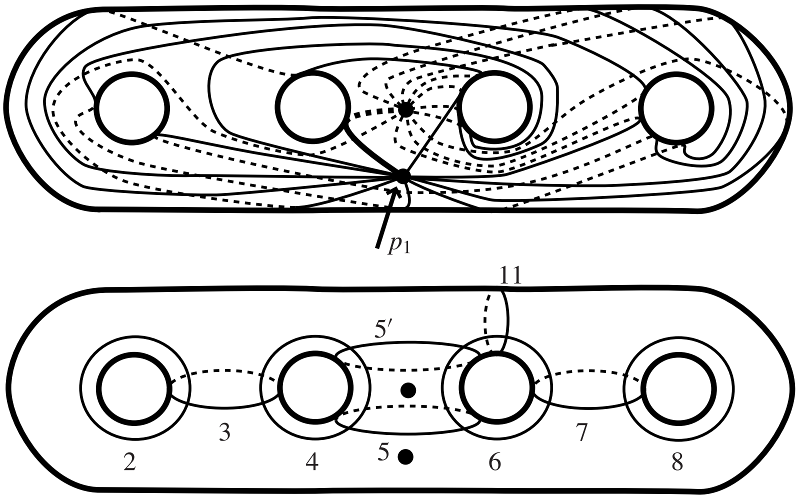


Fig. 22.

(these loops are shown in the bottom of Fig. 23). If we forget the vertex which is not p_1, then $7 = 7'$, so we conclude $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 11 = f_{4,7}$.

(2) not reducible with only one fixed point:

$f_{3,6} = (12, 1/12 + 1/4 + 2/3)$: The orbit space $\Sigma_3/f_{3,6}$ is a 2-sphere with three branch points, whose valencies are 1/12, 1/4 and 2/3. In Fig. 24, the vertex p is the preimage of the branch point with valency 1/12 by $\pi_{f_{3,6}}$, three other vertices are the preimages of the branch point with valency 1/4, and edges are the preimages of the arc connecting these branch points. We number the thick edge by 0, and other edges by I, II, …, XI clockwise around p. Let A_0 be the loop constructed from two edges 0 and IX connecting at p and the other end point, then A_0 is a loop with a base point p. This loop is denoted by $0 \to$ IX, and we use the same style of notations from here. Let A_1 be the loop I $\to$ X, A_2 be II $\to$ XI, A_3 be III $\to 0, \ldots,$ and A_{11} be XI $\to$ VIII. We perturb these loops so that these are intersect only at p and we use the same symbol for these loops got as a result. Let l be the loop in $\Sigma_3/f_{3,6}$ whose base point is the branch point with valency 1/12, and which bounds two 2-disks in $\Sigma_3/f_{3,6}$, one of which contains the branch point with valency 1/4 and the other of which contains the branch point with valency 2/3. Then $\pi_{f_{3,6}}^{-1}(l) = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_{11}$. Up to isotopy fixing p, $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8$ transforms A_0 to A_1, A_1 to $A_2, \ldots, A_{11}$ to A_0, so we conclude $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 8 = f_{3,6}$.

$f_{4,5} = (15, 1/15 + 1/3 + 3/5)$: In Fig. 25, the vertex p is the preimage of the branch point with valency 1/15 by $\pi_{f_{4,5}}$, five other vertices are the preimages of the branch point with valency 1/3, and edges are the preimages of arc connecting these branch points. We number the thick edge by 0, and other edges by I, II, …, XIV clockwise around p. Let A_0 be the loop $0 \to X$ whose base point is p, A_1 be I $\to$ XI,

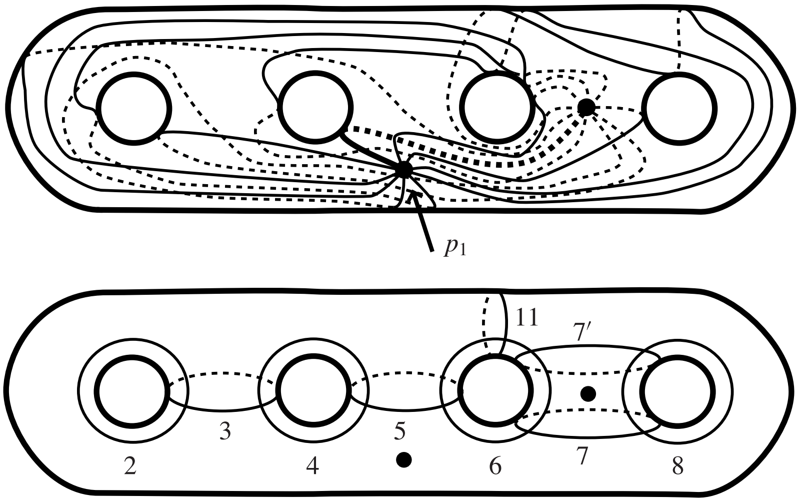


Fig. 23.

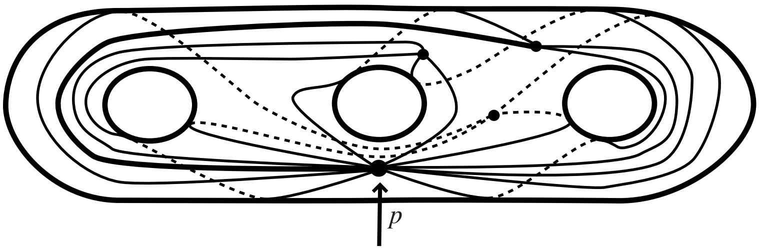


Fig. 24.

A_2 be II $\to$ XII, $\ldots$, and A_{14} be XIV $\to$ IX. Since $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11$ transforms A_0 to A_1, A_1 to A_2, $\cdots$, A_{14} to A_0, we conclude $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 11 = f_{4,5}$.

$f_{4,8} = (12, 1/12 + 1/6 + 3/4)$: In the top of Fig. 26, the vertex p is the preimage of the branch point with valency $1/12$ by $\pi_{f_{4,8}}$, three other vertices are the preimages of the branch point with valency $3/4$, and edges are the preimages of arc connecting these branch points. We number the thick edge by 0, and other edges by I, II, $\ldots$, XI clockwise around p. Let A_0 be the loop $0 \to$ III whose base point is p, A_1 be I $\to$ VI, A_2 be II $\to$ V, $\ldots$, and A_{11} be XI $\to$ II. Since $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 6 \cdot 11 \cdot 14$ (these loops are shown in the bottom of Fig. 26) transforms A_0 to A_1, A_1 to $A_2, \ldots, A_{11}$ to A_0, we conclude $3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 6 \cdot 11 \cdot 14 = f_{3,8}$.

(3) reducible:

$f_{4,9} = (6, 1/6 + 1/3 + 2/3 + 5/6)$: The bottom left hand of Fig. 27 illustrate the orbit space $\Sigma_4/f_{4,9}$. In Fig. 27, p_1 and p_2 is the inverse image of the branch points with valencies $1/6$ and $5/6$ respectively by $f_{4,9}$, the graph is the inverse image

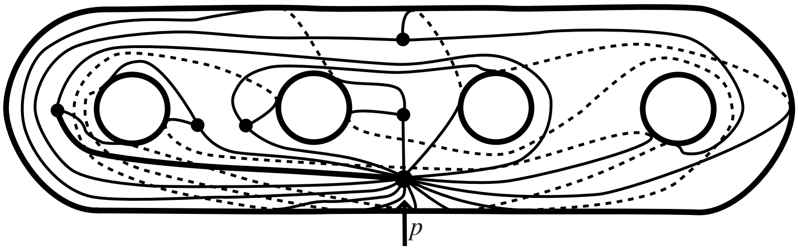


Fig. 25.

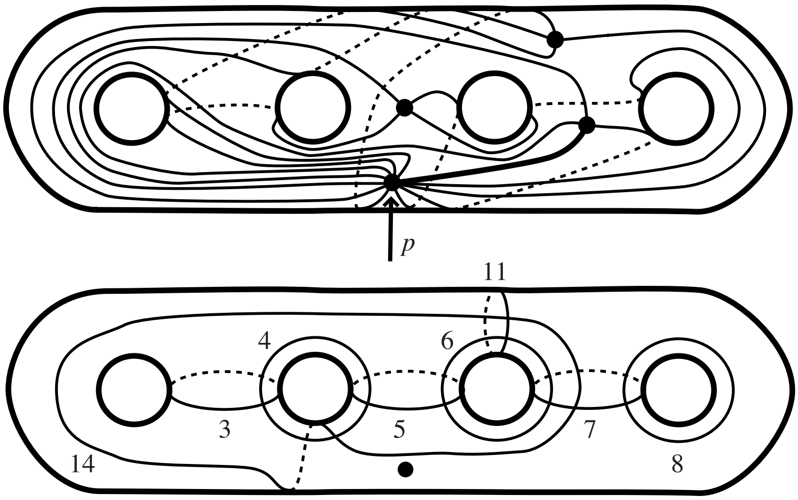


Fig. 26.

of the arc in the bottom left hand of Fig. 27, and the circle is the inverse image of the circle in the bottom left hand of Fig. 27. From the observation of the action of $1\cdot 2\cdot 3\cdot 4\cdot 10\cdot 9^{-1}\cdot 8^{-1}\cdot 7^{-1}\cdot 6^{-1}\cdot 11^{-1}$ (these loops are shown in the bottom right hand of Fig. 27), we conclude $1\cdot 2\cdot 3\cdot 4\cdot 10\cdot 9^{-1}\cdot 8^{-1}\cdot 7^{-1}\cdot 6^{-1}\cdot 11^{-1} = f_{4,9}$.

$f_{4,10} = (6, 1/3+1/3+1/3+1/2+1/2)$: The orbit space $\Sigma_4/f_{4,10}$ is as shown in the bottom left hand of Fig. 28. In the top of Fig. 28, the oriented simple closed curve is the preimage of a by $f_{4,10}$, the other separating simple closed curves is the preimage of b, and three other simple closed curves are the preimage of the arc connecting two branch points with valencies 1/2 and 1/2. We number the component of these three curves drawn with thick line by 0, and other components by I, II following the orientation of the curve which is a preimage of a. A product of twists $1\cdot 2\cdot 12^{-1}\cdot 4^{-1}\cdot 5^{-1}\cdot 6^{-1}\cdot 11^{-1}\cdot 8\cdot 9$ (these loops are as shown in a figure in genus 4 case of Theorem 3.2) fixes the preimages of a and b, and transforms 0 to I, I to II, II to 0 with opposite orientation. Therefore, we conclude $1\cdot 2\cdot 12^{-1}\cdot 4^{-1}\cdot 5^{-1}\cdot 6^{-1}\cdot 11^{-1}\cdot 8\cdot 9 = f_{4,10}$.

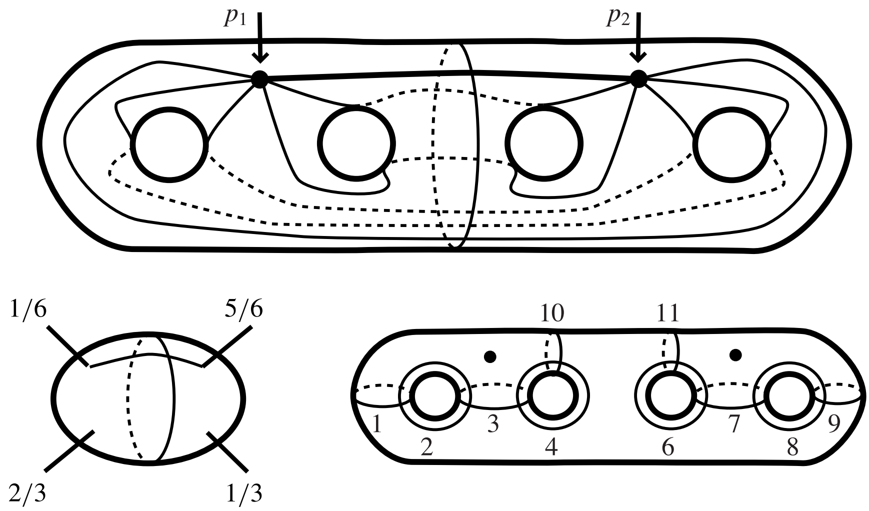


Fig. 27.

$f_{4,11} = (6, 1/2 + 1/2)$: The orbit space $\Sigma_4/f_{4,11}$ is as shown in the bottom right hand side of Fig. 28. In the top of Fig. 29, the oriented simple closed curve is the preimage of a by $\pi_{f_{4,11}}$, and the six other simple closed curves are the preimage of b. We number the component of the preimage of b drawn with thick line by 0, and other components by I, II, ..., V following the orientation of the curve which is a preimage of a. The product of twists $(2\cdot3\cdot4\cdot5\cdot6\cdot13\cdot7)^2\cdot9^{-1}$ (these loops are shown in a figure of genus 4 surface in Fig. 2) transforms 0 to I, I to II, ..., V to 0. In the bottom of Fig. 29, the oriented simple closed curve is the preimage of a by $\pi_{f_{4,11}}$, and the three other simple closed curves are the preimage of the arc connecting branch points on the orbit space shown in the bottom right hand of Fig. 28. We number the component of the preimage of the arc connecting branch points drawn with thick line by 0, and other component by I, II following the orientation of the curve which is a preimage of a. The product of twists $(2\cdot3\cdot4\cdot5\cdot6\cdot13\cdot7)^2\cdot9^{-1}$ transforms 0 to I, I to II, II to 0 with opposite orientation. The simple closed curves without arrows in Fig. 29 are disjoint each other and these circles and the preimage of a divide Σ_4 into disks. Therefore, we conclude $(2\cdot3\cdot4\cdot5\cdot6\cdot13\cdot7)^2\cdot9^{-1} = f_{4,11}$.

$f_{4,12} = (5, 1/5+2/5+3/5+4/5)$: The bottom left hand of Fig. 30 illustrate the orbit space $\Sigma_4/f_{4,12}$. In Fig. 30, p_1 and p_2 is the inverse images of the branch points with valencies 1/5 and 4/5 respectively by $\pi_{f_{4,12}}$, the graph is the inverse image of the arc in the bottom left hand of Fig. 30, and the circle is the inverse image of the circle in the bottom left hand of Fig. 30. From the observation of the action of $2\cdot3\cdot4\cdot12\cdot3\cdot4\cdot10\cdot3\cdot8^{-1}\cdot7^{-1}\cdot6^{-1}\cdot13^{-1}\cdot7^{-1}\cdot6^{-1}\cdot11^{-1}\cdot7^{-1}$ (these loops are as shown in the bottom right hand of Fig. 30), we conclude $2\cdot3\cdot4\cdot12\cdot3\cdot4\cdot10\cdot3\cdot8^{-1}\cdot7^{-1}\cdot6^{-1}\cdot13^{-1}\cdot7^{-1}\cdot6^{-1}\cdot11^{-1}\cdot7^{-1} = f_{4,9}$.

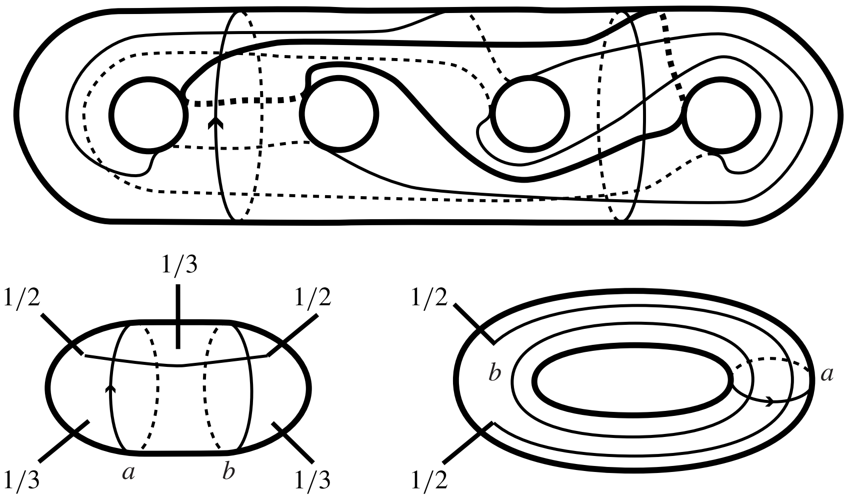


Fig. 28.

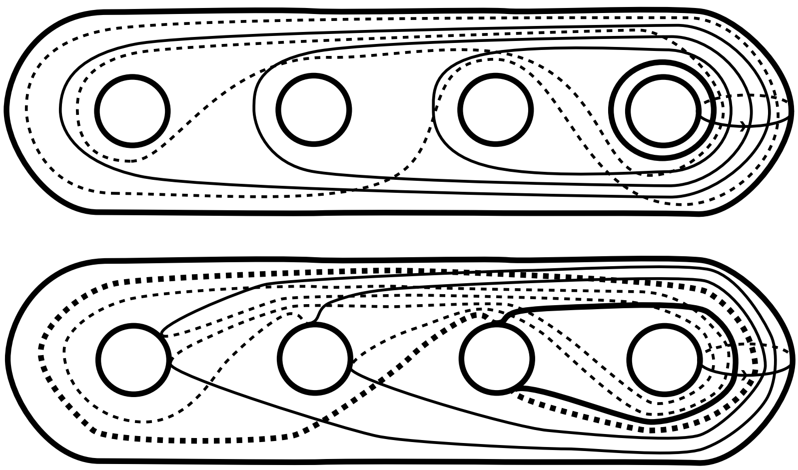


Fig. 29.

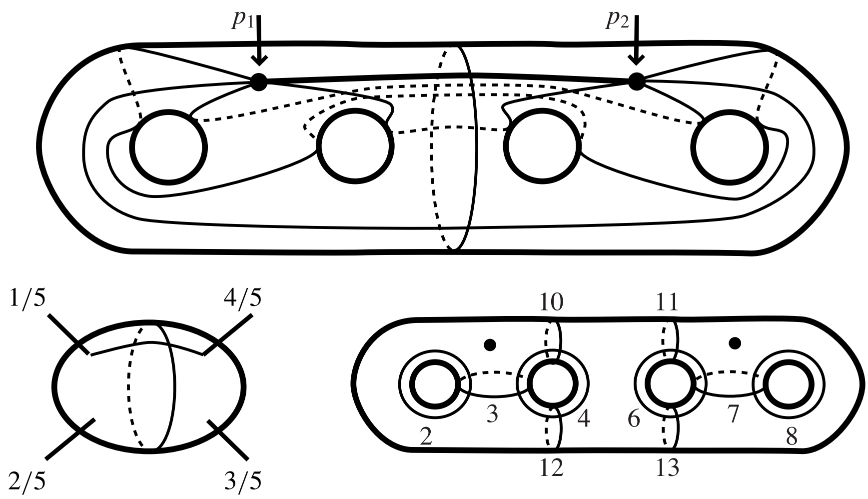


Fig. 30.

6.2. Tables of powers of periodic maps in Proposition 3.1. We can get the following table by using very simple program, for example, implemented for GAP.

(1) Genus 1 with multiple points:

$f_{1,1}$	$\left(6, \frac{1}{6}+\frac{1}{3}+\frac{1}{2}\right)$
$f_{1,1}^2$	$\left(3, \frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{1,1}^3$	$\left(2, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{1,1}^4$	$\left(3, \frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{1,1}^5$	$\left(6, \frac{1}{2}+\frac{2}{3}+\frac{5}{6}\right)$
$f_{1,1}^6$	id

$f_{1,2}$	$\left(4, \frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right)$
$f_{1,2}^2$	$\left(2, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{1,2}^3$	$\left(4, \frac{1}{2}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{1,2}^4$	id

(2) Genus 2:

$f_{2,1}$	$\left(10, \frac{1}{10}+\frac{2}{5}+\frac{1}{2}\right)$
$f_{2,1}^2$	$\left(5, \frac{1}{5}+\frac{2}{5}+\frac{2}{5}\right)$
$f_{2,1}^3$	$\left(10, \frac{1}{2}+\frac{7}{10}+\frac{4}{5}\right)$
$f_{2,1}^4$	$\left(5, \frac{1}{5}+\frac{1}{5}+\frac{3}{5}\right)$
$f_{2,1}^5$	$\left(2, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{2,1}^6$	$\left(5, \frac{2}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{2,1}^7$	$\left(10, \frac{1}{5}+\frac{3}{10}+\frac{1}{2}\right)$
$f_{2,1}^8$	$\left(5, \frac{3}{5}+\frac{3}{5}+\frac{4}{5}\right)$
$f_{2,1}^9$	$\left(10, \frac{1}{2}+\frac{3}{5}+\frac{9}{10}\right)$
$f_{2,1}^{10}$	id

$f_{2,2}$	$\left(8, \frac{1}{8}+\frac{3}{8}+\frac{1}{2}\right)$
$f_{2,2}^2$	$\left(4, \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}\right)$
$f_{2,2}^3$	$\left(8, \frac{1}{8}+\frac{3}{8}+\frac{1}{2}\right)$
$f_{2,2}^4$	$\left(2, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{2,2}^5$	$\left(8, \frac{1}{2}+\frac{5}{8}+\frac{7}{8}\right)$
$f_{2,2}^6$	$\left(4, \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}\right)$
$f_{2,2}^7$	$\left(8, \frac{1}{2}+\frac{5}{8}+\frac{7}{8}\right)$
$f_{2,2}^8$	id

$f_{2,3}$	$\left(6, \frac{1}{6}+\frac{1}{6}+\frac{2}{3}\right)$
$f_{2,3}^2$	$\left(3, \frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{2,3}^3$	$\left(2, \frac{1}{2}+\frac{1}{2}\right)$
$f_{2,3}^4$	$\left(3, \frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{2,3}^5$	$\left(6, \frac{1}{3}+\frac{5}{6}+\frac{5}{6}\right)$
$f_{2,3}^6$	id

$f_{2,4}$	$\left(6, \frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3}\right)$
$f_{2,4}^2$	$\left(3, \frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{2,4}^3$	$\left(2, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{2,4}^4$	$\left(3, \frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{2,4}^5$	$\left(6, \frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3}\right)$
$f_{2,4}^6$	id

(3) Genus 3:

$f_{3,1}$	$\left(14,\ \frac{1}{14}+\frac{3}{7}+\frac{1}{2}\right)$
$f_{3,1}^2$	$\left(7,\ \frac{1}{7}+\frac{3}{7}+\frac{3}{7}\right)$
$f_{3,1}^3$	$\left(14,\ \frac{1}{7}+\frac{5}{14}+\frac{1}{2}\right)$
$f_{3,1}^4$	$\left(7,\ \frac{4}{7}+\frac{5}{7}+\frac{5}{7}\right)$
$f_{3,1}^5$	$\left(14,\ \frac{3}{14}+\frac{2}{7}+\frac{1}{2}\right)$
$f_{3,1}^6$	$\left(7,\ \frac{1}{7}+\frac{1}{7}+\frac{5}{7}\right)$
$f_{3,1}^7$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,1}^8$	$\left(7,\ \frac{2}{7}+\frac{6}{7}+\frac{6}{7}\right)$
$f_{3,1}^9$	$\left(14,\ \frac{1}{2}+\frac{5}{7}+\frac{11}{14}\right)$
$f_{3,1}^{10}$	$\left(7,\ \frac{2}{7}+\frac{2}{7}+\frac{3}{7}\right)$
$f_{3,1}^{11}$	$\left(14,\ \frac{1}{2}+\frac{9}{14}+\frac{6}{7}\right)$
$f_{3,1}^{12}$	$\left(7,\ \frac{4}{7}+\frac{4}{7}+\frac{6}{7}\right)$
$f_{3,1}^{13}$	$\left(14,\ \frac{1}{2}+\frac{4}{7}+\frac{13}{14}\right)$
$f_{3,1}^{14}$	id

$f_{3,2}$	$\left(12,\ \frac{1}{12}+\frac{5}{12}+\frac{1}{2}\right)$
$f_{3,2}^2$	$\left(6,\ \frac{1}{6}+\frac{5}{6}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,2}^3$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,2}^4$	$\left(3,\ \frac{1}{2}+\frac{2}{3}\right)$
$f_{3,2}^5$	$\left(12,\ \frac{1}{12}+\frac{5}{12}+\frac{1}{2}\right)$
$f_{3,2}^6$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,2}^7$	$\left(12,\ \frac{1}{2}+\frac{7}{12}+\frac{11}{12}\right)$
$f_{3,2}^8$	$\left(3,\ \frac{1}{3}+\frac{2}{3}\right)$
$f_{3,2}^9$	$\left(4,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{3,2}^{10}$	$\left(6,\ \frac{1}{6}+\frac{1}{2}+\frac{1}{2}+\frac{5}{6}\right)$
$f_{3,2}^{11}$	$\left(12,\ \frac{1}{2}+\frac{7}{12}+\frac{11}{12}\right)$
$f_{3,2}^{12}$	id

$f_{3,3}$	$\left(8,\ \frac{1}{8}+\frac{1}{8}+\frac{3}{4}\right)$
$f_{3,3}^2$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{3,3}^3$	$\left(8,\ \frac{1}{4}+\frac{3}{8}+\frac{3}{8}\right)$
$f_{3,3}^4$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,3}^5$	$\left(8,\ \frac{5}{8}+\frac{5}{8}+\frac{3}{4}\right)$
$f_{3,3}^6$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{3,3}^7$	$\left(8,\ \frac{1}{4}+\frac{7}{8}+\frac{7}{8}\right)$
$f_{3,3}^8$	id

$f_{3,4}$	$\left(4,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{3,4}^2$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,4}^3$	$\left(4,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{3,4}^4$	id

$f_{3,6}$	$\left(12,\ \frac{1}{12}+\frac{1}{4}+\frac{2}{3}\right)$
$f_{3,6}^2$	$\left(6,\ \frac{1}{6}+\frac{1}{2}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{3,6}^3$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$
$f_{3,6}^4$	$\left(3,\ \frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{3,6}^5$	$\left(12,\ \frac{1}{4}+\frac{1}{3}+\frac{5}{12}\right)$
$f_{3,6}^6$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,6}^7$	$\left(12,\ \frac{7}{12}+\frac{2}{3}+\frac{3}{4}\right)$
$f_{3,6}^8$	$\left(3,\ \frac{2}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{3,6}^9$	$\left(4,\ \frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{3,6}^{10}$	$\left(6,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{5}{6}\right)$
$f_{3,6}^{11}$	$\left(12,\ \frac{1}{3}+\frac{3}{4}+\frac{11}{12}\right)$
$f_{3,6}^{12}$	id

$f_{3,7}$	$\left(8,\ \frac{1}{8}+\frac{1}{4}+\frac{5}{8}\right)$
$f_{3,7}^2$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$
$f_{3,7}^3$	$\left(8,\ \frac{3}{8}+\frac{3}{4}+\frac{7}{8}\right)$
$f_{3,7}^4$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{3,7}^5$	$\left(8,\ \frac{1}{8}+\frac{1}{4}+\frac{5}{8}\right)$
$f_{3,7}^6$	$\left(4,\ \frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{3,7}^7$	$\left(8,\ \frac{3}{8}+\frac{3}{4}+\frac{7}{8}\right)$
$f_{3,7}^8$	id

$f_{3,8}$	$\left(9,\ \frac{1}{9}+\frac{1}{3}+\frac{5}{9}\right)$
$f_{3,8}^2$	$\left(9,\ \frac{5}{9}+\frac{2}{3}+\frac{7}{9}\right)$
$f_{3,8}^3$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{2}{3}\right)$
$f_{3,8}^4$	$\left(9,\ \frac{1}{3}+\frac{7}{9}+\frac{8}{9}\right)$
$f_{3,8}^5$	$\left(9,\ \frac{1}{9}+\frac{2}{9}+\frac{2}{3}\right)$
$f_{3,8}^6$	$\left(3,\ \frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{3,8}^7$	$\left(9,\ \frac{2}{9}+\frac{1}{3}+\frac{4}{9}\right)$
$f_{3,8}^8$	$\left(9,\ \frac{4}{9}+\frac{2}{3}+\frac{8}{9}\right)$
$f_{3,8}^9$	id

$f_{3,9}$	$\left(7,\ \frac{1}{7}+\frac{2}{7}+\frac{4}{7}\right)$
$f_{3,9}^2$	$\left(7,\ \frac{1}{7}+\frac{2}{7}+\frac{4}{7}\right)$
$f_{3,9}^3$	$\left(7,\ \frac{3}{7}+\frac{5}{7}+\frac{6}{7}\right)$
$f_{3,9}^4$	$\left(7,\ \frac{1}{7}+\frac{2}{7}+\frac{4}{7}\right)$
$f_{3,9}^5$	$\left(7,\ \frac{3}{7}+\frac{5}{7}+\frac{6}{7}\right)$
$f_{3,9}^6$	$\left(7,\ \frac{3}{7}+\frac{5}{7}+\frac{6}{7}\right)$
$f_{3,9}^7$	id

(4) Genus 4:

$f_{4,1}$	$\left(18,\ \frac{1}{18}+\frac{4}{9}+\frac{1}{2}\right)$
$f_{4,1}^{2}$	$\left(9,\ \frac{1}{9}+\frac{4}{9}+\frac{4}{9}\right)$
$f_{4,1}^{3}$	$\left(6,\ \frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,1}^{4}$	$\left(9,\ \frac{2}{9}+\frac{2}{9}+\frac{5}{9}\right)$
$f_{4,1}^{5}$	$\left(18,\ \frac{1}{2}+\frac{11}{18}+\frac{8}{9}\right)$
$f_{4,1}^{6}$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,1}^{7}$	$\left(18,\ \frac{1}{2}+\frac{13}{18}+\frac{7}{9}\right)$
$f_{4,1}^{8}$	$\left(9,\ \frac{1}{9}+\frac{1}{9}+\frac{7}{9}\right)$
$f_{4,1}^{9}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,1}^{10}$	$\left(9,\ \frac{2}{9}+\frac{8}{9}+\frac{8}{9}\right)$
$f_{4,1}^{11}$	$\left(18,\ \frac{2}{9}+\frac{5}{18}+\frac{1}{2}\right)$
$f_{4,1}^{12}$	$\left(3,\ \frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,1}^{13}$	$\left(18,\ \frac{1}{9}+\frac{7}{18}+\frac{1}{2}\right)$
$f_{4,1}^{14}$	$\left(9,\ \frac{4}{9}+\frac{7}{9}+\frac{7}{9}\right)$
$f_{4,1}^{15}$	$\left(6,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3}+\frac{5}{6}\right)$
$f_{4,1}^{16}$	$\left(9,\ \frac{5}{9}+\frac{5}{9}+\frac{8}{9}\right)$
$f_{4,1}^{17}$	$\left(18,\ \frac{1}{2}+\frac{5}{9}+\frac{17}{18}\right)$
$f_{4,1}^{18}$	id

$f_{4,2}$	$\left(16,\ \frac{1}{16}+\frac{7}{16}+\frac{1}{2}\right)$
$f_{4,2}^{2}$	$\left(8,\ \frac{1}{8}+\frac{1}{2}+\frac{1}{2}+\frac{7}{8}\right)$
$f_{4,2}^{3}$	$\left(16,\ \frac{1}{2}+\frac{11}{16}+\frac{13}{16}\right)$
$f_{4,2}^{4}$	$\left(4,\ \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}\right)$
$f_{4,2}^{5}$	$\left(16,\ \frac{1}{2}+\frac{11}{16}+\frac{13}{16}\right)$
$f_{4,2}^{6}$	$\left(8,\ \frac{3}{8}+\frac{1}{2}+\frac{1}{2}+\frac{5}{8}\right)$
$f_{4,2}^{7}$	$\left(16,\ \frac{1}{16}+\frac{7}{16}+\frac{1}{2}\right)$
$f_{4,2}^{8}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,2}^{9}$	$\left(16,\ \frac{1}{2}+\frac{9}{16}+\frac{15}{16}\right)$
$f_{4,2}^{10}$	$\left(8,\ \frac{3}{8}+\frac{1}{2}+\frac{1}{2}+\frac{5}{8}\right)$
$f_{4,2}^{11}$	$\left(16,\ \frac{3}{16}+\frac{5}{16}+\frac{1}{2}\right)$
$f_{4,2}^{12}$	$\left(4,\ \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}\right)$
$f_{4,2}^{13}$	$\left(16,\ \frac{3}{16}+\frac{5}{16}+\frac{1}{2}\right)$
$f_{4,2}^{14}$	$\left(8,\ \frac{1}{8}+\frac{1}{2}+\frac{1}{2}+\frac{7}{8}\right)$
$f_{4,2}^{15}$	$\left(16,\ \frac{1}{2}+\frac{9}{16}+\frac{15}{16}\right)$
$f_{4,2}^{16}$	id

$f_{4,3}$	$\left(10,\ \frac{1}{10}+\frac{1}{10}+\frac{4}{5}\right)$
$f_{4,3}^{2}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,3}^{3}$	$\left(10,\ \frac{3}{5}+\frac{7}{10}+\frac{7}{10}\right)$
$f_{4,3}^{4}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,3}^{5}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,3}^{6}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,3}^{7}$	$\left(10,\ \frac{3}{10}+\frac{3}{10}+\frac{2}{5}\right)$
$f_{4,3}^{8}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,3}^{9}$	$\left(10,\ \frac{1}{5}+\frac{9}{10}+\frac{9}{10}\right)$
$f_{4,3}^{10}$	id

$f_{4,4}$	$\left(10,\ \frac{2}{5}+\frac{1}{2}+\frac{1}{2}+\frac{3}{5}\right)$
$f_{4,4}^{2}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,4}^{3}$	$\left(10,\ \frac{1}{5}+\frac{1}{2}+\frac{1}{2}+\frac{4}{5}\right)$
$f_{4,4}^{4}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,4}^{5}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,4}^{6}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,4}^{7}$	$\left(10,\ \frac{1}{5}+\frac{1}{2}+\frac{1}{2}+\frac{4}{5}\right)$
$f_{4,4}^{8}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,4}^{9}$	$\left(10,\ \frac{2}{5}+\frac{1}{2}+\frac{1}{2}+\frac{3}{5}\right)$
$f_{4,4}^{10}$	id

$f_{4,5}$	$\left(15,\ \frac{1}{15}+\frac{1}{3}+\frac{3}{5}\right)$
$f_{4,5}^{2}$	$\left(15,\ \frac{8}{15}+\frac{2}{3}+\frac{4}{5}\right)$
$f_{4,5}^{3}$	$\left(5,\ \frac{1}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,5}^{4}$	$\left(15,\ \frac{4}{15}+\frac{1}{3}+\frac{2}{5}\right)$
$f_{4,5}^{5}$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,5}^{6}$	$\left(5,\ \frac{3}{5}+\frac{4}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,5}^{7}$	$\left(15,\ \frac{1}{3}+\frac{4}{5}+\frac{13}{15}\right)$
$f_{4,5}^{8}$	$\left(15,\ \frac{2}{15}+\frac{1}{5}+\frac{2}{3}\right)$
$f_{4,5}^{9}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{2}{5}\right)$
$f_{4,5}^{10}$	$\left(3,\ \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,5}^{11}$	$\left(15,\ \frac{3}{5}+\frac{2}{3}+\frac{11}{15}\right)$
$f_{4,5}^{12}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{4}{5}\right)$
$f_{4,5}^{13}$	$\left(15,\ \frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$
$f_{4,5}^{14}$	$\left(15,\ \frac{2}{5}+\frac{2}{3}+\frac{14}{15}\right)$
$f_{4,5}^{15}$	id

$f_{4,6}$	$\left(12,\ \frac{1}{12}+\frac{1}{3}+\frac{7}{12}\right)$
$f_{4,6}^{2}$	$\left(6,\ \frac{1}{6}+\frac{1}{6}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,6}^{3}$	$\left(4,\ \frac{1}{4}+\frac{3}{4}\right)$
$f_{4,6}^{4}$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,6}^{5}$	$\left(12,\ \frac{5}{12}+\frac{2}{3}+\frac{11}{12}\right)$
$f_{4,6}^{6}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,6}^{7}$	$\left(12,\ \frac{1}{12}+\frac{1}{3}+\frac{7}{12}\right)$
$f_{4,6}^{8}$	$\left(3,\ \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,6}^{9}$	$\left(4,\ \frac{1}{4}+\frac{3}{4}\right)$
$f_{4,6}^{10}$	$\left(6,\ \frac{2}{3}+\frac{2}{3}+\frac{5}{6}+\frac{5}{6}\right)$
$f_{4,6}^{11}$	$\left(12,\ \frac{5}{12}+\frac{2}{3}+\frac{11}{12}\right)$
$f_{4,6}^{12}$	id

$f_{4,7}$	$\left(10,\ \frac{1}{10}+\frac{3}{10}+\frac{3}{5}\right)$
$f_{4,7}^{2}$	$\left(5,\ \frac{1}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}\right)$
$f_{4,7}^{3}$	$\left(10,\ \frac{1}{10}+\frac{1}{5}+\frac{7}{10}\right)$
$f_{4,7}^{4}$	$\left(5,\ \frac{3}{5}+\frac{4}{5}+\frac{4}{5}+\frac{4}{5}\right)$
$f_{4,7}^{5}$	$\left(2,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,7}^{6}$	$\left(5,\ \frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{2}{5}\right)$
$f_{4,7}^{7}$	$\left(10,\ \frac{3}{10}+\frac{4}{5}+\frac{9}{10}\right)$
$f_{4,7}^{8}$	$\left(5,\ \frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{4}{5}\right)$
$f_{4,7}^{9}$	$\left(10,\ \frac{2}{5}+\frac{7}{10}+\frac{9}{10}\right)$
$f_{4,7}^{10}$	id

$f_{4,8}$	$\left(12,\ \frac{1}{12}+\frac{1}{6}+\frac{3}{4}\right)$
$f_{4,8}^2$	$\left(6,\ \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{2}\right)$
$f_{4,8}^3$	$\left(4,\ \frac{1}{4}+\frac{1}{2}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right)$
$f_{4,8}^4$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,8}^5$	$\left(12,\ \frac{5}{12}+\frac{3}{4}+\frac{5}{6}\right)$
$f_{4,8}^6$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,8}^7$	$\left(12,\ \frac{1}{6}+\frac{1}{4}+\frac{7}{12}\right)$
$f_{4,8}^8$	$\left(3,\ \frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,8}^9$	$\left(4,\ \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{3}{4}\right)$
$f_{4,8}^{10}$	$\left(6,\ \frac{1}{2}+\frac{5}{6}+\frac{5}{6}+\frac{5}{6}\right)$
$f_{4,8}^{11}$	$\left(12,\ \frac{1}{4}+\frac{5}{6}+\frac{11}{12}\right)$
$f_{4,8}^{12}$	id

$f_{4,9}$	$\left(6,\ \frac{1}{6}+\frac{1}{3}+\frac{2}{3}+\frac{5}{6}\right)$
$f_{4,9}^2$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,9}^3$	$\left(2,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,9}^4$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,9}^5$	$\left(6,\ \frac{1}{6}+\frac{1}{3}+\frac{2}{3}+\frac{5}{6}\right)$
$f_{4,9}^6$	id

$f_{4,10}$	$\left(6,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,10}^2$	$\left(3,\ \frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$
$f_{4,10}^3$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,10}^4$	$\left(3,\ \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,10}^5$	$\left(6,\ \frac{1}{2}+\frac{1}{2}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right)$
$f_{4,10}^6$	id

$f_{4,11}$	$\left(6,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,11}^2$	(3,)
$f_{4,11}^3$	$\left(2,\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)$
$f_{4,11}^4$	(3,)
$f_{4,11}^5$	$\left(6,\ \frac{1}{2}+\frac{1}{2}\right)$
$f_{4,11}^6$	id

$f_{4,12}$	$\left(5,\ \frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)$
$f_{4,12}^2$	$\left(5,\ \frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)$
$f_{4,12}^3$	$\left(5,\ \frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)$
$f_{4,12}^4$	$\left(5,\ \frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)$
$f_{4,12}^5$	id

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