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COMPARING HYPERBOLIC DISTANCE WITH KRA'S DISTANCE ON THE UNIT DISK

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Abstract

In this paper, Kra’s distance $d_K$ and the hyperbolic distance $d_B$ are compared on the unit disk $\mathbb{D}$. It is shown that $2d_K < d_B < (\pi^2/8) \exp d_K$ on $\mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\}$, where the constants 2 and $\pi^2/8$ are sharp. As a consequence, this result gives a negative answer to a question posed by Martin [7] in a stronger sense.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{|z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\rho(z)|dz|$ denote the hyperbolic metric, i.e.,

$$
\rho(z)|dz| = \frac{1}{1 - |z|^2}|dz|, \quad z \in \mathbb{D}.
$$

Then the hyperbolic distance $d_\mathbb{D}(z_1, z_2)$ between two points $z_1, z_2$ induced by $\rho(z)$ is

$$
d_\mathbb{D}(z_1, z_2) = \frac{1}{2} \log \frac{1 + |(z_1 - z_2)/(1 - \bar{z}_1z_2)|}{1 - |(z_1 - z_2)/(1 - \bar{z}_1z_2)|}.
$$

Let $R$ be a hyperbolic Riemann surface covered by $\mathbb{D}$. Let $\omega: \mathbb{D} \to R$ be the canonical holomorphic universal covering of $R$. Then $d_\mathbb{D}$ induces a quotient hyperbolic distance $d_R$ on $R$ that satisfies

$$
d_R(\omega(a), q) = \min\{d_\mathbb{D}(z, a) : \omega(z) = q\}
$$

for all $a \in \mathbb{D}$ and $q \in R$.

A Teichmüller shift mapping on $R$ is the uniquely extremal quasiconformal mapping $T_{p_1, p_2}$ which sends $p_1$ to $p_2$ and is homotopic to the identity mapping modulo the ideal boundary $\partial R$. It is a Teichmüller mapping with Beltrami coefficient $\mu_{p_1, p_2}$ such that, for $p_1 = p_2$, $\mu_{p_1, p_2} = 0$, while for $p_1 \neq p_2$, $\mu_{p_1, p_2} = k_{p_1, p_2} |\phi_{p_1, p_2}|/|\phi_{p_1, p_2}|$.

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where \( k_{p_1, p_2} \in (0, 1) \) is a constant and \( \phi_{p_1, p_2} \) is a holomorphic quadratic differential in \( R - \{p_1\} \), which has a first order pole at \( p_1 \) and has unit \( L^1 \)-norm.

When studying the self-maps of Riemann surfaces and the geometry of Teichmüller spaces, Kra [4] introduced a distance \( d_K \) on every hyperbolic Riemann surface \( R \) by the Teichmüller shift mapping, which is defined as follows: for any two points \( p_1 \) and \( p_2 \) in \( R \),

\[
d_K(p_1, p_2) = \frac{1}{2} \log \frac{1 + k_{p_1, p_2}}{1 - k_{p_1, p_2}}.
\]

Kra [4] compared \( d_R \) with \( d_K \) for certain Riemann surfaces:

**Theorem A.** When \( R \) is of analytic finite type and is not conformally equivalent to \( \mathbb{C} \setminus \{0, 1\} \), there exists a universal constant \( c > 0 \) such that

\[
(1.1) \quad cd_R < d_K < d_R,
\]
on \( R \times R \setminus \{\text{diagonal}\} \).

Earle and Lakic [2] proved

**Theorem B.** If \( R \) is not conformally equivalent to \( \mathbb{C} \setminus \{0, 1\} \), then the identity map \( \text{id}: (R, d_R) \to (R, d_K) \) is not an isometry, moreover, \( d_K < d_R \) on \( R \times R \setminus \{\text{diagonal}\} \).

**Remark.** Liu [5] proved Theorem B for all hyperbolic Riemann surfaces with three exceptions: \( \mathbb{D}, \mathbb{D}^* = \mathbb{D} \setminus \{0\}, \) or an annulus.

In this paper, we compare \( d_R \) with \( d_K \) on the unit disk and give sharp inequalities between them.

**Theorem 1.** For the unit disk \( \mathbb{D} \), the hyperbolic distance \( d_\mathbb{D} \) and Kra’s distance satisfy

\[
(1.2) \quad 2d_K < d_\mathbb{D} < \frac{\pi^2}{8} \exp d_K
\]
on \( \mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\} \), where the constants \( 2 \) and \( \pi^2/8 \) are sharp.

We now introduce a basic concept. A sense preserving homeomorphism \( f \) of a domain \( \Omega \subset \mathbb{C} \) is called \( K \)-quasiconformal \((1 \leq K < \infty)\), if \( f \) is an \( L^2 \)-solution of the equation

\[
\bar{\partial} f = \mu \partial f,
\]
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where \( \mu \) is a measurable function with

\[
\| \mu \|_\infty \leq \frac{K - 1}{K + 1} < 1.
\]

There is a classical result of Teichmüller's concerning the distortion of normalized quasiconformal mappings [9]. We state Teichmüller's theorem as follows.

**Theorem C.** Let \( \rho(z, w) \) denote the hyperbolic metric of constant curvature \(-4\) in the three punctured sphere \( \mathbb{C} \setminus \{0, 1\} \). We have

(a) if \( f \) is a \( K \)-quasiconformal mapping of the Riemann sphere fixing \( 0, 1 \) and \( \infty \), then for any \( z \in \mathbb{C} \setminus \{0, 1\} \),

\[
\rho(z, f(z)) \leq \log K,
\]

(b) if \( z, w \in \mathbb{C} \setminus \{0, 1\} \) satisfy \( \rho(z, w) \leq \log K \), then there is a \( K \)-quasiconformal map of the Riemann sphere fixing \( 0, 1 \) and \( \infty \) such that \( w = f(z) \).

In [7], Martin used holomorphic motions to extend the (b) part of Teichmüller's theorem to any planar domain. He obtained the following theorem.

**Theorem D.** Let \( \Omega \) be a planar domain with at least three boundary points and let \( \rho_{\Omega}(z, w) \) be the hyperbolic metric of \( \Omega \) with constant curvature \(-1\). Suppose \( z, w \in \Omega \) and

\[
\rho_{\Omega}(z, w) \leq \log K.
\]

Then there is a \( K \)-quasiconformal self-homeomorphism \( f \) of \( \Omega \) such that

1. \( f(\zeta) = \zeta \) for all \( \zeta \in \partial \Omega \).
2. \( f(z) = w \).

Martin also asked if the (a) part of the theorem can be extended likewise. His question is precisely described as follows.

Let \( R \) be a planar domain with at least three boundary points and suppose that \( f \) is a \( K \)-quasiconformal mapping of \( R \) such that \( f(\zeta) = \zeta \) for all \( \zeta \in \partial R \). Does it follow that \( 2d_R(z, f(z)) \leq \log K \) for all \( z \in R \)? (Notice that the curvature of the hyperbolic metric determined by \( d_R \) is \(-4\).)

In [3], Huang and Cho gave a negative answer to this question for any planar simply-connected domain. Actually, Martin’s question can be reduced to whether \( d_R \leq d_K \) holds on \( R \times R \). Evidently it has a negative answer by Theorem B. When \( R = \mathbb{D} \), our Theorem 1 implies a negative answer in a stronger sense.
Theorem 2. For any given \( c > 0 \) and \( z \in \mathbb{D} \), there exists a \( K \)-quasiconformal mapping \( f \) of \( \mathbb{D} \) fixing all boundary points of \( \mathbb{D} \) such that

\[
d_\mathbb{D}(z, f(z)) > c \log K,
\]

where \( K \) depends only on \( c \).

We note that it might be hard, but would be very interesting to compare \( d_\mathbb{D} \) and \( d_K \) on \( \mathbb{D}^* \).

2. \( 2d_K < d_\mathbb{D} \)

In fact, on the unit disk, we have the following exact formula:

\[
\log \frac{\exp d_K + 1}{\exp d_K - 1} = \mu \left( \frac{\exp(2d_\mathbb{D}) - 1}{\exp(2d_\mathbb{D}) + 1} \right),
\]

where \( \mu(r) \) is the conformal module of the Grötzsch ring domain whose boundary components are the unit circle and the line segment \( \{ x : 0 \leq x \leq r \} \). Since \( d_K \) and \( d_\mathbb{D} \) are invariant under Möbius transformations, we only need to prove that

\[
2d_K(0, r) < d_\mathbb{D}(0, r)
\]

for \( r \in (0, 1) \).

By the result in [6], \( \mu(r) \) satisfies

\[
\log \frac{(1 + \sqrt{1 - r^2})^2}{r} < \mu(r) < \log \frac{4}{r}.
\]

Therefore, \( \mu(r) \) has the asymptotic behavior: as \( r \to 0 \),

\[
\mu(r) = \log \frac{4}{r} + s(r),
\]

where

\[
0 > s(r) > \log \frac{(1 + \sqrt{1 - r^2})^2}{r} - \log \frac{4}{r} > -\frac{r^2}{2} + o(r^3).
\]

Thus, we obtain the asymptotic behavior of \( d_K(0, r) \):

\[
d_K(0, r) = \log \frac{\exp \mu(r) + 1}{\exp \mu(r) - 1} = \log \frac{(4/r) \exp s(r) + 1}{(4/r) \exp s(r) - 1}
\]

\[
= \log \frac{\exp s(r) + r/4}{\exp s(r) - r/4} = \log \frac{1 + r/4 + s(r) + o(r^3)}{1 - r/4 + s(r) + o(r^3)}
\]
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\[ d_{K}(0, r) = \frac{r}{2} + O(r^3), \quad \text{as } r \to 0. \]

On the other hand, it is easy to check that

\[ d_{D}(0, r) = \frac{1}{2} \log \frac{1 + r}{1 - r} = r + \frac{r^3}{3} + o(r^3), \quad \text{as } r \to 0. \]

Thus, we have

\[ \lim_{r \to 0^+} \frac{d_{K}(0, r)}{d_{D}(0, r)} = \frac{1}{2}. \]

So, for any given \( c > 1/2 \), there exists some \( r(c) \in (0, 1) \) such that

\[ d_{K}(0, r) < cd_{D}(0, r) \]

holds whenever \( r \in (0, r(c)) \). Now, we show that (2.7) holds for all \( r \in (0, 1) \). Let \( OA \) denote the line segment \( \{x: 0 \leq x \leq r\} \) in \( \mathbb{D} \), where \( O \) is the origin \( z = 0 \) and \( A \) is the endpoint \( z = r \). Choose orderly \( n + 1 \) (sufficiently large) points \( A_0, A_1, \ldots, A_n \) in \( OA \) from \( O \) to \( A \) such that \( O = A_0, A = A_n \) and

\[ d_{D}(A_k, A_{k+1}) < d_{D}(0, r(c)) \]

for \( k = 0, 1, \ldots, n - 1 \). By the invariance of \( d_{K} \) and \( d_{D} \) under Möbius transformations and inequality (2.7), we have

\[ d_{K}(A_K, A_{K+1}) < cd_{D}(A_K, A_{K+1}). \]

Thus,

\[ d_{K}(0, r) = d_{K}(O, A) \leq \sum_{k=0}^{n-1} d_{K}(A_k, A_{k+1}) \]

\[ < c \sum_{k=0}^{n-1} d_{D}(A_k, A_{k+1}) = cd_{D}(O, A) = cd_{D}(0, r). \]
Since \( c \) is arbitrarily chosen in \((1/2, \infty)\), we conclude that

\[(2.9)\]
\[2d_K(0, r) \leq d_B(0, r).\]

Observe that

\[2d_K(0, r) \leq 2d_K(0, r') + 2d_K(r', r) \leq d_B(0, r') + d_B(r', r) = d_B(0, r).\]

If the equality in (2.9) holds for some \( r \in (0, 1) \), then

\[(2.10)\]
\[2d_K(0, x) = d_B(0, x)\]

for all \( x \in (0, r] \). This gives

\[\mu(x) = \log \frac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}, \quad x \in (0, r]\]

in terms of (2.1). However, it is impossible because the representation of \( \mu(r) \) is not an elementary function in \((0, r)\). Thus, we obtain \( 2K < d_B \) on \( \mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\} \).

Finally, it follows that the constant 2 is sharp from (2.6).

Examining the argument above carefully, we actually prove that the hyperbolic distance has the maximal property in the following sense.

**Theorem 3.** Let \( d(\cdot, \cdot) \) be a distance function defined on \( \mathbb{D} \times \mathbb{D} \). If \( d(\cdot, \cdot) \) is invariant under Möbius transformations of \( \mathbb{D} \) and satisfies

\[(2.11)\]
\[\lim_{r \to 0^+} \sup_{r} \frac{d(0, r)}{d_B(0, r)} = \lambda > 0,\]

then

\[(2.12)\]
\[d(z, w) \leq \lambda d_B(z, w),\]

for all \((z, w) \in \mathbb{D} \times \mathbb{D} \).

**3.** \( d_B < (\pi^2/8) \exp d_K \)

It suffices to show that

\[(3.1)\]
\[d_B(0, r) < \frac{\pi^2}{8} \exp d_K(0, r)\]

for \( r \in (0, 1) \).

We need two lemmas.
Lemma 1. \( g(r) = \mu(r) d_D(0, r) \) is an increasing function from \((0, 1)\) onto \((0, \pi^2/4)\).

Proof. Observe \( g(r) = \mu(r) \log((1 + r)/(1 - r))/2 \). Theorem 11.21 in [1] indicates that \( g(r) \) satisfies the desired condition. \(\Box\)

Lemma 2. \( h(r) = 1/(\mu(r) \exp d_K(0, r)) \) is an increasing function from \((0, 1)\) onto \((0, 1/2)\).

Proof. Observe
\[
 h(r) = \frac{1}{\mu(r)} \frac{\exp \mu(r) - 1}{\exp \mu(r) + 1}.
\]
Consider two auxiliary functions \( x = \mu(r) \) and
\[
 \tilde{h}(x) = \frac{1}{x} \frac{\exp x - 1}{\exp x + 1}, \quad x \in (0, \infty).
\]
We have
\[
 \tilde{h}'(x) = \frac{1 + 2x \exp x - \exp(2x)}{(x + x \exp x)^2}.
\]
It is not difficult to verify that
\[
 1 + 2x \exp x - \exp(2x) < 0, \quad x \in (0, \infty),
\]
and hence \( h(x) \) is a decreasing function in \((0, \infty)\). On the other hand, it is well-known that \( x = \mu(r) \) is a decreasing function from \((0, 1)\) onto \((0, \infty)\). Thus, \( h(r) \) is an increasing function in \((0, 1)\). In addition,
\[
 \lim_{r \downarrow 0} h(r) = \lim_{x \uparrow \infty} \tilde{h}(x) = 0
\]
and
\[
 \lim_{r \uparrow 1} h(r) = \lim_{x \downarrow 0} \tilde{h}(x) = \frac{1}{2}.
\]
This completes the proof of this lemma. \(\Box\)

Combining Lemmas 1 and 2, we get

Theorem 4. \( F(r) = g(r)h(r) = d_D(0, r)/\exp d_K(0, r) \) is an increasing function from \((0, 1)\) onto \((0, \pi^2/8)\).
Now, we obtain \( d_D < (\pi^2/8) \exp d_K \) on \( \mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\} \), where \( \pi^2/8 \) is sharp. Moreover, Theorem 2 is naturally derived from Theorem 4 and the definition of Teichmüller shift mapping.

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