Title	Comparing hyperbolic distance with Kra's distance on the unit disk
Author(s)	Yao, Guowu
Citation	Osaka Journal of Mathematics. 2012, 49(2), p. 349-356
Version Type	VoR
URL	https://doi.org/10.18910/4887
rights	
Note	

Yao, G.
Osaka J. Math.
49 (2012), 349–356

COMPARING HYPERBOLIC DISTANCE WITH KRA'S DISTANCE ON THE UNIT DISK

GUOWU YAO

(Received June 9, 2009, revised October 27, 2010)

Abstract

In this paper, Kra's distance d_K and the hyperbolic distance $d_{\mathbb{D}}$ are compared on the unit disk $\mathbb{D}$. It is shown that $2d_K < d_{\mathbb{D}} < (\pi^2/8)\exp d_K$ on $\mathbb{D}\times\mathbb{D}\setminus\{\text{diagonal}\}$, where the constants 2 and $\pi^2/8$ are sharp. As a consequence, this result gives a negative answer to a question posed by Martin [7] in a stronger sense.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{|z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\rho(z)|dz|$ denote the hyperbolic metric, i.e.,

$$\rho(z)|dz| = \frac{1}{1-|z|^2}|dz|, \quad z \in \mathbb{D}.$$

Then the hyperbolic distance $d_{\mathbb{D}}(z_1, z_2)$ between two points z_1, z_2 induced by $\rho(z)$ is

$$d_{\mathbb{D}}(z_1, z_2) = \frac{1}{2}\log\frac{1+|(z_1-z_2)/(1-\bar{z}_1z_2)|}{1-|(z_1-z_2)/(1-\bar{z}_1z_2)|}.$$

Let R be a hyperbolic Riemann surface covered by $\mathbb{D}$. Let $\omega\colon \mathbb{D}\to R$ be the canonical holomorphic universal covering of R. Then $d_{\mathbb{D}}$ induces a quotient hyperbolic distance d_R on R that satisfies

$$d_R(\omega(a), q) = \min\{d_{\mathbb{D}}(z, a)\colon \omega(z) = q\}$$

for all $a \in \mathbb{D}$ and $q \in R$.

A Teichmüller shift mapping on R is the uniquely extremal quasiconformal mapping T_{p_1,p_2} which sends p_1 to p_2 and is homotopic to the identity mapping modulo the ideal boundary ∂R. It is a Teichmüller mapping with Beltrami coefficient μ_{p_1,p_2} such that, for $p_1 = p_2$, $\mu_{p_1,p_2} = 0$, while for $p_1 \neq p_2$, $\mu_{p_1,p_2} = k_{p_1,p_2}|\phi_{p_1,p_2}|/\phi_{p_1,p_2}$,

2010 Mathematics Subject Classification. 30C62, 30F60, 32G15.

The author was supported by a Foundation for the Author of National Excellent Doctoral Dissertation (Grant No. 200518) and the National Natural Science Foundation of PR China.

where $k_{p_1,p_2} \in (0, 1)$ is a constant and ϕ_{p_1,p_2} is a holomorphic quadratic differential in $R - \{p_1\}$, which has a first order pole at p_1 and has unit L^1-norm.

When studying the self-maps of Riemann surfaces and the geometry of Teichmüller spaces, Kra [4] introduced a distance d_K on every hyperbolic Riemann surface R by the Teichmüller shift mapping, which is defined as follows: for any two points p_1 and p_2 in R,

$$d_K(p_1, p_2) = \frac{1}{2} \log \frac{1 + k_{p_1,p_2}}{1 - k_{p_1,p_2}}.$$

Kra [4] compared d_R with d_K for certain Riemann surfaces:

Theorem A. *When R is of analytic finite type and is not conformally equivalent to $\mathbb{C} \setminus \{0, 1\}$, there exists a universal constant $c > 0$ such that*

$$cd_R < d_K < d_R, \tag{1.1}$$

on $R \times R \setminus \{diagonal\}$.

Earle and Lakic [2] proved

Theorem B. *If R is not conformally equivalent to $\mathbb{C} \setminus \{0,1\}$, then the identity map $id\colon (R, d_R) \to (R, d_K)$ is not an isometry, moreover, $d_K < d_R$ on $R \times R \setminus \{diagonal\}$.*

REMARK. Liu [5] proved Theorem B for all hyperbolic Riemann surfaces with three exceptions: $\mathbb{D}$, $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, or an annulus.

In this paper, we compare d_R with d_K on the unit disk and give sharp inequalities between them.

Theorem 1. *For the unit disk $\mathbb{D}$, the hyperbolic distance $d_{\mathbb{D}}$ and Kra's distance satisfy*

$$2d_K < d_{\mathbb{D}} < \frac{\pi^2}{8} \exp d_K \tag{1.2}$$

on $\mathbb{D} \times \mathbb{D} \setminus \{diagonal\}$, where the constants 2 and $\pi^2/8$ are sharp.

We now introduce a basic concept. A sense preserving homeomorphism f of a domain $\Omega \subset \mathbb{C}$ is called K-quasiconformal ($1 \leq K < \infty$), if f is an L^2-solution of the equation

$$\bar{\partial} f = \mu\, \partial f,$$

where μ is a measurable function with

$$\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1.$$

There is a classical result of Teichmüller's concerning the distortion of normalized quasiconformal mappings [9]. We state Teichmüller's theorem as follows.

Theorem C. *Let $\rho(z, w)$ denote the hyperbolic metric of constant curvature -4 in the three punctured sphere $\mathbb{C} \setminus \{0, 1\}$. We have*
(a) *if f is a K-quasiconformal mapping of the Riemann sphere fixing 0, 1 and ∞, then for any $z \in \mathbb{C} \setminus \{0, 1\}$,*

$$\rho(z, f(z)) \leq \log K, \tag{1.3}$$

(b) *if $z, w \in \mathbb{C} \setminus \{0, 1\}$ satisfy $\rho(z, w) \leq \log K$, then there is a K-quasiconformal map of the Riemann sphere fixing 0, 1 and ∞ such that $w = f(z)$.*

In [7], Martin used holomorphic motions to extend the (b) part of Teichmüller's theorem to any planar domain. He obtained the following theorem.

Theorem D. *Let Ω be a planar domain with at least three boundary points and let $\rho_\Omega(z, w)$ be the hyperbolic metric of Ω with constant curvature -1. Suppose $z, w \in \Omega$ and*

$$\rho_\Omega(z, w) \leq \log K.$$

Then there is a K-quasiconformal self-homeomorphism f of Ω such that
(1) $f(\zeta) = \zeta$ *for all* $\zeta \in \partial\Omega$,
(2) $f(z) = w$.

Martin also asked if the (a) part of the theorem can be extended likewise. His question is precisely described as follows.

Let R be a planar domain with at least three boundary points and suppose that f is a K-quasiconformal mapping of R such that $f(\zeta) = \zeta$ for all $\zeta \in \partial R$. Does it follow that $2d_R(z, f(z)) \leq \log K$ for all $z \in R$? (Notice that the curvature of the hyperbolic metric determined by d_R is -4.)

In [3], Huang and Cho gave a negative answer to this question for any planar simply-connected domain. Actually, Martin's question can be reduced to whether $d_R \leq d_K$ holds on $R \times R$. Evidently it has a negative answer by Theorem B. When $R = \mathbb{D}$, our Theorem 1 implies a negative answer in a stronger sense.

Theorem 2. *For any given $c > 0$ and $z \in \mathbb{D}$, there exists a K-quasiconformal mapping f of $\mathbb{D}$ fixing all boundary points of $\mathbb{D}$ such that*

$$d_{\mathbb{D}}(z, f(z)) > c \log K, \tag{1.4}$$

where K depends only on c.

We note that it might be hard, but would be very interesting to compare $d_{\mathbb{D}^*}$ and d_K on $\mathbb{D}^*$.

2. $2d_K < d_{\mathbb{D}}$

In fact, on the unit disk, we have the following exact formula:

$$\log \frac{\exp d_K + 1}{\exp d_K - 1} = \mu\left(\frac{\exp(2d_{\mathbb{D}}) - 1}{\exp(2d_{\mathbb{D}}) + 1}\right), \tag{2.1}$$

where $\mu(r)$ is the conformal module of the Grötzsch ring domain whose boundary components are the unit circle and the line segment $\{x : 0 \le x \le r\}$. Since d_K and $d_{\mathbb{D}}$ are invariant under Möbius transformations, we only need to prove that

$$2d_K(0, r) < d_{\mathbb{D}}(0, r)$$

for $r \in (0, 1)$.

By the result in [6], $\mu(r)$ satisfies

$$\log \frac{(1 + \sqrt{1 - r^2})^2}{r} < \mu(r) < \log \frac{4}{r}. \tag{2.2}$$

Therefore, $\mu(r)$ has the asymptotic behavior: as $r \to 0$,

$$\mu(r) = \log \frac{4}{r} + s(r), \tag{2.3}$$

where

$$0 > s(r) > \log \frac{(1 + \sqrt{1 - r^2})^2}{r} - \log \frac{4}{r} > -\frac{r^2}{2} + o(r^3). \tag{2.4}$$

Thus, we obtain the asymptotic behavior of $d_K(0, r)$:

$$\begin{aligned} d_K(0, r) &= \log \frac{\exp \mu(r) + 1}{\exp \mu(r) - 1} = \log \frac{(4/r) \exp s(r) + 1}{(4/r) \exp s(r) - 1} \\ &= \log \frac{\exp s(r) + r/4}{\exp s(r) - r/4} = \log \frac{1 + r/4 + s(r) + o(r^3)}{1 - r/4 + s(r) + o(r^3)} \end{aligned}$$

$$
\begin{aligned}
&= \left[\left(\frac{r}{4}+s(r)\right)-\frac{1}{2}\left(\frac{r}{4}+s(r)\right)^2+\frac{1}{3}\left(\frac{r}{4}+s(r)\right)^3+o(r^3)\right] \\
&\quad -\left[\left(-\frac{r}{4}+s(r)\right)-\frac{1}{2}\left(-\frac{r}{4}+s(r)\right)^2+\frac{1}{3}\left(-\frac{r}{4}+s(r)\right)^3+o(r^3)\right] \\
&= \frac{r}{2}-\frac{r}{2}s(r)+\frac{r^3}{96}+o(r^3), \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Using (2.4), we obtain

$$d_K(0,r)=\frac{r}{2}+O(r^3), \quad \text{as} \quad r\to 0.$$

On the other hand, it is easy to check that

$$d_{\mathbb{D}}(0,r)=\frac{1}{2}\log\frac{1+r}{1-r}=r+\frac{r^3}{3}+o(r^3), \quad \text{as} \quad r\to 0. \tag{2.5}$$

Thus, we have

$$\lim_{r\to 0^+}\frac{d_K(0,r)}{d_{\mathbb{D}}(0,r)}=\frac{1}{2}. \tag{2.6}$$

So, for any given $c>1/2$, there exists some $r(c)\in(0,1)$ such that

$$d_K(0,r)<cd_{\mathbb{D}}(0,r) \tag{2.7}$$

holds whenever $r\in(0,r(c))$. Now, we show that (2.7) holds for all $r\in(0,1)$. Let OA denote the line segment $\{x : 0\le x\le r\}$ in $\mathbb{D}$, where O is the origin $z=0$ and A is the endpoint $z=r$. Choose orderly $n+1$ (sufficiently large) points $A_0, A_1, \dots, A_n$ in OA from O to A such that $O=A_0$, $A=A_n$ and

$$d_{\mathbb{D}}(A_k, A_{k+1})<d_{\mathbb{D}}(0,r(c)) \tag{2.8}$$

for $k=0,1,\dots,n-1$. By the invariance of d_K and $d_{\mathbb{D}}$ under Möbius transformations and inequality (2.7), we have

$$d_K(A_K, A_{K+1})<cd_{\mathbb{D}}(A_K, A_{K+1}).$$

Thus,

$$
\begin{aligned}
d_K(0,r)=d_K(O,A)&\le\sum_{k=0}^{n-1}d_K(A_k,A_{k+1}) \\
&<c\sum_{k=0}^{n-1}d_{\mathbb{D}}(A_k,A_{k+1})=cd_{\mathbb{D}}(O,A)=cd_{\mathbb{D}}(0,r).
\end{aligned}
$$

Since c is arbitrarily chosen in $(1/2, \infty)$, we conclude that

$$2d_K(0, r) \leq d_{\mathbb{D}}(0, r). \tag{2.9}$$

Observe that

$$\begin{aligned} 2d_K(0, r) &\leq 2d_K(0, r') + 2d_K(r', r) \\ &\leq d_{\mathbb{D}}(0, r') + d_{\mathbb{D}}(r', r) = d_{\mathbb{D}}(0, r). \end{aligned}$$

If the equality in (2.9) holds for some $r \in (0, 1)$, then

$$2d_K(0, x) = d_{\mathbb{D}}(0, x) \tag{2.10}$$

for all $x \in (0, r]$. This gives

$$\mu(x) = \log \frac{\sqrt[4]{1+x} + \sqrt[4]{1-x}}{\sqrt[4]{1+x} - \sqrt[4]{1-x}}, \quad x \in (0, r]$$

in terms of (2.1). However, it is impossible because the representation of $\mu(r)$ is not an elementary function in $(0, r)$. Thus, we obtain $2_K < d_{\mathbb{D}}$ on $\mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\}$. Finally, it follows that the constant 2 is sharp from (2.6).

Examining the argument above carefully, we actually prove that the hyperbolic distance has the maximal property in the following sense.

Theorem 3. *Let $d(\cdot\,,\,\cdot)$ be a distance function defined on $\mathbb{D} \times \mathbb{D}$. If $d(\cdot\,,\,\cdot)$ is invariant under Möbius transformations of $\mathbb{D}$ and satisfies*

$$\limsup_{r \to 0^+} \frac{d(0, r)}{d_{\mathbb{D}}(0, r)} = \lambda > 0, \tag{2.11}$$

then

$$d(z, w) \leq \lambda d_{\mathbb{D}}(z, w), \tag{2.12}$$

for all $(z, w) \in \mathbb{D} \times \mathbb{D}$.

3. $d_{\mathbb{D}} < (\pi^2/8) \exp d_K$

It suffices to show that

$$d_{\mathbb{D}}(0, r) < \frac{\pi^2}{8} \exp d_K(0, r) \tag{3.1}$$

for $r \in (0, 1)$.

We need two lemmas.

Lemma 1. *$g(r) = \mu(r)\, d_{\mathbb{D}}(0, r)$ is an increasing function from $(0, 1)$ onto $(0, \pi^2/4)$.*

Proof. Observe $g(r) = \mu(r)\log((1+r)/(1-r))/2$. Theorem 11.21 in [1] indicates that $g(r)$ satisfies the desired condition. □

Lemma 2. *$h(r) = 1/(\mu(r)\exp d_K(0,r))$ is an increasing function from $(0,1)$ onto $(0, 1/2)$.*

Proof. Observe

$$h(r) = \frac{1}{\mu(r)}\frac{\exp\mu(r) - 1}{\exp\mu(r) + 1}.$$

Consider two auxiliary functions $x = \mu(r)$ and

$$\tilde{h}(x) = \frac{1}{x}\frac{\exp x - 1}{\exp x + 1}, \quad x \in (0, \infty).$$

We have

$$\tilde{h}'(x) = \frac{1 + 2x\exp x - \exp(2x)}{(x + x\exp x)^2}.$$

It is not difficult to verify that

$$1 + 2x\exp x - \exp(2x) < 0, \quad x \in (0, \infty),$$

and hence $h(x)$ is a decreasing function in $(0, \infty)$. On the other hand, it is well-known that $x = \mu(r)$ is a decreasing function from $(0, 1)$ onto $(0, \infty)$. Thus, $h(r)$ is an increasing function in $(0, 1)$. In addition,

$$\lim_{r\downarrow 0} h(r) = \lim_{x\uparrow\infty} \tilde{h}(x) = 0$$

and

$$\lim_{r\uparrow 1} h(r) = \lim_{x\downarrow 0} \tilde{h}(x) = \frac{1}{2}.$$

This completes the proof of this lemma. □

Combining Lemmas 1 and 2, we get

Theorem 4. *$F(r) = g(r)h(r) = d_{\mathbb{D}}(0, r)/\exp d_K(0, r)$ is an increasing function from $(0, 1)$ onto $(0, \pi^2/8)$.*

Now, we obtain $d_{\mathbb{D}} < (\pi^2/8) \exp d_K$ on $\mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\}$, where $\pi^2/8$ is sharp. Moreover, Theorem 2 is naturally derived from Theorem 4 and the definition of Teichmüller shift mapping.

ACKNOWLEDGEMENTS. The author wishes to thank the referee for his/her helpful comments. The referee also noticed that Martin's result has been generalized to the setting of mappings of finite distortion on the unit disk [8].

References

[1] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen: Conformal Invariants, Inequalities, and Quasiconformal Maps, Wiley, New York, 1997.
[2] C.J. Earle and N. Lakic: *Variability sets on Riemann surfaces and forgetful maps between Teichmüller spaces*, Ann. Acad. Sci. Fenn. Math. **27** (2002), 307–324.
[3] H. Xinzhong and N.E. Cho: *On the distortion theorem for quasiconformal mappings with fixed boundary values*, J. Math. Anal. Appl. **256** (2001), 694–697.
[4] I. Kra: *On the Nielsen–Thurston–Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.
[5] L.X. Liu: *Invariant metrics in infinite-dimensional Teichmüller space*, Complex Variables Theory Appl. **25** (1994), 337–349.
[6] O. Lehto and K.I. Virtanen: Quasiconformal Mappings in the Plane, second edition, Springer, New York, 1973.
[7] G.J. Martin: *The distortion theorem for quasiconformal mappings*, Schottky's theorem and holomorphic motions, Proc. Amer. Math. Soc. **125** (1997), 1095–1103.
[8] G.J. Martin: *The Teichmüller problem for mean distortion*, Ann. Acad. Sci. Fenn. Math. **34** (2009), 233–247.
[9] O. Teichmüller: *Untersuchungen über konforme und quasiconforme Abbildung*, Deutsche Math. **3** (1938), 621–678.

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
P.R. China
e-mail: gwyao@math.tsinghua.edu.cn