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Author(s)	Čadek, Martin; Vanžura, Jiří
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ALMOST QUATERNIONIC STRUCTURES ON EIGHT-MANIFOLDS

Martin ČADEK and Jiří VANŽURA

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1. Introduction

$Sp(n)$ is the group of the quaternionic linear automorphisms acting from the left on a right quaternionic n-dimensional vector space preserving a positive definite Hermitian form on it. $Sp(n) \cdot Sp(1)$ is the group $Sp(n) \times Sp(1)/\{(1,1),(-1,-1)\}$. If we identify $\mathbb{H}^n$ with $\mathbb{R}^{4n}$, the following left action on a right quaternionic n-dimensional space $\mathbb{H}^n$

$$(A,\alpha)v = Av\bar{\alpha}, \quad A \in Sp(n), \alpha \in Sp(1),$$

where $\bar{\alpha}$ is the quaternionic conjugate to α, induces an inclusion $Sp(n) \cdot Sp(1) \hookrightarrow SO(4n)$.

Let ξ be an oriented real vector bundle of dimension $4n$. We will say that ξ has an $Sp(n) \cdot Sp(1)$-structure iff its structure group $SO(4n)$ can be reduced to $Sp(n) \cdot Sp(1)$. Such a structure was treated i.e. in [2], [13], [16]. In the case of the tangent bundle of a smooth manifold it is common to talk about almost quaternionic structure. (See [1], [13].) The prototype of a manifold with such an almost quaternionic structure is the quaternionic projective space $\mathbb{H}P^n$. Examples of manifolds with almost quaternionic structure are quaternionic-Kähler manifolds whose holonomy group is by definition a subgroup of $Sp(n) \cdot Sp(1)$ ([1], [18], [13]).

This paper is devoted to $Sp(n) \cdot Sp(1)$ for $n = 2$. (The case $n = 1$ is not interesting since the group $Sp(1) \cdot Sp(1)$ is isomorphic to $SO(4)$.) Our aim is to find nontrivial sufficient and in some cases also necessary conditions for the existence of an $Sp(2) \cdot Sp(1)$-structure in oriented 8-dimensional vector bundles over oriented 8-manifolds in terms of characteristic classes and cohomology of the base manifold. Analogous results for the almost complex structure in dimensions 8 and 10 were obtained in [15] and [20]. One of the corollaries of our main results in Section 7 reads as

Theorem 1.1. *Let M be an oriented closed connected smooth manifold of*

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dimension 8. *If*
(A) $w_2(M)=0$
(B) $w_6(M)=0$
(C) $4p_2(M)-p_1^2(M)-8e(M)=0$
(D) $\{p_1^2(M)+4e(M)\}[M]\equiv 0 \bmod 16$,
then M *has an almost quaternionic structure.*

The starting point for our considerations is the following proposition proved in [9] (see Theorem 3.2).

Proposition 1.2. *Let* X *be a CW-complex and let* ξ *be an oriented* 8-*dimensional vector bundle over* X. *Then* ξ *has an* $Sp(2)\cdot Sp(1)$-*structure if and only if it has a spinor structure* $\bar{\xi}$ *and the vector bundle* $\pi_*(\kappa\lambda)_*(\bar{\xi})$, *where* $\kappa\lambda$ *is a certain outer automorphism of* $Spin(8)$ *and* $\pi : Spin(8)\to SO(8)$ *is a standard double covering, has an oriented* 3-*dimensional subbundle.*

What is known in this respect are the results of Crabb and Steer [10] which answer the question whether a given 3-dimensional vector bundle η can be a subbundle of a given $4k$-dimensional vector bundle ζ over a $4k$-manifold. The necessary and sufficient conditions are given in terms of characteristic classes of η and ζ. However, what we need in order to apply Proposition 1.2, is the answer to the question whether a given 8-dimensional vector bundle has a 3-dimensional subbundle. To reach this purpose we carry out the following steps:

(i) In Section 3 we describe those cohomology classes which can appear as characteristic classes of a 3-dimensional spin vector bundle over a given CW-complex of dimension 8. Here a certain tertiary cohomology operation Φ and a secondary operation Σ appear.
(ii) Next we compute the operations Φ and Σ. For this aim we derive necessary and sufficient conditions for the existence of 3 linearly independent sections in an 8-dimensional spin vector bundle over a CW-complex of the same dimension in terms of characteristic classes and the higher order cohomology operations Σ and Φ (Section 4). Comparing this result with the known results in [10] and [11] derived by different methods, we get a formula for Φ and Σ on spin manifolds (Section 5).
(iii) Now, using [10] we can answer the question whether an 8-dimensional vector bundle has a 3-dimensional spin subbundle (Section 6) and apply Proposition 1.2 to obtain nontrivial sufficient conditions for the existence of an $Sp(2)\cdot Sp(1)$-structure (Section 7).

The reason why our conditions for manifolds satisfying $H^2(M;\mathbb{Z}_2)\neq 0$ are only sufficient ones consists in the fact that we are not able describe characteristic classes of all 3-dimensional vector bundles over M, but only the spin ones.

We do not know either how to avoid the usage of higher order cohomology operations and obtain our results only by the methods of the index theory used in [10] and [11].

2. Notation and preliminaries

In this section we introduce notation and recall also some facts about the singular cohomology of classifying spaces.

We suppose that all manifolds and vector bundles are oriented. We will use $w_m(\xi)$ for the m-th Stiefel–Whitney class of the vector bundle ξ, $p_m(\xi)$ for the m-th Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle ξ the symbol $c_m(\xi)$ denotes the m-th Chern class. The letters w_m, p_m, e and c_m will stand for the characteristic classes of the universal vector bundles over the classifying spaces $BSO(n)$, and $BU(n)$, respectively. The pullbacks of the Stiefel–Whitney, Pontrjagin and Euler classes in $H^*(BSpin(n))$ will be denoted by the same letters.

The mapping $\delta : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Mappings $i_* : H^*(X,\mathbb{Z}_2) \to H^*(X,Z_4)$ and $\rho_m : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_m)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and the reduction mod m, respectively. We will also use the Steenrod operations $Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ and $P_3^i : H^n(X;\mathbb{Z}_3) \to H^{n+4i}(X;\mathbb{Z}_3)$.

We say that $x \in H^*(X;\mathbb{Z})$ is an element of order n ($n = 2,3,4,\ldots$) if and only if $x \neq 0$ and n is the least positive integer such that $nx = 0$ (if it exists).

The Eilenberg–MacLane space with the n-th homotopy group G will be denoted $K(G,n)$, and ι_n will stand for the fundamental class in $H^n(K(G,n);G)$. Writing the fundamental class, it will be always clear which group G we have in mind.

Now we summarize some results on the cohomologies of $BSpin(n)$. We consider always the group $Spin(n)$ in the standard way as a subgroup of the Clifford algebra C_{n-1}. Using the standard forms of the Clifford algebras, we have $C_2 = \mathbb{H}$, $C_4 = \mathbb{H}(2)$, and the inclusion $\nu : C_2 \hookrightarrow C_4$ of the form

$$\nu(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{for } \alpha \in \mathbb{H}.$$

For the later use we shall introduce one more monomorphism of groups $\mu : Sp(1) \hookrightarrow Sp(2)$ by

$$\mu(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } \alpha \in \mathbb{H}, \quad |\alpha| = 1.$$

Using the above form of the Clifford algebras we can immediately see that $Spin(3) \cong Sp(1) \subset \mathbb{H}$, $Spin(5) \cong Sp(2) \subset \mathbb{H}(2)$, and ν, μ define monomorphisms $\nu, \mu : Spin(3) \hookrightarrow Spin(5)$. Let us notice that the factor $Spin(5)/\nu(Spin(3))$ is the Stiefel manifold $V_{5,2}$ while $Spin(5)/\mu(Spin(3))$ is the sphere S^7. Both these monomorphisms induce fibrations of classifying spaces

$$V_{5,2} \longrightarrow BSpin(3) \xrightarrow{\nu} BSpin(5), \quad S^7 \longrightarrow BSpin(3) \xrightarrow{\mu} BSpin(5).$$

Let us recall now the cohomology rings of $BSpin(3)$ and $BSpin(5)$.

Lemma 2.1. *The cohomology ring of $BSpin(3)$ is*

$$H^*(BSpin(3);\mathbb{Z}) \cong \mathbb{Z}[r]$$

where

$$p_1 = 4r.$$

The cohomology ring of $BSpin(5)$ is

$$H^*(BSpin(5);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2]$$

where q_1 and q_2 are defined by the relations

$$p_1 = 2q_1, \quad p_2 = q_1^2 + 4q_2.$$

Moreover

$$\rho_2 q_1 = w_4.$$

REMARK 2.2. Let us mention here that $r = e_1$, where $e_1 \in H^4(BSpin(3);\mathbb{Z})$ is the first symplectic Pontrjagin class of the universal $\mathbb{H}$-vector bundle over the classifying space $BSpin(3) = BSp(1)$. Similarly, $q_1 = e_1$ and $q_2 = -e_2$, where $e_1 \in H^4(BSpin(5);\mathbb{Z})$ and $e_2 \in H^8(BSpin(5);\mathbb{Z})$ is the first and the second symplectic Pontrjagin class of the universal $\mathbb{H}$-vector bundle over the classifying space $BSpin(5) = BSp(2)$, respectively.

Using the classical result by Borel and Hirzebruch (see [3], Theorem 10.3), we get easily the following lemma.

Lemma 2.3. *For the cohomology homomorphisms $\nu^*, \mu^* : H^*(BSpin(5);\mathbb{Z}) \to H^*(BSpin(3);\mathbb{Z})$ there is*

$$\nu^* q_1 = 2r, \quad \nu^* q_2 = -r^2, \quad \mu^* q_1 = r, \quad \mu^* q_2 = 0.$$

Let $\upsilon : BSpin(5) \to BSpin(8)$ be the fibration induced by the canonical inclusion $Spin(5) \hookrightarrow Spin(8)$.

Lemma 2.4. *The cohomology rings of $BSpin(8)$ are*

$$H^*(BSpin(8);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon]$$

and

$$H^*(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e, \delta w_6]/\langle 2\delta w_6\rangle$$

where q_1, q_2 and ε are defined by the relations

$$p_1 = 2q_1, \quad p_2 = q_1^2 + 2e + 4q_2, \quad \rho_2 q_2 = \varepsilon.$$

Moreover,

$$\rho_2 q_1 = w_4, \quad \rho_2 e = w_8$$

and

$$v^*(q_1) = q_1, \quad v^*(q_2) = q_2, \quad v^*(e) = 0.$$

Proof. See [17] and [8]. □

Let ξ be an oriented 8-dimensional vector bundle over a CW-complex X given by the homotopy class of some mapping $\xi : X \to BSO(8)$. ξ has a spinor structure iff $w_2(\xi) = 0$. If some lifting $\bar{\xi} : X \to BSpin(8)$ is fixed we can define spin characteristic classes

$$q_1(\xi) = \bar{\xi}^* q_1, \quad q_2(\xi) = \bar{\xi}^* q_2.$$

The first spin characteristic class is always independent of the choice of $\bar{\xi}$. Moreover, if $H^4(X;\mathbb{Z})$ has no element of order 4, then it is uniquely determined by the relations

$$2q_1(\xi) = p_1(\xi), \quad \rho_2 q_1(\xi) = w_4(\xi).$$

The second spin characteristic class is independent of the spinor structure $\bar{\xi}$ if X is simply connected or $H^8(X;\mathbb{Z}) \cong \mathbb{Z}$. In the case of an 8-dimensional manifold $q_2(\xi)$ is uniquely determined by the relation

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi).$$

See [8].

3. Higher order cohomology operations

We shall introduce four special higher order cohomology operations Σ, Ψ, Φ and Ω which will appear when building the Postnikov towers for the fibrations $r : BSpin(3) \to K(\mathbb{Z},4)$ and $q_1 : BSpin(5) \to K(\mathbb{Z},4)$ corresponding to the elements $r \in H^4(BSpin(3);\mathbb{Z})$ and $q_1 \in H^4(BSpin(5);\mathbb{Z})$. (See [19] and [21].)

Consider the fibration $K(\mathbb{Z}_2,5) \xrightarrow{j_1} Y_1 \xrightarrow{\pi_1} K(\mathbb{Z},4)$ induced from the path fibration $PK(\mathbb{Z}_2,6) \to K(\mathbb{Z}_2,6)$ by the mapping $Sq^2\rho_2\iota_4 : K(\mathbb{Z},4) \to K(\mathbb{Z}_2,6)$. The Serre exact sequence for this fibration implies that $H^7(Y_1;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Its generator σ satisfies

$$j_1^*(\sigma) = Sq^2\iota_5.$$

DEFINITION 3.1. Let Σ denote the secondary cohomology operation associated with the relation

$$Sq^2 \circ Sq^2\rho_2 = 0$$

in dimension 4.

Let X be a CW-complex. The operation Σ is defined on the set $\mathrm{Def}(\Sigma, X) = \{x \in H^4(X;\mathbb{Z}); Sq^2\rho_2 x = 0\}$. Its value $\Sigma(x)$ is the subset of $H^7(X;\mathbb{Z}_2)$ with the indeterminacy $\mathrm{Indet}(\Sigma, X) = Sq^2H^5(X;\mathbb{Z}_2)$. Moreover, it can be shown that

$$\Sigma(x+y) = \Sigma(x) + \Sigma(y)$$

for all $x, y \in \mathrm{Def}(\Sigma, X)$.

From the Serre exact sequence for the fibration π_1 we get easily that the group $H^8(Y_1;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has the three generators $\pi_1^*\rho_2\iota_4^2$, $Sq^1\sigma$ and ψ, the last one with the property

$$j_1^*(\psi) = Sq^2Sq^1\iota_5.$$

Unfortunately, the last requirement does not determine ψ uniquely. To fix it, we build the Postnikov tower for the fibration $r : BSpin(3) \to K(\mathbb{Z},4)$. Using the long homotopy sequence we find easily that its fibre F is 4-connected and $\pi_5(F) \cong \mathbb{Z}_2$, $\pi_6(F) \cong \mathbb{Z}_2$ and $\pi_7(F) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4$. Hence, the first Postnikov invariant is $Sq^2\rho_2\iota_4 \in H^6(K(\mathbb{Z},4);\mathbb{Z}_2)$ and the first stage of the Postnikov tower is just Y_1. Thus, we get the following commutative diagram.

$$\begin{tikzcd}
\bar{F}_1 \arrow[r] & F \arrow[r] \arrow[d] & K(\mathbb{Z}_2,5) \arrow[d, "j_1"] & \\
F_1 \arrow[r] & BSpin(3) \arrow[r, "r_1"] \arrow[d, "r"] & Y_1 \arrow[d, "\pi_1"] & \\
& K(\mathbb{Z},4) \arrow[r, equal] & K(\mathbb{Z},4) \arrow[r, "Sq^2\rho_2\iota_4"] & K(\mathbb{Z}_2,6)
\end{tikzcd}$$

Since $H^7(BSpin(3);\mathbb{Z}_2) \cong 0$, the next invariant is $\sigma \in H^7(Y_1;\mathbb{Z}_2)$. In dimension 8 we have $H^8(BSpin(3);\mathbb{Z}_2) \cong \mathbb{Z}_2$ with generator $\rho_2 r^2$ and $r_1^*\pi_1^*\rho_2\iota_4^2 = \rho_2 r^2$ and $r_1^* Sq^1\sigma = 0$. This shows that there is a unique element $\psi \in H^8(Y_1;\mathbb{Z}_2)$ such that

$$j_1^*\psi = Sq^2Sq^1\iota_5 \quad \text{and} \quad r_1^*(\psi) = 0$$

These considerations justify the following definition.

DEFINITION 3.2. Denote Ψ the secondary cohomology operation associated with the relation

$$Sq^2Sq^1 \circ Sq^2\rho_2 = 0$$

in dimension 4 uniquely determined by the property

$$\Psi(r) = 0$$

in $H^*(BSpin(3))$.

The operation Ψ is defined on $\mathrm{Def}(\Psi, X) = \mathrm{Def}(\Sigma, X)$. The value $\Psi(x)$ is a subset of $H^8(X;\mathbb{Z}_2)$ with the indeterminacy $\mathrm{Indet}(\Psi, X) = Sq^2Sq^1H^5(X;\mathbb{Z}_2)$.

Lemma 3.3. *Let X be a CW-complex. Then*

$$\Psi(x+y) = \Psi(x) + \Psi(y) + \rho_2(xy)$$

for all $x, y \in \mathrm{Def}(\Psi, X)$, and

$$\Psi(2x) = \rho_2 x^2 + \mathrm{Indet}(\Psi, X)$$

for all $x \in H^4(X;\mathbb{Z})$.

Proof. Since $\pi_1^*(Sq^2\rho_2\iota_4)\otimes 1 + 1 \otimes \pi_1^*(Sq^2\rho_2\iota_4) = 0$ in $H^6(Y_1 \times Y_1;\mathbb{Z}_2)$, there is a mapping $f_1 : Y_1 \times Y_1 \to Y_1$ such that the following diagram is commutative

$$\begin{tikzcd}
& Y_1 \times Y_1 \arrow[rr, dashed, "f_1"] \arrow[d, "\pi_1 \times \pi_1"] & & Y_1 \arrow[d, "\pi_1"] & \\
X \arrow[ur, "x_1 \times y_1"] \arrow[r, "x \times y"'] & K(\mathbb{Z},4) \times K(\mathbb{Z},4) \arrow[rr, "\iota_4 \otimes 1 + 1 \otimes \iota_4"] & & K(\mathbb{Z},4) \arrow[r, "Sq^2\rho_2\iota_4"] & K(\mathbb{Z}_2,6)
\end{tikzcd}$$

where the mappings x and y represent $x, y \in H^4(X;\mathbb{Z})$ and $x_1 : X \to Y_1, y : X \to Y_1$ their liftings in the fibration π_1. Hence we get

$$\begin{aligned} f_1^*\psi = a\psi \otimes 1 + b1 \otimes \psi + c\pi_1^*\rho_2\iota_4 \otimes \pi_1^*\rho_2\iota_4 + a'\pi_1^*\rho_2\iota_4^2 \otimes 1 \\ + b'1 \otimes \pi_1^*\rho_2\iota_4^2 + a''Sq^1\sigma \otimes 1 + b''1 \otimes Sq^1\sigma \end{aligned}$$

for some $a, b, b', b'', c, c', c'' \in \{0, 1\}$ and consequently

$$\begin{aligned} \Psi(x+y) = a\Psi(x) + b\Psi(y) + c\rho_2(xy) + a'\rho_2(x^2) + \\ + b'\rho_2(y^2) + a''Sq^1\Sigma(x) + b''Sq^1\Sigma(y). \end{aligned}$$

Taking $X = Y_1$, $x = \pi_1^*\iota_4$ and $y = 0$, having in mind that $\mathrm{Indet}(\Sigma, Y_1) = \mathrm{Indet}(\Psi, Y_1) = 0$, we get

$$\psi = \Psi(\pi_1^*\iota_4) = a\psi + a'\rho_2\pi_1\rho_2\iota_4^2 + a''Sq^1\sigma,$$

which implies $a = 1$ and $a' = a'' = 0$. Similarly we get $b = 1$ and $b' = b'' = 0$. Following Brown and Peterson (see [6], Lemma 2.2), we can show that ψ is not primitive. This implies that $c = 1$, which finishes the proof. □

Lemma 3.4. *For $q_1 \in H^4(BSpin(5);\mathbb{Z})$ we have*

$$\Psi(q_1) = \rho_2 q_2.$$

Proof. Since $\mathrm{Indet}(\Psi, BSpin(5)) = 0$, we have

$$\Psi(q_1) = a\rho_2 q_2 + b\rho_2 q_1^2$$

where $a, b \in \{0, 1\}$. If we apply μ^* on both sides, we get according to Lemma 2.3 and Definition 3.2

$$0 = \Psi(r) = \Psi(\mu^* q_1) = \mu^*\Psi(q_1) = \mu^*(a\rho_2 q_2 + b\rho_2 q_1^2) = b\rho_2 r^2$$

and hence $b = 0$. Next apply ν^*. Using Lemma 3.3 we have

$$\rho_2 r^2 = \Psi(2r) = \Psi(\nu^* q_1) = \nu^*\Psi(q_1) = \nu^*(a\rho_2 q_2) = a\rho_2 r^2,$$

which implies $a = 1$. $\square$

The Serre spectral sequence for the fibration π_1 gives that $H^9(Y_1;\mathbb{Z}_2) \cong \mathbb{Z}_2$ with the generator $Sq^1\psi = Sq^2\sigma$ since $j_1^* Sq^2\sigma = Sq^3Sq^1\iota_5 = j_1^* Sq^1\psi \neq 0$. Finally, considering the cohomology exact sequence corresponding to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \longrightarrow 0$, we find that $H^8(Y_1;\mathbb{Z}_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the generators $\pi_1^*\rho_4\iota_4^2$, $\rho_4\delta\sigma$ and $i_*\psi$.

Consider the fibration $K(\mathbb{Z}_2,6) \xrightarrow{j_2} Y_2 \xrightarrow{\pi_2} Y_1$ induced from the path fibration $PK(\mathbb{Z}_2,7) \to K(\mathbb{Z}_2,7)$ by the mapping $\sigma : Y_1 \to K(\mathbb{Z}_2,7)$. The transgression of the element $i_*Sq^2\iota_6 \in H^8(K(\mathbb{Z}_2,6);\mathbb{Z}_4)$ is

$$\tau(i_*Sq^2\iota_6) = i_*Sq^2\sigma = i_*Sq^1\psi = 0.$$

Hence there is an element $\varphi \in H^8(Y_2;\mathbb{Z}_4)$ such that

$$j_2^*\varphi = i_*Sq^2\iota_6.$$

But this property does not determine the element φ uniquely. In order to compute $H^8(Y_2;\mathbb{Z}_4)$ we apply the Serre exact sequence for the fibration $K(\mathbb{Z}_2,6) \longrightarrow Y_2 \longrightarrow Y_1$ with the coefficients $\mathbb{Z}_4$.

$$\begin{aligned} H^7(K(\mathbb{Z}_2,6);\mathbb{Z}_4) \xrightarrow{\tau} H^8(Y_1;\mathbb{Z}_4) &\longrightarrow H^8(Y_2;\mathbb{Z}_4) \\ &\longrightarrow H^8(K(\mathbb{Z}_2,6);\mathbb{Z}_4) \xrightarrow{\tau} H^9(Y_1;\mathbb{Z}_4) \end{aligned}$$

Let us mention first that $H^7(K(\mathbb{Z}_2,6);\mathbb{Z}_4) \cong \mathbb{Z}_2$ with the generator $\rho_4\delta\iota_6$, and $H^8(K(\mathbb{Z}_2,6);\mathbb{Z}_4) \cong \mathbb{Z}_2$ with the generator $i_*Sq^2\iota_6$. Further, $\tau(\rho_4\delta\iota_6) = \rho_4\delta\sigma$, $\tau(i_*Sq^2\iota_6) = 0$. This shows that $H^8(Y_2;\mathbb{Z}_4)$ fits into the exact sequence

$$0 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_2 \longrightarrow H^8(Y_2;\mathbb{Z}_4) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

This gives us for the group $H^8(Y_2;\mathbb{Z}_4)$ the possibilities $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. But anyhow the group $H^8(Y_2;\mathbb{Z}_4)$ has 16 elements. Returning now back to the Postnikov tower for $r : BSpin(3) \to K(\mathbb{Z},4)$, we see that Y_2 is its second stage.

$$\begin{tikzcd} \bar{F}_2 \arrow[r] & F_1 \arrow[r] \arrow[d] & K(\mathbb{Z}_2,6) \arrow[d, "j_2"] & \\ F_2 \arrow[r] & BSpin(3) \arrow[r, "r_2"] \arrow[d, "r_1"] & Y_2 \arrow[d, "\pi_2"] & \\ & Y_1 \arrow[r, equal] & Y_1 \arrow[r, "\sigma"] & K(\mathbb{Z}_2,7) \end{tikzcd}$$

From the Serre exact sequence for the fibration $F_2 \to BSpin(3) \to Y_2$ we get immediately that $H^8(Y_2;\mathbb{Z}_4)$ fits also into the exact sequence

$$0 \longrightarrow \mathbb{Z}_4 \longrightarrow H^8(Y_2;\mathbb{Z}_4) \longrightarrow \mathbb{Z}_4 \longrightarrow 0.$$

This shows that the only possibility is $H^8(Y_2;\mathbb{Z}_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Reconsidering with this information the Serre exact sequence for the fibration $K(\mathbb{Z}_2,6) \to Y_2 \to Y_1$, we can see that $H^8(Y_2;\mathbb{Z}_4)$ has generators $\pi_2^*\pi_1^*\rho_4\iota_4^2$ and φ, where φ can be chosen in such a way that

$$j_2^*\varphi = i_*Sq^2\iota_6, \quad r_2^*\varphi = 0.$$

Moreover, for such φ it holds

$$2\varphi = i_*\pi_2^*\psi, \quad \rho_2\varphi = \pi_2^*\psi.$$

Unfortunately, the above conditions still do not determine φ uniquely. (But there is only one more element with the same properties, namely $-\varphi$.)

Let $s : BSpin(3) \to K(\mathbb{Z},4)$ be a mapping representing the element $2r \in H^4(BSpin(3);\mathbb{Z})$. Since $Sq^2\rho_2 2r = 0$ and $\Sigma(2r) = 0$, this mapping can be lifted to $s_1 : BSpin(3) \to Y_1$ and $s_2 : BSpin(3) \to Y_2$. Both these mappings are uniquely determined up to homotopy. According to Lemma 3.3 we have

$$\rho_2 s_2^*(\varphi) = s_2^*(\psi) = \Psi(2r) = \rho_2 r^2.$$

Hence $s_2^*(\varphi) = \pm\rho_4 r^2$. This shows that there is a unique element $\varphi \in H^8(Y_2;\mathbb{Z}_4)$ such that

$$j_2^*\varphi = i_*Sq^2\iota_6, \quad r_2^*\varphi = 0, \quad s_2^*\varphi = -\rho_4 r^2.$$

DEFINITION 3.5. Let Φ be the tertiary cohomology operation associated with the relation

$$i_*Sq^2 \circ \Sigma = 0$$

in dimension 4, and uniquely determined by the properties

$$\Phi(r) = 0$$
$$\Phi(2r) = -\rho_4 r^2$$

for $r \in H^4(BSpin(3);\mathbb{Z})$.

The tertiary cohomology operation Φ is defined on $\mathrm{Def}(\Phi, X) = \{x \in H^4(X;\mathbb{Z});\ Sq^2\rho_2 x = 0, \Sigma(x) \ni 0\}$. Now, we will deal with the indeterminacy of the operation Φ on a CW-complex X.

Consider the fibration $K(\mathbb{Z}_2,6) \xrightarrow{j} E \xrightarrow{\pi} K(\mathbb{Z}_2,5)$ induced from the path fibration over $K(\mathbb{Z}_2,7)$ by the mapping $Sq^2\iota_5$. Notice that this fibration is a restriction of the fibration $\pi_2 : Y_2 \to Y_1$ induced by the inclusion $j_1 : K(\mathbb{Z}_2,5) \hookrightarrow Y_1$ so that the diagram

$$\begin{tikzcd} K(\mathbb{Z}_2,6) \arrow[r, "j"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "l"] & K(\mathbb{Z}_2,5) \arrow[d, "j_1"] \\ K(\mathbb{Z}_2,6) \arrow[r, "j_2"] & Y_2 \arrow[r, "\pi_2"] & Y_1 \end{tikzcd}$$

commutes. From the Serre exact sequence and the commutativity of the diagram we get that $H^8(E;\mathbb{Z}_4) \cong \mathbb{Z}_4$ with generator $\omega = l^*\varphi$. Moreover, $j^*\omega = i_*Sq^2\iota_6$ and $2\omega = i_*Sq^2Sq^1\pi^*\iota_5$.

DEFINITION 3.6. Let Ω be the secondary cohomology operation associated with the relation

$$i_*Sq^2 \circ Sq^2 = 0$$

in dimension 5.

Let X be a CW-complex. The operation Ω is defined on the set $\mathrm{Def}(\Omega, X) = \{x \in H^5(X;\mathbb{Z}_2); Sq^2x = 0\}$ with the indeterminacy $\mathrm{Indet}(\Omega, X) = i_*Sq^2H^6(X;\mathbb{Z}_2)$. It is not substantial, whether Ω is defined by ω or $-\omega$ since $2\omega = i_*Sq^2Sq^1\pi^*\iota_5 \in i_*Sq^2H^6(E;\mathbb{Z}_2)$.

Lemma 3.7. *The indeterminacy of the operation* Φ *is* $\mathrm{Indet}(\Phi, X) = \Omega\mathrm{Def}(\Omega, X)$.

Proof. Let $\lambda_1 : K(\mathbb{Z}_2,5) \times Y_1 \to Y_1$ and $\lambda_2 : K(\mathbb{Z}_2,6) \times Y_2 \to Y_2$ be the usual multiplications given by the composition of paths. It can be shown that there is a new multiplication $\lambda : E \times Y_2 \to Y_2$ such that the diagram

$$\begin{tikzcd} K(\mathbb{Z}_2,6) \times Y_2 \arrow[r, "\lambda_2"] \arrow[d, "j \times \mathrm{id}"'] & Y_2 \arrow[d, equal] \\ E \times Y_2 \arrow[r, "\lambda"] \arrow[d, "\pi \times \pi_2"'] & Y_2 \arrow[d, "\pi_2"] \\ K(\mathbb{Z}_2,5) \times Y_1 \arrow[r, "\lambda_1"] & Y_1 \end{tikzcd}$$

commutes. This implies that

$$\lambda^*(\varphi) = 1 \otimes \varphi + \omega \otimes 1.$$

Let x_2 and $\tilde{x}_2 : X \to Y_2$ be two liftings of a mapping $x : X \to K(\mathbb{Z}, 4)$. Put $x_1 = \pi_2 \circ x_2$, $\tilde{x}_1 = \pi_2 \circ \tilde{x}_2$. Since $\pi_1 \circ x_1 = x = \pi_1 \circ \tilde{x}_1$ there is $y_1 : X \to K(\mathbb{Z}_2, 5)$ such that

$$\tilde{x}_1 = \lambda_1 \circ (y_1, x_1).$$

Hence $\tilde{x}_1^*(\sigma) = x_1^*(\sigma) + y_1^* Sq^2 \iota_5$. Moreover, $\tilde{x}_1^*(\sigma) = x_1^*(\sigma) = 0$ because both maps have liftings. Consequently, $y_1^*(Sq^2\iota_5) = 0$ and y_1 can be lifted to $y : X \to E$. Now,

$$\pi_2 \circ \lambda \circ (y, x_2) = \lambda_1 \circ (y_1, x_1) = \tilde{x}_1 = \pi_2 \circ \tilde{x}_2.$$

Hence there is $y_2 : X \to K(\mathbb{Z}_2, 6)$ such that

$$\lambda \circ (y, x_2) = \lambda_2 \circ (y_2, \tilde{x}_2).$$

Applying the maps on both sides on $\varphi \in H^8(Y_2; \mathbb{Z}_4)$ we get

$$x_2^*(\varphi) + y^*(\omega) = \tilde{x}_2^*(\varphi) + i_* Sq^2 y_2^*(\iota_6).$$

That is why $\tilde{x}_2^*(\varphi) - x_2^*(\varphi) \in \Omega(y_1^*(\iota_5))$.

On the contrary, having $x : X \to K(\mathbb{Z}, 4)$, its lifting $x_2 : X \to Y_2$, $y_1 : X \to K(\mathbb{Z}_2, 5)$ and an element $z \in \Omega(y_1^*(\iota_5))$ we can easily find $\tilde{x}_2$ such that $\tilde{x}_2^*(\varphi) - x_2^*(\varphi) = z$. □

Lemma 3.8. *For $q_1 \in H^4(BSpin(5); \mathbb{Z})$ there is*

$$\Phi(q_1) = \rho_4 q_2.$$

Proof. Since $\mathrm{Indet}(\Phi, BSpin(5)) = 0$,

$$\Phi(q_1) = a\rho_4 q_2 + b\rho_4 q_1^2$$

where $a, b \in \{0, 1, 2, 3\}$. First, apply μ^* on both sides. According to Lemma 2.3 and Definition 3.5

$$0 = \Phi(r) = \Phi(\mu^* q_1) = \mu^* \Phi(q_1) = \mu^*(a\rho_4 q_2 + b\rho_4 q_1^2) = b\rho_4 r^2$$

and consequently $b = 0$. Next apply ν^*.

$$-\rho_4 r^2 = \Phi(2r) = \Phi(\nu^* q_1) = \nu^* \Phi(q_1) = \nu^*(a\rho_4 q_2) = -a\rho_4 r^2.$$

Hence $a = 1$. $\square$

Lemma 3.9. *Let X be a CW-complex. Then*

$$\Phi(x + y) = \Phi(x) + \Phi(y) - \rho_4(xy)$$

for all $x, y \in \mathrm{Def}(\Phi, X)$.

Proof. Consider $f_1 : Y_1 \times Y_1 \to Y_1$ from the proof of Lemma 3.3. Since $(\pi_2 \times \pi_2)^* f_1^*(\sigma) = 0$ in $H^7(Y_2 \times Y_2; \mathbb{Z}_2)$, there is a mapping $f_2 : Y_2 \times Y_2 \to Y_2$ such that we get the commutative diagram

$$\begin{tikzcd}
 & Y_2 \times Y_2 \arrow[r, dashed, "f_2"] \arrow[d, "\pi_2 \times \pi_2"] & Y_2 \arrow[d, "\pi_2"] & \\
X \arrow[ur, "x_2 \times y_2"] \arrow[r, "x_1 \times y_1"] \arrow[dr, "x \times y"'] & Y_1 \times Y_1 \arrow[r, "f_1"] \arrow[d, "\pi_1 \times \pi_1"] & Y_1 \arrow[r, "\sigma"] \arrow[d, "\pi_1"] & K(\mathbb{Z}_2, 7) \\
 & K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \arrow[r, "\iota_4 \otimes 1 + 1 \otimes \iota_4"] & K(\mathbb{Z}, 4) \arrow[r, "Sq^2 \rho_2 \iota_4"] & K(\mathbb{Z}_2, 6)
\end{tikzcd}$$

where the mappings x and y represent $x, y \in H^4(X; \mathbb{Z})$, $x_1 : X \to Y_1$, $y : X \to Y_1$ their liftings in the fibration π_1 such that $x_1^*(\sigma) = 0$, $x_2^*(\sigma) = 0$ and $x_2 : X \to Y_2$, $y_2 : X \to Y_2$ the liftings of x_1 and y_1 in the fibration π_2, respectively. Hence we get

$$\begin{aligned} f_2^*(\varphi) = {} & a\varphi \otimes 1 + b 1 \otimes \varphi + c\pi_2^*\pi_1^*(\rho_4\iota_4) \otimes \pi_2^*\pi_1^*(\rho_4\iota_4) \\ & + a'\pi_2^*\pi_1^*\rho_4\iota_4^2 \otimes 1 + b' 1 \otimes \pi_2^*\pi_1^*\rho_4\iota_4^2 \end{aligned}$$

for some $a, a', b, b', c, \in \{0, 1, 2, 3\}$. Consequently

$$\Phi(x + y) = a\Phi(x) + b\Phi(y) + c\rho_4(xy) + a'\rho_4 x^2 + b'\rho_4 y^2.$$

Taking $X = Y_2$, $x = \pi_2^*\pi_1^*\iota_4$, $y = 0$ and having in mind that $\mathrm{Indet}(\Phi, Y_2) = 0$, we have

$$\varphi = \Phi(\pi_2^*\pi_1^*\iota_4) = a\Phi(\pi_2^*\pi_1^*\iota_4) + a'\pi_2^*\pi_1^*\rho_4\iota_4^2,$$

which implies $a = 1$ and $a' = 0$. Similarly, we get $b = 1$ and $b' = 0$. Finally, we take $X = BSpin(3)$, $x = y = r$. Since $\mathrm{Indet}(\Phi, BSpin(3)) = 0$, we get

$$-\rho_4 r^2 = \Phi(2r) = \Phi(r) + \Phi(r) + c\rho_4 r^2 = c\rho_4 r^2,$$

which gives $c = -1$. $\square$

From Lemma 2.1 we can see that the first Pontrjagin class of a 3-dimensional spin vector bundle is divisible by 4. A kind of converse to this assertion is the following theorem which will play an important role in deriving of sufficient conditions for the existence of an $Sp(2) \cdot Sp(1)$-structure.

Theorem 3.10. *Let X be an 8-dimensional CW-complex, and let $a \in H^4(X;\mathbb{Z})$. Then there exists an oriented 3-dimensional vector bundle η over X with $w_2(\eta) = 0$ and $p_1(\eta) = 4a$ if and only if the following conditions are satisfied*
(i) $Sq^2\rho_2 a = 0$,
(ii) $0 \in \Sigma(a)$,
(iii) $P_3^1\rho_3 a + \rho_3 a^2 = 0$,
(iv) $0 \in \Phi(a)$.

Proof. We shall use the fibration $F \longrightarrow BSpin(3) \xrightarrow{r} K(\mathbb{Z},4)$, which has already appeared before. The element $a \in H^4(X;\mathbb{Z})$ can be considered as a mapping $a : X \longrightarrow K(\mathbb{Z},4)$, and it is obvious that there exists a 3-dimensional spin vector bundle η with the desired properties if and only if the mapping a can be lifted in the fibration r.

$$\begin{tikzcd} & BSpin(3) \arrow[d, "r"] \\ X \arrow[ur, dashed, "\eta"] \arrow[r, "a"] & K(\mathbb{Z},4) \end{tikzcd}$$

We shall investigate the existence of the lifting η by constructing the Postnikov tower for the fibration r. We already know that the first invariant of this tower is $Sq^2\rho_2\iota_4 \in H^6(K(\mathbb{Z},4);\mathbb{Z}_2)$. It determines the first stage Y_1 with the second invariant $\sigma \in H^7(Y_1;\mathbb{Z}_2)$. So the next stage is Y_2 (see the diagram before Definition 3.5). From the knowledge of $H^8(Y_2;\mathbb{Z}_4)$ we get that the $\mathbb{Z}_4$-invariant is φ.

It suffices to determine the $\mathbb{Z}_3$-invarint in $H^*(Y_2;\mathbb{Z}_3)$. For this purpose we shall investigate the Serre exact sequence for the fibration $F_2 \to BSpin(3) \to Y_2$ with the coefficients $\mathbb{Z}_3$.

$$0 = H^7(BSpin(3);\mathbb{Z}_3) \longrightarrow H^7(F_2;\mathbb{Z}_3) \xrightarrow{\tau} H^8(Y_2;\mathbb{Z}_3) \longrightarrow H^8(BSpin(3);\mathbb{Z}_3),$$

where $H^7(F_2;\mathbb{Z}_3) \cong \mathbb{Z}_3$, the generator being the fundamental class. We can use the Serre exact sequence for the fibration $K(\mathbb{Z}_2,5) \longrightarrow Y_1 \longrightarrow K(\mathbb{Z},4)$ with coefficients $\mathbb{Z}_3$. Let us remark that $H^8(K(\mathbb{Z},4);\mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ with the generators $\rho_3\iota_4^2$ and $P_3^1\rho_3\iota_4$. From this sequence, having in mind that $H^7(K(\mathbb{Z}_2,6);\mathbb{Z}_3) \cong H^8(K(\mathbb{Z}_2,6);\mathbb{Z}_3) \cong 0$, we get $H^8(Y_1;\mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ with the generators $\pi_1^*\rho_3\iota_4^2$ and $\pi_1^*P_3^1\rho_3\iota_4$. Next, from the Serre sequence for the fibration $K(\mathbb{Z}_2,6) \to Y_2 \to Y_1$

with the coefficients $\mathbb{Z}_3$, we get $H^8(Y_2;\mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ with the generators $\pi_2^*\pi_1^*\rho_3\iota_4^2$ and $\pi_2^*\pi_1^*P_3^1\rho_3\iota_4$. Finally, let us mention that $H^8(BSpin(3);\mathbb{Z}_3) \cong \mathbb{Z}_3$, the generator being $\rho_3 r^2$. We have

$$r_2^*\pi_2^*\pi_1^*\rho_3\iota_4^2 = r^2, \quad r_2^*\pi_2^*\pi_1^*P_3^1\rho_3\iota_4 = P_3^1\rho_3 r = 2r^2.$$

For the last result see [4]. Therefore the Serre sequence for the fibration $F_2 \longrightarrow BSpin(3) \longrightarrow Y_2$ gives us the invariant $\pi_2^*\pi_1^*\rho_3\iota_4^2 + \pi_2^*\pi_1^*P_3^1\rho_3\iota_4$.

This shows that a can be lifted to the third stage Y_3 of the Postnikov tower if and only if the conditions (i) – (iv) are satisfied. But because $\dim X \leq 8$, we can see that (i) – (iv) are necessary and sufficient conditions for the existence of a lift of a to $BSpin(3)$ in the fibration r. □

4. Existence of 3-fields

In this section we will use the tertiary cohomology operation Φ and the secondary operation Σ to find necessary and sufficient conditions for the existence of three linearly independent sections in an oriented 8-dimensional spin vector bundle over a CW-complex X of the same dimension. However, first of all the following theorem will serve us as an important tool for the computation of Φ and Σ in the next section.

Theorem 4.1. *Let ξ be an 8-dimensional oriented vector bundle over a CW-complex of dimension ≤ 8 with $w_2(\xi) = 0$. Then ξ has three linearly independent sections if and only if the following conditions are satisfied*
(1) $w_6(\xi) = 0$,
(2) $0 \in \Sigma(q_1(\xi))$,
(3) $e(\xi) = 0$,
(4) $\rho_4 q_2(\xi) \in \Phi(q_1(\xi))$.

REMARK 4.2. Both operations Σ and Φ on a closed connected smooth spin manifold M will be computed in the next Section.

Proof. We shall build the Postnikov tower for the fibration $V_{8,3} \longrightarrow BSpin(5) \xrightarrow{v} BSpin(8)$. The Stiefel manifold $V_{8,3}$ is 4-connected, $\pi_5(V_{8,3}) \cong \mathbb{Z}_2$, $\pi_6(V_{8,3}) \cong \mathbb{Z}_2$, and $\pi_7(V_{8,3}) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. The Serre exact sequence for this fibration shows immediately that the first Postnikov invariant is $w_6 \in H^6(BSpin(8);\mathbb{Z}_2)$. Thus we get the first stage of the tower in the following form.

$$\begin{tikzcd}
\bar{V}_1 \arrow[r] & V_{8,3} \arrow[r] \arrow[d] & K(\mathbb{Z}_2,5) \arrow[d, "i_1"] & \\
V_1 \arrow[r] & BSpin(5) \arrow[r, "\gamma_1"] \arrow[d, "v"] & E_1 \arrow[d, "v_1"] & \\
 & BSpin(8) \arrow[r, equal] & BSpin(8) \arrow[r, "w_6"] & K(\mathbb{Z}_2,6)
\end{tikzcd}$$

The fibre V_1 is 5-connected, $\pi_6(V_1) \cong \mathbb{Z}_2$ and $\pi_7(V_1) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. Moreover, the Serre exact sequence for the fibration v_1 shows that $H^7(E_1;\mathbb{Z}_2) \cong \mathbb{Z}_2$. We shall denote the unique generator by k. Taking into account the universal example for the operation Σ, we have the following commutative diagram.

$$\begin{tikzcd}
K(\mathbb{Z}_2,5) \arrow[r, dashed, "="] \arrow[d, "i_1"] & K(\mathbb{Z}_2,5) \arrow[d, "j_1"] & \\
E_1 \arrow[r, dashed, "f_1"] \arrow[d, "v_1"] & Y_1 \arrow[d, "\pi_1"] & \\
BSpin(8) \arrow[r, "q_1"] & K(\mathbb{Z},4) \arrow[r, "Sq^2\rho_2"] & K(\mathbb{Z}_2,6)
\end{tikzcd}$$

The mapping f_1 exists due to the fact that $Sq^2\rho_2 v_1^* q_1 = 0$. Since $i_1^* k = Sq^2 \iota_5$, there is $f_1^*\sigma = k$, or equivalently $k = \Sigma(v_1^* q_1)$.

The Serre exact sequence for the fibration γ_1 implies that the second Postnikov invariant is k. Consequently, the second stage of the Postnikov tower has the following form.

$$\begin{tikzcd}
\bar{V}_2 \arrow[r] & V_1 \arrow[r] \arrow[d] & K(\mathbb{Z}_2,6) \arrow[d, "i_2"] & \\
V_1 \arrow[r] & BSpin(5) \arrow[r, "\gamma_2"] \arrow[d, "\gamma_1"] & E_2 \arrow[d, "v_2"] & \\
 & E_1 \arrow[r, equal] & E_1 \arrow[r, "k"] & K(\mathbb{Z}_2,7)
\end{tikzcd}$$

Further invariants lie in $H^8(E_2;\mathbb{Z})$ and $H^8(E_2;\mathbb{Z}_4)$. The cohomology of E_2 can be computed from the Serre exact sequence for the fibration

$$K(\mathbb{Z}_2,6) \xrightarrow{i_2} E_2 \xrightarrow{v_2} E_1.$$

But for this sake we must know the cohomologies of E_1 first. We have

$H^8(E_1;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^5$ with the generators $v_1^*w_8$, $v_1^*w_4^2$, $v_1^*\varepsilon$, Sq^1k and l,
$H^8(E_1;\mathbb{Z}) \cong (\mathbb{Z})^3 \oplus \mathbb{Z}_2$ with the generators v_1^*e, $v_1^*q_1^2$, $v_1^*q_2$ and δk,
$H^8(E_1;\mathbb{Z}_4) \cong (\mathbb{Z}_4)^3 \oplus (\mathbb{Z}_2)^2$ with the generators $\rho_4 v_1^*q_1^2$, $\rho_4 v_1^*q_2$, $\rho_4 v_1^*e$, $\rho_4\delta k$ and i_*l,

where $l = f_1^*\psi = \Psi(v_1^*q_1)$ (see the last but one diagram above). Moreover, there is $i_1^*l = Sq^2Sq^1\iota_5$.

Further, we obtain

$H^8(E_2;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^4$ with the generators $v_2^*v_1^*w_4^2$, $v_2^*v_1^*\varepsilon$, $v_2^*v_1^*w_8$, v_2^*l,
$H^8(E_2;\mathbb{Z}) \cong (\mathbb{Z})^3$ with the generators $v_2^*v_1^*q_1^2$, $v_2^*v_1^*q_2$, $v_2^*v_1^*e$,
$H^8(E_2;\mathbb{Z}_4) \cong (\mathbb{Z}_4)^4$ with the generators $v_2^*v_1^*\rho_4 q_1^2$, $v_2^*v_1^*\rho_4 q_2$, $v_2^*v_1^*\rho_4 e$, m,

where $m = f_2^*\varphi = \Phi(v_2^*v_1^*q_1)$ (see the diagram below).

$$\begin{tikzcd}
K(\mathbb{Z}_2,6) \arrow[r, dashed, "="] \arrow[d] & K(\mathbb{Z}_2,6) \arrow[d] & \\
E_2 \arrow[r, dashed, "f_2"] \arrow[d, "v_2"] & Y_2 \arrow[d, "\pi_2"] & \\
E_1 \arrow[r, "f_1"] & Y_1 \arrow[r, "\sigma"] & K(\mathbb{Z}_2,7)
\end{tikzcd}$$

The fibre V_2 is 6-connected, and $\pi_7(V_2) \cong \mathbb{Z}\oplus\mathbb{Z}_4$. This means that on this stage we have two Postnikov invariants. The Serre exact sequence with the coefficients $\mathbb{Z}$ shows that the first of them is $v_2^*v_1^*e \in H^8(E_2;\mathbb{Z})$. The Serre exact sequence with the coefficients $\mathbb{Z}_4$ has the form

$$0 \longrightarrow H^7(V_2;\mathbb{Z}_4) \xrightarrow{\tau} H^8(E_2;\mathbb{Z}_4) \xrightarrow{\gamma_2^*} H^8(BSpin(5);\mathbb{Z}_4).$$

Using Lemma 3.8, it is easy to see that $\ker\gamma_2^* \cong \mathbb{Z}_4\oplus\mathbb{Z}_4$ with the generators $\rho_4 v_2^*v_1^*e$ and $m - \rho_4 v_2^*v_1^*q_2$. This shows that for the second of the two Postnikov invariants we can take $\Phi(v_2^*v_1^*q_1) - \rho_4 v_2^*v_1^*q_2 \in H^8(E_2;\mathbb{Z}_4)$.

Now, because $\dim X \leq 8$, we can immediately see that the vector bundle $\xi : X \longrightarrow BSpin(8)$ has three linearly independent sections if and only if the conditions of the theorem are satisfied. $\square$

5. Computation of Φ and Σ

This section is devoted to the computation of the cohomology operations Φ and Σ on closed connected smooth spin manifolds of dimension 8. Briefly said, it is carried out by comparing the theorems on the existence of three linearly independent sections in vector bundles proved in [10] and [11] with our Theorem 4.1.

Let $m \equiv 0 \bmod 4$ and let ξ be an oriented m-dimensional vector bundle over an oriented closed connected smooth manifold M of dimension m. In [10] Crabb and Steer defined

$$S(\xi) = \{2^{m/2}\hat{A}(M)\hat{B}(\xi)\}[M]$$

where $\hat{A}$ is the $\hat{A}$-genus given by $\prod_{s=1}^{m/2} y_s(\sinh(1/2)y_s)^{-1}$, $\hat{B}$ is given by $\prod_{s=1}^{m/2}\cosh(1/2)y_s$ and the Pontrjagin classes are the elementary symmetric polynomials in the squares y_s^2.

The signature defined in this way plays the role of an obstruction when we deal with the existence of 2 or 3 linearly independent sections of ξ as well as in the case of tangent bundles.

Proposition 5.1 ([10, Theorem 4.10 and 4.4. (iii)]). *Let $m \equiv 0 \bmod 4$. Let ξ be an oriented m-dimensional vector bundle over an oriented closed connected smooth m-manifold M, and η an oriented vector bundle of dimension 3 over M with $w_2(\eta) = w_2(\xi) + w_2(M)$. Suppose that η is a subbundle of ξ over the $(m-1)$-skeleton of M. Then the obstructions for η to be a subbundle of ξ over the whole manifold are*
(a) $e(\xi) = 0$
(b) $S(\xi - \eta) \equiv 0 \bmod 8$.

It is only a matter of computation (see [9]) to show that for $m = 8$ and a closed connected smooth spin manifold M

$$\begin{aligned} S(\xi) &\equiv \frac{1}{45\cdot 8}\{60p_2(\xi) + 15p_1^2(\xi) - 30p_1(M)p_1(\xi)\}[M] \equiv \\ &\equiv \frac{1}{3}\{q_1^2(\xi) - q_1(M)q_1(\xi) + 2q_2(\xi)\}[M] \bmod 8. \end{aligned} \tag{5.2}$$

Since for $z \in H^4(M;\mathbb{Z})$

$$\rho_2(zq_1(M) - z^2) = w_4(M)\rho_2 z - \rho_2 z^2 = Sq^4\rho_2 z - \rho_2 z^2 = 0$$

and $H^8(M;\mathbb{Z}) \cong \mathbb{Z}$, there is just one $y \in H^8(M;\mathbb{Z})$ such that $2y = zq_1(M) - z^2$. So, we will use notation $(1/2)(zq_1(M) - z^2)$ for this y.

Theorem 5.3. *Let M be a closed connected smooth spin manifold of dimension 8. Then* $\operatorname{Indet}(\Phi, M) = 0$ *and*

$$\Phi(z) = \rho_4\frac{1}{2}\{zq_1(M) - z^2\}$$

for every $z \in \operatorname{Def}(\Phi, M)$.

Proof. Let $z \in \mathrm{Def}(\Phi, M)$. Choose any element $y \in \Phi(z)$. Since $H^8(M;\mathbb{Z}) \cong \mathbb{Z}$ and $(3,4)=1$, there is $x \in H^8(M;\mathbb{Z})$ such that

$$\rho_4 x = y$$

and

$$\rho_3 x = P_3^1 \rho_3 z + \rho_3 z^2.$$

According to Theorem 2 in [7] there is an 8-dimensional oriented vector bundle ξ over M with $w_2(\xi) = 0$, $q_1(\xi) = z$, $e(\xi) = 0$ and $q_2(\xi) = x$. Moreover, for such a vector bundle $w_6(\xi) = Sq^2 \rho_2 z = 0$ and $0 \in \Sigma(z)$.

Then Theorem 4.1 claims that the vector bundle ξ has three linearly independent sections. Using formula (5.2), Proposition 5.1 for η trivial implies that

$$y = \rho_4 x = \rho_4 q_2(\xi) = \rho_4 \frac{1}{2}\{q_1(\xi) q_1(M) - q_1^2(\xi)\} = \rho_4 \frac{1}{2}\{z q_1(M) - z^2\}$$

which completes the proof. □

In a similar way we can compute the secondary operation Σ. For this purpose we need the following proposition.

Proposition 5.4 (See the last remark in [11] and [9, Proposition 5.2]). *Let ξ be an oriented m-dimensional vector bundle over a closed connected smooth manifold M of the same dimension $m \equiv 0 \bmod 4$, and let $w_2(\xi) = w_2(M)$. If ξ has three linearly independent sections over the $(m-2)$-skeleton of M then the obstruction to deforming them (relative to the $(m-3)$-skeleton of M) into sections which have three linearly independent extensions over $(m-1)$-skeleton of M is zero.*

Theorem 5.5. *Let M be a closed connected smooth spin manifold of dimension 8. Then*

$$\Sigma(z) = Sq^2 H^5(M;\mathbb{Z}_2)$$

for every $z \in \mathrm{Def}(\Sigma, M)$.

Proof. Let $z \in \mathrm{Def}(\Sigma, M)$. Since $H^8(M;\mathbb{Z}) \cong \mathbb{Z}$, there is $x \in H^8(M;\mathbb{Z})$ such that

$$\rho_3 x = P_3^1 \rho_3 z + \rho_3 z^2.$$

According to Theorem 2 in [7] there is an 8-dimensional oriented vector bundle ξ over M with $w_2(\xi) = 0$, $q_1(\xi) = z$, $e(\xi) = 0$ and $q_2(\xi) = x$. Moreover, for such a vector bundle $w_6(\xi) = Sq^2 \rho_2 z = 0$.

Then according to Proposition 5.4 the vector bundle ξ has three linearly independent sections over 7-skeleton. Then Theorem 4.1 implies that

$$0 \in \Sigma(q_1(\xi)) = \Sigma(z).$$

Consequently, $\Sigma(z) = \mathrm{Indet}(\Sigma, M) = Sq^2 H^5(M;\mathbb{Z}_2)$. $\square$

6. Existence of 3-dimensional subbundles

The following theorem on the existence of 3-dimensional subbundles in oriented 8-dimensional spin vector bundles is the last but one step to complete the proof of our main results on the existence of an $Sp(2)\cdot Sp(1)$-structure.

Theorem 6.1. *Let ζ be an oriented 8-dimensional vector bundle over a closed connected smooth spin 8-manifold M with $w_2(\zeta)=0$ and let $R \in H^4(M;\mathbb{Z})$. Then ζ has an oriented 3-dimensional subbundle η with $w_2(\eta)=0$ and $p_1(\eta)=4R$ if and only if*
(i) $Sq^2\rho_2 R = 0$
(ii) $P_3^1\rho_3 R + \rho_3 R^2 = 0$
(iii) $\{Rq_1(M) - R^2\}[M] \equiv 0 \bmod 8$
(iv) $w_6(\zeta) = 0$
(v) $e(\zeta) = 0$
(vi) $\{q_1^2(\zeta) - q_1(M)q_1(\zeta) + 2q_2(\zeta) + 2R^2 + 2Rq_1(\zeta) + 2Rq_1(M)\}[M] \equiv 0 \bmod 8.$

Proof. According to Theorem 5.3 the condition (iii) is equivalent to $\Phi(R)=0$. Then Theorem 3.10 and 5.5 say that the conditions (i) – (iii) are necessary and sufficient for the existence of an oriented 3-dimensional vector bundle η over M with $w_2(\eta)=0$ and $p_1(\eta)=4R$.

Now we show that the conditions (i) and (iv) are necessary and sufficient for η to be a subbundle of ζ over a 7-skeleton of M. This can be done by constructing the first stage of the Postnikov tower for the fibration $\theta : B(Spin(5)\times Spin(3)) \to BSpin(8)$ determined by the homomorphism $Spin(5)\times Spin(3) \to Spin(8)$ induced from the standart inclusion $SO(5)\times SO(3) \to SO(8)$. We will not go into details which are similar to the procedure used in the proof of Theorem 4.1. We note only that there are two obstructions in $H^6(M;\mathbb{Z}_2)$ and two in $H^7(M;\mathbb{Z}_2)$ and that

$$H^*(B(Spin(5)\times Spin(3));\mathbb{Z}) \cong \mathbb{Z}[r, q_1, q_2]$$

with the properties

$$Sq^2\rho_2 r = 0, \quad Sq^2\rho_2 q_1 = 0, \quad \Sigma(r) = 0, \quad \Sigma(q_1) = 0,$$

and

$$\theta^* q_1 = q_1 + 2r.$$

This yields the conditions (i) and (iv) on the manifold M.

Finally, we show that in the given case the condition (b) in Proposition 5.1 reads as (vi). Since

$$\begin{aligned}p_1(\zeta-\eta) &= p_1(\zeta)-p_1(\eta)\\ p_2(\zeta-\eta) &= p_2(\zeta)-p_1(\zeta)p_1(\eta)+p_1^2(\eta),\end{aligned}$$

formula (5.2) gives

$$\begin{aligned}S(\zeta-\eta) &\equiv \frac{1}{45\cdot 8}\{60p_2(\zeta)+15p_1^2(\zeta)-30p_1(M)p_1(\zeta)\\ &\quad +75p_1^2(\eta)-90p_1(\zeta)p_1(\eta)+30p_1(M)p_1(\eta)\}[M]\\ &\equiv \frac{1}{3\cdot 8}\{p_1^2(\zeta)-2p_1(M)p_1(\zeta)+4p_2(\zeta)\\ &\quad +5p_1^2(\eta)-6p_1(\eta)p_1(\zeta)+2p_1(M)p_1(\eta)\}[M] \bmod 8.\end{aligned}$$

Substituting $p_1(\zeta) = 2q_1(\zeta)$, $p_2(\zeta) = q_1^2(\zeta)+4q_2(\zeta)$ (we suppose that $e(\zeta) = 0$, which is simultaneously the condition (v) of Theorem 6.1 and the condition (a) of Proposition 5.1) and $p_1(\eta) = 4R$, we get that (a) and (b) of Proposition 5.1 are equivalent to (v) and (vi). □

8. Existence of almost quaternionic structure

Now we state our main result on the existence of an $Sp(2)\cdot Sp(1)$-structure in 8-dimensional vector bundles over 8-manifolds.

Theorem 8.1. *Let ξ be an oriented 8-dimensional vector bundle over a closed connected smooth spin manifold M. If there is $R \in H^4(M;\mathbb{Z})$ such that the conditions*

(1) $Sq^2\rho_2 R = 0$
(2) $\{Rp_1(M)-2R^2\}[M] \equiv 0 \bmod 16$
(3) $w_2(\xi) = 0$
(4) $w_6(\xi) = 0$
(5) $4p_2(\xi)-p_1^2(\xi)-8e(\xi) = 0$
(6) $\{p_1^2(\xi)-p_1(M)p_1(\xi)-8e(\xi)+8R^2+4Rp_1(\xi)+4Rp_1(M)\}[M] \equiv 0 \bmod 32$

are satisfied, then the structure group of ξ can be reduced to $Sp(2)\cdot Sp(1)$. If $H^2(M;\mathbb{Z}_2)=0$, then all the previous conditions are also necessary.

REMARK 8.2. The conditions (3) and (5) are necessary even if $H^2(M;\mathbb{Z}_2)\neq 0$.

Proof. Proposition 1.2 asserts that a vector bundle ξ has an $Sp(2)\cdot Sp(1)$-structure if and only if it has a spinor structure $\bar{\xi} \in [M, BSpin(8)]$ and the vector bundle

$$\zeta = \pi_*(\kappa\lambda)_*(\bar{\xi})$$

has an oriented 3-dimensional subbundle. According to Lemma 4.2 in [9]

$$q_1(\zeta) = q_1(\xi), \quad e(\zeta) = -q_2(\xi), \quad q_2(\zeta) = -e(\xi).$$

So, the existence of a three dimensional subbundle η with $p_1(\eta) = 4R$ for some $R \in H^4(M;\mathbb{Z})$ in the vector bundle ζ is sufficient for the existence of an $Sp(2)\cdot Sp(1)$-structure in the vector bundle ξ. Hence we show that our conditions imply the conditions (i) – (vi) of Theorem 6.1 for ζ and some R.

The condition (i) is the same as (1). (iii) is equivalent to (2). (4) of Theorem 7.1 yields $w_2(\zeta) = w_2(\xi) = 0$. Since $q_1(\zeta) = q_1(\xi)$, (iv) is equivalent to (4). (5) means $q_2(\xi) = 0$, which reads as (v) of Theorem 6.1. Rewriting (6) in terms of ζ, we get (vi).

It remains to prove the condition (ii) of Theorem 6.1. It need not be satisfied for a given R but it is certainly satisfied for $\bar{R} = -15R$. Moreover, if R satisfies the conditions (i), (iii) and (vi) of Theorem 6.1, then $\bar{R}$ satisfies them as well since

$$-15 \equiv (-15)^2 \equiv 1 \bmod 16.$$

This completes the proof. □

The application of Theorem 7.1 to tangent bundles yields

Corollary 8.3. *Let M be a an oriented closed connected smooth manifold of dimension* 8. *If*
(a) $w_2(M) = 0$
(b) $w_6(M) = 0$
(c) $4p_2(M) - p_1^2(M) - 8e(M) = 0$
and there is $R \in H^4(M;\mathbb{Z})$ such that
(d) $Sq^2\rho_2 R = 0$
(e) $\{Rp_1(M) - 2R^2\}[M] \equiv 0 \bmod 16$
(f) $\{R^2 + Rp_1(M) - e(M)\}[M] \equiv 0 \bmod 4$,
then M has an almost quaternionic structure. Conditions (a) *and* (c) *are always necessary for the existence of this structure while the remaining ones are necessary if $H^2(M;\mathbb{Z}_2) = 0$.*

Proof. The conditions in Theorem 7.1 correspond with the conditions (a) – (f) of this corollary. □

We can give also nontrivial sufficient conditions for the existence of an almost quaternionic structure only in terms of characteristic classes without any reference to an element $R \in H^4(M;\mathbb{Z})$. See Theorem 1.1.

Proof of Theorem 1.1. Let the assumptions of Theorem 1.1 be satisfied. The first three conditions of Corollary 7.3 are the same as the corresponding conditions of Theorem 1.1. Put $R = q_1(M)$. Then the condition (d) follows from (B), (e) is obviously satisfied, and (f) is a consequence of (D). So the assumptions of Corollary 7.3 are satisfied. □

REMARK 8.4. If we put $R = 0$ in Corollary 7.3 we get (A), (B), (C) and
(D) $e(M)[M] \equiv 0 \bmod 4$
as sufficient conditions for the existence of an almost quaternionic structure. According to Corollary 5.5 in [9], these are necessary and sufficient conditions for the existence of an $Sp(2)$-structure.

9. Examples

Now we will demonstrate the above statements on several examples.

EXAMPLE 9.1. The quaternionic projective space $\mathbb{H}P^2$ is known to be a quaternion-Kähler manifold so it must have an almost quaternionic structure. We will show that all the assumptions of Theorem 1.1 are satisfied. It can be seen from the following characteristic classes computed in [3] :

$$p_1(\mathbb{H}P^2) = 2u, \quad p_2(\mathbb{H}P^2) = 7u^2, \quad e(\mathbb{H}P^2) = 3u^2$$

where $u \in H^4(\mathbb{H}P^2;\mathbb{Z})$ and $H^*(\mathbb{H}P^2;\mathbb{Z}) = \mathbb{Z}[u]/\langle u^3\rangle$.

EXAMPLE 9.2. The complex Grassmann manifold $G_{4,2}(\mathbb{C})$ is also a quaternion-Kähler manifold. From [3] we know that

$$H^*(G_{4,2}(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[u,v]/\langle u^3 - 2uv, v^2 - u^2v\rangle$$

where $u \in H^2(G_{4,2}(\mathbb{C});\mathbb{Z})$ and $v \in H^4(G_{4,2}(\mathbb{C});\mathbb{Z})$ and

$$\begin{aligned} c_1(G_{4,2}(\mathbb{C})) &= -4u & \qquad c_2(G_{4,2}(\mathbb{C})) &= 7u^2 \\ c_3(G_{4,2}(\mathbb{C});\mathbb{Z}) &= -12uv & \qquad c_4(G_{4,2}(\mathbb{C})) &= 6u^2v, \end{aligned}$$

which yields $p_1(G_{4,2}(\mathbb{C})) = 2u^2$, $p_2(G_{4,2}(\mathbb{C})) = 14u^2v$, $e(G_{4,2}(\mathbb{C})) = 6u^2v$. Hence all the conditions of Theorem 1.1 are satisfied.

EXAMPLE 9.3. $G_2/SO(4)$ is the third 8-dimensional homogeneous space which is a quaternion-Kähler manifold. So it has an almost quaternionic structure. But in [3] it is proved that $w_6(G_2/SO(4)) \neq 0$, which shows that the condition (b) in Corollary 7.3 is not necessary.

EXAMPLE 9.4. The complex projective surfaces

$$V_d = \{(z_0, z_1, \ldots, z_5) \in \mathbb{C}P^5; z_0^d + z_1^d + \ldots + z_5^d = 0\}$$

considered as closed oriented smooth manifolds of real dimension 8 satisfy the necessary condition (C) of Theorem 1.1 only for $d = 2, 6$. (See [9].) Since $V_2 = G_{4,2}(\mathbb{C})$, we will deal only with $d = 6$. We get

$$p_1(V_6) = -30c^2, p_2(V_6) = 1095c^4, e(V_6) = 435c^4, c_1(V_6) = 0, c_3(V_6) = -70c^3,$$

where $c \in H^2(V_6; \mathbb{Z})$ and $c^4[V_6] = 6$. Hence all the assumptions of Theorem 1.1 are satisfied and V_6 has an almost quaternionic structure.

EXAMPLE 9.5. Let M_1 and M_2 be two closed simply connected smooth 4-manifolds with $w_2(M_1) = 0$, $w_2(M_2) = 0$. According to the remark after Rochlin Theorem in [12] the condition $w_2(M_s) = 0$ is equivalent to the fact that the intersection form ω_s of M_s is even. Then Rochlin Theorem ([12, Theorem 1.2]) asserts that the signature of both forms is divisible by 16 and Donaldson Theorem ([12, Theorem 1.3]) says that ω_s is indefinite. Using the classification of indefinite forms over $\mathbb{Z}$ we get

$$\omega_s = -2n_s E_8 \oplus m_s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $m_s \in \mathbb{N}$, $n_s \in \mathbb{Z}$, E_8 being described in [12], $\operatorname{rank} E_8 = 8$, $\operatorname{sign}(E_8) = 8$. Then the signature of M_s is $S(M_s) = -16n_s$ and the Euler characteristic is $16n_s + 2m_s + 2$. Moreover, the Signature Theorem yields

$$p_1(M_s)[M_s] = 3S(M_s) = -48n_s$$

for $s = 1, 2$. Next

$$\begin{aligned} e(M_1 \times M_2)[M_1 \times M_2] &= (16n_1 + 2m_1 + 2)(16n_2 + 2m_2 + 2) \\ p_1^2(M_1 \times M_2)[M_1 \times M_2] &= 2 \cdot 48^2 n_1 n_2 \\ p_2(M_1 \times M_2)[M_1 \times M_2] &= 48^2 n_1 n_2. \end{aligned}$$

The nontrivial sufficient condition for the existence of an almost quaternionic structure on $M_1 \times M_2$ is

$$144 n_1 n_2 = (8n_1 + m_1 + 1)(8n_2 + m_2 + 1).$$

This is the condition (C) of Theorem 1.1. The remaining conditions (A), (B) and (D) are satisfied.

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M. Čadek
Department of Algebra and
Geometry, Masaryk University
Janáčkovo nám. 2a
662 95 Brno, Czech Repubblic
e-mail: cadek@math.muni.cz

J. Vanžura
Academy of Sciences of
the Czech Republic
Institute of Mathematics
Žižkova 22
616 62 Brno, Czech Republic
e-mail: vanzura@ipm.cz