THE IMBEDDING PROBLEM OF 3-MANIFOLDS INTO 4-MANIFOLDS

Dedicated to Professor Hiroshi Toda on his 60th birthday

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We consider mainly the case $n=3$ of the following general *Imbedding Problem* in the topological category:

Under what relations between an n-manifold M and an $(n+1)$-manifold W, both closed, connected and oriented, does there exist an imbedding from M to W?

Since the problem is trivial for $n \leq 2$, the case $n=3$ is the first appearing non-trivial case. In general, for any n, there are two kinds of imbeddings from M to W. An imbedding f from M to W is said to be of *type* I or II, according to whether $W-fM$ is connected or not. If such an imbedding f exists, then we say that M *is type* I *or* II *imbedded in* W. If f is of type II, then $W-fM$ is seen to have exactly two components, since the boundary map $\partial: H_1(W, W-fM; Z_2) \to \tilde{H}_0(W-fM; Z_2)$ is onto and there is a duality isomorphism $H_1(W, W-fM; Z_2) \cong H^n(fM; Z_2)$ ($\cong Z_2$) (cf. Spanier [Sp; p. 342]). It is possible to characterize the type of an imbedding $f: M \to W$ in terms of homology. In fact, f is of type II or I according to whether the homomorphism $f_*: H_n(M; Z_2) \to H_n(W; Z_2)$ is trivial or not. This is proved by examining the following commutative diagram:

$$H^n(W; Z_2) \xrightarrow{i^*} H^n(fM; Z_2)$$
$$\cong \uparrow \qquad\qquad \uparrow \cong$$
$$H_1(W; Z_2) \xrightarrow{j_*} H_1(W, W-fM; Z_2) \xrightarrow{\partial} \tilde{H}_0(W-fM; Z_2) \to 0 \,,$$

where the vertical maps are the duality isomorphisms (cf. [Sp]). For example, if $\beta_1(W; Z)=0$, then we see from the Poincaré duality and the universal coefficient theorem that any imbedding from M to W is of type II. A typical example of a type I imbedding is $M \xrightarrow{\cong} 1 \times M \subset S^1 \times M = W$. Let $n=3$. First we show that there is an estimate of $\beta_2(W; Z)$ by $\beta_1(M; Z)$ or by certain integral invariants of an infinite cyclic covering of M, provided that M is topologically type

II imbedded in W. By this estimate, we find *infinitely many M which are smoothly type I imbedded in some smooth 4-manifolds having the Q-homology of $S^1 \times S^3$, but not topologically type II imbeddable in any W with $\beta_2(W; Z) < r$, for each $r > 0$* (See Theorem 2.5). This suggsets that the treatment of type I imbeddings is more difficult than that of type II imbeddings, because if M is type II imbedded in W, then M is also type I imbedded in some W' with $\beta_2(W'; Z) = \beta_2(W; Z)$ [For example, take $W' = W \# S^1 \times S^3$]. We can avoid this difficulty by considering punctured imbeddings instead of type I imbeddings. We denote by M^o a compact punctured manifold of M. Then our main result is that *there is an estimate of $\beta_2(W; Z_2)$ by $\beta_1(M; Z)$ or by certain integral invariants of an infinite cyclic covering of the double DM^o, provided that M^o is topologically imbedded in W.* This estimate enables us to find *infinitely many M such that M^o are not topologically imbeddable in any W with $\beta_2(W; Z_2) < r$, for each $r > 0$* (See Theorem 3.2). This research was initially planned in the piecewise-linear category (cf. [K, 1], [K, 2]), but after Freedman's work [F], it became a standard fact that there is a great difference between the piecewise-linear and topological imbeddabilities. In fact, Freedman showed that all homology 3-spheres are imbedded in S^4 by locally flat topological imbeddings, but, as it is well-known, not by piecewise-linear imbeddings. This is the reason why we are converted to the topological category.

In §1 we describe briefly the signature theorem for an infinite cyclic covering of a compact oriented $4m$-manifold with boundary, given in [K, 4]. From this, we derive an estimate of the $4m$-manifold by integral invariants of an infinite cyclic covering of the boundary. Several properties on an infinite cyclic covering of a closed $(4m-1)$-manifold are also given here. In §2 we discuss the estimate of a type II imbedding and its consequence, and in §3, the estimate of a punctured imbedding and its consequence. In §4 we remark that similar results hold in the case $n = 4m-1$ $(m > 1)$.

1. The signature theorem for an infinite cyclic covering

Consider a pair $(B, \dot{\gamma})$ where B is a compact oriented $(4m-1)$-manifold and $\dot{\gamma} \in H^1(B; Z)$. Using the infinite cyclic covering space $\tilde{B}$ of B associated with $\dot{\gamma}$, we have defined in [K, 3] integral invariants, $\sigma_a^{\dot{\gamma}}(B)$, $a \in [-1, 1]$, of the proper oriented homotopy equivalence class of $(B, \dot{\gamma})$. The invariant $\sigma_a^{\dot{\gamma}}(B)$ is called the *local signature* of $(B, \dot{\gamma})$ at a and vanishes except a finite number of a. The sum $\sum_{a \in [-1, 1]} \sigma_a^{\dot{\gamma}}(B)$ is called the *signature* of $(B, \dot{\gamma})$ and denoted by $\sigma^{\dot{\gamma}}(B)$. Next, consider a pair (X, γ) where X is a compact oriented $4m$-manifold and $\gamma \in H^1(X; Z)$. Using the infinite cyclic covering space $\tilde{X}$ of X associated with γ, we have also defined in [K, 4] two kinds of integral invariants, $\tau_{a-0}^{\gamma}(X)$ for $a \in (-1, 1]$ and $\tau_{a+0}^{\gamma}(X)$ for $a \in [-1, 1)$, of the proper oriented homotopy equivalence

class of (X, γ). The following theorem, which we call the *signature theorem*, was proved in [K, 4]:

Theorem 1.1. *Assume that $(B, \dot{\gamma})$ is the boundary of (X, γ) with a compact oriented 4m-manifold X and $\gamma \in H^1(X; Z)$. Then*

$$\tau^\gamma_{a-0}(X) - \operatorname{sign} X = \sum_{x \in [a, 1]} \sigma^{\dot\gamma}_x(B) \quad and \quad \tau^\gamma_{a+0}(X) - \operatorname{sign} X = \sum_{x \in (a, 1]} \sigma^{\dot\gamma}_x(B).$$

Note that $\sigma^{\dot\gamma}_{-1}(B)$ does not appear in the above identities. To simplify the notations, we denote $\tau^\gamma_{a+0}(X)$ by $\tau^\gamma_a(X)$ and the sum $\sum_{x \in (a, 1]} \sigma^{\dot\gamma}_x(B)$ by $\tau^{\dot\gamma}_a(B)$. Let $\tau^\gamma_1(X)$ $= \lim_{a \to 1-0} \tau^\gamma_a(X)$ and $\tau^{\dot\gamma}_1(B) = \lim_{a \to 1-0} \tau^{\dot\gamma}_a(B) \ (=\sigma^{\dot\gamma}_1(B))$. Then the signature theorem implies the identity

$$\tau^\gamma_a(X) - \operatorname{sign} X = \tau^{\dot\gamma}_a(B)$$

for all $a \in [-1, 1]$. Note that $\sigma^{\dot\gamma}_{-1}(B) + \tau^{\dot\gamma}_{-1}(B) = \sigma^{\dot\gamma}(B)$. Let (Y, A) be a pair such that Y is a compact manifold and A is a compact submanifold. Let $(\tilde{Y}, \tilde{A})$ be the infinite cyclic covering space pair of (Y, A) associated with an element $\gamma \in H^1(Y; Z)$. Let $\langle t \rangle$ be the covering transformation group with a specified generator t. Let $\Lambda = Z \langle t \rangle$ and $\Gamma = Q \langle t \rangle$. Since $H_*(\tilde{Y}, \tilde{A}; Z)$ is a finitely generated Λ-module and Λ is Noetherian, we see that the kernel of $t - 1 : H_*(\tilde{Y}, \tilde{A}; Z)$ $\to H_*(\tilde{Y}, \tilde{A}; Z)$ is a finitely generated abelian group. We denote this rank by κ^γ_* $(Y, A; Z)$. It also equals the Q-dimension of the kernel of $t - 1 : H_*(\tilde{Y}, \tilde{A}; Q)$ $\to H_*(\tilde{Y}, \tilde{A}; Q)$. The following is easily obtained (cf. [K, 1; Lemma 1.1]):

Lemma 1.2. *For any integer $d \neq 0$, $\kappa^{d\gamma}_*(Y, A) = \kappa^\gamma_*(Y, A)$.*

Let $TH_*(\tilde{Y}, \tilde{A}; Q)$ be the Γ-torsion part of $H_*(\tilde{Y}, \tilde{A}; Q)$, which is a finitely generated Γ-module, and $BH_*(\tilde{Y}, \tilde{A}; Q) = H_*(\tilde{Y}, \tilde{A}; Q)/TH_*(\tilde{Y}, \tilde{A}; Q)$, which is Γ-free. We denote this rank by $\beta^\gamma_*(Y, A; Q)$. We use the signature theorem to prove the following:

Lemma 1.3. *Let B be a closed oriented $(4m-1)$-manifold and $\dot{\gamma} \in H^1(B; Z)$ and d be a non-zero integer.*

(1) *For a real number θ such that $\cos d\theta \neq \pm 1$ and $\sigma^{\dot\gamma}_{\cos d\theta}(B) = 0$, we have $\tau^{d\dot\gamma}_{\cos \theta}(B)$ $= \tau^{\dot\gamma}_{\cos d\theta}(B)$ and $\sigma^{d\dot\gamma}_{\cos \theta}(B) = 0$,*

(2) *$\sigma^{d\dot\gamma}(B) = \sigma^{\dot\gamma}(B)$ (if d is odd) or 0 (if d is even).*

The following is direct from Lemma 1.3:

Corollary 1.4. (1) *$\tau^{d\dot\gamma}_1(B) = \tau^{\dot\gamma}_1(B)$,*

(2) *$\tau^{d\dot\gamma}_{-1}(B) = \tau^{\dot\gamma}_{-1}(B)$ (if d is odd) or $\tau^{\dot\gamma}_1(B)$ (if d is even),*

(3) *$\sigma^{d\dot\gamma}_{-1}(B) = \sigma^{\dot\gamma}_{-1}(B)$ (if d is odd) or $-\sigma^{\dot\gamma}_1(B)$ (if d is even),*

(4) *If $\cos d\theta \neq \pm 1$, then $\sigma^{d\dot\gamma}_{\cos \theta}(B) = \operatorname{sign}(\sin \theta \sin |d| \theta) \sigma^{\dot\gamma}_{\cos d\theta}(B)$,*

(5) *If* $\cos d\theta = \pm 1$ *but* $\cos \theta \neq \pm 1$, *then* $\sigma^{d\dot{\gamma}}_{\cos \theta}(B) = 0$.

1.5. Proof of Lemma 1.3. First, assume that $(B, \dot{\gamma})$ is the boundary of a pair (X, γ). Let $\tilde{X}$ and $\tilde{X}^{(d)}$ be the infinite cyclic covering spaces of X associated with γ and $d\gamma$, respectively. Let $A(t)$ be a t-Hermitian matrix, which is the Γ-intersection matrix associated with a Γ-basis $e_1, e_2, \cdots, e_r$ of $BH_{2m}(\tilde{X}; Q)$. By [K, 1; Lemma 1.1], we can consider $e_1, e_2, \cdots, e_r$ as a Γ-basis for $BH_{2m}(\tilde{X}^{(d)}; Q)$, associated with which the Γ-intersection matrix is $A(t^d)$. Since $\sigma^{\dot{\gamma}}_{\cos d\theta}(B) = 0$, it follows from the signature theorem that

$$
\begin{aligned}
\tau^{\dot{\gamma}}_{\cos d\theta}(B) &= \tau^{\gamma}_{\cos d\theta \pm 0}(X) - \operatorname{sign} X \\
&= \lim_{dv \to d\theta \pm 0} \operatorname{sign} A(e^{idv}) - \operatorname{sign} X \\
&= \lim_{v \to \theta \pm 0} \operatorname{sign} A((e^{iv})^d) - \operatorname{sign} X \\
&= \tau^{d\gamma}_{\cos \theta \pm 0}(X) - \operatorname{sign} X \\
&= \tau^{d\dot{\gamma}}_{\cos \theta}(B) \quad \text{and} \quad \sigma^{d\dot{\gamma}}_{\cos \theta}(B) = 0,
\end{aligned}
$$

showing (1). For (2) note that $\sigma^{\dot{\gamma}}(B)$ is the α-invariant of the double covering space of B associated with the Z_2-reduction $\dot{\gamma}(2) \in H^1(B; Z_2)$ of $\dot{\gamma}$ (See [K, 4; Lemma 4.3]). Since it is similar for $\sigma^{d\dot{\gamma}}(B)$, we see that $\sigma^{d\dot{\gamma}}(B) = \sigma^{\dot{\gamma}}(B)$ (if d is odd) or 0 (if d is even), showing (2). If $(B, \dot{\gamma})$ is not a boundary, then some multiple $N(B, \dot{\gamma})$ $(N > 0)$ is a boundary (cf. [K, 4; Remark 1.6]) and we obtain the identities (1), (2) on $N(B, \dot{\gamma})$ in place of $(B, \dot{\gamma})$. Dividing them by N, we obtain the desired (1), (2). This completes the proof.

For an abelian group H, let tH be the torsion part and $bH = H/tH$. Let X be a compact oriented $4m$-manifold with boundary B. Let $\hat{\beta}_*(X; Z)$ be the rank of the cokernel of the natural homomorphism $H_*(B; Z) \to H_*(X; Z)$. Note that any intersection matrix on $bH_{2m}(X; Z)$ has the rank $\hat{\beta}_{2m}(X; Z)$, by Poincaré duality.

Theorem 1.6. *Assume that for some non-zero integer d, $(B, d\dot{\gamma})$ is the boundary of a pair (X, γ) with a compact oriented $4m$-manifold X and $\gamma \in H^1(X; Z)$. Then for all a,*

$$
|\tau^{\dot{\gamma}}_a(B)| - \kappa^{\dot{\gamma}}_{2m-1}(B) \leq \hat{\beta}_{2m}(X; Z) + |\operatorname{sign} X| \, .
$$

Proof. By Lemma 1.2, $\kappa^{\dot{\gamma}}_{2m-1}(B) = \kappa^{d\dot{\gamma}}_{2m-1}(B)$. By Lemma 1.3(1),

$$
\max_{a \in [-1,1]} \tau^{\dot{\gamma}}_a(B) = \max_{a \in [-1,1]} \tau^{d\dot{\gamma}}_a(B) \quad \text{and} \quad \min_{a \in [-1,1]} \tau^{\dot{\gamma}}_a(B) = \min_{a \in [-1,1]} \tau^{d\dot{\gamma}}_a(B) \, .
$$

Thus, we may assume $d = 1$. Let $(\tilde{X}, \tilde{B})$ be the infinite cyclic covering space pair of (X, B) associated with γ. Let $\hat{\beta}^{\gamma}_*(X; Q)$ be the Γ-rank of the cokernel of the natural homomorphism $H_*(\tilde{B}; Q) \to H_*(\tilde{X}; Q)$. By the exact sequence of

$(\tilde{X}, \tilde{B})$, we have

$$\hat{\beta}_{2m}^{\gamma}(X; Q) = \Sigma_{q=0}^{2m}(-1)^q \beta_q^{\gamma}(X, B; Q) + \Sigma_{q=0}^{2m-1}(-1)^q \beta_q^{\dot{\gamma}}(B; Q)$$
$$+ \Sigma_{q=0}^{2m-1}(-1)^{q+1} \beta_q^{\gamma}(X; Q) .$$

From the Wang exact sequence

$$\rightarrow H_q(\tilde{X}, \tilde{B}; Q) \xrightarrow{t-1} H_q(\tilde{X}, \tilde{B}; Q) \xrightarrow{p_*} H_q(X, B; Q) \rightarrow H_{q-1}(\tilde{X}, \tilde{B}; Q)$$

$$\xrightarrow{t-1} H_{q-1}(\tilde{X}, \tilde{B}; Q) \rightarrow ,$$

we see that $\beta_q(X, B; Z) = \beta_q^{\gamma}(X, B; Q) + \kappa_q^{\gamma}(X, B) + \kappa_{q-1}^{\gamma}(X, B)$. Similarly, β_q $(B; Z) = \beta_q^{\dot{\gamma}}(B; Q) + \kappa_q^{\dot{\gamma}}(B) + \kappa_{q-1}^{\dot{\gamma}}(B)$ and $\beta_q(X; Z) = \beta_q^{\gamma}(X; Q) + \kappa_q^{\gamma}(X) + \kappa_{q-1}^{\gamma}(X)$. Note that $\hat{\beta}_{2m}(X; Z) = \Sigma_{q=0}^{2m}(-1)^q \beta_q(X, B; Z) + \Sigma_{q=0}^{2m-1}(-1)^q \beta_q(B; Z) + \Sigma_{q=0}^{2m-1}$ $(-1)^{q+1} \beta_q(X; Z)$. Then we have

$$\hat{\beta}_{2m}^{\gamma}(X; Q) = \hat{\beta}_{2m}(X; Z) - \kappa_{2m}^{\gamma}(X, B) + \kappa_{2m-1}^{\dot{\gamma}}(B) - \kappa_{2m-1}^{\gamma}(X)$$
$$\leq \hat{\beta}_{2m}(X; Z) + \kappa_{2m-1}^{\dot{\gamma}}(B) .$$

The inequality $|\tau_a^{\gamma}(X)| \leq \hat{\beta}_{2m}^{\gamma}(X; Q)$ is directly obtained from the definition of $\tau_a^{\gamma}(X)$ (cf. [K, 4]). Therefore, by the signature theorem,

$$|\tau_a^{\dot{\gamma}}(B)| \leq |\tau_a^{\gamma}(X)| + |\operatorname{sign} X|$$
$$\leq \hat{\beta}_{2m}^{\gamma}(X; Q) + |\operatorname{sign} X|$$
$$\leq \hat{\beta}_{2m}(X; Z) + \kappa_{2m-1}^{\dot{\gamma}}(B) + |\operatorname{sign} X| .$$

This completes the proof.

Corollary 1.7. *Under the assumption of Theorem 1.6,*

$$|\tau_a^{\dot{\gamma}}(B)| \leq \beta_{2m}(X; Z) + |\operatorname{sign} X|$$

for all a.

Proof. By the proof of Theorem 1.6, $|\tau_a^{\dot{\gamma}}(B)| \leq |\tau_a^{\gamma}(X)| + |\operatorname{sign} X|$ and $|\tau_a^{\gamma}$ $(X)| \leq \hat{\beta}_{2m}^{\gamma}(X; Q) \leq \beta_{2m}^{\gamma}(X; Q) \leq \beta_{2m}^{\gamma}(X; Q) + \kappa_{2m}^{\gamma}(X) + \kappa_{2m-1}^{\gamma}(X) = \beta_{2m}(X; Z)$, completing the proof.

REMARK 1.8. In Theorem 1.6 and Corollary 1.7, if we replace $\tau_a^{\dot{\gamma}}(B)$ with $\sigma^{\dot{\gamma}}(B)$, then the resulting inequalities do not hold in general. Some counterexample was given in [K, 4; 4.5].

2. Type II imbeddings

Theorem 2.1. *Assume that M is topologically type II imbedded in W.* Then $\beta_1(M; Z) \leq \beta_2(W; Z)/2$ *or there is an indivisible element* $\dot{\gamma} \in H^1(M; Z)$ *such*

that for all a

$$|\tau_a^{\dot\gamma}(M)| - \kappa_1^{\dot\gamma}(M) \leq \beta_2(W; Z) + |\operatorname{sign} W| \leq 2\beta_2(W; Z).$$

Proof. Assume that $\beta_1(M; Z) > \beta_2(W; Z)/2$. Regard $f: M \subset W$. Since it is of type II and $H_1(W, W-M; Z) \cong H^3(M; Z) \cong Z$, the boundary map $\partial: H_1(W, W-M; Z) \to \tilde{H}_0(W-M; Z)$ is an isomorphism, so that the natural homomorphism $H_1(W-M; Z) \to H_1(W; Z)$ is onto. Using Quinn's handle straightening lemma [Q], we can kill $H_1(W; Q)$ without changing $\beta_2(W; Z)$ by a surgery on $W-M$. We assume $\beta_1(W; Z) = 0$. By Ancel/Cannon [A/C], the imbedding $f_P = f \times \operatorname{id}: M_P = M \times CP^2 \subset W \times CP^2 = W_P$ is homotopic to a bi-collared imbedding $f_P': M_P \to W_P$, which is also of type II. Let $f_P' M_P = M_P'$. M_P' splits W_P into two compact connected submanifolds E', E''. To see that $\beta_1(W_P - M_P'; Z) \neq 0$, suppose that $H_1(W_P - M_P'; Q) = 0$. Then $H_1(E'; Q) = H_1(E''; Q) = 0$ and $\beta_2(E', M_P'; Z) \geq \beta_1(M_P'; Z)$ and $\beta_2(E'', M_P'; Z) \geq \beta_1(M_P'; Z)$. Hence $\beta_2(W, M; Z) = \beta_2(W_P, M_P'; Z) = \beta_2(E', M_P'; Z) + \beta_2(E'', M_P'; Z) \geq 2\beta_1(M_P'; Z) = 2\beta_1(M; Z)$. Since $H_3(W; Q) = H_1(W; Q) = 0$, we see from the exact sequence of (W, M) that $\beta_2(W, M; Z) = \beta_1(M; Z) + \beta_2(W; Z) - \beta_2(M; Z) = \beta_2(W; Z)$, so that $\beta_2(W; Z) \geq 2\beta_1(M; Z)$, contradicting our assumption. Therefore, $\beta_1(W_P - M_P'; Z) = \beta_1(E'; Z) + \beta_1(E''; Z) \neq 0$. Say $\beta_1(E'; Z) \neq 0$. Let $\gamma \in H^1(E'; Z)$ be any non-zero element. Since the natural map $H_1(M_P'; Q) \to H_1(E'; Q)$ is onto, $\dot\gamma_P = \gamma | M_P' \in H^1(M_P'; Z)$ is not zero. Write $\dot\gamma_P = d\dot\gamma_P$ for an integer $d \neq 0$ and an indivisible element $\dot\gamma_P$. By Theorem 1.6,

$$|\tau_a^{\dot\gamma_P}(M_P')| - \kappa_3^{\dot\gamma_P}(M_P') \leq \hat\beta_4(E'; Z) + |\operatorname{sign} E'|.$$

Let $\dot\gamma \in H^1(M; Z)$ correspond to $\dot\gamma_P$. Directly, $\kappa_3^{\dot\gamma_P}(M_P') = \kappa_1^{\dot\gamma}(M)$. By [K, 3], $\tau_a^{\dot\gamma_P}(M_P') = \tau_a^{\dot\gamma}(M)$. Let $H' \subset H_4(E'; Q)$ and $H'' \subset H_4(E''; Q)$ be Q-subspaces of dimensions $\hat\beta_4(E'; Z)$ and $\hat\beta_4(E''; Z)$ on which Q-intersection matrices are non-singular, respectively.

Lemma 2.2. *The composite* $H' \oplus H'' \subset H_4(E'; Q) \oplus H_4(E''; Q) \overset{i'_* + i''_*}{\to} H_4(W_P; Q) \overset{projection}{\to} H_2(W; Q) \otimes H_2(CP^2; Z)$ *is injective, where* i'_* *and* i''_* *are natural maps.*

Assuming this lemma, we have $\hat\beta_4(E'; Z) + \hat\beta_4(E''; Z) \leq \beta_2(W; Z)$. By the Novikov addition theorem, $\operatorname{sign} E' + \operatorname{sign} E'' = \operatorname{sign} W_P = \operatorname{sign} W$. Since $|\operatorname{sign} E''| \leq \hat\beta_4(E''; Z)$, it follows that

$$|\tau_a^{\dot\gamma}(M)| - \kappa_1^{\dot\gamma}(M) \leq \hat\beta_4(E'; Z) + |\operatorname{sign} E'| \leq \beta_2(W; Z) - \hat\beta_4(E''; Z) +$$
$$|\operatorname{sign} W| + |\operatorname{sign} E''| \leq \beta_2(W; Z) + |\operatorname{sign} W|.$$

This completes the proof except the proof of Lemma 2.2.

2.3. Proof of Lemma 2.2. Using the intersection pairing Int_{W_P} on $H_4(W_P; Q)$, we see that $i'_* + i''_* | H' \oplus H''$ is injective, whose image we denote by H. Let $x \in H$ be non-zero and write $x = x_0 + x_2 + x_4$ with $x_i \in H_i(W; Q) \otimes H_{4-i}(CP^2; Z)$. If $x_4 \neq 0$, then there is an element $x'_0 \in H_0(W; Q) \otimes H_4(CP^2; Z)$ with $\text{Int}_{W_P}(x_4, x'_0) \neq 0$. Then $\text{Int}_{W_P}(x, x'_0) = \text{Int}_{W_P}(x_4, x'_0) \neq 0$. But, x'_0 is represented by a cycle in M_P' and hence $\text{Int}_{W_P}(H, x'_0) = 0$, which is a contradiction. Thus, $x_4 = 0$ and $x = x_0 + x_2$. Note that there is an element $x' = x'_0 + x'_2$ in H with $\text{Int}_{W_P}(x, x') \neq 0$. Then $\text{Int}_{W_P}(x, x') = \text{Int}_{W_P}(x_2, x'_2) \neq 0$, and $x_2 \neq 0$. This completes the proof of Lemma 2.2.

Since any imbedding from M to W with $\beta_1(W; Z) = 0$ is of type II, the following is direct from Theorem 2.1:

Corollary 2.4. *If M is topologically imbedded in any W with $H_*(W; Q) \cong H_*(S^4; Q)$ and $\beta_1(M; Z) \neq 0$, then there is an indivisible element $\dot{\gamma} \in H^1(M; Z)$ such that $|\tau_a^{\dot{\gamma}}(M)| \leq \kappa_1^{\dot{\gamma}}(M)$ for all a.*

This answers in part Problem 3.20 of Kirby's Problem List [Ki] (cf. [G/L]). Note that there are many M which are smoothly imbedded in S^4 and have $|\tau_a^{\dot{\gamma}}(M)| = \kappa_1^{\dot{\gamma}}(M) \neq 0$ for an indivisible $\dot{\gamma}$ and all a. For example, let M be the torus bundle over S^1 with monodromy matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\dot{\gamma}$, the element represented by the bundle projection. Directly, we see that M is smoothly imbedded in S^4 and $|\tau_a^{\dot{\gamma}}(M)| = \kappa_1^{\dot{\gamma}}(M) = 1$ for all a.

Theorem 2.5. *For any positive integers r, r', there are infinitely many M having all of the following properties (0)–(4):*
(0) $H_*(M; Z) \cong H_*(\underset{r}{\#} S^1 \times S^2; Z)$ and $\kappa_1^{\dot{\gamma}}(M) = 0$ and $|\tau_{-1}^{\dot{\gamma}}(M)| \geq r'$ for all indivisible elements $\dot{\gamma} \in H^1(M; Z)$,
(1) *M is smoothly type II imbedded in $\underset{r}{\#} S^2 \times S^2$,*
(2) *M is smoothly type I imbedded in a smooth 4-manifold W^* with $tH_q(W^*; Z) \cong Z_2$(if $q = 1, 2$) or 0(if $q \neq 1, 2$) and $bH_*(W^*; Z) \cong H_*(S^1 \times S^3; Z)$,*
(3) *M° is smoothly imbedded in a smooth 4-manifold W^{**} with $tH_q(W^{**}; Z) \cong Z_2$ (if $q = 1, 2$) or 0(if $q \neq 1, 2$) and $bH_*(W^{**}; Z) \cong H_*(S^4; Z)$,*
(4) *M is not topologically type II imbeddable in any W with $\beta_2(W; Z) < 2r$ and $\beta_2(W; Z) + |\text{sign } W| < r'$.*

REMARK 2.6. We can conclude from Theorem 2.5 that Theorem 2.1 can not apply to type I imbeddings and if $\beta_1(M; Z) \leq \beta_2(W; Z)/2$, then $|\tau_a^{\dot{\gamma}}(M)| - \kappa_1^{\dot{\gamma}}(M)$, $a \in [-1, 1]$, do not, in general, restrict $\beta_2(W; Z)$ in Theorem 2.1. Cooper [C] has obtained a result corresponding to (4) in the piecewise-linear category.

2.7. Proof of Theorem 2.5. Let k be any invertible knot in S^3 with $|\sigma(k)| \geq r'$, where $\sigma(k)$ denotes the signature of the knot k. Let $M(k)$ be the 0-surgery manifold of k. Note that $\sigma^{\dot\gamma*}(M(k)) = \sigma(k)$ and $\sigma^{\dot\gamma*}_{-1}(M(k)) = \kappa_1^{\dot\gamma*}(M(k)) = 0$ for any generator $\dot\gamma* \in H^1M(k); Z) \cong Z$. By Lemma 1.3, $\tau^{d\dot\gamma*}_{-1}(M(k)) = \tau^{\dot\gamma*}_{-1}(M(k)) = \sigma(k)$ (if d is odd) or 0 (if d is even), for $\sigma^{d\gamma*}_{-1}(M(k)) = \sigma^{\gamma*}_{-1}(M(k)) = 0$. Let M be the r-fold connected sum of $M(k)$. Then $H_*(M; Z) \cong H_*(\#_r S^1 \times S^2; Z)$ and $\kappa_1^{\dot\gamma}(M) = 0$ and $|\tau^{\dot\gamma}_{-1}(M)| \geq s|\sigma(k)| \geq r'$ for any indivisible element $\dot\gamma \in H^1(M; Z)$, where s is the number of the summands $M(k)$ of M such that $\dot\gamma|M(k)$ is an odd multiple of $\dot\gamma*$. This shows (0). For (1) note that there is a piecewise-linearly imbedded 2-sphere $S^2(k)$ in $S^2 \times S^2$ which is homotopic to $S^2 \times q$ and has just one non-locally flat point represented by the knot k (See Suzuki [Su]). Since $S^2(k)$ has the self-intersection number 0, we see that the boundary of a (smooth) regular neighborhood of $S^2(k)$ in $S^2 \times S^2$ is diffeomorphic to $M(k)$, so that M is smoothly imbedded in $\#_r S^2 \times S^2$, showing (1). For (2) we use that k is invertible.

From this, we have an orientation-preserving diffeomorphism h of $M(k)$ with $h_* = -1$ on $H_1(M(k); Z)$. Let W be the mapping torus of h. Then $tH_q(W; Z) \cong Z_2$ (if $q = 1, 2$) or 0 (if $q \neq 1, 2$) and $bH_*(W; Z) \cong H_*(S^1 \times S^3; Z)$. We may consider that h sends a 3-disk D^3 in $M(k)$ to itself by the identity. Let W^{**} be a closed 4-manifold obtained from W by replacing $S^1 \times D^3 \subset W$ by $D^2 \times \partial D^3$. We have $tH_q(W^{**}; Z) \cong Z_2$(if $q = 1, 2$) or 0(if $q \neq 1, 2$) and $bH_*(W^{**}; Z) \cong H_*(S^4; Z)$. $M(k)^o = M(k)$-Int D^3 is smoothly imbedded in W^{**} and the connected sum $M(k) \# T_g$, T_g the solid torus of genus g, is smoothly imbedded in $M(k)^o$ and hence in W^{**}. A boundary-disk sum, M^o, of r copies of $M(k)^o$ is smoothly imbedded in $(M(k) \# T_g) \times [0, 1]$ with $g = r - 1$. Thus, using a collar of $M(k) \# T_g$ in W^{**}, we see that M^o is smoothly imbedded in W^{**}. Let W^* be a closed 4-manifold obtained from W^{**} by replacing a tubular neighborhood $T(\partial M^o) = S^2 \times D^2$ of ∂M^o in W^{**} by $D^3 \times \partial D^2$, where the framing of $T(\partial M^o) = S^2 \times D^2$ is chosen so that some $S^2 \times p$ ($p \in \partial D^2$) is a boundary-parallel 2-sphere in M^o. We see that M is smoothly type I imbedded in W^* and $tH_q(W^*; Z) \cong Z_2$(if $q = 1, 2$) or 0 (if $q \neq 1, 2$) and $bH_*(W^*; Z) \cong H_*(S^1 \times S^3; Z)$, showing (2) and (3). For (4) suppose that M is topologically type II imbedded in W with $\beta_2(W; Z) < 2r$ and $\beta_2(W; Z) + |\text{sign } W| < r'$. Since $\beta_1(M; Z) = r > \beta_2(W; Z)/2$, we have, by (0) and Theorem 2.1, an indivisible element $\dot\gamma \in H^1(M; Z)$ such that

$$r' \leq |\tau^{\dot\gamma}_{-1}(M)| - \kappa_1^{\dot\gamma}(M) \leq \beta_2(W; Z) + |\text{sign } W| < r',$$

which is a contradiction. This completes the proof.

3. Punctured imbeddings

Let α be the standard reflection on the double DM^o of M^o.

DEFINITION. An element $\dot{\gamma} \in H^1(DM^o; Z)$ is Z_2-*asymmetric* if the Z_2-reduction $\dot{\gamma}(2) \in H^1(DM^o; Z_2)$ of $\dot{\gamma}$ has $\alpha^*(\dot{\gamma}(2)) \neq \dot{\gamma}(2)$.

Theorem 3.1. *Assume that M^o is topologically imbedded in W. Then $\beta_1(M; Z) \leq \beta_2(W; Z_2)/2$ or there is a Z_2-asymmetric indivisible element $\dot{\gamma} \in H^1(DM^o; Z)$ such that for all a*

$$|\tau_a^{\dot{\gamma}}(DM^o)| - \kappa_1^{\dot{\gamma}}(DM^o) \leq \beta_2(W; Z) + |\text{sign } W| \leq 2\beta_2(W; Z).$$

Proof. Assume that $\beta_1(M; Z) > \beta_2(W; Z_2)/2$. Regard $f: M^o \subset W$. Since $H_1(W, W - M^o; Z) \cong H^3(M^o; Z) = 0$, the natural homomorphism $H_1(W - M^o; Z) \to H_1(W; Z)$ is onto. By [Q], we can kill $H_1(W; Q)$ without changing $\beta_2(W; Z)$ by a surgery on $W - M^o$. We assume $\beta_1(W; Z) = 0$. Choose mutually disjoint $S^1 \times D_i^3$, $i = 1, 2, \cdots, s$, in $W - M^o$ (by using [Q]) so that the cores $S^1 \times 0_i$, $i = 1, 2, \cdots, s$, represent a basis for $H_1(W; Z_2)$. Let $F = W - \cup_{i=1}^s S^1 \times D_i^3$. By [A/C] and a boundary collar technique, the imbedding $f_P = f \times \text{id}: M_P^o = M^o \times CP^2 \subset F \times CP^2 = F_P$ is homotopic to a bi-collared imbedding $f_P': M_P^o \to F_P$. Let $N = M^o \times CP^2 \times [0, 1]$ be a collar of $f_P' M_P^o$ in F_P. Construct $W^* = F \cup_{i=1}^s D^2 \times S_i^2$ identifying $S^1 \times \partial D_i^3$ with $\partial D^2 \times S_i^2$ for all i. Then $\beta_2(W^*; Z) = \beta_2(W; Z_2)$ and $\beta_1(W^*; Z_2) = 0$. Let $W_P = W \times CP^2$, $W_P^* = W^* \times CP^2$, $E = W_P$-Int N and $E^* = W_P^*$-Int N. Note that there is an epimorphism $\mu: H_1(E; Z) \to H_1(E^*; Z)$. We show that $\beta_1(E^*; Z) \neq 0$. Suppose $H_1(E^*; Q) = 0$. By Poincaré duality, $H_7(E^*, \partial E^*; Q) = 0$. But, $H_7(E^*, \partial E^*; Q) \cong H_7(W_P^*, N; Q) \cong H_7(W_P^*, M_P^o; Q) \cong H_3(W^*, M^o; Q) \otimes H_4(CP^2; Z)$. Thus, $H_3(W^*, M^o; Q) = 0$. Since $\partial: H_2(E^*, \partial E^*; Q) \to H_1(\partial E^*; Q)$ is onto and $H_2(E^*, \partial E^*; Q) \cong H_2(W_P^*, N; Q) \cong H_2(W_P^*, M_P^o; Q) \cong H_2(W^*, M^o; Q) \otimes H_0(CP^2; Z)$ and $\partial E^* \cong DM^o \times CP^2$, we see that $\beta_2(W^*, M^o; Z) \geq \beta_1(DM^o; Z) = 2\beta_1(M; Z)$. Using $H_3(W^*, M^o; Q) = 0$, we obtain from the exact sequence of (W^*, M^o) that $\beta_2(W; Z_2) = \beta_2(W^*; Z) = \beta_2(W^*, M^o; Z) \geq 2\beta_1(M; Z)$, contradicting our assumption. Therefore, $\beta_1(E^*; Z) \neq 0$. Take any indivisible element $\gamma^* \in H^1(E^*; Z)$. Then $\gamma^*(2) \in H^1(E^*; Z_2)$ is not zero. Note that $\partial E^* = \partial E = \partial N = DM_P^o (= DM^o \times CP^2)$. By the Mayer/Vietoris sequence, the natural homomorphism $H^1(N; Z_2) \oplus H^1(E^*; Z_2) \to H^1(DM_P^o; Z_2)$ is injective, for $H^1(W_P^*; Z_2) = 0$. Thus, $\gamma^*(2)|DM_P^o \in H^1(DM_P^o; Z_2)$ is non-zero and there are an odd integer d and an indivisible element $\dot{\gamma}_P \in H^1(DM_P^o; Z)$ such that $d\dot{\gamma}_P = \gamma^*|DM_P^o$. Let $\dot{\gamma} \in H^1(DM^o; Z)$ be an indivisible element corresponding to $\dot{\gamma}_P$. We show that $\dot{\gamma}$ is Z_2-asymmetric. If $\alpha^*(\dot{\gamma}(2)) = \dot{\gamma}(2)$, then $\dot{\gamma}_P(2) = d\dot{\gamma}_P(2)$ lies in the image of the natural homomorphism $H^1(N; Z_2) \to H^1(DM_P^o; Z_2)$, so that the natural homomorphism $H^1(N; Z_2) \oplus H^1(E^*; Z_2) \to H^1(DM_P^o; Z_2)$ is not injective, a contradiction. Hence $\dot{\gamma}$ is Z_2-asymmetric. Let $\gamma = \mu^\sharp(\gamma^*) \in H^1(E; Z)$. Then $\gamma|DM_P^o = d\dot{\gamma}_P$. By Theorem 1.6, $|\tau_a^{\dot{\gamma}_P}(DM_P^o)| - \kappa_3^{\dot{\gamma}_P}(DM_P^o) \leq \hat{\beta}_4(E; Z) + |\text{sign } E|$ for all a. By [K, 3], $\tau_a^{\dot{\gamma}_P}(DM_P^o) = \tau_a^{\dot{\gamma}}(DM^o)$. Directly, $\kappa_3^{\dot{\gamma}_P}(DM_P^o) = \kappa_1^{\dot{\gamma}}(DM^o)$. By Lemma 2.2, $\hat{\beta}_4(E; Z) \leq \beta_2(W; Z)$. By the Novikov addition theorem,

sign $E=$sign $W_P=$sign W, for sign $N=0$. It follows that $|\tau_a^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}(DM^o)$
$\leq\beta_2(W;Z)+|$sign $W|$ for all a. This completes the proof.

Theorem 3.2. *For any positive integers r, r', there are infinitely many M having all of the following properties* (0)–(4):

(0) $H_*(M;Z)\cong H_*(\#_r S^1\times S^2;Z)$ *and* $\kappa_1^{\dot\gamma}(DM^o)=0$ *and* $|\tau_{-1}^{\dot\gamma}(DM^o)|\geq r'$ *for all* Z_2-*asymmetric indivisible elements* $\dot\gamma\in H^1(DM^o;Z)$,

(1)   *M is smoothly type II imbedded in* $\#_r S^2\times S^2$,

(2)   *M is smoothly type I imbedded in a smooth 4-manifold W^* with* $tH_q(W^*;Z)$ $\cong\oplus_r Z_2$ *(if $q=1$, 2) or 0 (if $q\neq1$, 2) and $bH_*(W^*;Z)\cong H_*(S^1\times S^3;Z)$,*

(3)   *M^o is smoothly imbedded in a smooth 4-manifold W^{**} with* $tH_q(W^{**};Z)\cong$ $\oplus_r Z_2($if $q=1$, 2$)$ *or 0 (if $q\neq1$, 2) and $bH_*(W^{**};Z)\cong H_*(S^4;Z)$,*

(4)   *M^o is not topologically imbeddable in any W with $\beta_2(W;Z_2)<2r$ and β_2 $(W;Z)+|$sign $W|<r'$.*

REMARK 3.3. We can conclude from Theorem 3.2 that $|\tau_a^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}$ (DM^o), $a\in[-1, 1)$, do not restrict $\beta_2(W;Z)$ if $\beta_1(M;Z)\leq\beta_2(W;Z_2)/2$ and this inequality can not be replaced by $\beta_1(M;Z)\leq\beta_2(W;Z)/2$, in Theorem 3.1.

3.4.   **Proof of Theorem 3.2.** Let k_i, $i=1, 2, \cdots, r$, be invertible knots in S^3 such that $|\sigma(k_j)|\geq r'+\Sigma_{i=1}^{j-1}|\sigma(k_i)|$, $j=1, 2, \cdots, r$. Let $M_i=M(k_i)$ and $M=\#_{i=1}^r M_i$. Then $H_*(M;Z)\cong H_*(\#_r S^1\times S^2;Z)$. Let $\dot\gamma\in H^1(DM^o;Z)$ be any Z_2-asymmetric indivisible element. Directly, $\kappa_1^{\dot\gamma}(DM^o)=0$. Write $DM^o=(M_1\#\bar M_1)\#(M_2\#\bar M_2)$ $\#\cdots\#(M_r\#\bar M_r)$, where $\bar M_i^o=\alpha M_i^o$. Let $M_{i(j)}\#\bar M_{i(j)}$, $j=1, 2, \cdots, s$, be all of the summands of DM^o such that $\dot\gamma\,|\,M_{i(j)}\#\bar M_{i(j)}$ is still Z_2-asymmetric, where $1\leq i(1)$ $<i(2)<\cdots<i(s)\leq r$. Since $\tau_{-1}^{d\dot\gamma^*}(M_i)=-\tau_{-1}^{d\dot\gamma^*}(\bar M_i)=\sigma(k_i)$ (if d is odd) or 0 (if d is even) for a generator $\dot\gamma^*\in H^1(M_i;Z)$ (cf. Lemma 1.3), it follows that $\tau_{-1}^{\dot\gamma}(DM^o)$ $=\varepsilon_1\sigma(k_{i(1)})+\varepsilon_2\sigma(k_{i(2)})+\cdots+\varepsilon_s\sigma(k_{i(s)})$, $\varepsilon_j=\pm1$, and $|\tau_{-1}(DM^o)|\geq|\sigma(k_{i(s)})|-$ $\Sigma_{j=1}^{s-1}|\sigma(k_{i(j)})|\geq r'$, showing (0). Since M_i is smoothly imbedded in $S^2\times S^2$ (cf. 2.7), M is smoothly imbedded in $\#_r S^2\times S^2$ (by a type II imbedding), showing

(1). Since k_i are invertible, there is an orientation-preserving diffeomorphism h of M with $h_*=-1$ on $H_1(M;Z)$. Let W^* be the mapping torus of h. We have $tH_q(W^*;Z)\cong\oplus_r Z_2$ (if $q=1$, 2) or 0 (if $q\neq1$, 2) and $bH_*(W^*;Z)\cong H_*(S^1\times S^3;Z)$, showing (2). For (3) we can kill $bH_1(W^*;Z)$ by a surgery on W^*-M^o to obtain a desired W^{**}. For (4) suppose that there is an imbedding from M^o to W with $\beta_2(W;Z_2)<2r$ and $\beta_2(W;Z)+|$sign $W|<r'$. Since $\beta_1(M;Z)=r>\beta_2(W;Z_2)/2$, we see from (0) and Theorem 3.1 that there is a Z_2-asymmetric indivisible element $\dot\gamma\in H^1(DM^o;Z)$ such that $r'\leq|\tau_{-1}^{\dot\gamma}(DM^o)|-\kappa_1^{\dot\gamma}(DM^o)\leq\beta_2(W;Z)+|$sign $W|<r'$, which is a contradiction. This completes the proof of Theorem 3.2.

4. Higher dimensional analogues

We consider the case $n=4m-1$ $(m>1)$ only. The argument of this case is simpler than the case $n=3$, because any topological imbedding from a compact oriented n-manifold to W is homotopic to a bi-collared imbedding by $[A/C]$.

Theorem 4.1. *Assume that M is topologically type II imbedded in W. Then $\beta_1(M;Z)\leq\beta_2(W;Z)$ or there is an indivisible element $\dot\gamma\in H^1(M;Z)$ such that for all a*

$$|\tau_a^{\dot\gamma}(M)|-\kappa_{2m-1}^{\dot\gamma}(M)\leq\beta_{2m}(W;Z)+|\text{ sign } W|\leq2\beta_{2m}(W;Z).$$

Proof. The proof is analogous to that of Theorem 2.1. We give the outline only. Regard $M\subset W$. We can assume that $H_1(W;Q)=0$ by a surgery on $W-M$ and M splits W into two manifolds E', E''. Assume $\beta_1(M;Z)>\beta_2$ $(W;Z)$. By the Mayer/Vietoris sequence, we have $\beta_1(E';Z)+\beta_1(E'';Z)>0$. Say $\beta_1(E';Z)>0$. Let $\gamma\in H^1(E';Z)$ be any non-zero element. Then $\gamma|M\in H^1(M;Z)$ is non-zero, since the natural map $H_1(M;Q)\rightarrow H_1(E';Q)$ is onto. The rest of the proof follows from Theorem 1.6, the inequalities $\hat\beta_{2m}(E';Z)+\hat\beta_{2m}$ $(E'';Z)\leq\beta_{2m}(W;Z)$, $|\text{ sign } E''|\leq\hat\beta_{2m}(E'';Z)$ and the Novikov addition theorem. This completes the proof.

Theorem 4.2. *For any positive integers r, r', there are infinitely many smooth M having all of the following properties (0)–(4):*
(0) $H_*(M;Z)\cong H_*(\#_r S^1\times S^{n-1};Z)$ *and* $\kappa_{2m-1}^{\dot\gamma}(M)=0$ *and* $|\tau_{-1}^{\dot\gamma}(M)|\geq r'$ *for all indivisible elements* $\dot\gamma\in H^1(M;Z)$,
(1) *M is smoothly type II imbedded in a smooth $(n+1)$-manifold homotopy equivalent to $\#_r S^2\times S^{n-1}$,*
(2) *M is smoothly type I imbedded in a smooth $(n+1)$-manifold W^* with tH_q $(W^*;Z)\cong Z_2$ (if $q=1,n-1$) or 0 (if $q\neq1,n-1$) and $bH_*(W^*;Z)\cong H_*(S^1\times S^n;Z)$,*
(3) *M° is smoothly imbedded in a smooth $(n+1)$-manifold W^{**} with $tH_q(W^{**};Z)$ $\cong Z_2$(if $q=1,n-1$) or 0 (if $q\neq1,n-1$) and $bH_*(W^{**};Z)\cong H_*(S^{n+1};Z)$,*
(4) *M is not topologically type II imbeddable in any W with $\beta_2(W;Z)<r$ and $\beta_{2m}(W;Z)+|\text{ sign } W|<r'$.*

Proof. We take any invertible smooth $(n-2)$-knot K in S^n with $|\sigma(K)|\geq r'$ (cf. Levine [L]). Construct $F=D^{n+1}\cup D^{n-1}\times D^2$ identifying a tubular neighborhood $T(K)=S^{n-2}\times D^2$ of K in $S^n=\partial D^{n+1}$ with $\partial D^{n-1}\times D^2$. Let $M(K)=\partial F$. Then $H_*(M(K);Z)\cong H_*(S^1\times S^{n-1};Z)$ and the double DF is homotopy equivalent to $S^2\times S^{n-1}$. It is now an easy exercise (cf. 2.7) that the r-fold connected sum, M, of $M(K)$ has (0)–(4) by using Theorem 4.1 for (4). This completes the

proof.

In the following, we can take M^o to be any compact connected oriented n-manifold such that ∂M^o is non-empty connected and $\beta_1(\partial M^o; Z)=0$:

Theorem 4.3. *Assume that M^o is topologically imbedded in W. Then β_1 $(M^o; Z) \leq \dim_{Z_2} H^2(W; Z) \otimes Z_2$ or there is a Z_2-asymmetric indivisible element $\dot{\gamma} \in H^1(DM^o; Z)$ such that for all a*

$$|\tau_a^{\dot{\gamma}}(DM^o)| - \kappa_{2m-1}^{\dot{\gamma}}(DM^o) \leq \beta_{2m}(W; Z) + |\operatorname{sign} W| \leq 2\beta_{2m}(W; Z) .$$

Proof. Regard $M^o \subset W$. We can assume without changing $\beta_{2m}(W; Z)$ and $\dim_{Z_2} H^2(W; Z) \otimes Z_2$ that $\beta_1(W; Z_2)=0$ by a surgery on $W - M^o$ (then $\beta_2(W; Z)$ $= \dim_{Z_2} H^2(W; Z) \otimes Z_2$) and M^o has a collar $N \cong M^o \times [0, 1]$ in W. Let $E = M -$ Int N. Then $\partial E = DM^o$. Assume that $\beta_1(M^o; Z) > \beta_2(W; Z)$. By the Mayer/Vietoris sequence, the natural homomorphism $H_1(DM^o; Q) \to H_1(N; Q) \oplus H_1(E; Q)$ is onto. Since $\beta_1(DM^o; Z) = 2\beta_1(M^o; Z)$ and $\beta_1(N; Z) = \beta_1(M^o; Z)$ and the kernel of this epimorphism is the image of $\partial: H_2(W; Q) \to H_1(DM^o; Q)$, we see that $\beta_1(E; Z) > 0$. Let $\gamma \in H^1(E; Z)$ be any indivisible element. Since the natural homomorphism $H^1(N; Z_2) \oplus H^1(E; Z_2) \to H^1(DM^o; Z_2)$ is injective, we see that $\gamma | DM^o \in H^1(DM^o; Z)$ is Z_2-asymmetric. The desired inequality now follows from Theorem 1.6, the inequality $\hat{\beta}_{2m}(E; Z) \leq \beta_{2m}(W; Z)$ and the Novikov addition theorem, completing the proof.

Theorem 4.4. *For any positive integers r, r', there are infinitely many smooth M having all of the following properties (0)–(4):*

(0) $H_*(M; Z) \cong H_*(\sharp_r S^1 \times S^{n-1}; Z)$ *and $\kappa_{2m-1}^{\dot{\gamma}}(DM^o)=0$ and $|\tau_{-1}^{\dot{\gamma}}(DM^o)| \geq r'$ for all Z_2-asymmetric indivisible elements $\dot{\gamma} \in H^1(DM^o; Z)$,*

(1) *M is smoothly type II imbedded in a smooth $(n+1)$-manifold homotopy equivalent to $\sharp_r S^2 \times S^{n-1}$,*

(2) *M is smoothly type I imbedded in a smooth $(n+1)$-manifold W^* with tH_q $(W^*; Z) \cong \oplus_r Z_2$ (if $q=1, n-1$) or 0 (if $q \neq 1, n-1$) and $bH_*(W^*; Z) \cong H_*(S^1 \times S^n; Z)$,*

(3) *M^o is smoothly imbedded in a smooth $(n+1)$-manifold W^{**} with $tH_q(W^{**}; Z)$ $\cong \oplus_r Z_2$ (if $q=1, n-1$) or 0 (if $q \neq 1, n-1$) and $bH_*(W^{**}; Z) \cong H_*(S^{n+1}; Z)$,*

(4) *M^o is not topologically imbeddable in any W with $\dim_{Z_2} H^2(W; Z) \otimes Z_2 < r$ and $\beta_{2m}(W; Z) + |\operatorname{sign} W| < r'$.*

Proof. Take any invertible smooth $(n-2)$-knots K_i, $i=1, 2, \cdots, r$, in S^n such that $|\sigma(K_j)| \geq r' + \Sigma_{i=1}^{j-1} |\sigma(K_i)|$, $j=1, 2, \cdots, r$. The connected sum $M = M(K_1) \sharp M(K_2) \sharp \cdots \sharp M(K_r)$ is proved to have (0)–(4), by using Theorem 4.3 for (4) (cf. 3.4). This completes the proof.

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