<table>
<thead>
<tr>
<th>Title</th>
<th>On the existence of harmonic functions on Riemann surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kuramochi, Zenjiro</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Mathematical Journal. 7(1) P.23–P.28</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1955-06</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/4897">https://doi.org/10.18910/4897</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/4897</td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>

*Osaka University Knowledge Archive : OUKA*

https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
On the Existence of Harmonic Functions on Riemann Surfaces

By Zenjiro Kuramoto

H. L. Royden proved by the use of the theory of Banach algebra that the class of $O_{HD}$ or $O_{HDN}$ is invariant under a quasi-conformal mapping whose dilatation quotient is bounded. The purpose of this article is to give a function-theoretic proof of the theorem.

Let $F$ be an abstract Riemann surface, let $\{F_n\}$ be its exhaustion and let $G$ be a non-compact subdomain whose relative boundary $\partial G$ consists of at most an enumerably infinite number of analytic curves clustering nowhere in $F$.

Theorem. (Extension of L. Myrberg's Theorem). Let $U(p) : p \in F$ be a harmonic function on an abstract Riemann surface $F$ such that $D_p(U(p)) < \infty$ and suppose that the universal covering surface $F^\infty$ of $F$ is mapped onto the unit-circle $|z| < 1$. Then $U(p)$ is represented by Poisson's integral.

Lemma. Let $V(z)$ be a continuous sub-harmonic function on $|z| \leq 1$ such that $\int |V(e^{i\theta})|d\theta \leq M$ and let $G$ be a simply connected domain in $|z| < 1$ with a rectifiable boundary $\partial G$. Then $\int_{\partial G} V(z) d\omega \leq M$, where $\omega$ is the harmonic measure of $\partial G$ with respect to $G$.

Proof of the lemma. Denote by $V^*(z)$ the upper envelope of sub-harmonic functions $\{V^i(z)\}$ such that $O \leq V^i(z) \leq V(z)$ on the complementary set of $G$. Then $V^*(z) = |V(z)|$ on $\partial G$ and sub-harmonic in $|z| < 1$ and is harmonic in $G$. Let $V^{**}(z)$ be a harmonic function in $|z| < 1$ with the boundary value $|V(e^{i\theta})|$. Then

$$M \geq \int |V(e^{i\theta})| d\theta = \int V^{**}(e^{i\theta}) d\theta = V^{**}(O) \geq V^*(O) = \int_{\partial G} V^*(z) d\omega.$$

Map the universal covering surface $F^\infty$ of $F$ onto $|z| < 1$. Then the circle $|z| < \rho$ ($\rho < 1$) is contained in the image of $F^\infty_n$ for sufficiently large number $n$. Denote by $G_n(p, \rho)$ the Green's function of $F^\infty_n$ and denote by $h_n(p, \rho_0)$ its conjugate function, where $\rho_0$ is the image of $z = 0$. Put $e^{-G_n - ih_n} = r_n e^{i\varphi_n}$. Then
\[ D_{F_n}(U(p)) = \frac{1}{2} \int_{\partial F_n} \frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} r_n d\varphi_n \leq M' \]

and
\[
\int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n = 2\pi U^2(0) + \int F_n 2 \text{ grad } U^2(p) \log r_n dr_n d\varphi_n \\
+ \frac{\partial U^2(r_n e^{i\varphi_n})}{\partial r_n} \log r_n r_n d\varphi_n \leq M',
\]

whence
\[
\int_{\partial F_n} U^2(r_n e^{i\varphi_n}) d\varphi_n \leq M'',
\]

where \( M'' \) is independent of \( n \).

Map \( F_n \to |\xi| < 1 \) and let \( G_\rho \) be the image of \( |z| < \rho \) by this mapping. Since the connectivity of \( F_n \) is finite, \( \partial F_n \) is mapped in \( |\xi| = 1 \) except possibly a set of linear measure zero, and further \( d\varphi_n \) corresponds to \( ds \) on \( |\xi| = 1 \), where \( ds \) consists of at most an enumerably infinite of arcs. Since \( U^2(\xi) \) is sub-harmonic, we have by the lemma

\[
M'' \geq \int_{|\xi| = 1} U^2(\xi) d\xi \geq \int_{|\xi| = 1} U^2(\xi) d\omega = \int_{|z| = \rho_n} U(\rho_n e^{i\theta}) d\theta
\]

Let \( \rho_n \to 1 \). Then by Fatou's theorem \( U(\rho) \) is represented by Poisson's integral in \( |z| < 1 \).

Let \( \bar{U}(\rho) : \rho \in F \) be a harmonic function of Dirichlet bounded. Then there exist subdomains \( G_i \) (\( i = 1, 2 \)) in \( F \) with the property as follows: there exist harmonic functions of Dirichlet bounded such that \( \bar{U}_i(\rho) \geq O \), \( \bar{U}_i(\rho) = O \) on \( \partial G_i \) and \( G_1 \cap G_2 = O \).

Let \( G \) be one of them and let \( \bar{U}(\rho) \) be one of \( \bar{U}_i(\rho) \) in the sequel. Map the universal covering surface \( G^\infty \) of \( G \) onto \( |z| < 1 \). Then there exists a constant \( \delta (\delta > 0) \) and a set \( E_\delta \) of positive measure on \( |z| = 1 \) such that \( \bar{U}(\rho) \) has angular limits larger than \( \delta \), because \( \bar{U}(\rho) \) is represented by Poisson's integral on \( |z| < 1 \). Let \( G' \) be the subdomain where \( \bar{U}(\rho) > \frac{\delta}{2} \). Then \( G' \) determines a set \( B^\delta \) of the ideal boundary.

Let \( V_{n+1}(\rho) \) be a harmonic function in \( (F_{n+1} \cap G) - (G' \cap (F_{n+1} - F_n)) = H_{n+1} \) such that \( V_{n+1}(\rho) = 0 \) on \( \partial G' + (\partial F_{n+1} \cap (G - G')) \) and \( V_{n+1}(\rho) = 1 \) on \( \partial F_n \cap G' \). Then \( V(\rho) = \lim_{n \to \infty} V_{n+1}(\rho) \geq \omega_\delta(z) \), where \( \omega_\delta(z) \) is the harmonic measure of \( E_\delta \), with respect to \( G^\infty \). Hence \( V(\rho) \) is non-
On the Existence of Harmonic Functions on Riemann Surfaces

constant. Let \( U_{n,n+i}(p) \) be a harmonic function in \( H_{n,n+i} \) such that \( U_{n,n+i}(p) = 0 \) on \( \partial G \), \( U_{n,n+i}(p) = 1 \) on \( (\partial G' \cap (F_{n+i} - F_n)) + (\partial F_n \cap G') \) and \( \frac{\partial U_{n,n+i}}{\partial n} = 0 \) on \( \partial F_{n+i} \backslash (G - G') \). It is clear

\[
D_G(U(p)) \geq D_{H_n,n+i}(U_{n,n+i}(p))
\]

and

\[
D_{H_n,n+i}(U_{n,n+i+1}(p)) \geq D_{H_n,n+i}(U_{n,n+i}(p)).
\]

Hence \( U_{n,n+i}(p) \) converges to \( U_n(p) \) in mean. Since \( U_{n,n+i}(p) \geq V_{n,n+i}(p) \), \( U(p) \) is non-constant.

Let \( G' \) be the niveau curve of \( U_n(p) \) with height \( \delta' \) (\( 0 < \delta' < 1 \)) and put

\[
 '\delta = \delta' - \varepsilon, \quad \delta = \varepsilon, \quad \delta = \varepsilon
\]

respectively.

Let \( U_{n,n+i}(p) \) be a harmonic function in \( G' \cap F_{n+i} \) such that \( U_{n,n+i}(p) = 0 \) on \( \partial G' \), \( U_{n,n+i}(p) = \delta' \) on \( G' \cap (F_{n+i} - F_n) + (\partial F_n \cap G') \) and \( \frac{\partial U_{n,n+i}}{\partial n} = 0 \) on \( \partial F_{n+i} \). Then we can prove as the previous manner the following

**Lemma.** \( U_{n,n+i}(p) \to U_n(p) \) in mean and

\[
\lim_{i \to \infty} D(U_{n,n+i}(p)) = D(U_n(p)).
\]

On the other hand, for given number \( \varepsilon \) and \( i \), we see easily that there exists a number \( j_0 \) such that

\[
D(U_{n,n+i}(p)) \leq D(U_{n,n+i+j}(p)) + \varepsilon < \delta' D(U_n(p)) + \varepsilon \quad \text{for } j \geq j_0.
\]

Let \( \varepsilon \to 0 \) and then \( i \to \infty \). Then we have

\[
D(U_n(p)) \leq \delta' D(U_n(p)).
\]

Let \( U''_{n,n+i}(p) \) be a harmonic function in \( G' \cap F_{n+i} \) such that \( U''_{n,n+i}(p) = 1 \) on \( \partial G' \), \( U''_{n,n+i}(p) = \delta' \) on \( (\partial G' \cap (F_{n+i} - F_n)) + (\partial F_n \cap G') \) and \( \frac{\partial U''_{n,n+i}}{\partial n} = 0 \) on \( \partial F_{n+i} \). Then by the same manner we have

\[
\lim_{i \to \infty} U''_{n,n+i}(p) = U_n''(p) = U_n(p) \quad \text{and} \quad D(U_n(p)) \leq (1 - \delta') D(U_n(p)),
\]

whence

\[
D(U_n(p)) = \delta' D(U_n(p)) \quad \text{on} \quad G - (G' \cap (F - F_n))
\]
Next we have by the same manner as the previous\(^0\) the following

**Lemma.**

\[ \int_{\partial G \cap F} \frac{\partial U_n}{\partial n} \, ds = D (U_n(p)) \bigg|_{G - (G \cap F_n)}, \]

for every number \( \epsilon \) except for at most two numbers \( \epsilon_1 \) and \( \epsilon_2 \).

We say that the sequence \( \{G' \cap (F - F_n)\} \) \( (n = 1, 2, \cdots) \) determines a set \( B_{G'} \) of the ideal boundary and call \( \lim_{\epsilon \to G'} D (U_n(p)) \) the capacity of \( B_{G'} \) with respect to \( G \). Then by the above lemmas, we can prove\(^0\) as the previous the following

**Theorem.** \( U_n(p) \) converges to \( U^0(p) \) in mean and

\[ D (U_n(p)) \bigg|_{G - (G \cap F_n)} \]

Since \( U^0(p) \geq V(p) \), \( U^0(p) \) is non-constant and since \( D_\partial (U(p)) \geq D_\partial (U^0(p)) \),

\[ \infty > \text{Cap}(B_{G'}) > 0. \]

**Proof of the Royden’s theorem.**

Let \( F^* \) be another Riemann surface such that \( F^* \ni p^* : p^* = T(p) : p \in F \), where \( T(p) \) is a quasi-conformal mapping whose dilatation quotient is bounded \( \leq K \). Put \( U^0_n(p) := U_n(T(p)) \). Then \( U^0_n(p) \) is not necessarily harmonic and

\[ \frac{1}{K} D_{H_{n+1}^0}(U_{n+1}(p)) \leq D_{T(H_{n+1}^0)}(U_{n+1}(p^*)) \leq KD_{H_{n+1}^0}(U_{n+1}(p)), \]

where \( T(G) \) is the image of \( G \) by the mapping \( p^* = T(p) \).

Let \( U_{n+1}(p^*) \) be a harmonic function in \( T(H_{n+1}) \) such that \( U_{n+1}(p^*) = 0 \) on \( \partial(T(G)) \), \( U_{n+1}(p^*) = 1 \) on \( \partial(T(F_{n+1}^0)) \) and \( \partial U_{n+1}(p^*) = 0 \) on \( \partial(T(F_{n+1}^0)) \). Then by the Dirichlet principle

\[ D_{T(H_{n+1})}(U_{n+1}(p^*)) \leq D_{T(H_{n+1})}(U_{n+1}(p^*)) \leq KD_{H_{n+1}^0}(U_{n+1}(p)). \]

Since the inverse mapping \( T^{-1}(p) \) of \( T(p) \) is also a quasi-conformal mapping,

\[ D_{H_{n+1}^0}(U_{n+1}(T^{-1}(p^*))) \leq D_{H_{n+1}^0}(U_{n+1}(T^{-1}(p^*))) \leq KD_{H_{n+1}^0}(U_{n+1}(T^{-1}(p))), \]

where \( U_{n+1}^0(T^{-1}(p^*)) = U_{n+1}^0(p) \). On the other hand, by the Dirichlet principle, we have
On the Existence of Harmonic Functions on Riemann Surfaces

\[ D_{H_n, n+1}(U_{n+1}(\mathcal{P})) \leq D_{H_n, n+1}(U_n(T^{-1}(\mathcal{P}))). \]

Hence

\[ D_{T; H_n, n+1}(U_{n+1}(\mathcal{P})) \leq \frac{1}{K} D_{H_n, n+1}(U_n(T^{-1}(\mathcal{P}))) \geq \frac{1}{K} D_{H_n, n+1}(U_n(\mathcal{P})). \]

\[ (U_{n+1}(\mathcal{P})). \]

\( T(G') \) also determines a set \( B_{TCG} \) of the ideal boundary of \( T(F) \).

Let \( i \to \infty \) and then \( n \to \infty \). Then

\[ KD_{G}(U^{0}(\mathcal{P})) \geq D_{T; G}(U^{0}(\mathcal{P})) \geq \frac{1}{K} D_{G}(U^{0}(\mathcal{P})). \]

Hence \( B_{TG'} \) is a set of positive capacity with respect to \( T(G) \) and

\[ \frac{1}{K} \text{Cap } B_{G} \leq \text{Cap } B_{TG'} \leq K \text{ Cap } B_{G}. \]

Thus we obtain our theorem by the method M. Parreau and A. Mori\(^ {100} \).

We construct an open Riemann surface \([ (G-G') \cap F_n \wedge ] \) by the process of the symmetrization with respect to \( \partial F_n \). Then \( (G-G') \cap F_n \) is a ring domain. Let \( n \to \infty \). Then \( (G-G') \cap F \) is a generalized semi-ring domain. Thus our theorem is an extension of the well known \( "\text{Modulsatz}" \).

Let \( F_0 \) be a compact set of \( F \) and let \( N_n(p, p_0) \) be a harmonic function in \( (F-F_0) \cap F_n \) such that \( N_n(p, p_0) = 0 \) on \( \partial F_0 \) and \( \frac{\partial N_n}{\partial n} = 0 \) on \( \partial F_n \) and has one logarithmic singularity at \( p_0 \). We can prove that \( N_n(p, p_0) \) converges to \( N(p, p_0) \), when \( n \to \infty \). Let \( \{ p_i \} \) be a sequence of points tending to the boundary. If a subsequence of \( \{ N(p, p_i) \} \) converges to \( N(p, \{ p'_i \}) \), we say that \( \{ p'_i \} \) determines an ideal point\(^ {100} \). We can introduce a topology of the ideal boundary by the use of \( N(p, \{ p_i \}) \) and prove that the above topology is invariant under the quasi-conformal mapping.

(Received March 17, 1955)
References


2) $O_{\text{HD}}$ and $O_{\text{HDN}}$ are the class of Riemann surfaces on which no harmonic function of Dirichlet bounded exists or the dimension of harmonic function of Dirichlet bounded is $N$ respectively.

3) In this article, we denote the relative boundary of $G$ by $\partial G$.


7), 8) and 9) See 6).

10) See 5).

11) A note on this topology will appear in this Journal.