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On the Existence of Harmonic Functions on Riemann Surfaces

By Zenjiro KURAMOCHI

H. L. Royden¹⁾ proved by the use of the theory of Banach algebra that the class of O_{HD} or O_{HDN} ²⁾ is invariant under a quasi-conformal mapping whose dilatation quotient is bounded. The purpose of this article is to give a function-theoretic proof of the theorem.

Let F be an abstract Riemann surface, let $\{F_n\}$ be its exhaustion and let G be a non-compact subdomain whose relative boundary ∂G ³⁾ consists of at most an enumerably infinite number of analytic curves clustering nowhere in F .

Theorem. (EXTENSION OF L. MYRBERG'S THEOREM⁴⁾). *Let $U(p) : p \in F$ be a harmonic function on an abstract Riemann surface F such that $D_F(U(p)) < \infty$ and suppose that the universal covering surface F^∞ of F is mapped onto the unit-circle : $|z| < 1$. Then $U(p)$ is represented by Poisson's integral.*

Lemma. *Let $V(z)$ be a continuous sub-harmonic function on $|z| \leq 1$ such that $\int |V(e^{i\theta})| d\theta \leq M$ and let G be a simply connected domain in $|z| < 1$ with a rectifiable boundary ∂G . Then $\int_{\partial G} |V(z)| d\omega \leq M$, where ω is the harmonic measure of ∂G with respect to G .*

Proof of the lemma. Denote by $V^*(z)$ the upper envelope of sub-harmonic functions $\{V^i(z)\}$ such that $0 \leq V^i(z) \leq |V(z)|$ on the complementary set of G . Then $V^*(z) = |V(z)|$ on ∂G and sub-harmonic in $|z| < 1$ and is harmonic in G . Let $V^{**}(z)$ be a harmonic function in $|z| < 1$ with the boundary value $|V(e^{i\theta})|$. Then

$$M \geq \int |V(e^{i\theta})| d\theta = \int V^{**}(e^{i\theta}) d\theta = V^{**}(O) \geq V^*(O) = \int_{\partial G} V^*(z) d\omega.$$

Map the universal covering surface F^∞ of F onto $|z| < 1$. Then the circle : $|z| < \rho$ ($\rho < 1$) is contained in the image of F_n^∞ for sufficiently large number n . Denote by $G_n(p, p_0)$ the Green's function of F_n and denote by $h_n(p, p_0)$ its conjugate function, where p_0 is the image of $z=0$. Put $e^{-G_n - ih_n} = r_n e^{i\varphi_n}$. Then

$$D_{F_n}(U(p)) = \frac{1}{2} \int_{\partial F_n} \frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} r_n d\varphi_n \leq M'$$

and

$$\begin{aligned} \int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n &= 2\pi U^2(O) + \iint_{F_n} 2 \operatorname{grad} U^2(p) \log r_n dr_n d\varphi_n \\ &+ \frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} \log r_n r_n d\varphi_n \leq M', \end{aligned}$$

whence

$$\int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n \leq M'',$$

where M'' is independent of n .

Map F_n^∞ onto $|\xi| < 1$ and let G_p be the image of $|z| < \rho$ by this mapping. Since the connectivity of F_n is finite, ∂F_n is mapped in $|\xi| = 1$ except possibly a set of linear measure zero, and further $d\varphi_n$ corresponds to ds on $|\xi| = 1$, where ds consists of at most an enumerably infinite of arcs. Since $U^2(\xi)$ is sub-harmonic, we have by the lemma

$$M'' \geq \int_{|\xi|=1} U^2(r_n e^{i\varphi_n}) d\varphi_n = \int_{\partial G_p} U^2(\xi) d\theta \geq \int_{\partial G_p} U^2(\xi) d\omega = \int_{|z|=\rho_n} U(\rho_n e^{i\theta_n}) d\theta_n$$

Let $\rho_n \rightarrow 1$. Then by Fatou's theorem $U(p)$ is represented by Poisson's integral in $|z| < 1$.

Let $\tilde{U}(p) : p \in F$ be a harmonic function of Dirichlet bounded. Then there exist subdomains⁵⁾ G_i ($i = 1, 2$) in F with the property as follows: there exist harmonic functions of Dirichlet bounded such that $\tilde{U}_i(p) \geq 0$, $\tilde{U}_i(p) = 0$ on ∂G_i and $G_1 \cap G_2 = 0$.

Let G be one of them and let $U(p)$ be one of $\tilde{U}_i(p)$ in the sequel. Map the universal covering surface G^∞ of G onto $|z| < 1$. Then there exists a constant δ ($\delta > 0$) and a set E_δ of positive measure on $|z| = 1$ such that $U(p)$ has angular limits larger than δ , because $U(p)$ is represented by Poisson's integral on $|z| < 1$. Let G' be the subdomain where $U(p) > \frac{\delta}{2}$. Then G' determines a set $B_{\frac{\delta}{2}}^{(6)}$ of the ideal boundary. Let $V_{n,n+i}(p)$ be a harmonic function in $(F_{n+i} \cap G) - (G' \cap (F_{n+i} - F_n)) = H_{n,n+i}$ such that $V_{n,n+i}(p) = 0$ on $\partial G' + (\partial F_{n-i} \cap (G - G'))$ and $V_{n,n+i}(p) = 1$ on $\partial F_n \cap G'$. Then $V(p) = \lim_n \lim_i V_{n,n+i}(p) \geq \omega_\delta(z)$, where $\omega_\delta(z)$ is the harmonic measure of E_δ , with respect to G^∞ . Hence $V(p)$ is non-

constant. Let $U_{n,n+i}(p)$ be a harmonic function in $H_{n,n+i}$ such that $U_{n,n+i}(p)=0$ on ∂G , $U_{n,n+i}(p)=1$ on $(\partial G' \cap (F_{n+i} - F_n)) + (\partial F_n \cap G')$ and $\frac{\partial U_{n,n+i}}{\partial n} = 0$ on $\partial F_{n+i} \cap (G - G')$. It is clear

$$D_G(U(p)) \geq D_{H_{n,n+i}}(U_{n,n+i}(p))$$

and

$$D_{H_{n,n+i+j}}(U_{n,n+i+j}(p)) \geq D_{H_{n,n+i}}(U_{n,n+i}(p)).$$

Hence $U_{n,n+i}(p)$ converges to $U_n(p)$ in mean. Since $U_{n,n+i}(p) \geq V_{n,n+i}(p)$, $U(p)$ is non-constant.

Let C_δ' be the niveau curve of $U_n(p)$ with height δ' ($0 < \delta' < 1$) and put

$${}_0G_{\delta'} = \varepsilon_p[0 < U_n(p) < \delta'] \text{ and } {}_1G_{\delta'} = \varepsilon_p[\delta' < U(p) < 1]$$

respectively.

Let $U'_{n,n+i}(p)$ be a harmonic function in ${}_0G_{\delta'} \cap F_{n+i}$ such that $U'_{n,n+i}(p)=0$ on ∂G , $U'_{n,n+i}(p)=\delta'$ on ${}_0G_{\delta'} \cap (F_{n+i} - F_n) + (\partial F_n \cap {}_0G_{\delta'})$ and $\frac{\partial U'_{n,n+i}}{\partial n} = 0$ on ∂F_{n+i} . Then we can prove as the previous manner⁷⁾ the following

Lemma. $U'_{n,n+i}(p) \rightarrow U_n(p)$ in mean and

$$\lim_i D_{\substack{0 \\ {}_0G_{\delta'} \cap F_{n+i}}}(U'_{n,n+i}(p)) = D_{\substack{0 \\ {}_0G_{\delta'}}}(U_n(p)).$$

On the other hand, for given number ε and i , we see easily that there exists a number j_0 such that

$$D_{\substack{0 \\ {}_0G_{\delta'} \cap F_{n+i}}}(U'_{n,n+i}(p)) \leq D_{\substack{0 \\ {}_0G_{\delta'} \cap F_{n+i+j}}}(U'_{n,n+i+j}(p)) + \varepsilon < \delta' D(U_n(p)) + \varepsilon \quad \text{for } j \geq j_0.$$

Let $\varepsilon \rightarrow 0$ and then $i \rightarrow \infty$. Then we have

$$D_{\substack{0 \\ {}_0G_{\delta'}}}(U_n(p)) \leq \delta' D(U_n(p)).$$

Let $U''_{n,n+i}(p)$ be a harmonic function in ${}_0G_{\delta'} \cap F_{n+i}$ such that $U''_{n,n+i}(p) = 1$ on $\partial G'$, $U''_{n,n+i}(p) = \delta'$ on $(\partial {}_1G_{\delta'} \cap (F_{n+i} - F_n)) + (\partial F_n \cap {}_1G_{\delta'})$ and $\frac{\partial U''_{n,n+i}}{\partial n} = 0$ on ∂F_{n+i} . Then by the same manner we have

$$\lim_i U''_{n,n+i}(p) = U''_n(p) = U_n(p) \text{ and } D_{\substack{1 \\ {}_1G_{\delta'}}}(U_n(p)) \leq (1 - \delta') D(U_n(p)),$$

whence

$$D_{\substack{0 \\ {}_0G_{\delta'}}}(U_n(p)) = \delta' D(U_n(p)).$$

Next we have by the same manner as the previous⁸⁾ the following

Lemma.

$$\int_{\partial_0 G \cap F} \frac{\partial U_n}{\partial n} ds = D_{G - (G' \cap F_n)}(U_n(p)),$$

for every number ε except for at most two numbers ε_1 and ε_2 .

We say that the sequence $\{G' \cap (F - F_n)\}$ ($n = 1, 2, \dots$) determines a set $B_{G'}$ of the ideal boundary and call $\lim_i D_{G - G'}(U_n(p))$ the capacity of $B_{G'}$ with respect to G . Then by the above lemmas, we can prove⁹⁾ as the previous the following

Theorem. $U_n(p)$ converges to $U^0(p)$ in mean and

$$D_{G - (F - F_n) \cap G'}(U_n(p)) \downarrow D_G(U^0(p)).$$

Since $U^0(p) \geq V(p)$, $U^0(p)$ is non-constant and since $D_G(U(p)) \geq D_G(U^0(p))$,

$$\infty > \text{Cap}(B_{G'}) > 0.$$

Proof of the Royden's theorem.

Let F^* be another Riemann surface such that $F^* \ni p^* : p^* = T(p) : p \in F$, where $T(p)$ is a quasi-conformal mapping whose dilatation quotient is bounded $\leq K$. Put $U_n^q(p) = U_n(T(p))$. Then $U_n^q(p)$ is not necessarily harmonic and

$$\frac{1}{K} D_{H_n, n}(U_{n, n+i}(p)) \leq D_{T(H_n, n+i)}(U_{n, n+i}^q(p^*)) \leq K D_{H_n, n+i}(U_{n, n+i}(p)),$$

where $T(G)$ is the image of G by the mapping $p^* = T(p)$.

Let $U_{n, n+i}(p^*)$ be a harmonic function in $T(H_n, n+i)$ such that $U_{n, n+i}(p^*) = 0$ on $\partial(T(G))$, $U_{n, n+i}(p^*) = 1$ on $\partial(T(F_{n+i})) + T(\partial G' \cap (F_{n+i} - F_n))$ and $\frac{\partial U_{n, n+i}(p^*)}{\partial n} = 0$ on $\partial(T(F_{n+i}))$. Then by the Dirichlet principle

$$D_{T(H_n, n+i)}(U_{n, n+i}(p^*)) \leq D_{T(H_n, n+i)}(U_{n, n+i}^q(p^*)) \leq K D_{H_n, n+i}(U_{n, n+i}(p)).$$

Since the inverse mapping $T^{-1}(p)$ of $T(p)$ is also a quasi-conformal mapping,

$$D_{H_n, n+i}(U_{n, n+i}^q(T^{-1}(p^*))) \leq K_{T(H_n, n+i)} D(U_{n, n+i}(p^*)) \leq K D_{H_n, n+i}(U_{n, n+i}(p)),$$

where $U_{n, n+i}^q(T^{-1}(p^*)) = U_{n, n+i}(p)$. On the other hand, by the Dirichlet principle, we have

$$D_{H_n, n+i}(U_{n, n+i}(p)) \leq D_{H_n, n+i}(U_{n, n+i}^q(T^{-1}(p^*))).$$

Hence

$$D_{T(H_n, n+i)}(U_{n, n+i}(p^*)) \geq \frac{1}{K} D_{H_n, n+i}(U_{n, n+i}^q(T^{-1}(p^*))) \geq \frac{1}{K} D_{H_n, n+i}(U_{n, n+i}(p)).$$

$T(G')$ also determines a set $B_{T(G')}$ of the ideal boundary of $T(F)$. Let $i \rightarrow \infty$ and then $n \rightarrow \infty$. Then

$$KD_G(U^0(p)) \geq D_{T(G)}(U^0(p^*)) \geq \frac{1}{K} D_G(U^0(p)).$$

Hence $B_{T(G')}$ is a set of positive capacity with respect to $T(G)$ and

$$\frac{1}{K} \operatorname{Cap} B_{G'} \leq \operatorname{Cap} B_{T(G')} \leq K \operatorname{Cap} B_{G'}.$$

Therefore on $T(G)$, there exists a Dirichlet bounded harmonic function with boundary value 0 on $T(\partial G)$. Thus we obtain our theorem by the method M. Parreau and A. Mori¹⁰⁾.

We construct an open Riemann surface $[(G-G') \cap F_n]^\wedge$ by the process of the symmetrization with respect to ∂F_n . Then $((G-G') \cap F_n) + [(G-G') \cap F_n]^\wedge$ is a ring domain. Let $n \rightarrow \infty$. Then $(G-G') \cap F$ is a generalized semi-ring domain. Thus our theorem is an extension of the well known "Modulsatz".

Let F_0 be a compact set of F and let $N_n(p, p_0)$ be a harmonic function in $(F-F_0) \cap F_n$ such that $N_n(p, p_0) = 0$ on ∂F_0 and $\frac{\partial N_n}{\partial n} = 0$ on ∂F_n and has one logarithmic singularity at p_0 . We can prove that $N_n(p, p_0)$ converges to $N(p, p_0)$, when $n \rightarrow \infty$. Let $\{p_i\}$ be a sequence of points tending to the boundary. If a subsequence of $\{N(p, p_i)\}$ converges to $N(p, \{p_i\})$, we say that $\{p_i\}$ determines an ideal point¹¹⁾. We can introduce a topology of the ideal boundary by the use of $N(p, \{p_i\})$ and prove that the above topology is invariant under the quasi-conformal mapping.

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References

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- 2) O_{HD} and O_{HDN} are the class of Riemann surfaces on which no harmonic function of Dirichlet bounded exists or the dimension of harmonic function of Dirichlet bounded is N respectively.
- 3) In this article, we denote the relative boundary of G by ∂G .
- 4) L. Myrberg: *Bemerkungen zur Theorie der harmonischen Funktionen.* *Annales Acad. Fenn.* 1952.
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- 6) Z. Kuramochi: Harmonic measures and capacity of sets of the ideal boundary. I and II. *Proc. Jap. Acad.* **30**. 1954 and **31**. 1955.
- 7), 8) and 9) See 6).
- 10) See 5).
- 11) A note on this topology will appear in this Journal.