# STUDIES ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR QUASILINEAR ORDINARY DIFFERENTIAL SYSTEMS

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（準線形常微分方程式系の解の漸近的挙動に関する研究）

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CONTENTS

# CHAPTER 1

# INTRODUCTION

## 1.1. Quasilinear Ordinary Differential Systems and Fixed Point Theorems.

Many important classes of engineering and scientific problems related to linear or nonlinear phenomena are often described by ordinary differential systems. For instance, automatic control, optimal control, adaptive control, and electrical networks have been investigated by the qualitative methods of ordinary differential systems.

In the case of linear systems, the fundamental theory has been already established ( e.g., Arimoto [69], Brockett [64], Cesari [65], Kalman [66], Kimura [71], Kodama-Suda [72], Narendra-Taylor [67], Zubov [68] ). Van der Pol's equation, Duffing's equation, and Lienard's equation appear in nonlinear problems of electric engineering and they have been studied for a long time. General nonlinear systems have been partly discussed by Liapunov's direct method ( Lefschetz [73], Narendra-Taylor [67], and Zubov [68] ). However, it seems that systematic analyses have not been established yet in the case of general nonlinear systems.

The purpose of this thesis is to extend several criteria for the existence and asymptotic behavior of solutions for nonlinear systems as well as to use a unified method via fixed point theorems. Especially, conditions obtained in this thesis are explicit and quantitative. Moreover, they can be easily applied to engineering and scientific problems.

It is often that weakly nonlinear problems are considered as linear systems and qualitative theory for linear systems are applied in order to deal with the above problems. However, the weakly nonlinear phenomena should be described by the following quasilinear ordinary differential system

(N) $$x' = A(t,x)x + F(t,x)$$

associated with the following linear system

(L) $$x' = B(t)x.$$

Here $A(t,x)$ and $F(t,x)$ are both continuous on $R \times R^n$, $R = (-\infty, +\infty)$, and $B(t)$ is a real $n \times n$ matrix continuous on $R$.

The above quasilinear system constitutes quite a large class. When the following system

(E) $$x' = f(t,x), \quad f(t,0) \equiv 0$$

is considered and $f(t,x)$ is differentiable in $x$ and the Jacobian matrix $f_x(t,x)$ is continuous on $R \times R^n$, then

$$f(t,x) = \left( \int_0^1 f_x(t,sx)\,ds \right) x.$$

The above system (E) becomes

(Q) $$x' = A(t,x)x, \quad \text{where } A(t,x) = \int_0^1 f_x(t,sx)\,ds.$$

Recently the above system (Q) has been investigated in order to discuss nonlinear control systems (Hirai-Ikeda[70], Kartsatos[27] Landau-Tomizuka [74] ).

In this thesis we shall give quantitative conditions how the quasilinear system (N) is close to the linear system (L) and give extended criteria obtained before. It seems that the explicit conditions which obtained by using a systematic method via fixed point theorems are easily applicable to engineering and scientific problems.

Let $S$ be a set and $V : S \to S$ a mapping. An $x \in S$ is called to be a fixed point of $V$, if $V(x) = x$. A great number of theorems concerning the existence of a fixed point of a mapping are obtained by various mathematical analysis (e.g., Cronin [17], Istratescu [25], and Zeidler [63] ). Well-known fixed point theorems are as follows : Brouwer's fixed point theorem ( Dunford-Schwartz [19] ), Schauder's fixed point theorem ( Smart [58] ), and the contraction principle, i.e., Banach's fixed point theorem ( Smart [58] ).

**Schauder's Fixed Point Theorem.** *Let $S$ be a convex closed set in a Banach space. Suppose that $V : S \to S$ is continuous and the image $V(S)$ is relatively compact. Then there exists at least one fixed point of $V$ in $S$.*

**The Contraction Principle.** *Let $S$ be a closed set in a complete metric space with a metric $\rho$: $S \times S \to [0,+\infty)$ and a mapping $V : S \to S$. Suppose that there exists a positive number $k < 1$ such that*

$$\rho(V(x),V(y)) \leq k\rho(x,y)$$

*for any $x$, $y \in S$. Then there exists one and only one fixed point of $V$ in $S$.*

Fixed point theorems are most useful to discuss the following qualitative properties of solutions for differential systems : the existence, uniqueness of solutions for initial value problems or boundary value problems ( Reissig-Sansone-Conti [45] and Sansone -Conti [57] ), the existence, uniqueness of periodic solutions ( Yoshizawa [61, 62]), and the continuation of solutions for initial value problems ( Lakshmikantham-Leela [38] ).

In the following chapters the above fixed point theorems play an important role.

In Chapter 2 an existence theorem of periodic solutions for a periodic quasilinear system (N), which is associated with a periodic linear system (L), is obtained by using Schauder's fixed point theorem. Under additional conditions an existence and uniqueness theorem of periodic solutions is given by applying the contraction principle.

In Chapter 3 the continuous dependence of periodic solutions on a parameter $\lambda$ for the following periodic quasilinear ordinary differential system

$$(N') \qquad x' = A(t,x,\lambda)x + \lambda F(t,x,\lambda) + f(t)$$

associated with the following periodic linear system

$$(L') \qquad x' = B(t)x + f(t)$$

is treated. By Schauder's fixed point theorem the existence of periodic solutions for (N') is shown. And under Lipschitz conditions the continuous dependence of periodic solutions on $\lambda$ for (N') is proved by the contraction principle.

Boundary value problems (N) with a nonlinear boundary condition

$N(x) = 0$ on a finite interval $J=[0,T]$, $T > 0$, are discussed in Chapter 4. Corresponding to (N) a linear boundary value problem (L) with a linear boundary condition $L(x) = 0$ is considered. Schauder's fixed point theorem leads to an existence theorem of solutions for (N). Under Lipschitz conditions the existence and uniqueness of a solution for (N) are given by using the contraction principle. In the case that $A(t,x) = B(t)$ and $N(x) = L(x) - c$ ($c \in R^n$), an existence theorem and a uniqueness theorem of solutions are obtained by Schauder's fixed point theorem and by the contraction principle, respectively. They are applied to two points boundary value problems.

In Chapter 5 boundary value problems (N) with a nonlinear boundary condition $N(x) = 0$ on an infinite interval are dealt with. Corresponding to (N) a linear problem (L) with a linear boundary condition $L(x) = 0$ is considered. The existence of solutions for (N) is shown by Schauder's fixed point theorem. The existence and uniqueness of a solution of (N) are proved by the contraction principle.

The stability of solutions for (N) with $F(t,0) \equiv 0$ is considered in Chapter 6. Sufficient conditions for the uniformly asymptotic stability in the large of the zero solution for (N) are obtained by using Schauder's fixed point theorem. The method of this chapter is quite different from others. Moreover sufficient conditions for the exponentially asymptotic stability in the large of the zero solution for (N) are given by constructing a Liapunov's function.

Finally, the discussion on the asymptotic equivalence between (N) and (L) is given in Chapter 7, showing the uniform stability of the zero solution or the uniform boundedness of solutions for (N).

## 1.2. Periodic Solutions of Periodic Systems.

The problem of the existence of $T$-periodic solutions for the following $T$-periodic quasilinear ordinary differential system

$$x' = A(t,x)x + F(t,x) \tag{N}$$

is discussed by Kartsatos [27], where $A(t,x)$ is a real $n\times n$ matrix continuous on $\mathbf{R}\times\mathbf{R}^n$ and $T$-periodic in $t$, and $F(t,x)$ is an $\mathbf{R}^n$-valued function continuous on $\mathbf{R}\times\mathbf{R}^n$ and $T$-periodic in $t$. Together with the above, the following linear system

$$x' = B(t)x \tag{L}$$

is concerned, where $B(t)$ is a real $n\times n$ matrix continuous and $T$-periodic. It is assumed that the above (L) satisfies the following hypothesis.

**Hypothesis 1.2.1.** *There exist no T-periodic solutions of* (L) *except for the zero solution.*

In the case of the system

$$x' = B(t)x + F(t,x), \tag{AL}$$

many authors assume that $F(t,x)$ satisfies Lipschitz condition with respect to $x$ and study the existence of $T$-periodic solutions by using Brouwer's fixed point theorem ( for example, [12 - 14], [20], [23], [37], and [57] ). Under the same condition on $F(t,x)$, the existence of $T$-periodic solutions for (AL) is proved by the implicit function theorem ( [17] ) or by the contraction principle ( [22] ). Without assuming the uniqueness of solutions for initial value problems Güssefeld [21], and Rouche and Mawhin [46] consider the existence of $T$-periodic solutions by applying Leray-Schauder's fixed point theorem.

In the case of (N), Lasota and Opial [40] consider some implicit and qualitative hypothesis correspoding to Hypothesis 1.2.1 : if $y$ is a continuous and $T$-periodic function, then $A(\cdot, y(\cdot))$ belongs to $S$ where $S$ is a compact subset of continuous and $T$-periodic matrices whose systems satisfy Hypothesis 1.2.1. Moreover they require the following condition that

$$\text{(2.1)} \qquad \liminf_{r \to +\infty} \frac{1}{r} \int_0^T \sup_{\|x\| \leq r} \|F(t,x)\| dt = 0.$$

The above condition is too strong to ensure the existence of periodic solutions for (N).

In the case of the system

$$\text{(N')} \qquad x' = A(t,x)x + f(t),$$

where $f(t)$ is continuous and $T$-periodic, Kartsatos [27] treats this subject. He considers a quantitative condition on $A(t,x)$ that there exists a sufficient small $\delta_0 > 0$ such that

$$\|A(t,x) - B\| \leq \delta_0$$

for $t \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^n$, where $B$ is a constant matrix and the system

$$\text{(L')} \qquad x' = Bx$$

satisfies Hypothesis 1.2.1.

In Chapter 2, instead of the linear system (L'), the linear system (L), which satisfies Hypothesis 1.2.1, is considered ( not necessarily independent of $t$ )( see [50, 53]). The existence and uniqueness of $T$-periodic solutions for (N) are treated under the condition that $A(t,x)$ is closed to $B(t)$ in the sense as follows:

**Hypothesis 1.2.2.** *There exist* $\delta > 0$ *and* $r > 0$ *such that*

$$\int_0^T \|A(s,x) - B(s)\| ds \leq \delta$$

*for* $\|x\| \leq r$.

Theorem 2.3.1 is an existence theorem of periodic solutions for periodic linear systems which are close to (L). In Theorem 2.3.2 sufficient conditions on $F(t,x)$ for the existence of $T$-periodic solutions for (N), which are weaker than (2.1), are obtained. Under explicit and quantitative conditions on $A(t,x)$, the existence of $T$-periodic solutions for (N) is proved by using Schauder's fixed point theorem. Theorem 2.4.1, under additional assumptions, ensures the uniqueness of a $T$-periodic solution for (N). Here the contraction principle is applied.

## 1.3. Continuous Dependence on a Parameter of Periodic Solutions.

Let $A(t,x,\lambda)$ be a real $n\times n$ matrix continuous on $\mathbf{R}\times\mathbf{R}^n\times[-\lambda_0,\lambda_0]$ with $T$-periodicity in $t$, $F(t,x,\lambda)$ an $\mathbf{R}^n$-valued function continuous on $\mathbf{R}\times\mathbf{R}^n\times[-\lambda_0,\lambda_0]$ with $T$-periodicity in $t$, where $\lambda_0 > 0$, and $f(t)$ an $\mathbf{R}^n$-valued function continuous on $\mathbf{R}$ and $T$-periodic in $t$. The $T$-

periodic quasilinear ordinary differential system

$$(N) \qquad x' = A(t,x,\lambda)x + \lambda F(t,x,\lambda) + f(t)$$

is considered associated with the linear system

$$(L) \qquad x' = B(t)x + f(t),$$

which satisfies the following hypothesis.

**Hypothesis 1.3.1.** *There exists one and only one T-periodic solution of* (L).

Here $B(t)$ is a real $n \times n$ matrix continuous on $\boldsymbol{R}$ and $T$-periodic.

A great number of works have been done on the existence and the continuous dependence on a parameter of periodic solutions for the quasilinear ordinary differential system containing a parameter under Hypothesis 1.3.1 ( see [10], [17], [18], [22], [23], [41], [42], [46] ). Especially, in the case where $A(t,x,\lambda) \equiv B(t)$ and $f(t) \equiv 0$, Cronin [17] considers the following system

$$x' = B(t)x + \lambda F(t,x,\lambda)$$

under the condition that $F(t,x,\lambda)$ satisfies Lipschitz condition with respect to $x$ and obtains existence theorems of periodic solutions for the large value of $\lambda$ by using the degree theory. When $A(t,x,\lambda) \equiv B(t)$, the dependence on $\lambda$ of $T$-periodic solutions of

$$(AL) \qquad x' = B(t)x + \lambda F(t,x,\lambda) + f(t)$$

is discussed in [17], [23], [41], and [46]. Under the condition that all the solutions for (AL) are uniquely determined, the existence of $T$-periodic solutions for (AL) is proved for sufficiently small $\lambda$ by using the implicit function theorem. Then sufficient conditions for the existence of periodic solutions are not necessarily given explicitly (see [17], [42]). When the Lipschitz condition on $F(t,x,\lambda)$ with respect to $x$ is satisfied, Hale [23] applies the contraction principle and deals with the continuous dependence on $\lambda$ of $T$-periodic solutions for (AL) under some additional assumptions. Rouche and Mawhin [46] investigate the continuous dependence on $\lambda$ of $T$-periodic solutions for (AL) by a different method of functional analysis. They suppose that $\partial F/\partial x$ is continuous and the value of $\sup\{ \|\partial F/\partial x(t,\pi(t),0)\| : t \in \boldsymbol{R} \}$ is given, where $\pi(\cdot)$ is a unique $T$-periodic solution for (L).

In Chapter 3, the dependence on $\lambda$ of $T$-periodic solutions for (N) is obtained under Hypothesis 1.3.1 ( see [49],[56] ). Moreover it is assumed that $A(t,x)$ is closed to $B(t)$ in the following sense.

**Hypothesis 1.3.2.** *There exist* $\lambda_1 > 0$ ( $\lambda_1 \leq \lambda_0$ ), $r > 0$, *and* $\delta > 0$ *such that*

$$\int_0^T \|A(s,x,\lambda) - B(s)\| ds \leq \delta$$

*for* $|\lambda| \leq \lambda_1$ *and* $\|x\| \leq r$.

Theorem 3.3.1 is an existence theorem of $T$-periodic solutions for periodic linear systems which are close to (L). Theorem 3.3.2, in which explicit sufficient conditions for the existence of

$T$-periodic solutions for (N) are shown without assuming the uniqueness of solutions for initial value problems, is a strict extention of the above result for the existence of $T$-periodic solutions for (AL). Theorem 3.3.3, where sufficient conditions that there exists at least one $T$-periodic solution for (N) tending to the $T$-periodic solution for (L) are given, is proved by Schauder's fixed point theorem. In Theorem 3.3.4, explicit and sufficient conditions on $A(t,x,\lambda)$ and $F(t,x,\lambda)$ ensure the continuous dependence on $\lambda$ of $T$-periodic solutions for (N).

## 1.4. Boundary Value Problems on a Finite Interval.

The following nonlinear boundary value problem of the quasilinear ordinary differential system

$$\text{(N)} \qquad x' = A(t,x)x + F(t,x)$$

$$\text{(C)} \qquad N(x) = 0$$

is dealt with, where $A(t,x)$ and $F(t,x)$ are both continuous on $J \times \boldsymbol{R}^n$, and $N : C(J) \to \boldsymbol{R}^n$ is a continuous operator (not necessarily linear). Let $J = [0,T]$, where $T > 0$ and $C(J)$ be the space of $\boldsymbol{R}^n$-valued functions continuous on $J$.

The boundary value problem ((N),(C)) is treated by functional analysis ( [6], [15], and [27] ). Many authors suppose qualitative conditions on $A(t,x)$. Moreover Anichini [1,2], Conti-Iannacci [11], Kartsatos [31,32], and Opial [44] discuss the existence of solutions of the boundary value problem ((N),(C)), respectively, under the condition that

$$\liminf_{n \to +\infty} \frac{1}{n} \int_0^T \sup_{\|x\| \leq n} \|F(s,x)\| ds = 0.$$

In Chapter 4, the nonlinear problem ((N),(C)) is considered associated with the following linear problem

(L) $$x' = B(t)x$$

(LC) $$\mathcal{L}(x) = 0$$

( see [55] ). It is assumed that the following hypothesis holds.

**Hypothesis 1.4.1.** *There exist no solutions of* ((L),(LC)) *except for the zero solution.*

In our results an explicit and quantitative condition on $A(t,x)$ is given as follows:

**Hypothesis 1.4.2.** *There exist* $\delta > 0$ *and* $r > 0$ *such that*

$$\int_0^T \|A(s,x) - B(s)\| ds \leq \delta$$

*for* $\|x\| \leq r$.

And also much weaker condition on $F(t,x)$ is assumed here.

In Theorem 4.3.1 the existence of solutions of linear problems which are close to ((L),(LC)) is shown by applying a different approach ( see [53] ). In Theorem 4.3.2 the existence of solutions of ((N),(C)) is proved by Schauder's fixed point theorem. Theorem

4.3.3 is an existence theorem of solutions of ((N),(C)), where $N(x) = \mathcal{L}(x) - c$ and $c \in \boldsymbol{R}^n$ is arbitrarily given. And we improve Theorem 2 in [44]. Theorem 4.3.4 ensures the existence and uniqueness of a solution of ((N),(C)) by using the contraction principle.

In section 4.4, the following linear boundary value problem

$$\text{(AL)} \qquad x' = B(t)x + F(t,x)$$

$$\text{(NH)} \qquad \mathcal{L}(x) = c$$

is discussed by a similar approach used in the above problem ((N),(C)). Here $B(t)$ is a real $n\times n$ matrix continuous on $J$, $\mathcal{L} : C(J) \to \boldsymbol{R}^n$ is a bounded linear operator and $c \in \boldsymbol{R}^n$.

In section 5.5, the above results for ((AL),(NH)) are applied to the following second order ordinary differential equation

$$\text{(E)} \qquad u'' = f(t,u,u')$$

with boundary conditions

$$\text{(BC)} \qquad \begin{aligned} \alpha_1 u(0) + \alpha_2 u'(0) &= c_1 \\ \beta_1 u(T) + \beta_2 u'(T) &= c_2, \end{aligned}$$

where $f : J\times\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ is a continuous function, $\alpha_i$, $\beta_i$, and $c_i \in \boldsymbol{R}$ ( $i = 1,2$ ).

### 1.5. Boundary Value Problems on an Infinite Interval.

The following boundary value problem of the quasilinear ordinary differential system

$$(N) \qquad x' = A(t,x)x + F(t,x)$$

$$(C) \qquad N(x) = 0$$

is considered, where $N : C_r^{lim} \to \mathbf{R}^n$ is a continuous operator( not necessarily linear ). Here $C_r^{lim} = \{ x \in C(\mathbf{R}^+) : \lim_{t\to+\infty} x(t)$ exists and $\|x(t)\| \le r$ for $t \in \mathbf{R}^+ \}$ and $\mathbf{R}^+ = [0,+\infty)$.

There are many studies on the qualitative theory of quasilinear ordinary differential systems. Avramescue [4], Kartsatos [31], Vidossich [59] discuss the existence of solutions under the conditions that the right-hand side of (N) is differentiable. In Kartsatos [26, 27,28,30,31,33], the problem ((N),(C)) is considered associated with the linear problem

$$(L) \qquad x' = B(t)x$$

$$(LC) \qquad \mathcal{L}(x) = 0$$

and various existence theorems for ((N),(C)) are obtained under strong conditions on $F(t,x)$, where $\mathcal{L} : C^{lim} \to \mathbf{R}^n$ is a bounded linear operator and $C^{lim} = \{ x \in C(\mathbf{R}^+) : \lim_{t\to+\infty} x(t)$ exists $\}$. For example, the following operators are considered.

$$\mathcal{L}(x) = Px(0) - Q\lim_{t\to+\infty} x(t); \quad N(x) = \mathcal{L}(x) - c.$$

Here $P$ and $Q$ are both known constant $n\times n$ matrices and $c$ ia a fixed $\mathbf{R}^n$-vector. Now it is assumed that the following hypotheses hold.

**Hypothesis 1.5.1.** $\int_0^{+\infty} \|B(s)\| ds < +\infty$.

**Hypothesis 1.5.2.** *There exist no solutions for ((L),(LC)) except for the zero solution.*

**Hypothesis 1.5.3.** *There exist two numbers $\delta > 0$ and $r > 0$, and a summable function $m_1$ such that*

$$\|A(t,x) - B(t)\| \leq m_1(t)$$

*for $t \in \mathbf{R}^+$ and $\|x\| \leq r$, and that*

$$\int_0^{+\infty} m_1(s)ds \leq \delta.$$

Under the condition that Hypotheses 1.5.1, 1.5.2, and 1.5.3 hold and that $A(t,x)$, $N(x)$ are sufficiently close to $B(t)$, $L(x)$ in some sense, respectively, Kartsatos [28] assumes some qualitative conditions on $A(t,x)$ in (N) and proves the existence of solutions for ((N),(C)). However, the qualitative condition in [28] is a necessary condition when $A(t,x)$ is sufficiently close to $B(t)$.

Several results for the existence and uniqueness of solutions for ((N),(C)) are obtained under Hypotheses 1.5.1, 1.5.2, and 1.5.3 by applying a different appproach used in [53, 54]. In Theorem 5.3.2 the existence of solutions for ((N),(C)) is shown by using Schauder's fixed point theorem. We obtain explicit and quantitative conditions corresponding to qualitative conditions on $A(t,x)$ in [28] and weaker conditions on $F(t,x)$ than those in [28]. Here it is not considered any differentiability of right-hand side of (N). Moreover assuming Lipschitz conditions, the existence and uniqueness of a solution for ((N),(C)) are given by using the contraction principle in Theorem 5.4.1.

## 1.6. Stability of Solutions.

A large number of results have been obtained for the stability of the zero solution of ordinary differential systems

(S) $\quad x' = X(t,x), \qquad X(t,0) \equiv 0$

and

(P) $\quad x' = X(t,x) + F(t,x), \qquad F(t,0) \equiv 0$

Here $X(t,x)$ and $F(t,x)$ are both $R^n$-valued functions continuous on $R^+ \times R^n$. It is assumed that the following hypothesis holds.

**Hypothesis 1.6.1.** *The zero solution of* (S) *is uniformly asymptotically stable.*

In order to investigate the asymptotic behavior of the zero solution for (P), the following four methods are effective : Liapunov's second method([62]), the invariance principle([39]), the method via fundamental matrices of solutions([60]), and that via fixed point theorems([3]).

When $X(t,x)$ is Lipschitz continuous in $x$, Liapunov functions corresponding to (S) can be constructed by using converse theorems. Without assuming Lipschitz conditions on $X(t,x)$, Athanassov [3] discusses sufficient conditions for the asymptotic behavior of solutions, not in the sense of Liapunov, by using Schauder's fixed point theorem.

When $X(t,x) = A(t,x)x$, Kartsatos [27] establishes various kinds of sufficient conditions for the asymptotic behavior of solutions for

(N) $$x' = A(t,x)x + F(t,x).$$

He utilizes the method via fundamental matrices of solutions. However, fundamental matrices of solutions can not be solved explicitly in general.

In Chapter 6, it is assumed that $X(t,x) = B(t)x$ and that Hypothesis 1.6.1 holds ([47],[75]). Then, since the following system

(L) $$x' = B(t)x$$

is linear, Hypothesis 1.6.1 holds if and only if the zero solution of (L) is exponentially asymptotically stable in the large (see [61]). Moreover it is assumed that the following hypothesis holds.

**Hypothesis 1.6.2.** *There exists a constant* $\delta > 0$ *such that*

$$\int_0^{+\infty} \sup_{\|x\| \leqq r} \|A(s,x) - B(s)\| ds \leq \delta$$

*for any* $r \geq 0$.

Under Hypotheses 1.6.1 and 1.6.2, sufficient conditions for the global stability of the zero solution for (N) in the sense of Liapunov, are obtained by Schauder's fixed point theorem and Liapunov's second method. Here it is not assumed that the right-hand side of (N) is Lipschitz continuous with respect to $x$. However, the following hypothesis is assumed.

**Hypothesis 1.6.3.** *All the solutions of* (N) *for initial value problems are uniquely determined.*

In section 6.3 we consider the systems with the perturbed term $F(t,x)$ which is treated by Lasota and Opial [40] and Opial [44]. In Theorem 6.3.1, a sufficient condition that the zero solution of (N) is uniformly asymptotically stable in the large is obtained by Schauder's fixed point theorem. In section 6.4, a similar approach to [53 - 56] is used and Theorem 6.4.1 for the exponentially asymptotic stability in the large of the zero solution is given by applying Liapunov's second method.

## 1.7. Asymptotic Equivalence.

The following two ordinary differential systems

(S) $$x' = X(t,x)$$

and

(P) $$x' = X(t,x) + F(t,x)$$

are said to be asymptotically equivalent, if there exists a solution of (P) approaching to a given solution of (S) as $t \to +\infty$ and vice versa. Here $X(t,x)$ and $F(t,x)$ are both continuous on $R^+ \times R^n$.

A great number of works have been done on this subject. Assuming various kinds of stability the following methods are used : the comparison principle ( [7] ), especially, the method by Liapunov functions ( [35,36] ), the method via fundemental matrices of solutions ( [8], [29], [43] ), and the method via fixed point theorems ( [34] ).

In the case that (S) is nonlinear, Liapunov functions play an important role under the conditions that $X(t,x)$ and $F(t,x)$ satisfy Lipschitz continuity with respect to $x$. Liapunov functions corresponding to (S) are constructed by using converse theorems in order to discuss the asymptotic equivalence between (S) and (P). When $X(t,x) = B(t)x$ and fundamental matrices of solutions for (S) are given, various sufficient conditions for the asymptotic equivalence are obtained. However,fundamental matrices of solutions can not be easily solved in genaral. In [8], under the condition that (S) is linear and the zero solution of (S) is conditionally stable, the asymptotic equivalece is discussed. In [34], the case where $X(t,x) = A(t,x)x$ is dealt with and the authors consider a condition on $A(\cdot,x(\cdot))$, where $x(\cdot)$ is an element of a given set of functions.

In Chapter 7, the following linear ordinary differential system (L) and quasilinear system (N)

(L) $$x' = B(t)x$$

(N) $$x' = A(t,x)x + F(t,x)$$

are considered ( [48] ). The asymptotic equivalence between (L) and (N) is treated under the following hypothesis.

**Hypothesis 1.7.1.** *There exists a constant* $K \geq 1$ *such that*

$$\|X_B(t)X_B^{-1}(\tau)\| \leq K$$

*for* $t \geq \tau \geq 0$, *where* $X_B$ *is the fundamental matrix of solutions for* (L) *such that* $X_B(0) = I$. *Here* $I$ *is the identity matrix.*

Hypothesis 1.7.1 holds (i.e., the zero solution of (L) is uniformly stable in the sense of Liapunov) if and only if all the solutions of (L) are uniformly bounded ( [23] ). It is assumed that $A(t,x)$ is sufficiently close to $B(t)$ in the following sense.

**Hypothesis 1.7.2.** *There exists a constant* $\delta > 0$ *such that*

$$\int_0^{+\infty} \sup_{\|x\| \leq r} \|A(s,x) - B(s)\| ds \leq \delta$$

*for any* $r \geq 0$.

Here $A(t,x)x$ and $F(t,x)$ are not necessarily Lipschitz continuous. However, the following hypothesis is considered.

**Hypothesis 1.7.3.** *All the solutions of* (N) *for initial value problems are uniquely determined.*

In Chapter 7, sufficient conditions for the asymptotic equivalence are given by a similar approach to [47] and [54] as well as by using Schauder's fixed point theorem. In Theorem 7.3.1, under a condition which is concerned with the integral of $F(t,x)$ in the neighborhood of the origin, the uniform stability of the zero solution for (N) and the asymptotic equivalence between (L) and (N) are obtained. In Theorem 7.3.2, the condition on $F(t,x)$, which is considered in [40] and [44], ensures the uniform boundedness of the solutions for (N) and the asymptotic equivalence between the two systems.

# NOTATIONS

The symbol $\|\cdot\|$ will be a norm in $\boldsymbol{R}^n$ and corresponding norm for $n\times n$ matrices. Let $J$ be a set in $\boldsymbol{R}$ and $C(J)$ the space of $\boldsymbol{R}^n$-valued functions bounded and continuous on $J$ with the supremum norm $\|\cdot\|_\infty$. $M(J)$ is the space of real $n\times n$ matrices bounded and continuous on $J$ with the norm

$$\|B\|_\infty = \sup\{ \|B(t)\| : t \in J \}, \quad B \in M(J).$$

We denote

$$S_r = \{ x \in \boldsymbol{R}^n : \|x\| \leq r \}, \quad r > 0$$

and

$$\|\mathcal{L}\| = \sup\{ \|\mathcal{L}(x)\| : \|x\|_\infty = 1 \},$$

where $\mathcal{L} : C(J) \to \boldsymbol{R}^n$ is bounded linear operator. We put $\boldsymbol{R}^+ = [0,+\infty)$.

In this thesis the following notations and definitions are used.

**Chapter 2.**

Let $J = [0,T]$, $T > 0$. A function $x \in C(\boldsymbol{R})$ is said to be $T$-periodic if $T$ is the smallest number such that $x(t+T) = x(t)$ for $t \in \boldsymbol{R}$. $C_T$ is the space of $T$-periodic functions with the supremum norm. We denote

$$C_{T,r} = \{ x \in C_T : \|x\|_\infty \leq r \}, \ r > 0.$$

**Chapter 3.**

Let $M_T$ be the space of real $n \times n$ continuous, $T$-periodic matrices with supremum norm. $C_{T,r}$ is the same as the above chapter.

**Chapter 4.**

$N : C_r(J) \to \mathbf{R}^n$ is a continuous operator(not necessarily linear), where $C_r(J) = \{ x \in C(J) : \|x\|_\infty \leq r \}$, $r > 0$, and $J = [0,T]$, $T > 0$.

**Chapter 5.**

$L : C^{lim} \to \mathbf{R}^n$ is a bounded linear operator and $N : C_r^{lim} \to \mathbf{R}^n$ is a continuous operator ( not necessarily linear ), where

$$C^{lim} = \{ x \in C(\mathbf{R}^+) : \lim_{t \to +\infty} x(t) \ exists \}$$

and

$$C_r^{lim} = \{ x \in C^{lim} : \|x\|_\infty \leq r \}, \quad r > 0.$$

**Chapter 6.**

In this chapter the following definitions of the asymptotic stability of solutions for

$$\text{(S)} \qquad x' = X(t,x)$$

are used. A solution through a point $(\tau,\xi)$ in $\mathbf{R}^+ \times \mathbf{R}^n$ will be denoted by $x(\cdot;\tau,\xi)$.

**Definition 1.** The zero solution of (S), with $X(t,0) \equiv 0$, is **uniformly stable,** if for any $\varepsilon > 0$ there eixsts an $\eta > 0$ such that when $\tau \geq 0$ and $\|\xi\| \leq \eta$,

$$\|x(t;\tau,\xi)\| \leq \varepsilon$$

for all $t \geq \tau$.

**Definition 2.** The solutions of (S) are **uniformly bounded**, if any $\alpha > 0$ there exists a $\beta > 0$ such that when $\tau \geq 0$ and $\|\xi\| \leq \alpha$,

$$\|x(t;\tau,\xi)\| \leq \beta$$

for all $t \geq \tau$.

**Definition 3.** The zero solution of (S) is **uniformly attractive in the large**, if for any $\varepsilon > 0$ and $\alpha > 0$ there exists a $T > 0$ such that when $\tau \geq 0$ and $\|\xi\| \leq \alpha$,

$$\|x(t;\tau,\xi)\| \leq \varepsilon$$

for all $t \geq \tau + T$.

**Definition 4.** The zero solution of (S), with $X(t,0) \equiv 0$, is **uniformly asymptotically stable in the large**, if it is uniformly stable, uniformly attractive in the large and the solutions of (S) are uniformly bounded.

**Definition 5.** The zero solution of (S), with $X(t,0) \equiv 0$, is **exponentially asymptotically stable in the large**, if there exists a $c > 0$ such that for any $\alpha > 0$ there exists an $M > 0$ satisfying

$$\|x(t;\tau,\xi)\| \leq M\|\xi\|\exp(-c(t-\tau))$$

for all $t \geq \tau$ and $\|\xi\| \leq \alpha$.

Let $V(t,x)$: $R^{+}\times R^{n} \to R$ be continuous on $R^{+}\times R^{n}$ and locally Lipschitz continuous in $x$. We define the following notation.

$$V'_{(S)}(t,x) = \limsup_{h \to +0} h^{-1}\{ V(t+h,x+hf(t,x)) - V(t,x) \}.$$

**Chapter 7.**

The two ordinary differential systems

(S) $$x' = X(t,x)$$

and

(P) $$x' = X(t,x) + F(t,x)$$

are said to be **asymptotically equivalent**, if there exists a solution $x$ of (P) such that

$$\|x(t) - x_1(t)\| \to 0 \quad ( t \to +\infty ),$$

for a given solution $x_1$ of (S), and vice versa.

The definitions of the uniform stability of the zero solution and the uniform boundedness of solutions are the same as those in Chapter 6.

# CHAPTER 2

# PERIODIC SOLUTIONS OF PERIODIC SYSTEMS

## 2.1. Introduction.

The problem of the existence and uniqueness of $T$-periodic solutions for the following $T$-periodic quasilinear ordinary differential system

(N) $\qquad x' = A(t,x)x + F(t,x)$

is discussed in this chapter, where $A(t,x)$ is a real $n\times n$ matrix continuous on $\boldsymbol{R}\times\boldsymbol{R}^n$ and $T$-periodic in $t$, and $F(t,x)$ is an $\boldsymbol{R}^n$-valued function continuous on $\boldsymbol{R}\times\boldsymbol{R}^n$ and $T$-periodic in $t$. Together with the above system, the following linear system

(L) $\qquad x' = B(t)x$

is concerned, where $B(t)$ is a real $n\times n$ matrix continuous on $\boldsymbol{R}\times\boldsymbol{R}^n$ and $T$-periodic. It is assumed that the following hypotheses hold.

**Hypothesis 2.1.** *There exist no T-periodic solutions for* (L) *except for the zero solution.*

**Hypothesis 2.2.** *There exist* $\delta > 0$ *and* $r > 0$ *such that*

$$\int_0^T \|A(s,x) - B(s)\|ds \leq \delta$$

*for* $x \in S_r$.

In section 2.3 the existence of $T$-periodic solutions for (N) is shown by using Schauder's fixed point theorem. In section 2.4 the existence and uniqueness of a $T$-periodic solution for (N) are proved by applying the contraction principle.

## 2.2. Preliminaries.

Let $J = [0,T]$. Consider a bounded linear opeator $\mathcal{L} : C(J) \to \boldsymbol{R}^n$ such that

$$\mathcal{L}(x) = x(0) - x(T)$$

with $\|\mathcal{L}\| = \sup\{ \|\mathcal{L}(x(\cdot))\| : \|x\|_\infty = 1 \}$, where $C(J)$ is the space of $\boldsymbol{R}^n$-valued functions continuous on $J$ with the supremum norm $\|\cdot\|_\infty$. Then we have $\|\mathcal{L}\| = 2$. Let $X_B$ be the fundamental matrix of solutions for (L) such that $X_B(0) = I$. Put

$$U_B = I - X_B(T),$$

we have

$$\mathcal{L}(X_B(\cdot)x_0) = U_B x_0 \quad ( x_0 \in \boldsymbol{R}^n ) .$$

We also put

$$K = \exp(\int_0^T \|B(s)\| ds).$$

Since $X_B(t)X_B^{-1}(\tau) = I + \int_\tau^t B(s)X_B(s)X_B^{-1}(\tau)ds$ for $t, \tau \in J$, we have

$$\|X_B(t)X_B^{-1}(\tau)\| \le 1 + \left|\int_\tau^t \|B(s)\|\|X_B(s)X_B^{-1}(\tau)\|ds\right|.$$

Thus, by applying Gronwall's lemma, it follows that

(2.1)
$$\|X_B(t)X_B^{-1}(\tau)\| \le K$$

for $t$, $\tau \in J$, which implies that

(2.2)
$$\|X_B(t)\| \le K \quad \text{and} \quad \|X_B^{-1}(t)\| \le K.$$

The following lemma concerned with $U_B$ holds.

**Lemma 2.1.** *The statements* (i) - (iii) *are equivalent mutually.*

(i) Hypothesis 2.1 *holds;*

(ii) $U_B$ *is nonsingular;*

(iii) *For any* $f \in C_T$ *, there exists a unique T-periodic solution of*

(2.3)
$$x' = B(t)x + f(t),$$

*where* $C_T$ *is the space of functions continuous on* $\mathbf{R}$ *and T-periodic.*

The above lemma can be proved in the same way as the proof of Lemma 5.2 in Chapter 5.

From the elementary result in linear algebra, the following lemma is obtained.

**Lemma 2.2.** *Suppose that* Hypothesis 2.1 *holds. Then there exists a constant number* $\rho$ $(0 < \rho < 1)$ *such that*

(2.4)
$$\|U_B^{-1}\| \le 1/\rho.$$

Such a $\rho$ will be fixed throughout this chapter.

## 2.3. Existence of Periodic Solutions.

In this section the existence of $T$-periodic solutions for (N) is discussed by Schauder's fixed point theorem. Now consider the following linear problem

$$(2.5) \qquad x' = A(t,y(t))x + F(t,y(t))$$

$$(2.6) \qquad \mathcal{L}(x) = 0$$

for $y \in C_{T,r}$, where

$$(2.7) \qquad C_{T,r} = \{ x \in C_T : \|x\|_\infty \leq r \}.$$

Let $X_y$ be the fundamental matrix solutions of the linear homogeneous system corresponding to (2.5) such that $X_y(0) = I$. The existence theorem of ((2.5),(2.6)) is obtained as follows:

**Theorem 2.3.1.** *Suppose that* Hypotheses 2.1 - 2.2 *hold and there exists a non-negative number C satisfying the following condition* (2.8).

$$(2.8) \qquad \int_0^T \|F(s,x)\| ds \leq rC \quad \text{for } x \in S_r.$$

*Let the following relations* (2.9) - (2.10) *hold.*

$$(2.9) \qquad K^2\delta\cdot\exp(\delta) \leq \frac{\rho}{2\|U_B^{-1}\|};$$

$$(2.10) \qquad C \leq \frac{\rho(1-\rho)}{\{2K\cdot\exp(\delta) + \rho(1-\rho)\}K\cdot\exp(\delta)}.$$

*Then for any* $y \in C_{T,r}$, *there exists a nonsingular matrix* $U_y$ *such that*

$$(2.11) \qquad L(X_y(\cdot)x_0) = U_y x_0$$

*for* $x_0 \in \mathbf{R}^n$, *whose inverse satisfies*

$$(2.12) \qquad \|U_y^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho)\ \},$$

*and there exists one and only one solution* $x_y \in C_{T,r}$ *such that*

$$(2.13) \qquad x_y(t) = -\ U_y^{-1}[L(p_y)] + \int_0^t A(s,y(s))x_y(s)ds + \int_0^t F(s,y(s))ds,$$

*where* $\quad p_y(t) = \int_0^t X_y(t)X_y^{-1}(s)F(s,y(s))ds$ *for* $t \in J$.

*Proof.* In the same way as (2.1) it can be seen that

$$(2.14) \qquad \|X_y(t)X_y^{-1}(\tau)\| \leq K\cdot\exp(\delta)$$

for $t,\tau \in J$, so that

$$(2.15) \qquad \|X_y(t)\| \leq K\cdot\exp(\delta) \quad \text{and} \quad \|X_y^{-1}(t)\| \leq K\cdot\exp(\delta).$$

Since $X_y(t) = X_B(t) + X_B(t)\int_0^t X_B^{-1}(s)\{\ A(s,y(s))-B(s)\ \}X_y(s)ds$ for $t \in J$, by (2.1), Hypothesis 2.2, (2.9), and (2.15), we have

$$\|X_y(t) - X_B(t)\| \leq K^2\delta\cdot\exp(\delta)$$
$$\leq \rho/(2\|U_B^{-1}\|).$$

Then it follows that

$$\|(U_y - U_B)x_0\| = \|\mathcal{L}(X_y(\cdot) - X_B(\cdot))x_0\|$$
$$\le \rho\|x_0\|/\|U_B^{-1}\|$$

for $x_0 \in \boldsymbol{R}^n$. From (2.4) we get

$$\|U_y x_0\| \ge \rho(\ 1 - \rho\ )\|x_0\|.$$

Hence $U_y$ has the inverse satisfying (2.12).

By Lemma 2.1 the problem ((2.5),(2.6)) has one and only one solution $x_y$ satisfying (2.13). Since (2.5) is $T$-periodic and the uniqueness of solutions of (2.5) for initial value problems, $x_y$ is $T$-periodic solution for (2.5). We shall show that $\|x_y(t)\| \le r$ for $t \in J$. From (2.8), (2.12), (2.13) and (2.14) we obtain

$$\|x_y(t)\| \le \frac{2rCK\exp(\delta)}{\rho(\ 1 - \rho\ )} + rC + \int_0^t \|A(s,y(s))\|\|x_y(s)\|ds$$

for $t \in J$. By applying Gronwall's lemma and (2.10) we have

$$\|x_y(t)\| \le \left(\frac{2rCK\exp(\delta)}{\rho(\ 1 - \rho\ )} + rC\right)\exp\left(\int_0^t \|A(s,y(s))\|ds\right)$$
$$\le r.$$

Thus $x_y \in C_{T,r}$. This completes the proof. **Q.E.D.**

By applying Schauder's fixed point theorem we obtain the following theorem.

**Theorem 2.3.2.** *Suppose that the same assumption as* Theorem 2.3.1. *Then there exists at least one T-periodic solution for* (N) *which belongs to* $C_{T,r}$.

*Proof.* It is easy to show that the solution of ((2.5),(2.6)) can be expressed by

$$x_y(t) = -X_y(t)U_y^{-1}[L(p_y)] + p_y(t)$$

for $t \in J$, where $y \in C_{T,r}$. Define $V: C_{T,r} \to C_{T,r}$ by $V(y) = x_y$. Then $V$ maps the convex closed set $C_{T,r}$ into itself. It is easily seen that the compactness and continuity of $V$ are proved in the same way as the proof of Theorem 3.3.2 in Chapter 3.

According to Schauder's fixed point theorem, $V$ has at least one fixed point in $C_{T,r}$. Therefore there exists at least one solution of (N) and this completes the proof. **Q.E.D.**

## 2.4. Existence and Uniqueness of Periodic Solutions.

When $A(t,x)$, $F(t,x)$ satisfy Lipschitz conditions, respectively, the existence and uniqueness of a $T$-periodic solution for (N) are shown by the contraction principle as follows:

**Theorem 2.4.1.** *Suppose, under the assumptions in* Theorem 2.3.1, *that there exist a positive number* $L_1$ *such that*

$$\|A(t,x_1) - A(t,x_2)\| \leq L_1\|x_1 - x_2\|$$

*and*

$$\|F(t,x_1) - F(t,x_2)\| \leq L_1\|x_1 - x_2\|$$

*for* $t \in J$, $x_1, x_2 \in S_r$. *Let* $L_1$ *satisfy the following inequality.*

$$(2.17) \qquad L_1\left(\frac{b_1K^2\exp(2\delta)}{\rho(1-\rho)} + b_2\right)\left(1 + \frac{2K\exp(\delta)}{\rho(1-\rho)}\right)$$

$$+ \frac{2K\exp(\delta)}{\rho(1-\rho)} < 1,$$

*where* $b_1 = 2rCK\exp(\delta)$ *and* $b_2 = (T + rCK\exp(\delta))K\exp(\delta)$. *Then there exists one and only one T-periodic solution for* (N) *which belongs to* $C_{T,r}$.

*Proof.* Let $k$ be the left-hand side of (2.17). We shall show that the operator $V : C_{T,r} \to C_{T,r}$ such that

$$(2.18) \qquad [V(y)](t) = -X_y(t)U_y^{-1}[L(p_y)] + p_y(t)$$

is a contraction. Let $y_1, y_2 \in C_{T,r}$ and let $X_1, X_2$ be fundamental matrices of the following linear systems, respectively.

$$x' = A(t, y_1(t))x \quad ; \quad x' = A(t, y_2(t))x.$$

By the same argument as (2.1), we obtain for $t, \tau \in J$

$$\|X_i(t)X_i^{-1}(\tau)\| \leq K\exp(\delta) \quad (i = 1, 2),$$

which yields

$$\|p_i\|_\infty \leq rCK\exp(\delta),$$

where

$$p_i(t) = X_i(t)\int_0^t X_i^{-1}(s)F(s, y_i(s))ds$$

for $t \in J$. Since

$$[X_1(t)]' = A(t,y_2(t))X_1(t) + [\ A(t,y_1(t)) - A(t,y_2(t))\ ]X_1(t),$$

by the variation of parameters formula, we obtain for $t,\tau \in J$

$$X_1(t) = X_2(t)X_2^{-1}(\tau)X_1(\tau)$$
$$+ \int_\tau^t X_2(t)X_2^{-1}(s)[\ A(s,y_1(s)) - A(s,y_2(s))\ ]X_1(s)ds,$$

then

$$\|X_1(t)X_1^{-1}(\tau) - X_2(t)X_2^{-1}(\tau)\| \le L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

Hence we get

$$\|X_1 - X_2\|_\infty \le L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty$$

and

$$\|X_1^{-1} - X_2^{-1}\|_\infty \le L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

Let $U_i$ be matrices such that

$$\mathcal{L}(X_i(\cdot)x_0) = U_ix_0 \quad (i = 1,2)$$

for $x_0 \in \boldsymbol{R}^n$. We have

$$\|U_1 - U_2\| \le \|\mathcal{L}\|\|X_1 - X_2\|_\infty$$
$$\le 2L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

By the same argument as (2.12) it follows that

$$\|U_i^{-1}\| \le 1/\{\ \rho(\ 1 - \rho\ )\},$$

so that

$$\|U_1^{-1} - U_2^{-1}\| \le \|U_1^{-1}\|\|U_1 - U_2\|\|U_2^{-1}\|$$

$$\leq \left( \frac{2L_1 K^2 \exp(2\delta)}{\rho^2 (1-\rho)^2} \right) \|y_1 - y_2\|_\infty .$$

Moreover it follows, for $t \in J$, that

$$\|p_1(t) - p_2(t)\| \leq \int_0^T \|X_1(t)X_1^{-1}(s) - X_2(t)X_2^{-1}(s)\| \|F(s,y_1(s))\| ds$$

$$+ \int_0^T \|X_2(t)X_2^{-1}(s)\| \|F(s,y_1(s)) - F(s,y_2(s))\| ds$$

$$\leq L_1 b_2 \|y_1 - y_2\|_\infty .$$

From (2.18) it follows, for $t \in J$, that

$$\|[V(y_1)](t) - [V(y_2)](t)\| \leq \|X_1 - X_2\|_\infty \|U_1^{-1}\| \|\mathcal{L}\| \|p_1\|_\infty$$

$$+ \|X_2\|_\infty \|U_1^{-1} - U_2^{-1}\| \|\mathcal{L}\| \|p_1\|_\infty$$

$$+ \|X_2\|_\infty \|U_2^{-1}\| \|\mathcal{L}\| \|p_1 - p_2\|_\infty$$

$$+ \|p_1 - p_2\|_\infty$$

$$\leq \left( \frac{b_1 L_1 K^2 \exp(2\delta)}{\rho(1-\rho)} \right) \|y_1 - y_2\|_\infty + \left( \frac{\|\mathcal{L}\| b_1 L_1 K^3 \exp(3\delta)}{\rho^2 (1-\rho)^2} \right) \|y_1 - y_2\|_\infty$$

$$+ \left( \frac{K \exp(\delta)}{\rho(1-\rho)} \right) ( \|\mathcal{L}\| + \|\mathcal{L}\| L_1 b_2 ) \|y_1 - y_2\|_\infty + L_1 b_2 \|y_1 - y_2\|_\infty$$

$$= k \|y_1 - y_2\|_\infty .$$

Thus, from (2.17), $V$ is a contraction. This completes the proof.

**Q.E.D.**

# CHAPTER 3

## CONTINUOUS DEPENDENCE ON A PARAMETER OF PERIODIC SOLUTIONS

### 3.1. Introduction.

Let $A(t,x,\lambda)$ be a real $n\times n$ matrix continuous on $R\times R^n\times[-\lambda_0,\lambda_0]$ with $T$-periodicity in $t$, $F(t,x,\lambda)$ an $R^n$-valued function continuous on $R\times R^n\times[-\lambda_0,\lambda_0]$ with $T$-periodicity in $t$, where $\lambda_0 > 0$, and $f(t)$ an $R^n$-valued function continuous on $R$ and $T$-periodic in $t$. The $T$-periodic quasilinear ordinary differential system

(N) $$x' = A(t,x,\lambda)x + \lambda F(t,x,\lambda) + f(t)$$

is considered associated with the linear system

(L) $$x' = B(t)x + f(t),$$

which satisfies the following hypothesis.

**Hypothesis 3.1.** *There exists one and only one T-periodic solution for* (L).

Here $B(t)$ is a real $n\times n$ matrix continuous on $R$ and $T$-periodic. And it is assumed that $A(t,x)$ is closed to $B(t)$ in the following sense.

**Hypothesis 3.2.** *There exist* $\lambda_1 > 0$ $(\lambda_1 \leq \lambda_0)$, $r > 0$, *and* $\delta > 0$ *such that*

$$\int_0^T \|A(s,x,\lambda) - B(s)\|ds \leq \delta$$

*for* $x \in S_r$ *and* $|\lambda| \leq \lambda_1$.

In this chapter the following properties of $T$-periodic solutions for (N) are considered by Schauder's fixed point theorem and the contraction principle : the existence, the dependence on $\lambda$, and the continuous dependence on $\lambda$.

## 3.2. Preliminaries.

Define a bounded linear operator $\mathcal{L}$: $C[0,T] \to \boldsymbol{R}^n$ by

$$\mathcal{L}(x(\cdot)) = x(0) - x(T)$$

with the norm

$$\|\mathcal{L}\| = \sup\{ \|\mathcal{L}(x(\cdot))\| : \|x\|_\infty = 1 \}.$$

Then we have $\|\mathcal{L}\| = 2$. Let $X_B$ be the fundamental matrix of solutions for the homogeneous system corresponding to (L) such that $X_B(0) = I$, where $I$ is the identity matrix. Put $U_B = I - X_B(T)$. We have

$$\mathcal{L}(X_B(\cdot)x_0) = U_B x_0$$

for $x_0 \in \boldsymbol{R}^n$. The following lemmas are well known.

**Lemma 3.1.** Hypothesis 3.1 *is equivalent to* det $U_B \neq 0$ (*see*[23]).

**Lemma 3.2.** *If* det $U_B \neq 0$, *then we can choose a positive constant* $\rho$ ( $0 < \rho < 1$ ) *such that*

$$\|U_B^{-1}\| \leq 1/\rho. \tag{3.1}$$

A positive constant satisfying (3.1) will be fixed throughout this chapter. From the above lemmas the following lemma is obtained.

**Lemma 3.3.** *Suppose that* Hypothesis 3.1 *holds. Let* $\pi(\cdot)$ *be the* *T-periodic solution of* (L). *Then*

$$\|\pi\|_\infty \leq MK(\ 1 + 2K/\rho\ ),$$

*where* $M = \int_0^T \|f(s)\| ds$ *and* $K = \exp(\int_0^T \|B(s)\| ds)$.

*Proof.* Since $X_B(t)X_B^{-1}(\tau) = I + \int_\tau^t B(s)X_B(s)X_B^{-1}(\tau)ds$, by using Gronwall's lemma, it follows that

$$(3.2) \qquad \|X_B(t)X_B^{-1}(\tau)\| \leq K$$

for $t,\tau \in [0,T]$, which implies that $\|X_B(t)\| \leq K$ for $t \in [0,T]$.

A solution $x(\cdot)$ of (L) is $T$-periodic if and only if $\mathcal{L}(x(\cdot)) = 0$, so that

$$\pi(t) = -X_B(t)U_B^{-1}[\mathcal{L}(p(\cdot))] + p(t),$$

where $p(t) = X_B(t)\int_0^t X_B^{-1}(s)f(s)ds$ for $t \in \boldsymbol{R}$, with $\|p(t)\| \leq KM$ for $t \in [0,T]$. Therefore

$$\|\pi(t)\| \leq \|X_B(t)\|\|U_B^{-1}\|\|\mathcal{L}\|KM + \|p(t)\|$$
$$\leq KM(\ 1 + 2K/\rho\ ).$$

This completes the proof. **Q.E.D.**

In this chapter it is assumed that

$$r > r_0 = KM(\ 1 + 2K/\rho\ ).$$

### 3.3. Continuous Dependence on a Parameter of Periodic Solutions.

Consider the following periodic linear nonhomogeneous system

$$x' = A(t,y(t),\lambda)x + \lambda F(t,y(t),\lambda) + f(t) \tag{3.3}$$

for $y \in C_{T,r}$, together with a boundary condition

$$\mathcal{L}(x(\cdot)) = 0. \tag{3.4}$$

Here

$$C_{T,r} = \{\ x \in C_T\ :\ \|x\|_\infty \le r\ \},$$

and $C_T$ is the space of $T$-periodic functions of $C(\mathbf{R})$.
Put $U_y = I - X_y(T)$, where $X_y$ is the fundamental matrix of solutions of the linear homogeneous system corresponding to (3.3) such that $X_y(0) = I$. We have

$$\mathcal{L}(X_y(\cdot)x_0) = U_y x_0$$

for $x_0 \in \mathbf{R}^n$. By an analogous argument in Theorem 2.3.1 in Chapter 2 the following theorem is obtained.

**Theorem 3.3.1.** *Suppose that* Hypotheses 3.1 *and* 3.2 *holds and that there exist numbers* $\delta \ge 0$, $\Delta \ge 0$ *and* $\lambda_1 > 0$, $\lambda_1 \le \lambda_0$, *satisfying the conditions* (3.5) - (3.6) *below.*

$$(3.5) \qquad K^2\delta\exp(\delta) \le \frac{\rho}{2\|U_B^{-1}\|};$$

$$(3.6) \qquad \{ \lambda_1\Delta + M \}K\exp(\delta)\left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right] \le r.$$

*Let $F$ satisfy the condition* (3.7).

$$(3.7) \qquad \int_0^T \|F(s,x,\lambda)\|ds \le \Delta \quad for \quad x \in S_r, \ \lambda \in \Lambda_1.$$

*Here $\Lambda_1 = [-\lambda_1, \lambda_1]$. Then, for $y \in C_{T,r}$ and $\lambda \in \Lambda_1$, there exists the inverse of $U_y$ such that*

$$(3.8) \qquad \|U_y^{-1}\| \le 1/\{ \rho(1-\rho) \}$$

*and there exists one and only one solution $x_y \in C_{T,r}$ of* ((3.3),(3.4)) *such that*

$$x_y(t) = - U_y^{-1}[L(p_y(\cdot))] + \int_0^t A(s,y(s),\lambda)x_y(s)ds$$

$$(3.9) \qquad + \lambda\int_0^t F(s,y(s),\lambda)ds + \int_0^t f(s)ds,$$

*where* $p_y(t) = X_y(t)\int_0^t X_y^{-1}(s)\{ \lambda F(s,y(s),\lambda) + f(s) \}ds \quad for \ t \in \boldsymbol{R}.$

*Proof.* By the same argument used in (3.2), it can be seen that

(3.10) $$\|X_y(t)X_y^{-1}(\tau)\| \leq K\exp(\delta)$$

for $t$, $\tau \in [0,T]$. Since, by the variation of parameters formula,

$$X_y(t) = X_B(t) + X_B(t)\int_0^t X_B^{-1}(s)\{\ A(s,y(s),\lambda) - B(s)\ \}X_y(s)ds,$$

we obtain, from (3.2), Hypothesis 3.2, and (3.10),

$$\|X_y(t) - X_B(t)\| \leq \int_0^t \|X_B(t)X_B^{-1}(s)\|\|A(s,y(s),\lambda) - B(s)\|\|X_y(s)\|ds$$

$$\leq K^2\delta\exp(\delta)$$

for $t \in [0,T]$. This yields, by (3.5),

$$\|X_y(t) - X_B(t)\| \leq \rho/(\ 2\|U_B^{-1}\|\ ),$$

so that

$$\|(U_B - U_y)x_0\| \leq \rho\|x_0\|/\|U_B^{-1}\|$$

for $x_0 \in \boldsymbol{R}^n$. From (3.1) we have

$$\|U_y x_0\| \geq \rho(\ 1 - \rho\ )\|x_0\|.$$

Hence $U_y$ has the inverse and (3.8) holds.

The problem ((3.3),(3.4)) has a unique solution $x_y$ satisfying (3.9). Since $L(x_y(\cdot)) = 0$, $x_y$ belongs to $C_T$. We shall show that $\|x_y\|_\infty \leq r$. By the definition of $p_y$ we obtain

$$\|p_y(t)\| \leq K(\ \lambda_1\Delta + M\ )\exp(\delta)$$

for $t \in [0,T]$. It follows, from (3.7) and (3.9), that

$$\|x_y(t)\| \leq \frac{2K(\ \lambda_1\Delta + M\ )\exp(\delta)}{\rho(\ 1 - \rho\ )} + \int_0^t \|A(s,y(s),\lambda)\|\,\|x_y(s)\|ds + \lambda_1\Delta + M.$$

By using Gronwall's lemma, (3.6) and Hypothesis 3.2, we have

$$\|x_y(t)\| \leq (\ \lambda_1\Delta + M\ )\left[\ 1 + \frac{2K\exp(\delta)}{\rho(\ 1 - \rho\ )}\right]\exp(\ \int_0^t \|A(s,y(s),\lambda)\|ds\ )$$

$$\leq (\ \lambda_1\Delta + M\ )\left[\ 1 + \frac{2K\exp(\delta)}{\rho(\ 1 - \rho\ )}\right]K\exp(\delta)$$

$$\leq r.$$

Hence $x_y$ belongs to $C_{T,r}$. This completes the proof. **Q.E.D.**

By applying Schauder's fixed point theorem, an existence theorem of $T$-periodic solutions of (N) is obtained without assuming the uniqueness of solutions of (N) for initial value problems.

**Theorem 3.3.2.** *Suppose that the same assumption as* Theorem 3.3.1 *holds. Then, for any* $\lambda \in \Lambda_1$, *there exists at least one $T$-periodic solution of* (N).

*Proof.* Let $\lambda \in \Lambda_1$. From Theorem 3.3.1 we can define an operator $\mathcal{P} : C_{T,r} \to C_{T,r}$ by $\mathcal{P}(y) = x_y$, where $x_y$ is a unique $T$-periodic solution of (3.3) in $C_{T,r}$. We shall show that $\mathcal{P}$ is a continuous compact operator. For the continuity of $\mathcal{P}$ it suffices to prove that

$$\mathcal{P}(y_n) \to \mathcal{P}(y_0)\ (\ n \to \infty\ ),$$

as $y_n \to y_0$ ( $n \to \infty$ ) in $C_{T,r}$. Let $X_n$ ,$X_0$ be the fundamental matrices of solutions for the following linear systems, respectively.

$$x' = A(t, y_n(t), \lambda)x \quad ; \quad x' = A(t, y_0(t), \lambda)x.$$

It follows that

$$\|X_n(t)X_n^{-1}(\tau)\| \leq K\exp(\delta) \text{ and } \|X_0(t)X_0^{-1}(\tau)\| \leq K\exp(\delta)$$

for $t$, $\tau \in [0,T]$. By the variation of parameters formula we have

$$X_n(t) - X_0(t) = X_0(t)\int_0^t X_0^{-1}(s)\{A(s,y_0(s),\lambda)-A(s,y_n(s),\lambda)\}X_n(s)ds.$$

Then we obtain

$$\|X_n - X_0\|_\infty \leq K^2\exp(2\delta)\int_0^T \|A(s,y_n(s),\lambda) - A(s,y_0(s),\lambda)\|ds,$$

which implies that

(3.11) $$X_n \to X_0 \quad (n \to \infty).$$

Put $U_n = I - X_n(T)$ and $U_0 = I - X_0(T)$. Then

$$\|U_n - U_0\| \to 0 \quad (n \to \infty).$$

From Theorem 3.3.1 we get

$$\|U_n^{-1}\| \leq 1/\{\rho(1 - \rho)\} \text{ and } \|U_0^{-1}\| \leq 1/\{\rho(1 - \rho)\}.$$

We have

$$\|U_n^{-1} - U_0^{-1}\| \leq \|U_n - U_0\|/\{\rho^2(1 - \rho)^2\}.$$

Thus

$$\|U_n^{-1} - U_0^{-1}\| \to 0 \quad ( n \to \infty ).$$

Since

$$X_n^{-1}(t)-X_0^{-1}(t)=[\int_0^t X_n^{-1}(s)\{A(s,y_0(s),\lambda)-A(s,y_n(s),\lambda)\}X_0(s)ds]X_0^{-1}(t),$$

in the same way as (3.11), we obtain

$$X_n^{-1} \to X_0^{-1} \quad ( n \to \infty ). \tag{3.12}$$

Let
$$p_n(t) = X_n(t)\int_0^t X_n^{-1}(s)F(s,y_n(s),\lambda)ds$$

and
$$p_0(t) = X_0(t)\int_0^t X_0^{-1}(s)F(s,y_0(s),\lambda)ds.$$

Since all the following sequences $\{X_n(\cdot)\}$, $\{X_n^{-1}(\cdot)\}$, $\{F(\cdot,y_n(\cdot),\lambda)\}$ are uniformly convergent in $[0,T]$ as $n \to \infty$, respectively,

$$p_n \to p_0 \quad ( n \to \infty ).$$

Hence

$$\mathcal{P}(y_n) \to \mathcal{P}(y_0) \quad ( n \to \infty ).$$

In order to prove the compactness of $\mathcal{P}$ it suffices to show that the image $\mathcal{P}( C_{T,r} )$ is uniformly bounded and equicontinuous. By the definition of $\mathcal{P}$ we obtain

$$(3.13) \qquad \|\mathcal{P}(y)\|_\infty = \|x_y\|_\infty \leq r$$

for $y \in C_{T,r}$. Thus $\mathcal{P}( C_{T,r} )$ is uniformly bounded.

From (3.9) and (3.13) it follows that

$$\|[\mathcal{P}(y)](t_1) - [\mathcal{P}(y)](t_2)\|$$

$$\leq \left|\int_{t_1}^{t_2} \|A(s,y(s),\lambda)\| r ds\right| + |\lambda| \left|\int_{t_1}^{t_2} \|F(s,y(s),\lambda)\| ds\right|$$

for $t_1$, $t_2 \in [0,T]$, which implies that $\mathcal{P}( C_{T,r} )$ is equicontinuous.

According to Ascoli-Arzela's theorem $\mathcal{P}( C_{T,r} )$ is a relatively compact subset in $C_T$. Therefore $\mathcal{P}$ is a compact continuous operator from $C_{T,r}$ into $C_{T,r}$. And also $C_{T,r}$ is a convex closed subset in $C_T$. By using Schauder's fixed point theorem $\mathcal{P}$ has at least one fixed point $x$ in $C_{T,r}$, i.e., $x$ satisfies

$$x'(t) = A(t,x(t),\lambda)x(t) + \lambda F(t,x(t),\lambda) + f(t)$$

for $\lambda \in \Lambda_1$. This completes the proof. **Q.E.D.**

Suppose that the following hypothesis holds.

**Hypothesis 3.3.** *There exists a continuous and strictly increasing function* $\mu : [0,\lambda_2] \to \mathbf{R}^+$, $0 < \lambda_2 \leq \lambda_1$, *such that* $\mu(0) = 0$ *and that for* $(t,x,\lambda) \in [0,T]\times S_r \times \Lambda_2$

$$\|A(t,x,\lambda) - B(t)\| \leq \mu(|\lambda|),$$

*where* $\Lambda_2 = [-\lambda_2, \lambda_2]$ *and* $\mathbf{R}^+ = [0,+\infty)$.

Then we have the following theorem.

**Theorem 3.3.3.** *If, under the assumption in* Theorem 3.3.2, Hypothesis 3.3 *holds, then for any* $\varepsilon > 0$ *there exists an* $\eta(\varepsilon) > 0$ *such that for all* $\lambda$, $|\lambda| \leq \eta(\varepsilon)$, *there exists at least one T-periodic solution* $x(\cdot;\varepsilon,\lambda)$ *of* (N) *satisfying*

$$\|x(t;\varepsilon,\lambda) - \pi(t)\| \leq \varepsilon \tag{3.14}$$

*for* $t \in \mathbf{R}$.

*Proof.* Choose $\varepsilon$ such that $0 < \varepsilon < r - r_0$. Let $\eta = \eta(\varepsilon) > 0$ satisfy the following inequality.

$$\{ r_0 T\mu(\eta) + \eta\Delta \} K\exp(\delta) \left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right] \leq \varepsilon. \tag{3.15}$$

We denote $C_{T,\varepsilon} = \{ w \in C_T ; \|w\|_\infty \leq \varepsilon \}$. For each $\lambda$, $|\lambda| \leq \eta(\varepsilon)$, and $w \in C_{T,\varepsilon}$, we consider the following linear nonhomogeneous system

$$z' = A_w(t,w(t),\lambda)z + \lambda F_w(t,w(t),\lambda) + f_w(t,w(t),\lambda) \tag{3.16}$$

together with a boundary condition

$$\mathcal{L}(z(\cdot)) = 0. \tag{3.17}$$

Put $A_w(t,w(t),\lambda) = A(t,w(t)+\pi(t),\lambda)$, $F_w(t,w(t),\lambda) = F(t,w(t)+\pi(t),\lambda)$, and $f_w(t,w(t),\lambda) = \{ A(t,w(t)+\pi(t),\lambda) - B(t) \}\pi(t)$. Then the following relations (3.18) - (3.20) are obtained.

$$\int_0^T \|A_w(t,w(t),\lambda) - B(s)\|ds \leq \delta; \tag{3.18}$$

$$\int_0^T \|F_w(t,w(t),\lambda)\|ds \leq \Delta; \tag{3.19}$$

$$\int_0^T \|f_w(t,w(t),\lambda)\|ds \leq \int_0^T \|A(s,w(s)+\pi(s),\lambda) - B(s)\|\|\pi(s)\|ds$$

$$\leq r_0 T\mu(\eta). \tag{3.20}$$

We denote $Z_w$ by the fundamental matrix solutions for the linear homogeneous system corresponding to (3.16) such that $Z_w(0) = I$. Let $V_w = I - Z_w(T)$. We have $\mathcal{L}(Z_w(\cdot)x_0) = V_w x_0$ for $x_0 \in R^n$. Since

$Z_w(t)Z_w^{-1}(\tau) = I + \int_\tau^t A_w(t,w(t),\lambda)Z_w(s)Z_w^{-1}(\tau)ds$, we obtain

$$\|Z_w(t)Z_w^{-1}(\tau)\| \leq K\exp(\delta) \tag{3.21}$$

for $t$, $\tau \in [0,T]$. By the variation of parameters formula, we get

$$Z_w(t) = X_B(t) + X_B(t)\int_0^t X_B^{-1}(s)\{ A_w(s,w(s),\lambda) - B(s) \} Z_w(s)ds.$$

From (3.2), (3.5), (3.18) and (3.21)

$$\| Z_w(t) - X_B(t) \| \leq \rho/( 2\| U_B^{-1} \| ),$$

which implies that

$$\| (V_w - U_B)x_0 \| \leq \rho \| x_0 \| / \| U_B^{-1} \|.$$

Hence we have the following estimate.

$$\| V_w^{-1} \| \leq 1/\{ \rho( 1 - \rho ) \}. \tag{3.22}$$

There exists one and only one solution $z_w$ of ((3.16),(3.17)) such that

$$z_w(t) = - V_w^{-1}[\mathcal{L}(q_w(\cdot))] + \int_0^t A_w(s,w(s),\lambda) z_w(s)ds$$

$$+ \lambda\int_0^t F_w(s,w(s),\lambda)ds + \int_0^t f_w(s,w(s),\lambda)ds \tag{3.23}$$

for $t \in \boldsymbol{R}$, where

$$q_w(t) = Z_w(t)\int_0^t Z_w^{-1}(s)\{ \lambda F_w(s,w(s),\lambda) + f_w(s,w(s),\lambda) \} ds. \tag{3.24}$$

By using (3.19),(3.20) and (3.21), it follows that

$$\| q_w \|_\infty \leq K\{ \eta\Delta + r_0 T\mu(\eta) \}\exp(\delta). \tag{3.25}$$

It is clear that $z_w$ belongs to $C_T$. We shall show that $\| z_w \|_\infty \leq \varepsilon$.

From (3.19), (3.20), (3.22), (3.23) and (3.25), we have

$$\|z_w(t)\| \leq \frac{2K\{\ \eta\Delta + r_0 T\mu(\eta)\ \}\exp(\delta)}{\rho(\ 1 - \rho\ )} + \int_0^t \|A_w(s,w(s),\lambda)\|\|z_w(s)\|ds$$
$$+\ \eta\Delta + r_0 T\mu(\eta).$$

By using Gronwall's lemma and (3.15)

$$\|z_w(t)\| \leq \{\ \eta\Delta + r_0 T\mu(\eta)\ \}\left( 1 + \frac{2K\exp(\delta)}{\rho(\ 1 - \rho\ )}\right)\exp\left(\int_0^t \|A_w(s,w(s),\lambda)\|ds\right)$$
$$\leq \{\ \eta\Delta + r_0 T\mu(\eta)\ \}\left( 1 + \frac{2K\exp(\delta)}{\rho(\ 1 - \rho\ )}\right)K\exp(\delta)$$
$$\leq \varepsilon.$$

We can define an operator $Q : C_{T,\varepsilon} \to C_{T,\varepsilon}$ by $Q(z) = z_w$, where $z_w$ is a unique $T$-periodic solution of (3.16) in $C_{T,\varepsilon}$. By the same argument used in the proof of Theorem 3.3.2, $Q$ is a compact continuous operator. And also $C_{T,\varepsilon}$ is a convex closed subset in $C_T$. According to Schauder's fixed point theorem, for any $\varepsilon > 0$ and $\lambda$, $|\lambda| \leq \eta(\varepsilon)$, $Q$ has at least one fixed point $z(\cdot) = z(\cdot;\varepsilon,\lambda)$ in $C_{T,\varepsilon}$ satisfying

$$z'(t) = A(t,z(t)+\pi(t),\lambda)z(t) + \lambda F(t,z(t)+\pi(t),\lambda)$$
$$+ \{\ A(t,z(t)+\pi(t),\lambda) - B(t)\ \}\pi(t).$$

Let $x(\cdot;\varepsilon,\lambda) = z(\cdot;\varepsilon,\lambda) + \pi(\cdot)$. It can be seen that there exists at least one $T$-periodic solution $x(\cdot;\varepsilon,\lambda)$ of (N) satisfying (3.14). This completes the proof. **Q.E.D.**

When $A$, $F$ satisfy Lipschitz conditions with respect to $x$, respectively, we have the following theorem for the continuous dependence on $\lambda$ of periodic solutions of (N).

**Hypothesis 3.4.** *There exists a positive constant $L$ such that*

$$\|A(t,x_1,\lambda) - A(t,x_2,\lambda)\| \leq L\|x_1 - x_2\|$$

*and that*

$$\|F(t,x_1,\lambda) - F(t,x_2,\lambda)\| \leq L\|x_1 - x_2\|$$

*for any $t \in [0,T]$, $\lambda \in \Lambda_2$ and $x_i \in S_r$, $i = 1,2$.*

**Theorem 3.3.4.** *Suppose that the assumption in* Theorem 3.3.3 *and* Hypothesis 3.4 *hold. If*

(3.26) $$\lambda_2 \leq \lambda_2\Delta + M$$

*and*

(3.27) $$2rLT\left( K\exp(\delta) + \frac{r}{\lambda_2\Delta + M} \right) < 1,$$

*then, for any $\lambda \in \Lambda_2$, there exists one and only one $T$-periodic solution $x(\cdot;\lambda)$ of* (N). *Moreover*

(3.28) $$x(t;\lambda) \to \pi(t) \quad (\lambda \to 0)$$

*uniformly in $t \in \mathbf{R}$.*

*Remark.* From the second assertion of Theorem 3.3.4, the $T$-periodic solution for (N) is continuous in $\lambda$.

*Proof.* Choose $\lambda \in \Lambda_2$. By Theorem 3.3.2, we can define an operator $P$: $C_{T,r} \to C_{T,r}$ by $P(y) = x_y$ for $y \in C_{T,r}$, where $x_y$ is the $T$-periodic solution of (3.3) in $C_{T,r}$. Then we have

$$(3.29)\quad [\mathcal{P}(y)](t) = -\ X_y(t)U_y^{-1}[\mathcal{L}(p_y(\cdot))] + p_y(t).$$

Denote $k$ by the left-hand side of (3.27). We shall show that $\mathcal{P}$ is a contraction. Let $y_1, y_2 \in C_{T,r}$ and let $X_1, X_2$ be the fundamental matrices of the following linear systems, respectively.

$$x' = A(t, y_1(t), \lambda)x \quad ; \quad x' = A(t, y_2(t), \lambda)x.$$

By the same argument used in (3.2), we obtain

$$\|X_i(t)X_i^{-1}(\tau)\| \leq K\exp(\delta)$$

for $i = 1, 2$, where $t, \tau \in [0, T]$. This yields

$$\|p_i\|_\infty \leq K(\ \lambda_2\Delta + M\ )\exp(\delta),$$

where

$$(3.30)\qquad p_i(t) = X_i(t)\int_0^t X_i^{-1}(s)\{\ \lambda F(s, y_i(s), \lambda) + f(s)\ \}ds$$

for $t \in [0, T]$. Put $U_i = I - X_i(T)$ for $i = 1, 2$. In the same way as (3.8) it follows that $\|U_i^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho\ )\}$, so that

$$\|U_1^{-1} - U_2^{-1}\| \leq \|U_1 - U_2\|/\{\ \rho^2(\ 1 - \rho\ )^2\}.$$

Since

$$[X_1(t)]' = A(t, y_2(t), \lambda)X_1(t) + [\ A(t, y_1(t), \lambda) - A(t, y_2(t), \lambda)\ ]X_1(t),$$

by the variation of parameters formula, we obtain

$$\|X_1(t) - X_2(t)\| \leq K^2\exp(2\delta)\int_0^T \|A(s, y_1(s), \lambda) - A(s, y_2(s), \lambda)\|ds$$

$$\leq LTK^2\exp(2\delta)\|y_1 - y_2\|_\infty$$

for $t \in [0,T]$. Hence we get

(3.31) $$\|X_1 - X_2\|_\infty \leq LTK^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

In the same way as (3.31) we have

$$\|X_1^{-1} - X_2^{-1}\|_\infty \leq LTK^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

We shall show

(3.32) $$\|\mathcal{P}(y_1) - \mathcal{P}(y_2)\|_\infty \leq k\|y_1 - y_2\|_\infty$$

for $y_1$, $y_2 \in C_{T,r}$. From (3.29) it follows that

$$[\mathcal{P}(y_i)](t) = - X_i(t)U_i^{-1}[\mathcal{L}(p_i(\cdot))] + p_i(t)$$

for $i = 1, 2$, so that

$$\|[\mathcal{P}(y_1)](t) - [\mathcal{P}(y_2)](t)\| \leq E_1 + E_2 + E_3 + E_4.$$

Here

$$E_1 = 2\|X_1 - X_2\|_\infty\|U_1^{-1}\|\|p_1\|_\infty,$$

$$E_2 = 2\|X_2\|_\infty\|U_1^{-1} - U_2^{-1}\|\|p_1\|_\infty,$$

$$E_3 = 2\|X_2\|_\infty\|U_2^{-1}\|\|p_1 - p_2\|_\infty,$$

and

$$E_4 = \|p_1 - p_2\|_\infty.$$

We get

$$E_1 \leq LT\left(\frac{2K\exp(\delta)}{\rho(1 - \rho)}\right)(\lambda_2\Delta + M)K^2\exp(2\delta)\|y_1 - y_2\|_\infty$$

$$\leq LT\left(1 + \frac{2K\exp(\delta)}{\rho(1-\rho)}\right)(\lambda_1\Delta + M)K^2\exp(2\delta)\|y_1 - y_2\|_\infty,$$

by using (3.6), we obtain

$$E_1 \leq rLTK\exp(\delta)\|y_1 - y_2\|_\infty. \tag{3.33}$$

We have

$$E_2 \leq 2K\exp(\delta)\left(\frac{\|U_1 - U_2\|}{\rho^2(1-\rho)^2}\right)K\exp(\delta)(\lambda_2\Delta + M)$$

$$\leq 2K\exp(\delta)\left(\frac{2\|X_1 - X_2\|_\infty}{\rho^2(1-\rho)^2}\right)K\exp(\delta)(\lambda_2\Delta + M)$$

$$\leq 2K\exp(\delta)\left(\frac{2LTK^2\exp(2\delta)\|y_1 - y_2\|_\infty}{\rho^2(1-\rho)^2}\right)K\exp(\delta)(\lambda_2\Delta + M)$$

$$= LT\left(\frac{2K\exp(\delta)}{\rho(1-\rho)}\right)^2\{K\exp(\delta)\}^2(\lambda_2\Delta + M)\|y_1 - y_2\|_\infty,$$

by using (3.6), we get

$$E_2 \leq LT\left(\frac{r}{\lambda_1\Delta + M}\right)^2(\lambda_2\Delta + M)\|y_1 - y_2\|_\infty$$

$$\leq \left(\frac{r^2LT}{\lambda_2\Delta + M}\right)\|y_1 - y_2\|_\infty. \tag{3.34}$$

On the other hand, from (3.30) we have

$$E_4 \leq \|X_1 - X_2\|_\infty\int_0^T\|X_1^{-1}\|_\infty\{|\lambda|\|F(s,y_1(s),\lambda)\| + \|f(s)\|\}ds$$

$$+ \|X_2\|_\infty \int_0^T \|X_1^{-1} - X_2^{-1}\|_\infty \{ |\lambda| \|F(s,y_1(s),\lambda)\| + \|f(s)\| \} ds$$

$$+ \|X_2\|_\infty \int_0^T \|X_2^{-1}\|_\infty |\lambda| \|F(s,y_1(s),\lambda) - F(s,y_2(s),\lambda)\| ds$$

$$\leq LTK^3 \exp(3\delta)( \lambda_2\Delta+M ) \|y_1 - y_2\|_\infty + LTK^3 \exp(3\delta)( \lambda_2\Delta+M ) \|y_1 - y_2\|_\infty$$

$$+ LTK^2 \exp(2\delta) \lambda_2 \|y_1 - y_2\|_\infty,$$

by using (3.6), (3.26), $0 < \rho < 1$ and $0 < \lambda_1 \leq \lambda_2$, we have

(3.35) $$E_4 \leq rLTK\exp(\delta) \|y_1 - y_2\|_\infty .$$

This yields

$$E_3 \leq rLT \left( \frac{2K\exp(\delta)}{\rho( 1 - \rho )} \right) K\exp(\delta) \|y_1 - y_2\|_\infty$$

$$\leq rLT \left( 1 + \frac{2K\exp(\delta)}{\rho( 1 - \rho )} \right) K\exp(\delta) \|y_1 - y_2\|_\infty ,$$

by using (3.6), we have

(3.36) $$E_3 \leq \left( \frac{r^2 LT}{\lambda_2 \Delta + M} \right) \|y_1 - y_2\|_\infty$$

for $0 < \lambda_2 \leq \lambda_1$. From (3.33),(3.34),(3.35)and (3.36) we can see that (3.32) holds. Since $P$ is a contraction, for any $\lambda \in \Lambda_2$ there exists one and only one solution of (N), which belongs to $C_{T,r}$.

In order to show that (3.28) holds it suffices to prove that $Q$ is a contraction, where $Q$ is the operator defined in Theorem 3.3.3.

Let $\varepsilon$ such that $0 < 2\varepsilon < r - r_0$ and let $\eta = \eta(\varepsilon)$ satisfy (3.15). Choose $\lambda$ such that

$$(3.37) \qquad |\lambda| \leq \min(\ r - 2\varepsilon,\ \eta(\varepsilon)\ ).$$

Let $w_1, w_2 \in C_{T,\varepsilon}$ and let $Z_1, Z_2$ be the fundamental matrices of the following linear systems, respectively.

$$z' = A_1(t,w_1(t))z \quad ; \quad z' = A_2(t,w_2(t))z.$$

where $A_i(t,w_i(t)) = A(t,w_i(t)+\pi(t),\lambda)$ for $i = 1,2$. Put

$$q_i(t) = Z_i(t)\int_0^t Z_i^{-1}(s)\{\ \lambda F_i(s,w_i(s)) + f_i(s,w_i(s))\ \}ds$$

for $t \in \mathbf{R}$, where $F_i(t,w_i(t)) = F(t,w_i(t)+\pi(t),\lambda)$,

and $f_i(t,w_i(t)) = \{\ A(t,w_i(t)+\pi(t),\lambda) - B(t)\ \}\pi(t)$, and put

$V_i = I - Z_i(T)$ for $i = 1,2$. Then we have the following estimates.

$$\|Z_i(t)Z_i^{-1}(\tau)\| \leq K\exp(\delta) \qquad \textit{for } t,\tau \in [0,T];$$

$$\|Z_1 - Z_2\|_\infty \leq LTK^2\exp(2\delta)\|w_0\|_\infty \qquad (\ w_0 \equiv w_1 - w_2\ );$$

$$\|Z_1^{-1} - Z_2^{-1}\|_\infty \leq LTK^2\exp(2\delta)\|w_0\|_\infty;$$

$$\|q_i\|_\infty \leq K\tilde{\lambda}\exp(\delta) \qquad (\ \tilde{\lambda} \equiv |\lambda|\Delta + r_0 T\mu(|\lambda|\ )\ );$$

$$\|V_1 - V_2\| \leq 2LTK^2\exp(2\delta)\|w_0\|_\infty;$$

$$\|V_i^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho\ )\};$$

$$\|V_1^{-1} - V_2^{-1}\| \le \|V_1 - V_2\|/\{ \rho^2( 1 - \rho )^2\}$$

$$\le \left( \frac{2LTK^2\exp(2\delta)}{\rho^2( 1 - \rho )^2} \right)\|w_0\|_\infty .$$

Moreover it follows that

$$\|q_1 - q_2\|_\infty \le \|Z_1 - Z_2\|_\infty \int_0^T \|Z_1^{-1}\|_\infty \{ |\lambda| \|F_1(s,w_1(s))\| + \|f_1(s,w_1(s))\| \} ds$$

$$+ \|Z_2\|_\infty \int_0^T \|Z_1^{-1} - Z_2^{-1}\|_\infty \{ |\lambda| \|F_1(s,w_1(s))\| + \|f_1(s,w_1(s))\| \} ds$$

$$+ K\exp(\delta) \int_0^T \{ |\lambda| \|F_1(s,w_1(s)) - F_2(s,w_2(s))\| + \|f_1(s,w_1(s)) - f_2(s,w_2(s))\| \} ds.$$

Since $\int_0^T \|f_1(s,w_1(s)) - f_2(s,w_2(s))\| ds \le rLT\|w_0\|_\infty$, we have

$$\|q_1 - q_2\|_\infty \le 2LTK^3\exp(3\delta)\tilde{\lambda}\|w_0\|_\infty + LTK\exp(\delta)( |\lambda| + r )\|w_0\|_\infty$$

$$= LT\{ |\lambda| + 2K^2\exp(2\delta)\tilde{\lambda} \} K\exp(\delta)\|w_0\|_\infty + rLTK\exp(\delta)\|w_0\|_\infty .$$

It can be seen that

$$[Q(w_i)](t) = - Z_i(t)V_i^{-1}[L(q_i(\cdot))] + q_i(t)$$

for $i = 1, 2$, so that

$$\|[Q(w_1)](t) - [Q(w_2)](t)\| \le 2\|Z_1 - Z_2\|_\infty \|V_1^{-1}\| \|q_1\|_\infty + 2\|Z_2\|_\infty \|V_1^{-1} - V_2^{-1}\| \|q_1\|_\infty$$

$$+ 2\|Z_2\|_\infty \|V_2^{-1}\| \|q_1 - q_2\|_\infty + \|q_1 - q_2\|_\infty$$

$$\leq \left[ \frac{2LTK^3\exp(3\delta)\tilde{\lambda}}{\rho(1-\rho)} \right] \|w_0\|_\infty + \left[ \frac{4LTK^4\exp(4\delta)\tilde{\lambda}}{\rho^2(1-\rho)^2} \right] \|w_0\|_\infty$$

$$+ LT\left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right]\{ |\lambda| + 2K^2\exp(2\delta)\tilde{\lambda} \}K\exp(\delta)\|w_0\|_\infty$$

$$+ rLT\left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right]K\exp(\delta)\|w_0\|_\infty$$

$$= \left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right]LTK\exp(\delta)\left[ \frac{2K^2\exp(2\delta)\tilde{\lambda}}{\rho(1-\rho)} + |\lambda| + 2K^2\exp(2\delta)\tilde{\lambda} \right]\|w_0\|_\infty$$

$$+ rLT\left[ 1 + \frac{2K\exp(\delta)}{\rho(1-\rho)} \right]K\exp(\delta)\|w_0\|_\infty .$$

By using (3.15), (3.36) and (3.6), we obtain

$$\|Q(w_1) - Q(w_2)\|_\infty \leq \left[ \frac{rLT(2\varepsilon + |\lambda|)}{\lambda_1\Delta + M} \right]\|w_0\| + \left[ \frac{r^2LT}{\lambda_1\Delta + M} \right]\|w_0\|$$

$$\leq \left[ \frac{2r^2LT}{\lambda_2\Delta + M} \right]\|w_0\| .$$

From (3.27) $Q$ is a contraction. This completes the proof. Q.E.D.

CHAPTER 4

# BOUNDARY VALUE PROBLEMS ON A FINITE INTERVAL

## 4.1. Introduction.

In this chapter the following nonlinear boundary value problem of quasilinear ordinary differential system

(N) $$x' = A(t,x)x + F(t,x)$$

(C) $$N(x) = 0$$

is dealt with, where $A(t,x)$ and $F(t,x)$ are both continuous on $J \times R^n$, and $N : C(J) \to R^n$ is a continuous operator (not necessarily linear). Moreover the following linear boundary value problem

(AL) $$x' = B(t)x + F(t,x)$$

(NH) $$L(x) = c$$

is discussed by a similar approach used in the above problem. Here $B(t)$ is a real $n \times n$ matrix continuous on $J$, $L : C(J) \to R^n$ is a bounded linear operator and $c \in R^n$. Our results for ((AL),(NH)) are applied to the following second order ordinary differential equation

(E) $$u'' = f(t,u,u')$$

with boundary conditions

(BC) $$\alpha_1 u(0) + \alpha_2 u'(0) = c_1$$
$$\beta_1 u(T) + \beta_2 u'(T) = c_2,$$

where $f : J \times R \times R \to R$ is a continuous function, $\alpha_i$, $\beta_i$, and $c_i \in R$ ( $i = 1,2$ ).

In what follwos, the nonlinear problem ((N),(C)) is considered associated with the following linear problem

(L) $$x' = B(t)x$$

(LC) $$L(x) = 0.$$

It is assumed that the following hypothesis holds.

**Hypothesis 4.1.** *There exist no solutions of* ((L),(LC)) *except for the zero solution.*

In our results an explicit and quantitative condition on $A(t,x)$ as follows:

**Hypothesis 4.2.** *There exist* $\delta > 0$ *and* $r > 0$ *such that*

$$\int_0^T \|A(s,x) - B(s)\| ds \leq \delta$$

*for* $x \in S_r$.

In section 4.3 the existence, and the uniqueness of solutions for ((N),(C)) are proved by Schauder's fixed point theorem, and by the contraction principle, respectively. In section 4.4 the existence and uniqueness of solutions for ((AL),(NH)) are treated in a similar approach to section 4.3. The above results for ((AL),(NH)) are applied to ((E),(BC)) in section 4.5.

## 4.2. Preliminaries.

Let $X_B$ be the fundamental matrix solutions of (L) such that $X_B(0) = I$, where $I$ is the identity matrix. Since

$$X_B(t)X_B^{-1}(\tau) = I + \int_\tau^t B(s)X_B(s)X_B^{-1}(\tau)ds,$$

for $t,\tau \in J$, we have

$$\|X_B(t)X_B^{-1}(\tau)\| \leq 1 + \left|\int_\tau^t \|B(s)\|\|X_B(s)X_B^{-1}(\tau)\|ds\right|.$$

Thus, by applying Gronwall's lemma, it follows that

$$\|X_B(t)X_B^{-1}(\tau)\| \leq K \tag{4.1}$$

for $t$, $\tau \in J$, which implies that

$$\|X_B(t)\| \leq K \quad \text{and} \quad \|X_B^{-1}(t)\| \leq K, \tag{4.2}$$

where

$$K = \exp(\int_0^T \|B(s)\|ds).$$

Let $U_B$ be the constant matrix such that

$$\mathcal{L}(X_B(\cdot)x_0) = U_B x_0 \tag{4.3}$$

for $x_0 \in \boldsymbol{R}^n$. The following lemma concerned with $U_B$ holds.

**Lemma 4.1.** *The statements* (i) - (iii) *are equivalent mutually.*

(i) Hypothesis 4.1 *holds;*

(ii) $U_B$ *is nonsingular;*

(iii) *For any* $f \in C(J)$ *and* $c \in R^n$, *there exists a unique solution* $x \in C(J)$ *of*

$$x' = B(t)x + f(t), \quad \mathcal{L}(x) = c.$$

The above lemma can be proved in the same as the proof of Lemma 5.2 in Chapter 5.

From the elementary result in linear algebra, the following lemma is obtained.

**Lemma 4.2.** *Suppose that* Hypothesis 4.1 *holds. Then there exists a constant number* $\rho$ ( $0 < \rho < 1$ ) *such that*

$$\|U_B^{-1}\| \leq 1/\rho. \tag{4.4}$$

Such a $\rho$ will be fixed throughout this chapter.

## 4.3. Existence and Uniqueness of Solutions for Nonlinear Problems.

In this section the problem ((N),(C)) is discussed by applying Schauder's fixed point theorem and the contraction principle. Consider the following problem

$$x' = A(t,y(t))x + F(t,y(t)) \tag{4.5}$$

$$\mathcal{L}(x) = \mathcal{L}(y) - \mathcal{N}(y) \tag{4.6}$$

for $y \in C_r$, where

$$C_r = \{ x \in C(J) : \|x\|_\infty \leq r \}.$$

Let $X_y$ be the fundamental matrix solutions of the linear homogeneous system corresponding to (4.5) such that $X_y(0) = I$. An existence theorem of ((4.5),(4.6)) is obtained as follows.

**Theorem 4.3.1.** *Suppose that* Hypotheses 4.1 - 4.2 *hold and there exist non-negative numbers $C$ and $a$ satisfying the following conditions* (4.7) - (4.8), *respectively.*

$$(4.7) \qquad \int_0^T \|F(s,x)\| ds \leq rC \quad \text{for } x \in S_r;$$

$$(4.8) \qquad \|\mathcal{L}(x) - \mathcal{N}(x)\| \leq ar \quad \text{for } x \in C_r.$$

*Let the above numbers satisfy the following relations* (4.9)-(4.10).

$$(4.9) \qquad K^2\delta\exp(\delta) \leq \frac{\rho}{\|\mathcal{L}\|\|U_B^{-1}\|};$$

$$(4.10) \qquad \begin{aligned} & aK\exp(\delta) < \rho(1-\rho) \text{ and} \\ & C \leq \frac{\rho(1-\rho) - aK\exp(\delta)}{\{K\exp(\delta)\|\mathcal{L}\| + \rho(1-\rho)\}K\exp(\delta)}. \end{aligned}$$

*Then for any $y \in C_r$, there exists a nonsingular matrix $U_y$ such that*

$$(4.11) \qquad \mathcal{L}(X_y(\cdot)x_0) = U_y x_0$$

*for $x_0 \in \mathbf{R}^n$, whose inverse satisfies*

$$(4.12) \qquad \|U_y^{-1}\| \leq 1/\{\rho(1-\rho)\},$$

*and there exists one and only one solution $x_y \in C_r$ such that*

$$(4.13) \qquad \begin{aligned} x_y(t) &= U_y^{-1}[\mathcal{L}(y) - \mathcal{N}(y) - \mathcal{L}(p_y)] \\ &+ \int_0^t A(s,y(s))x_y(s)ds + \int_0^t F(s,y(s))ds, \end{aligned}$$

where $p_y(t) = \int_0^t X_y(t)X_y^{-1}(s)F(s,y(s))ds$ for $t \in J$.

*Proof.* In the same as (4.1) it can be seen that

$$\|X_y(t)X_y^{-1}(\tau)\| \leq K\exp(\delta) \tag{4.14}$$

for $t,\tau \in J$, so that

$$\|X_y(t)\| \leq K\exp(\delta) \quad \text{and} \quad \|X_y^{-1}(t)\| \leq K\exp(\delta). \tag{4.15}$$

Since $X_y(t) = X_B(t) + X_B(t)\int_0^t X_B^{-1}(s)\{\ A(s,y(s))-B(s)\ \}X_y(s)ds$ for $t \in J$, by (4.1), Hypothesis 4.2, (4.9), and (4.15), we have

$$\begin{aligned}\|X_y(t) - X_B(t)\| &\leq K^2\delta\exp(\delta)\\ &\leq \rho/(\|\mathcal{L}\|\|U_B^{-1}\|).\end{aligned}$$

Then it follows that

$$\begin{aligned}\|(U_y - U_B)x_0\| &= \|\mathcal{L}(X_y(\cdot) - X_B(\cdot))x_0\|\\ &\leq \rho\|x_0\|/\|U_B^{-1}\|\end{aligned}$$

for $x_0 \in \mathbf{R}^n$. From (4.4) we get

$$\|U_yx_0\| \geq \rho(\ 1 - \rho\ )\|x_0\|.$$

Hence $U_y$ has the inverse satisfying (4.12).

By Lemma 4.1 the problem ((4.5),(4.6)) has one and only one solution $x_y$ satisfying (4.13). We shall show that $\|x_y(t)\| \leq r$ for

$t \in J$. From (4.7), (4.8), (4.12), (4.13) and (4.14), we obtain

$$\|x_y(t)\| \leq \frac{ar + \|\mathcal{L}\| rCK\exp(\delta)}{\rho(1-\rho)} + rC + \int_0^t \|A(s,y(s))\| \|x_y(s)\| ds$$

for $t \in J$. By applying Gronwall's lemma and (4.10) we have

$$\|x_y(t)\| \leq \left( \frac{ar + \|\mathcal{L}\| rCK\exp(\delta)}{\rho(1-\rho)} + rC \right) \exp\left( \int_0^t \|A(s,y(s))\| ds \right)$$

$$\leq r.$$

Thus $x_y \in C_r$. This completes the proof. **Q.E.D.**

By applying Schauder's fixed point theorem we obtain the following theorem.

**Theorem 4.3.2.** *Suppose that the same assumption as* Theorem 4.3.1. *Then there exists at least one solution of* ((N),(C)).

*Proof.* It is easy to show that the solution of ((4.5),(4.6)) can be expressed by

$$x_y(t) = X_y(t)U_y^{-1}[\mathcal{L}(y) - \mathcal{N}(y) - \mathcal{L}(p_y)] + p_y(t)$$

for $t \in J$, where $y \in C_r$. Define $V: C_r \to C_r$ by $V(y) = x_y$. Then $V$ maps the convex closed set $C_r$ into itself. It is easily seen that the compactness and continuity of $V$ are proved in the same way as the proof of Theorem 3.3.2 in Chapter 3.

According to Schauder's fixed point theorem, $V$ has at least one fixed point in $C_r$. Therefore there exists at least one solution of

((N),(C)) and this completes the proof. **Q.E.D.**

By using the above theorem, an existence result for the linear problem ((N),(C)) is obtained.

**Theorem 4.3.3.** *Suppose that* Hypothesis 4.1 *and* (4.9) *hold, and that there exists a constant* $C_0 > 0$ *such that*

$$C_0 < \frac{\rho(1-\rho)}{[\|L\|K\exp(\delta) + \rho(1-\rho)]K\exp(\delta)}.$$

*Let* $A(t,x)$ *satisfy the following condition for any* $r \geq 0$.

$$\int_0^T \|A(s,x) - B(s)\|ds \leq \delta$$

*for* $x \in S_r$. *And let* $F(t,x)$ *satisfy the following condition.*

$$\liminf_{n \to +\infty} \frac{1}{n} \int_0^T \sup_{\|x\| \leq n} \|F(s,x)\|ds \leq C_0. \tag{4.16}$$

*Then, for any* $c \in \boldsymbol{R}^n$, *there exists at least one solution of* ((N),(C)).

*Proof.* There exists an $a > 0$ such that

$$C_0 = \frac{\rho(1-\rho) - aK\exp(\delta)}{[\|L\|K\exp(\delta) + \rho(1-\rho)]K\exp(\delta)}.$$

For any $c \in \boldsymbol{R}^n$, we put the following operator $N$

$$N(x) = L(x) - c.$$

From (4.16), there exists a sufficiently large $r > 0$ satisfying

$$\|c\| \leq ar$$

and

$$\int_0^T \|F(s,x)\|ds \leq rC_0$$

for $x \in S_r$. By Theorem 4.3.2, ((N),(C)) has at least one solution in $C_r$. This completes the proof. **Q.E.D.**

When $A(t,x)$, $F(t,x)$ and $N(x)$ satisfy Lipschitz conditions, respectively, the existence and uniqueness of a solution for ((N),(C)) are shown by the contraction principle as follows:

**Theorem 4.3.4.** *Suppose, under the assumptions in* Theorem 4.3.1, *that there exist a positive number $L_1$ such that*

$$\|A(t,x_1) - A(t,x_2)\| \leq L_1\|x_1 - x_2\|$$

*and*

$$\|F(t,x_1) - F(t,x_2)\| \leq L_1\|x_1 - x_2\|$$

*for $t \in J$, $x_1$, $x_2 \in S_r$, and a positive number $L_2$ such that*

$$\|N(y_1) - N(y_2)\| \leq L_2\|y_1 - y_2\|_\infty$$

*for $y_1$, $y_2 \in C_r$. Let $L_1$ and $L_2$ satisfy the following inequality.*

$$L_1\left(\frac{b_1K^2\exp(2\delta)}{\rho(1-\rho)} + b_2\right)\left(1 + \frac{\|\mathcal{L}\|K\exp(\delta)}{\rho(1-\rho)}\right) + \frac{K\exp(\delta)(\|\mathcal{L}\| + L_2)}{\rho(1-\rho)} < 1, \tag{4.17}$$

*where* $b_1 = r(a + \|\mathcal{L}\|CK\exp(\delta))$ *and* $b_2 = (T + rCK\exp(\delta))K\exp(\delta)$.

*Then there exists one and only one solution of* ((N),(C)).

*Proof.* Let $k$ be the left-hand side of (4.17). We shall show that the operator $V : C_r \to C_r$ such that

$$(4.18) \qquad [V(y)](t) = X_y(t)U_y^{-1}[L(y) - N(y) - L(p_y)] + p_y(t)$$

is a contraction. Let $y_1, y_2 \in C_r$ and let $X_1, X_2$ be fundamental matrices of the following linear systems, respectively.

$$x' = A(t,y_1(t))x \quad ; \quad x' = A(t,y_2(t))x.$$

By the same argument as (4.1), we obtain for $t,\tau \in J$

$$\|X_i(t)X_i^{-1}(\tau)\| \le K\exp(\delta) \quad (i = 1,2),$$

which yields

$$\|p_i\|_\infty \le rCK\exp(\delta),$$

where

$$p_i(t) = X_i(t)\int_0^t X_i^{-1}(s)F(s,y_i(s))ds$$

for $t \in J$. Since

$$[X_1(t)]' = A(t,y_2(t))X_1(t) + [\, A(t,y_1(t)) - A(t,y_2(t))\,]X_1(t),$$

by the variation of parameters formula, we obtain for $t,\tau \in J$

$$X_1(t) = X_2(t)X_2^{-1}(\tau)X_1(\tau)$$
$$+ \int_\tau^t X_2(t)X_2^{-1}(s)[\, A(s,y_1(s)) - A(s,y_2(s))\,]X_1(s)ds,$$

then

$$\|X_1(t)X_1^{-1}(\tau) - X_2(t)X_2^{-1}(\tau)\| \leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty .$$

Hence we get

$$\|X_1 - X_2\|_\infty \leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty$$

and

$$\|X_1^{-1} - X_2^{-1}\|_\infty \leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty .$$

Let $U_i$ be matrices such that

$$\mathcal{L}(X_i(\cdot)x_0) = U_ix_0 \qquad ( i = 1, 2 )$$

for $x_0 \in \boldsymbol{R}^n$. We have

$$\begin{aligned}\|U_1 - U_2\| &\leq \|\mathcal{L}\|\|X_1 - X_2\|_\infty \\ &\leq \|\mathcal{L}\|L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty .\end{aligned}$$

By the same argument as (4.12) it follows that

$$\|U_i^{-1}\| \leq 1/\{ \rho( 1 - \rho )\},$$

so that

$$\begin{aligned}\|U_1^{-1} - U_2^{-1}\| &\leq \|U_1^{-1}\|\|U_1 - U_2\|\|U_2^{-1}\| \\ &\leq \left( \frac{\|\mathcal{L}\|L_1K^2\exp(2\delta)}{\rho^2( 1 - \rho )^2} \right)\|y_1 - y_2\|_\infty .\end{aligned}$$

Moreover it follows, for $t \in J$, that

$$\|p_1(t) - p_2(t)\| \leq \int_0^T \|X_1(t)X_1^{-1}(s) - X_2(t)X_2^{-1}(s)\|\|F(s,y_1(s))\|ds$$

$$+ \int_0^T \|X_2(t)X_2^{-1}(s)\| \|F(s,y_1(s)) - F(s,y_2(s))\| ds$$

$$\leq L_1 b_2 \|y_1 - y_2\|_\infty.$$

From (4.18) it follows, for $t \in J$, that

$$\|[V(y_1)](t) - [V(y_2)](t)\|$$

$$\leq \|X_1 - X_2\|_\infty \|U_1^{-1}\| [\|\mathcal{L}(y_1) - N(y_1)\| + \|\mathcal{L}\| \|p_1\|_\infty]$$

$$+ \|X_2\|_\infty \|U_1^{-1} - U_2^{-1}\| [\|\mathcal{L}(y_1) - N(y_1)\| + \|\mathcal{L}\| \|p_1\|_\infty]$$

$$+ \|X_2\|_\infty \|U_2^{-1}\| [\|\mathcal{L}(y_1) - \mathcal{L}(y_2)\| + \|N(y_1) - N(y_2)\| + \|\mathcal{L}\| \|p_1 - p_2\|_\infty]$$

$$+ \|p_1 - p_2\|_\infty$$

$$\leq \left( \frac{b_1 L_1 K^2 \exp(2\delta)}{\rho(1 - \rho)} \right) \|y_1 - y_2\|_\infty + \left( \frac{\|\mathcal{L}\| b_1 L_1 K^3 \exp(3\delta)}{\rho^2 (1 - \rho)^2} \right) \|y_1 - y_2\|_\infty$$

$$+ \left( \frac{K\exp(\delta)}{\rho(1 - \rho)} \right) (\|\mathcal{L}\| + L_2 + \|\mathcal{L}\| L_1 b_2) \|y_1 - y_2\|_\infty + L_1 b_2 \|y_1 - y_2\|_\infty$$

$$= k \|y_1 - y_2\|_\infty.$$

Thus, from (4.17), $V$ is a contraction. This completes the proof.

**Q.E.D.**

## 4.4. Existence and Uniqueness of Solutions for Linear Problems.

Consider the following linear problem

(AL) $$x' = B(t)x + F(t,x)$$

(NH) $$\mathcal{L}(x) = c,$$

where $c \in R^n$. By the same approach as section 4.3 we obtain the following results.

**Theorem 4.4.1.** *Suppose that* Hypothesis 4.1 *holds and that there exists a positive number* $C_1$ *satisfying*

$$\liminf_{n \to +\infty} \frac{1}{n} \int_0^T \sup_{\|x\| \leq n} \|F(s,x)\| ds \leq C_1$$

*and*

(4.19) $$C_1 < 1/[K(K_1 \|U_B^{-1}\| \|\mathcal{L}\| + 1)],$$

*where* $K_1 = \max\{\|X_B(t)X_B^{-1}(s)\| : 0 \leq s \leq t \leq T\}$. *Then, for any* $c \in R^n$, *there exists at least one solution of* ((AL),(NH)).

*Proof.* From (4.19) there exists a number $a_1 > 0$ such that

$$C_1 = \frac{1 - K\|U_B^{-1}\|a_1}{K(K_1\|U_B^{-1}\|\|\mathcal{L}\| + 1)}.$$

For any $c \in R^n$, we put a positive number $r$ satisfying

$$\|c\| \leq a_1 r$$

and

$$\int_0^T \|F(s,x)\|ds \leq rC_1$$

for $x \in S_r$.

Consider the following linear problem

(4.20) $$x' = B(t)x + F(t,y(t))$$

(NH) $$\mathcal{L}(x) = c$$

for $y \in C_r$. The above problem ((4.20),(NH)) has one and only one solution $x_y$ such that

$$x_y(t) = U_B^{-1}[c - \mathcal{L}(q_y)] + \int_0^t B(s)x_y(s)ds + \int_0^t F(s,y(s))ds,$$

where $q_y(t) = \int_0^t X_B(t)X_B^{-1}(s)F(s,y(s))ds$, with $\|q_y(t)\| \leq K_1 rC_1$ for $t \in J$. We shall show that

$$\|x_y(t)\| \leq r$$

for $t \in J$. It follows that

$$\|x_y(t)\| \leq \|U_B^{-1}\|(a_1 r+\|\mathcal{L}\|\|q_y\|_\infty) + \int_0^t \|B(s)\|\|x_y(s)\|ds + rC_1,$$

by using Gronwall's lemma, so that

$$\|x_y(t)\| \leq r(\|U_B^{-1}\|a_1 + \|U_B^{-1}\|\|\mathcal{L}\|K_1C_1 + C_1)\exp\left(\int_0^t \|B(s)\|ds\right)$$

$$\leq r[\ \|U_B^{-1}\|a_1 + \{\ (\|U_B^{-1}\|\|\mathcal{L}\|K_1 + 1)\ \}C_1\ ]K$$
$$\leq r.$$

Therefore we can define an operator $V_1 : C_r \to C_r$ by $V_1(y) = x_y$. $V_1$ is continuous on $C_r$ and the image $V_1(\ C_r\ )$ is a compact subset in $C_r$. Thus, by Schauder's fixed point theorem, $V_1$ has at least one fixed point in $C_r$, that is, ((AL),(NH)) has at least one solution in $C_r$. This completes the proof. **Q.E.D.**

By using the contraction principle we can show the existence and uniqueness of a solution of ((AL),(NH)) as follows:

**Theorem 4.4.2.** *Suppose that Hypothesis 4.1 holds and that there exists a positive number $L_3$ satisfying*

$$\|F(t,x) - F(t,y)\| \leq L_3\|x - y\|$$

*for $t \in J$, $x \in \boldsymbol{R}^n$ and $y \in \boldsymbol{R}^n$. If*

$$(4.21) \qquad (\|U_B^{-1}\|\|\mathcal{L}\|K_1 + 1)K_1L_3T < 1,$$

*then, for any $c \in \boldsymbol{R}^n$, there exists one and only one solution of* ((AL),(NH)).

*Proof.* Let $c \in \boldsymbol{R}^n$ be given arbitrarily. Consider the above linear problem ((4.20),(NH)) for $y \in C(J)$. It can be seen that for $y$ in $C(J)$ there exists one and only one solution $x_y$ of ((4.20),(NH)). We can define an operator $V_2 : C(J) \to C(J)$ by $V_2(y) = x_y$. It can be expressed as follows

$$[V_2(y)](t) = X_B(t)U_B^{-1}[c - \mathcal{L}(q_y)] + q_y(t).$$

We have

$$\|V_2(y) - V_2(z)\|_\infty \le (\|U_B^{-1}\|\|\mathcal{L}\|K_1 + 1)K_1 L_3 T\|y - z\|_\infty$$

for $y, z \in C(J)$. From (4.21), $V_2$ is a contraction. Therefore ((AL),(NH)) has one and only one solution. This completes the proof.

**Q.E.D.**

## 4.5. Two Points Boundary Value Problems.

Consider the second order ordinary differential equation (E) with boundary conditions (BC) or the equivalent linear problem

(4.22) $$x' = Bx + F(t,x)$$

(NH) $$\mathcal{L}(x) = c.$$

Here we put as follows. $x = {}^t(u,v) \in \mathbf{R}^2$ with $\|x\| = |u| + |v|$.

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad F(t,x) = \begin{pmatrix} 0 \\ f(t,u,v) \end{pmatrix} ;$$

$$\mathcal{L}\begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} = \begin{pmatrix} \alpha_1 u(0) + \alpha_2 v(0) \\ \beta_1 u(T) + \beta_2 v(T) \end{pmatrix} ; \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then we have

$$X_B(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_B = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_1 T + \beta_2 \end{pmatrix}.$$

When $\Delta = \alpha_1(\beta_1 T + \beta_2) - \alpha_2\beta_1 \neq 0$, we obtain

$$U_B^{-1} = \frac{1}{\Delta}\begin{pmatrix} \beta_1 T + \beta_2 & -\alpha_2 \\ -\beta_1 & \alpha_1 \end{pmatrix}$$

and

$$\|U_B^{-1}\| = \max[\ (|\beta_1 T + \beta_2|+|\beta_1|)/|\Delta|,\ (|\alpha_1|+|\alpha_2|)/|\Delta|\ ],$$

so that Hypothesis 4.1 holds. Moreover we get $K_1 = 1 + T$, $K = \exp(T)$ and $\|L\| \leq \alpha + \beta$, where $\alpha = \max(|\alpha_1|,|\alpha_2|)$ and $\beta = \max(|\beta_1|,|\beta_2|)$.

By applying Theorem 4.4.1, an existence result for ((E),(BC)) is obtained.

**Example 1.** *Suppose that* $\Delta \neq 0$ *holds and that there exists a positive number* $C_2$ *satisfying*

$$\liminf_{n \to +\infty} \frac{1}{n}\int_0^T \sup_{\|x\|\leqq n} |f(s,u,v)|\,ds \leq C_2$$

*and*

$$C_2 < 1/[\ \exp(T)\{\ (1+T)\|U_B^{-1}\|(\alpha+\beta) + 1\ \}\ ].$$

*Then, for any* ${}^t(c_1,c_2) \in \boldsymbol{R}^2$, *there exists at least one solution of* ((E),(BC)).

By Theorem 4.4.2, the following result for the existence and uniqueness of a solution of ((E),(BC)) is given.

**Example 2.** *Suppose that* $\Delta \neq 0$ *holds and that there exists a positive number* $L$ *such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq L( |u_1 - u_2| + |v_1 - v_2| )$$

*for* $t \in J$, ${}^t(u_1,v_1) \in \boldsymbol{R}^2$ *and* ${}^t(u_2,v_2) \in \boldsymbol{R}^2$. *If*

$$[ \|U_B^{-1}\|(\alpha+\beta)(1+T) + 1 ](1+T)LT < 1,$$

*then, for any* ${}^t(c_1,c_2) \in \boldsymbol{R}^2$, *there exists one and only one solution of* ((E),(BC)).

# CHAPTER 5

## BOUNDARY VALUE PROBLEMS ON AN INFINITE INTERVAL

### 5.1. Introduction.

The following boundary value problem of the quasilinear ordinary differential system

(N) $$x' = A(t,x)x + F(t,x)$$

(C) $$N(x) = 0$$

is considered, where $A(t,x)$ and $F(t,x)$ are both continuous on $R^+\times R^n$, and $N : C_r^{lim} \to R^n$ is a continuous operator( not necessarily linear ). Here

$$C_r^{lim} = \{ x \in C(R^+) : \lim_{t\to+\infty} x(t) \text{ exists and } \|x\|_\infty \leq r \},\ r > 0.$$

The above problem ((N),(C)) is considered associated with the linear problem

(L) $$x' = B(t)x$$

(LC) $$L(x) = 0,$$

where $B(t)$ is a real $n\times n$ matrix continuous on $R^+$ and $L : C^{lim} \to R^n$ is a bounded linear operator. $C^{lim} = \{ x\in C(R^+) : \lim_{t\to+\infty} x(t) \text{ exists} \}$.

It is assumed that the following hypotheses hold.

**Hypothesis 5.1.** $\int_0^{+\infty} \|B(s)\| ds < +\infty.$

**Hypothesis 5.2.** *There exist no solutions for ((L),(LC)) except for the zero solution.*

**Hypothesis 5.3.** *There exists two numbers $\delta > 0$ and $r > 0$, and a summable function $m_1$ such that*

$$\|A(t,x) - B(t)\| \leq m_1(t)$$

*for any $t \in R^+$ and $x \in S_r$, and that*

$$\int_0^{+\infty} m_1(s) ds \leq \delta.$$

In section 5.3 the existence of solutions for ((N),(C)) is shown by Schauder's fixed point theorem. In section 5.4 , under Lipschitz conditions the existence and uniqueness of a solution for ((N),(C)) are proved by the contraction principle.

## 5.2. Preliminaries.

Let $X_B$ be the fundamental matrix solutions for (L) such that $X_B(0) = I$, where $I$ is the identity matrix. Under Hypothesis 5.1 we obtain the following lemma.

**Lemma 5.1.** *Suppose that* Hypothesis 5.1 *holds. Then*

(5.1) $$\|X_B(t)X_B^{-1}(\tau)\| \leq K$$

*for* $t,\tau \in \mathbf{R}^+$ *and there exist* $\lim_{t\to+\infty} X_B(t)$ *and* $\lim_{t\to+\infty} X_B^{-1}(t)$, *where*

$$K = \exp\left(\int_0^{+\infty} \|B(s)\| ds\right).$$

*Proof.* Since $X_B(t)X_B^{-1}(\tau) = I + \int_\tau^t B(s)X_B(s)X_B^{-1}(\tau)ds$ for $t,\tau \in \mathbf{R}^+$ so that

$$\|X_B(t)X_B^{-1}(\tau)\| \leq 1 + \int_0^t \|B(s)\| \|X_B(s)X_B^{-1}(\tau)\| ds,$$

by using Gronwall's lemma, which implies that (5.1) holds. Hence we have for $t \in \mathbf{R}^+$

$$\|X_B(t)\| \leq K, \qquad \|X_B^{-1}(t)\| \leq K.$$

Since $X_B^{-1}(t) = I - \int_0^t X_B^{-1}(s)B(s)ds$, it follows that

$$\|X_B^{-1}(t_1) - X_B^{-1}(t_2)\| \leq \left| \int_{t_1}^{t_2} \|X_B^{-1}(s)\| \|B(s)\| ds \right|$$

$$\leq \left| K\int_{t_1}^{t_2} \|B(s)\| ds \right|$$

for $t_1, t_2 \in \mathbf{R}^+$. By Hypothesis 5.1 there exists $\lim_{t\to+\infty} X_B^{-1}(t)$. Since $X_B(t)X_B^{-1}(t) = I$, it follows that there exists $\lim_{t\to+\infty} X_B(t)$. This completes the proof. **Q.E.D.**

From the above lemma there exists a unique constant matrix $U_B$ such that

$$\mathcal{L}(X_B(\cdot)x_0) = U_B x_0 \tag{5.2}$$

for $x_0 \in \mathbf{R}^n$. Under Hypothesis 5.1 we get the following lemma concerned with $U_B$.

**Lemma 5.2.** *Suppose that* Hypothesis 5.1 *holds. Then the following statements* (i) - (iii) *are equivalent mutually.*

(i) Hypothesis 5.2 *holds;*

(ii) $U_B$ *is nonsingular;*

(iii) *For each continuous* $\mathbf{R}^n$*-valued function* $f$ *such that* $\int_0^{+\infty} \|f(s)\| ds < +\infty$ *and each* $c \in \mathbf{R}^n$, *there exists one and only one solution* $x \in C^{lim}$ *of the following linear problem*

$$x' = B(t)x + f(t) \tag{5.3}$$

$$\mathcal{L}(x) = c. \tag{5.4}$$

*Proof.* (i) $\rightarrow$ (ii). Let Hypothesis 5.1 hold. Then there exist no vectors $x_0 \in \mathbf{R}^n$ satisfying $\mathcal{L}(X_B(\cdot)x_0) = 0$ except for the zero vector. By (5.2) $U_B$ is nonsingular.

(ii) $\rightarrow$ (iii). Let $U_B$ be nonsingular. Put

$$x(t) = X_B(t)U_B^{-1}[c - \mathcal{L}(p)] + p(t),$$

where $c \in \mathbf{R}^n$ and

$$p(t) = \int_0^t X_B(t)X_B^{-1}(s)f(s)ds$$

for $t \in \mathbf{R}^+$, then it can be seen that $x(\cdot)$ is a solution of ((5.3), (5.4)). From the uniqueness of solutions of initial value problems for (5.3), $x(\cdot)$ is a unique solution of ((5.3),(5.4)).

(iii) → (i). It is clear. This completes the proof. **Q.E.D.**

*Remark.* Consider the case where $L(x) = \lim_{t\to+\infty} x(t)$. It can be easily seen that Hypothesis 5.2 holds under Hypothesis 5.1. The following linear problem

$$x' = (\sin t/t)x, \; L(x) = 0$$

shows that Hypothesis 5.1 is not necessarily satisfied under Hypothesis 5.2.

It is clear that the inequality (5.1) holds under Hypothesis 5.2. The scalar equation $x' = (\sin t)x$ implies that Hypothesis 5.2 does not necessarily hold even if (5.1) is satisfied.

The following lemma is an elementary result in linear algebra.

**Lemma 5.3.** *Let $U_B$ be nonsingular. Then there exists a positive number $\rho < 1$ such that*

$$\|U_B^{-1}\| \leq 1/\rho. \tag{5.5}$$

Under Hypotheses 5.1 and 5.2 there exists a $\rho$ in (5.5). Such a positive number $\rho$ is fixed throughtout this chapter.

A set $S$ in $C_r^{lim}$ is said to be **equiconvergent** if for any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $\|f(t) - \lim_{\tau\to+\infty} f(\tau)\| < \varepsilon$ for all $f \in S$

and all $t \geq T(\varepsilon)$. In the same way as the well known proof of Ascoli-Arzela theorem, we have the following lemma.

**Lemma 5.4.** *If a set $S$ in $C_r^{lim}$ is uniformly bounded, equicontinuous and equiconvergent, then $S$ is relatively compact in $C_r^{lim}$.*

## 5.3. Existence of Solutions.

Consider the following linear problem

$$x' = A(t,y(t))x + F(t,y(t)) \tag{5.6}$$

$$\mathcal{L}(x) = \mathcal{L}(y) - \mathcal{N}(y) \tag{5.7}$$

for $y \in C_r^{lim}$. Let $X_y$ be the fundametal matrix of solutions of the linear homogeneous system corresponding to (5.6) such that $X_y(0)=I$. An existence theorem for ((5.6),(5.7)) is obtained as follows.

**Theorem 5.3.1.** *Suppose that* Hypotheses 5.1 *and* 5.2 *hold. Let non-negative number $\delta$, $C$, $a$ and a summable function $m_2$ satisfy the following conditions* (5.8) - (5.10).

$$K^2\delta\exp(\delta) \leq \frac{\rho}{\|\mathcal{L}\|\|U_B^{-1}\|}; \tag{5.8}$$

$$aK\exp(\delta) < \rho(1-\rho) \text{ and}$$

$$C \leq \frac{\rho(1-\rho) - aK\exp(\delta)}{\{K\exp(\delta)\|\mathcal{L}\| + \rho(1-\rho)\}K\exp(\delta)}; \tag{5.9}$$

$$\int_0^{+\infty} m_2(s)ds \leq rC. \tag{5.10}$$

*Let* $F(t,x)$ *and* $N(x)$ *satisfy the following conditions* (5.11) - (5.12), (5.12), *respectively.*

$$(5.11) \qquad \|F(t,x)\| \le m_2(t) \quad \text{for } (t,x) \in R^+ \times S_r;$$

$$(5.12) \qquad \|\mathcal{L}(x) - N(x)\| \le ar \quad \text{for } x \in C_r^{lim}.$$

*Then for any* $y \in C_r^{lim}$, *there exists a nonsingular matrix* $U_y$ *such that*

$$(5.13) \qquad \mathcal{L}(X_y(\cdot)x_0) = U_y x_0$$

*for* $x_0 \in R^n$, *whose inverse satisfies*

$$(5.14) \qquad \|U_y^{-1}\| \le 1/\{\ \rho(\ 1 - \rho)\ \},$$

*and there exists one and only one solution* $x_y \in C_r^{lim}$ *for* ((5.6),(5.7)) *such that*

$$x_y(t) = U_y^{-1}[\mathcal{L}(y) - N(y) - \mathcal{L}(p_y)]$$

$$(5.15) \qquad + \int_0^t A(s,y(s))x_y(s)ds + \int_0^t F(s,y(s))ds,$$

*where* $\quad p_y(t) = \int_0^t X_y(t)X_y^{-1}(s)F(s,y(s))ds$ *for* $t \in R^+$.

*Proof.* From Hypotheses 5.1 and 5.2 it follows that

$$\int_0^{+\infty} \|A(s,y(s))\|ds < +\infty$$

for $y \in C_r^{lim}$. Using Lemma 5.1, we have $U_y$ satisfying (5.13). By the same argument used in (5.1) it can be seen that

$$(5.16) \qquad \|X_y(t)X_y^{-1}(\tau)\| \leq K\exp(\delta)$$

for $t,\tau \in \mathbf{R}^+$, so that

$$\|X_y(t)\| \leq K\exp(\delta) \text{ and } \|X_y^{-1}(t)\| \leq K\exp(\delta).$$

Since, by the variation of parameters formula,

$$X_y(t) = X_B(t) + X_B(t)\int_0^t X_B^{-1}(s)\{ A(s,y(s)) - B(s) \}X_y(s)ds,$$

we get

$$\begin{aligned}\|X_y(t) - X_B(t)\| &\leq \int_0^t \|X_B(t)X_B^{-1}(s)\|\|A(s,y(s)) - B(s)\|\|X_y(s)\|ds \\ &\leq K^2\delta\exp(\delta).\end{aligned}$$

This yields, by (5.8),

$$\|X_y(t) - X_B(t)\| \leq \rho/\{ \|L\|\|U_B^{-1}\| \}$$

so that for $x_0 \in \mathbf{R}^n$

$$\begin{aligned}\|(U_B - U_y)x_0\| &= \|L[(X_B(\cdot) - X_y(\cdot))x_0]\| \\ &\leq \|L\|\|X_B - X_y\|_\infty\|x_0\| \\ (5.17) \qquad &\leq \rho\|x_0\|/\|U_B^{-1}\|.\end{aligned}$$

We have, from (5.17) and (5.5), for $x_0 \in \mathbf{R}^n$

$$\begin{aligned}\rho\|x_0\| &\geq \|U_B^{-1}\|\|(U_B - U_y)x_0\| \\ &\geq \|x_0\| - \|U_B^{-1}\|\|U_yx_0\|\end{aligned}$$

$$\geq \|x_0\| - \|U_y x_0\|/\rho,$$

which implies that

$$\|U_y x_0\| \geq \rho(\ 1 - \rho\ )\|x_0\|.$$

Hence $U_y$ has the inverse satisfying (5.14).

By Lemma 5.2, the linear problem ((5.6),(5.7)) has a unique solution $x_y$ satisfying (5.15). We shall show that $\|x_y\|_\infty \leq r$. From definition of $p_y$ we obtain for $t \in R^+$

(5.18) $$\|p_y(t)\| \leq rCK\exp(\delta).$$

It follows, from (5.14), (5.12) and (5.18), that

$$\|U_y^{-1}[\mathcal{L}(y) - N(y) - \mathcal{L}(p_y)]\| \leq \|U_y^{-1}\|[\|\mathcal{L}(y) - N(y)\| + \|\mathcal{L}\|\|p_y\|_\infty]$$

$$\leq \frac{ar + \|\mathcal{L}\|rCK\exp(\delta)}{\rho(\ 1 - \rho\ )}$$

so that, by (5.15), (5.11) and (5.10),

$$\|x_y(t)\| \leq \frac{ar + \|\mathcal{L}\|rCK\exp(\delta)}{\rho(\ 1 - \rho\ )} + rC + \int_0^t \|A(s,y(s))\|\|x_y(s)\|ds.$$

By Gronwall's lemma we have

$$\|x_y(t)\| \leq \left\{ \frac{ar + \|\mathcal{L}\|rCK\exp(\delta)}{\rho(\ 1 - \rho\ )} + rC \right\}\exp\left(\int_0^t \|A(s,y(s))\|ds\right).$$

Since

$$\exp\left(\int_0^{+\infty} \|A(s,y(s))\|ds\right) \leq \exp\left(\int_0^{+\infty} \|A(s,y(s)) - B(s)\|ds + \int_0^{+\infty} \|B(s)\|ds\right)$$

$$\leq K\exp(\delta),$$

it follows, by (5.9), that $\|x_y(t)\| \leq r$ for $t \in \mathbf{R}^+$. The existence of $\lim_{t\to+\infty} x_y(t)$ is easily proved. Hence $x_y$ belongs to $C_r^{lim}$. This completes the proof. **Q.E.D.**

By applying Schauder's fixed point theorem we obtain the following theorem.

**Theorem 5.3.2.** *Suppose that the same assumption in* Theorem 5.3.1 *holds. Then there exists at least one solution of* ((N),(C)), *which belongs to* $C_r^{lim}$.

*Proof.* It is easy to show that the solution $x_y$ of ((5.6),(5.7)) can be expressed by

$$(5.19)\quad x_y(t) = X_y(t)U_y^{-1}[L(y) - N(y) - L(p_y)] + p_y(t)$$

for $t \in \mathbf{R}^+$, where $y \in C_r^{lim}$. From Theorem 5.3.1 there exists one and only one solution $x_y(\cdot)$ of ((5.6),(5.7)) for any $y \in C_r^{lim}$, so that we can define an operator $V : C_r^{lim} \to C_r^{lim}$ by $V(y) = x_y$ for $y \in C_r^{lim}$. Then $V$ maps the convex closed set $C_r^{lim}$ into itself.

We shall show that $V$ is a continuous compact operator. For the continuity of $V$ it suffices to prove that

$$V(y_n) \to V(y_0) \quad (n \to \infty),$$

as $y_n \to y_0$ $(n \to \infty)$ in $C_r^{lim}$. Let $X_n$, $X_0$ be fundamental matrices of the following linear systems, respectively,

$$x' = A(t,y_n(t))x \quad ; \quad x' = A(t,y_0(t))x.$$

It follows that for $t,\tau \in \mathbf{R}^+$

$$\|X_n(t)X_n^{-1}(\tau)\| \leq K\exp(\delta) \text{ and } \|X_0(t)X_0^{-1}(\tau)\| \leq K\exp(\delta).$$

Since

$$[X_n(t)]' = A(t,y_0(t))X_n(t) + [\ A(t,y_n(t)) - A(t,y_0(t))\ ]X_n(t),$$

by the variation of parameters formula, we obtain

$$\|X_n - X_0\|_\infty \leq K^2\exp(2\delta)\int_0^{+\infty}\|A(s,y_n(s)) - A(s,y_0(s))\|ds.$$

From Lebesgue's convergence theorem we have

(5.20) $$X_n \to X_0 \quad (\ n \to \infty\ )$$

in $M(\mathbf{R}^+)$. Let $U_n$, $U_0$ be constant matrices satisfying the following relations, respectively.

$$\mathcal{L}(X_n(\cdot)x_0) = U_n x_0 \ ; \ \mathcal{L}(X_0(\cdot)x_0) = U_0 x_0.$$

By (5.20) we get

$$\|U_n - U_0\| \to 0 \quad (\ n \to \infty\ ).$$

From Theorem 5.3.1 we obtain

$$\|U_n^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho\ )\ \} \text{ and } \|U_0^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho\ )\ \}.$$

We have

$$\|U_n^{-1} - U_0^{-1}\| \leq \|U_n^{-1}\|\|U_0 - U_n\|\|U_0^{-1}\|$$

$$\leq \|U_n - U_0\|/\{ \rho^2( 1 - \rho )^2 \}.$$

Thus

$$\|U_n^{-1} - U_0^{-1}\| \to 0 \quad ( n \to \infty ).$$

By the same way as (5.20) we get

$$X_n^{-1} \to X_0^{-1} \quad ( n \to \infty )$$

in $M(\boldsymbol{R}^+)$. Hence it follows that, by Lebesgue's convergence theorem,

$$\int_0^t X_n^{-1}(s)F(s,y_n(s))ds \to \int_0^t X_0^{-1}(s)F(s,y_0(s))ds \ ( n \to \infty )$$

uniformly with respect to $t \in \boldsymbol{R}^+$. Therefore

$$V(y_n) \to V(y_0) \quad ( n \to \infty ).$$

In order to prove the compactness of $V$ it suffices to show that the image $V( C_r^{lim} )$ is uniformly bounded, equicontinuous and equiconvergent. By the definition of $V$ we obtain for $y \in C_r^{lim}$

(5.21) $$\|V(y)\|_\infty = \|x_y\|_\infty \leq r.$$

Thus $V( C_r^{lim} )$ is uniformly bounded. From (5.15) and (5.21) it follows that for $t_1, t_2 \in \boldsymbol{R}^+$

$$\| (V(y))(t_1) - (V(y))(t_2) \|$$

$$\leq \left| \int_{t_1}^{t_2} \{ m_1(s) + \|B(s)\| \} r ds \right| + \left| \int_{t_1}^{t_2} m_2(s) ds \right|,$$

which implies that $V(C_r^{lim})$ is equicontinuous. By the same argument about the equicontinuity, $V(C_r^{lim})$ is equiconvergent. Thus, by Lemma 5.4, $V(C_r^{lim})$ is a relatively compact set in $C_r^{lim}$.

According to Schauder's fixed point theorem, $V$ has at least one fixed point in $C_r^{lim}$. Therefore there exists at least one solution for ((N),(C)) and this completes the proof. **Q.E.D.**

## 5.4. Existence and Uniqueness of Solutions.

When $A(t,x)$, $F(t,x)$ and $N(x)$ satisfy Lipschitz conditions, respectively, the existence and uniqueness of a solution for ((N),(C)) are obtained by the contraction principle as follows:

**Theorem 5.4.1.** *Suppose, under the assumptions in* Theorem 5.3.2, *that there exists a summable function* $\eta$ *satisfying*

$$\|A(t,x_1) - A(t,x_2)\| \leq \eta(t)\|x_1 - x_2\|$$

*and*

$$\|F(t,x_1) - F(t,x_2)\| \leq \eta(t)\|x_1 - x_2\|$$

*for* $t \in R^+$, $x_1$, $x_2 \in S_r$, *with* $L_1 = \int_0^{+\infty}\eta(s)ds$, *and that a positive number* $L_2$ *satisfies*

$$\|N(y_1) - N(y_2)\| \leq L_2\|y_1 - y_2\|_\infty$$

*for* $y_1$, $y_2 \in C_r^{lim}$. *Let* $L_1$ *and* $L_2$ *satisfy the following inequality.*

$$L_1\left(\frac{b_1K^2\exp(2\delta)}{\rho(1-\rho)} + b_2\right)\left(1 + \frac{\|\mathcal{L}\|K\exp(\delta)}{\rho(1-\rho)}\right)$$

$$+ \frac{K\exp(\delta)\{\|\mathcal{L}\| + L_2\}}{\rho(1-\rho)} < 1, \tag{5.22}$$

*where* $b_1 = aC + \|\mathcal{L}\|CK\exp(\delta)$ *and* $b_2 = \{1 + rCK\exp(\delta)\}K\exp(\delta)$. *Then there exists one and only one solution of* ((N),(C)), *which belongs to* $C_r^{lim}$.

*Proof.* From Theorem 5.3.1, we can define an operator $V : C_r^{lim} \to C_r^{lim}$ by $V(y) = x_y$ for $y \in C_r^{lim}$, where $x_y$ is a unique solution for ((5.6),(5.7)) in $C_r^{lim}$. From (5.19) we get for $t \in \boldsymbol{R}^+$

$$(5.23) \quad [V(y)](t) = X_y(t)U_y^{-1}[L(y) - N(y) - L(p_y)] + p_y(t).$$

Let $k$ be the left-hand side of (5.22). We shall show that $V$ is a contraction. Let $y_1, y_2 \in C_r^{lim}$ and let $X_1, X_2$ be the fundamental matrices of the following linear systems, respectively.

$$x' = A(t, y_1(t))x \quad ; \quad x' = A(t, y_2(t))x.$$

By the same argument used in (5.1), we obtain for $t, \tau \in \boldsymbol{R}^+$

$$\|X_i(t)X_i^{-1}(\tau)\| \leq K\exp(\delta) \quad (i = 1, 2),$$

which yields

$$\|p_i\|_\infty \leq rCK\exp(\delta),$$

where

$$p_i(t) = X_i(t)\int_0^t X_i^{-1}(s)F(s, y_i(s))ds$$

for $t \in \boldsymbol{R}^+$. Since

$$[X_1(t)]' = A(t, y_2(t))X_1(t) + [\, A(t, y_1(t)) - A(t, y_2(t))\, ]X_1(t),$$

by the variation of parameters formula, we obtain for $t, \tau \in \boldsymbol{R}^+$

$$X_1(t) = X_2(t)X_2^{-1}(\tau)X_1(\tau)$$

$$+ \int_\tau^t X_2(t)X_2^{-1}(s)[\ A(s,y_1(s)) - A(s,y_2(s))\ ]X_1(s)ds.$$

Then

$$\|X_1(t)X_1^{-1}(\tau) - X_2(t)X_2^{-1}(\tau)\| \leq K^2\exp(2\delta)\int_0^{+\infty}\eta(s)\|y_1(s) - y_2(s)\|ds$$

$$\leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

Hence we get

$$\|X_1 - X_2\|_\infty \leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty$$

and

$$\|X_1^{-1} - X_2^{-1}\|_\infty \leq L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

Let $U_i$ be matrices such that

$$\mathcal{L}(X_i(\cdot)x_0) = U_ix_0 \quad (i=1,2)$$

for $x_0 \in \boldsymbol{R}^n$. We have

$$\|U_1 - U_2\| \leq \|\mathcal{L}\|\|X_1 - X_2\|_\infty$$

$$\leq \|\mathcal{L}\|L_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty.$$

By the same argument as (5.14) it follows that

$$\|U_i^{-1}\| \leq 1/\{\ \rho(\ 1 - \rho\ )\},$$

so that

$$\|U_1^{-1} - U_2^{-1}\| \leq \|U_1^{-1}\|\|U_1 - U_2\|\|U_2^{-1}\|$$

$$\leq \left( \frac{\|\mathcal{L}\|L_1 K^2 \exp(2\delta)}{\rho^2(1-\rho)^2} \right) \|y_1 - y_2\|_\infty .$$

Moreover it follows that

$$\|p_1 - p_2\|_\infty \leq \int_0^{+\infty} \|X_1(t)X_1^{-1}(s) - X_2(t)X_2^{-1}(s)\|\|F(s,y_1(s))\|ds$$

$$+ \int_0^{+\infty} \|X_2(t)X_2^{-1}(s)\|\|F(s,y_1(s)) - F(s,y_2(s))\|ds$$

$$\leq rCL_1K^2\exp(2\delta)\|y_1 - y_2\|_\infty + L_1K\exp(\delta)\|y_1 - y_2\|_\infty$$

$$= L_1 b_2 \|y_1 - y_2\|_\infty .$$

From (5.2) it follows that for $t \in R^+$

$$\|[V(y_1)](t) - [V(y_2)](t)\|$$

$$\leq \|X_1 - X_2\|_\infty \|U_1^{-1}\|[\|\mathcal{L}(y_1) - N(y_1)\| + \|\mathcal{L}\|\|p_1\|_\infty]$$

$$+ \|X_2\|_\infty \|U_1^{-1} - U_2^{-1}\|[\|\mathcal{L}(y_1) - N(y_1)\| + \|\mathcal{L}\|\|p_1\|_\infty]$$

$$+ \|X_2\|_\infty \|U_2^{-1}\|[\|\mathcal{L}(y_1) - \mathcal{L}(y_2)\| + \|N(y_1) - N(y_2)\| + \|\mathcal{L}\|\|p_1 - p_2\|_\infty]$$

$$+ \|p_1 - p_2\|_\infty$$

$$\leq \left( \frac{b_1 L_1 K^2 \exp(2\delta)}{\rho(1-\rho)} \right) \|y_1 - y_2\|_\infty + \left( \frac{\|\mathcal{L}\| b_1 L_1 K^3 \exp(3\delta)}{\rho^2 (1-\rho)^2} \right) \|y_1 - y_2\|_\infty$$

$$+ \left( \frac{K\exp(\delta)}{\rho(1-\rho)} \right) ( \|\mathcal{L}\| + L_2 + \|\mathcal{L}\| L_1 b_2 ) \|y_1 - y_2\|_\infty + L_1 b_2 \|y_1 - y_2\|_\infty$$

$$= k \|y_1 - y_2\|_\infty .$$

Therefore, from (5.22), $V$ is a contraction. This completes the proof.

**Q.E.D.**

# CHAPTER 6

## STABILITY OF SOLUTIONS

### 6.1. Introduction.

The following quasilinear ordinary differential system

$$(N) \qquad x' = A(t,x)x + F(t,x), \qquad F(t,0) \equiv 0$$

is considered in this chapter. Here $A(t,x)$ is a real $n\times n$ matrix continuous on $\mathbf{R}^+\times\mathbf{R}^n$ and $F(t,x)$ is an $\mathbf{R}^n$-valued function continuous on $\mathbf{R}^+\times\mathbf{R}^n$. Together with the above system, the following linear system

$$(L) \qquad x' = B(t)x$$

is concerned, where $B(t)$ is a real $n\times n$ matrix continuous on $\mathbf{R}^+$. The stability of the zero solution of (N) is discussed under the following hypothesis.

**Hypothesis 6.1.** *The zero solution of* (L) *is uniformly asymptotically stable in the large.*

Hypothesis 6.1 holds if and only if the zero solution of (L) is exponentially asymptotically stable in the large ( see [62] ).

In order to investigate the global behavior of solutions of (N), Schauder's fixed point theorem will be applied under the following hypothesis.

**Hypothesis 6.2.** *All the solutions of* (N) *for initial value problems are uniquely determined.*

In what follows we assume that $A(t,x)$ is close to $B(t)$ in the following sense.

**Hypothesis 6.3.** *There exists a constant* $\delta > 0$ *such that*

$$\int_0^{+\infty} \sup_{\|x\| \leqq r} \|A(s,x) - B(s)\| ds \leq \delta$$

*for any* $r \geq 0$.

In Theorem 6.3.1, a sufficient condition that the zero solution of (N) is uniformly asymptotically stable in the large is obtained by Schauder's fixed point theorem. A similar approach to [53 - 56] is used, and Theorem 6.4.1 for the exponentially asymptotic stability in the large of the zero solution for (N) is given by applying Liapunov's second method.

## 6.2. Preliminaries.

**Lemma 6.2.1.** Hypothesis 6.1 *holds if and only if there exist* $K \geq 1$ *and* $\lambda > 0$ *such that*

$$(6.1) \qquad \|X_B(t)X_B^{-1}(\tau)\| \leq K\exp(-\lambda(t-\tau))$$

*for* $t \geq \tau$, *where* $X_B$ *is the fundamental matrix of solutions of* (L) *such that* $X_B(0) = I$. *Here* $I$ *is the identity matrix* (see [23]).

**Lemma 6.2.2.** *Suppose that* Hypotheses 6.1 *and* 6.3 *hold. Then,* *for* $r \geq 0$ *and* $y$ *in* $C(\boldsymbol{R}^+)$ *such that* $\|y\|_\infty \leq r$, *we have*

$$(6.2) \qquad \|X_y(t)X_y^{-1}(\tau)\| \leq K\exp(K\delta - \lambda(t - \tau))$$

*for* $t \geq \tau$, *where* $X_y$ *is a fundamental matrix solutions of* $x' = A(t,y(t))x$ *and* $X_y(0) = I$.

*Proof.* Let $r \geq 0$ and $y \in C(\boldsymbol{R}^+)$ such that $\|y\|_\infty \leq r$. By the variation of parameters formula, we have

$$X_y(t) = X_B(t)X_B^{-1}(\tau)X_y(\tau) + \int_\tau^t X_B(t)X_B^{-1}(s)\{ A(s,y(s)) - B(s) \}X_y(s)ds$$

for $t \geq \tau$. From (6.1) it follows that

$$\|X_y(t)X_y^{-1}(\tau)\|\exp(\lambda t) \leq K\exp(\lambda\tau) + K\int_\tau^t \|A(s,y(s)) - B(s)\|\|X_y(s)X_y^{-1}(\tau)\|\exp(\lambda s)ds$$

for $t \geq \tau$. Thus, by applying Gronwall's lemma, we have

$$\|X_y(t)X_y^{-1}(\tau)\|\exp(\lambda t) \leq K\exp(\lambda\tau + K\int_\tau^t \|A(s,y(s)) - B(s)\|ds),$$

which implies that (6.2) holds. This completes the proof. **Q.E.D.**

## 6.3. Uniformly Asymptotic Stability in the Large.

In this section we consider the below condition (6.3) on $F(t,x)$, which is appeared in [40] and [44]. Under additional conditions a stability theorem is given as follows:

**Theorem 6.3.1.** *Suppose that* Hypotheses 6.1 - 6.3 *hold and that there exists a non-negative number* $C < 1/\{K\exp(K\delta)\}$ *satisfying the following conditions* (6.3) *and* (6.4).

$$(6.3)\qquad \liminf_{r \to +\infty} \frac{1}{r} \int_0^{+\infty} \sup_{\|x\| \leqq r} \|F(s,x)\| ds \leq C;$$

$$(6.4)\qquad \liminf_{r \to +0} \frac{1}{r} \int_0^{+\infty} \sup_{\|x\| \leqq r} \|F(s,x)\| ds \leq C.$$

*Then the zero solution of* (N) *is uniformly asymptotically stable in the large.*

*Proof.* (Uniform Boundedness) Let $\alpha > 0$ be given arbitrarily. By (6.3), there exists a $\beta > \alpha$ satisfying

$$(6.5)\qquad \beta \geq \frac{K\alpha\exp(K\delta)}{1 - CK\exp(K\delta)}$$

and

$$(6.6)\qquad \int_0^{+\infty} \sup_{\|x\| \leqq \beta} \|F(s,x)\| ds \leq \beta C.$$

For $\tau \geq 0$, $\|\xi\| \leq \alpha$, and $i \in \boldsymbol{N}$, we put

$$m_\beta(i) = \max\{\|A(t,x)\|\beta + \|F(t,x)\| : t \in J_i,\ \|x\| \leq \beta\},$$

where $J_i = [\tau, \tau + i]$. We consider the following subset $D_\beta(i)$ in $C(J_i)$, where $C(J_i)$ is the space of continuous functions on $J_i$ with the supremum norm.

$D_\beta(i) = \{y \in C(J_i)$: *y satisfies conditions* (6.7)-(6.9) *below* $\}$.

(6.7) $\quad y(\tau) = \xi$;

(6.8) $\quad \|y(t)\| \leq \beta \quad for\ t \in J_i$;

(6.9) $\quad \|y(t) - y(s)\| \leq m_\beta(i)|t - s| \quad for\ t, s \in J_i$.

It follows that $D_\beta(i)$ is convex and closed. By Ascoli-Arzela's theorem, $D_\beta(i)$ is a compact subset in $C(J_i)$.

Consider an initial value problem

$(N_y)$ $$x' = A(t,y(t))x + F(t,y(t)), \quad x(\tau) = \xi$$

for $y \in D_\beta(i)$. It is easily seen that for $y \in D_\beta(i)$ there exists one and only one solution $x_y$ of $(N_y)$ such that

(6.10) $$x_y(t) = X_y(t)X_y^{-1}(\tau)\xi + \int_\tau^t X_y(t)X_y^{-1}(s)F(s,y(s))ds$$

for $t \in J$. From (6.2), (6.5), (6.6), and (6.10), we obtain $x_y(\tau) = \xi$ and

$$\|x_y(t)\| \leq K\exp(K\delta)\alpha + K\exp(K\delta)\int_0^{+\infty} \sup_{\|y\| \leq \beta}\|F(s,y)\|ds$$

$$\leq K\exp(K\delta)(\alpha + \beta C)$$

$$\leq \beta$$

for $t \in J_i$. Since $x_y(t) = \xi + \int_\tau^t [A(s,y(s))x_y(s) + F(s,y(s))]ds$,

we have

$$\|x_y(t) - x_y(s)\| \leq m_\beta(i)|t - s|$$

for $t,s \in J_i$. Thus, $x_y$ belongs to $D_\beta(i)$ for $y \in D_\beta(i)$.

We can define an operator $V : D_\beta(i) \to D_\beta(i)$ by $V(x) = x_y$. It can be expressed as follows:

$$[V(y)](t) = X_y(t)X_y^{-1}(\tau)\xi + \int_\tau^t X_y(t)X_y^{-1}(s)F(s,y(s))ds$$

for $t \in J_i$. In a similar way to the proof of an analogous part in Theorem 3.3.2 in Chapter 3, we can show that $V$ is continuous on $D_\beta(i)$. By applying Schauder's fixed point theorem, $V$ has at least one fixed point in $D_\beta(i)$. Therefore there exists at least one solution $x(\cdot)$ of (N), which belongs to $D_\beta(i)$. For any $i \in \boldsymbol{N}$, we can choose a solution $x_i(\cdot)$ in $D_\beta(i)$, so that we obtain a sequence $\{x_i\}$, where $x_i$ satisfy the following conditions (6.11) - (6.13).

(6.11) *$x_i$ belongs to $D_\beta(i)$ for $i \in \boldsymbol{N}$;*

(6.12) *$x_i(t)$ satisfies* (N) *for $t \in J_i$;*

(6.13) *$x_i(t) = x_i(\tau+i)$ for $t \geq \tau + i$.*

Let $J = [0,+\infty)$. It is clear that $\{x_i\}$ is uniformly bounded and equicontinuous on any compact interval in $J$. In a similar way to the proof of Ascoli-Arzela's theorem, we obtain some subsequenc of $\{x_i\}$ which converges uniformly on any compact interval in $J$, the limit of which is a solution of (N) passing through $\xi$ at $\tau$. From

Hypothesis 6.2, it follows that for any $\alpha > 0$, there exists a $\beta > 0$ such that if $\tau \geq 0$ and $\|\xi\| \leq \alpha$, then $\|x(t)\| \leq \beta$ for $t \geq \tau$, where $x(\cdot)$ is a unique solution of (N) passing through $\xi$ at $\tau$. This implies that the solutions of (N) are uniformly bounded.

(Uniform Stability) Let $\varepsilon > 0$ be given arbitrarily. From (6.4) there exist $\eta > 0$ and $\eta_1 > 0$, $\eta_1 \leq \eta \leq \varepsilon$, satisfying

$$\int_0^{+\infty} \sup_{\|x\| \leq \eta} \|F(s,x)\| ds \leq \eta C \quad \text{and} \quad \eta_1 \leq \frac{\eta(1 - CK\exp(K\delta))}{K\exp(K\delta)}.$$

For $\tau \geq 0$, $\|\xi\| \leq \eta_1$, and $i \in \boldsymbol{N}$, we consider a constant $m_\eta(i)$ and a subset $D_\eta(i)$ in $C(J_i)$. In a similar to the proof of the uniform boundedness, we can show that the zero solution of (N) is uniformly stable.

(Uniform Attractivity in the Large) For any $\alpha > 0$, we choose a $\beta$ satisfying (6.5) and (6.6). We denote $x(\cdot)$ by the solution of (N) passing through $\xi$ at $\tau$, where any $\tau \geq 0$ and $\|\xi\| \leq \alpha$. Then $\|x(t)\| \leq \beta$ for $t \geq \tau$. Since $x' = B(t)x + [A(t,x) - B(t)]x + F(t,x)$,

$$x(t) = X_B(t)X_B^{-1}(\tau)\xi + \int_\tau^t X_B(t)X_B^{-1}(s)[A(s,x(s)) - B(s)]x(s)ds$$

$$+ \int_\tau^t X_B(t)X_B^{-1}(s)F(s,x(s))ds$$

for $t \geq \tau$. From (6.1) and (6.3), for any sufficiently small $\varepsilon > 0$ there exists a large $T_1 > 0$ satisfying

$$\|X_B(t)X_B^{-1}(\tau)\xi\| \leq \varepsilon/[3\exp(K\delta)]$$

and

$$K\int_{\tau+T_1}^{t}\|F(s,x(s))\|ds \leq \varepsilon/[3\exp(K\delta)]$$

for $t \geq \tau + T_1$. We have

$$K\exp(-\lambda t)\int_{\tau}^{\tau+T_1}\exp(\lambda s)\|F(s,x(s))\|ds \leq \varepsilon/[3\exp(K\delta)]$$

for $t \geq \tau + T_2$, where $T_2 \geq T_1$. Hence

$$\|x(t)\| \leq \varepsilon/\exp(K\delta) + K\int_{\tau}^{t}\|A(s,x(s)) - B(s)\|\|x(s)\|ds$$

for $t \geq \tau + T_2$. By Gronwall's lemma and Hypothesis 6.3, we obtain

$$\|x(t)\| \leq \varepsilon\exp(K\int_{0}^{t}\|A(s,x(s)) - B(s)\|ds - K\delta)$$

$$\leq \varepsilon$$

for $t \geq \tau + T_2$. This completes the proof. **Q.E.D.**

## 6.4. Exponentially Asmyptotic Stability in the Large.

By using Liapunov's second method as well as Schauder's fixed point the following theorem is obtained.

**Theorem 6.4.1.** *Suppose that* Hypotheses 6.1 - 6.3 *hold and that*

$$\int_{0}^{+\infty}\sup_{\|x\|\leq r}\|F(s,x)\|ds \leq rC \tag{6.14}$$

*for all* $r \geq 0$, *where* $C$ *is the same as in* Theorem 6.3.1. *Then the zero solution of* (N) *is exponentially asymptotically stable in the large.*

*Proof.* By (6.14), the inequality (6.3) holds. Let $\alpha > 0$ be given arbitrarily. When $\tau \geq 0$ and $\|\xi\| \leq \alpha$, in a similar to the argument used in the uniform boundedness of Theorem 6.3.1, it can be seen that $\|x(t)\| \leq b$ for $t \geq \tau$, where $x(\cdot)$ is the solution of (N) passing through $\xi$ at $\tau$ and

$$b = \frac{\|\xi\| K\exp(K\delta)}{1 - CK\exp(K\delta)}.$$

We denote

$$\eta(t) = \|A(t,x(t)) - B(t)\|$$

for $t \geq \tau$ and

$$U(t,y) = \sup\{ \ \|\phi(t+s;t,y)\|\exp(\lambda s) \ : \ s \geq 0 \ \}$$

for $t \geq \tau$ and $\|y\| \leq b$, where $\phi(\cdot;t,y)$ is the solution of (L) passing through $y$ at $t$. From Theorem 19.1 in [62], we have

$$\|y\| \leq U(t,y) \leq K\|y\|,$$

$$|U(t,y_1) - U(t,y_2)| \leq K\|y_1 - y_2\|,$$

and

$$U'_{(L)}(t,y) \leq - \lambda U(t,y)$$

for $t \geq \tau$, and $\|y\|$, $\|y_1\|$, $\|y_2\| \leq b$.

Consider the following function

$$W(t,y) = U(t,y)\exp\left( - K\int_{\tau}^{t}\eta(s)ds \right)$$

for $t \geq \tau$ and $\|y\| \leq b$. By the same way as the proof of Theorem 24.1 in [62], we have

$$\exp( - K\delta )\|y\| \leq W(t,y) \leq K\|y\|,$$

$$|W(t,y_1) - W(t,y_2)| \leq K\|y_1 - y_2\|$$

for $t \geq \tau$ and $\|y\|$, $\|y_1\|$, $\|y_2\| \leq b$. Moreover it can be shown that $W(t,x)$ is continuous in $(t,x)$. Since $x' = B(t)x + [A(t,x) - B(t)]x + F(t,x)$, we have

$$W'_{(N)}(t,x(t)) \leq \exp\left(-K\int_{\tau}^{t}\eta(s)ds\right)[ - K\eta(t)U(t,x(t)) + U'_{(L)}(t,x(t))$$

$$+ K\|A(t,x(t)) - B(t)\|\|x(t)\| + K\|F(t,x(t))\| ]$$

$$\leq \exp\left(-K\int_{\tau}^{t}\eta(s)ds\right)( - K\eta(t)U(t,x(t)) - \lambda U(t,x(t))$$

$$+ K\eta(t)U(t,x(t)) + K\|F(t,x(t))\| )$$

$$\leq - \lambda W(t,x(t)) + K\|F(t,x(t))\|$$

for $t \geq \tau$, and hence

$$W(t,x(t)) \leq \left[ W(\tau,\xi) + \frac{\|\xi\|CK^2\exp(K\delta)}{1 - CK\exp(K\delta)} \right]\exp(- \lambda(t - \tau)).$$

Therefore

$$\|x(t)\| \leq \exp(K\delta)\left( K + \frac{CK^2\exp(K\delta)}{1 - CK\exp(K\delta)} \right)\|\xi\|\exp(-\lambda(t-\tau))$$

$$= \frac{K\exp(K\delta)\|\xi\|}{1 - CK\exp(K\delta)}\exp(-\lambda(t-\tau))$$

for $t \geq \tau$, which implies that the zero solution of (N) is exponentially asymptotically stable in the large. This completes the proof. **Q.E.D.**

# CHAPTER 7

## ASYMPTOTIC EQUIVALENCE

### 7.1. Introduction.

The following linear ordinary differential system (L) and quasilinear system (N)

(L) $$x' = B(t)x$$

(N) $$x' = A(t,x)x + F(t,x)$$

are considered in this chapter. Here $B(t)$ is a real $n\times n$ matrix continuous on $\mathbf{R}^+$, and $A(t,x)$ and $F(t,x)$ are both continuous on $\mathbf{R}^+\times\mathbf{R}^n$. The asymptotic equivalence between (L) and (N) is treated under the following hypothesis.

**Hypothesis 7.1.** *There exists a constant $K \geq 1$ such that*

(7.1) $$\|X_B(t)X_B^{-1}(\tau)\| \leq K$$

*for $t \geq \tau \geq 0$, where $X_B$ is the fundamental matrix of* (L) *such that $X_B(0) = I$. Here $I$ is the identity matrix.*

Hypothesis 7.1 holds (i.e., the zero solution of (L) is uniformly stable) if and only if all the solutions of (L) are uniformly bounded (see [23]). It is assumed that $A(t,x)$ is sufficiently close to $B(t)$ in the following sense.

**Hypothesis 7.2.** *There exists a constant $\delta > 0$ such that*

$$\int_0^{+\infty} \sup_{\|x\| \leq r} \|A(s,x) - B(s)\| ds \leq \delta$$

*for any* $r \geq 0$.

Here $A(t,x)$ and $F(t,x)$ are not necessarily Lipschitz continuous. However, the following hypothesis is assumed.

**Hypothesis 7.3.** *All the solutions of* (N) *for initial value problems are uniquely determined.*

In what follows sufficient conditions for the asymptotic equivalence are given by a similar approach to [47] and [55] as well as by using Schauder's fixed point theorem. In Theorem 7.3.1, under a condition which is concerned with the integral of $F(t,x)$ in the neighborhood of the origin, the uniform stability of the zero solution for (N) and the asymptotic equivalence between (L) and (N) are proved. In Theorem 7.3.2 the condition on $F(t,x)$, which is considered for the existence of solutions of (N) in [40] and [44], ensures the uniform boundedness of solutions for (N) and the asymptotic equivalence between the two systems.

## 7.2. Preliminaries.

**Lemma 7.2.1.** *Suppose that* Hypotheses 7.1 - 7.2 *hold. Then, for any* $r \geq 0$ *and* $y \in C(\boldsymbol{R}^+)$ such that $\|y\|_\infty \leq r$, *we have*

(7.2) $$\|X_y(t) X_y^{-1}(\tau)\| \leq K\exp(K\delta)$$

*for* $t \geq \tau \geq 0$, *where* $X_y$ *is a fundamental matrix solutions of*

$x' = A(t,y(t))x$ *and* $X_y(0) = I$.

*Proof.* Since $X_y'(t) = B(t)X_y(t) + [A(t,y(t)) - B(t)]X_y(t)$, by the variation of parameters formula, we have

$$X_y(t) = X_B(t)X_B^{-1}(\tau)X_y(\tau)$$

$$+ X_B(t)\int_\tau^t X_B^{-1}(s)[A(s,y(s)) - B(s)]X_y(s)ds$$

for $t \geq \tau \geq 0$. It follows that, by (7.1),

$$\|X_y(t)X_y^{-1}(\tau)\| \leq K + K\int_\tau^t \|A(s,y(s)) - B(s)\|\|X_y(s)X_y^{-1}(\tau)\|ds$$

for $t \geq \tau \geq 0$. From Gronwall's lemma, we get

$$\|X_y(t)X_y^{-1}(\tau)\| \leq K\exp\left( K\int_\tau^t \|A(s,y(s)) - B(s)\|ds \right)$$

$$\leq K\exp(K\delta)$$

for $t \geq \tau \geq 0$. This completes the proof. **Q.E.D.**

By Schauder's fixed point theorem the uniform stability of the zero solution of (N), with $F(t,0) \equiv 0$, is shown as follows:

**Lemma 7.2.2.** *Let* Hypotheses 7.1 - 7.3 *hold. Suppose that* $F(t,0) \equiv 0$ *and that there exists a positive number* $C < 1/\{K\exp(K\delta)\}$ *such that*

$$\liminf_{r \to +0} \frac{1}{r} \int_0^{+\infty} \sup_{\|x\| \leq r} \|F(s,x)\|ds \leq C. \tag{7.3}$$

*Then the zero solution of* (N) *is uniformly stable.*

The above lemma can be proved in the same way as the proof of Theorem 6.3.1 in Chapter 6. Furthermore the following lemma is obtained.

**Lemma 7.2.3.** *Let* Hypotheses 7.1 - 7.3 *hold. Suppose that the following condition is satisfied.*

$$(7.4) \qquad \liminf_{r \to +\infty} \frac{1}{r} \int_0^{+\infty} \sup_{\|x\| \leq r} \|F(s,x)\| ds \leq C,$$

*where* $C < 1/\{K\exp(K\delta)\}$. *Then the solutions of* (N) *are uniformly bounded.*

### 7.3. Asymptotic Equivalence.

In the following theorem the asymptotic equivalence between (L) and (N), zero solutions of which are uniformly stable, respectively, is shown by Schauder's fixed point theorem.

**Theorem 7.3.1.** *Suppose that the assumption in* Lemma 7.2.2 *holds. Then* (L) *and* (N) *are asymptotically equivalent.*

*Proof.* Let $a_0 > 0$ satisfy $a_0 < 1 - CK\exp(K\delta)$. From Hypothesis 7.1 and (7.3), there exist positive numbers $a_1$ and $\eta$ ($a_1 > \eta$) such that, when $\tau \geq 0$ and $\|\xi\| \leq \eta$,

$$(7.5) \qquad \|x_1(t)\| \leq \frac{a_1(1 - CK\exp(K\delta) - a_0)}{K\exp(K\delta)}$$

for $t \geq \tau$, and that

$$(7.6) \qquad \int_0^{+\infty} \sup_{\|x\| \leq a_1} \|F(s,x)\| ds \leq a_1 C.$$

Here $x_1(\cdot)$ is the solution of (L) passing through $\xi$ at $\tau$. We shall show that there exists a solution $x(\cdot)$ of (N) such that

$$\|x(t) - x_1(t)\| \to 0$$

as $t \to +\infty$, and $\|x(t)\| \leq a_1$ for $t \geq \tau$. Let $\{ \varepsilon_i > 0 \}$ be a sequence such that

$$(7.7) \qquad \varepsilon_0 \leq \min\left( a_1 - \eta, \frac{a_0 a_1}{K\exp(K\delta)} \right) \text{ and that } \varepsilon_i \leq \frac{\varepsilon_{i-1}}{3K}$$

for $i \in \boldsymbol{N}$. From Hypothesis 7.1 and (7.3), there exists a divergent sequence $\{ t_i \geq \tau : t_0 = \tau, t_i > t_{i-1} \text{ for } i \in \boldsymbol{N} \}$ such that

$$(7.8) \qquad a_1 \int_{t_{i-1}}^{+\infty} \sup_{\|x\| \leq a_1} \|A(s,x) - B(s)\| ds \leq \varepsilon_i$$

and that

$$(7.9) \qquad \int_{t_{i-1}}^{+\infty} \sup_{\|x\| \leq a_1} \|F(s,x)\| ds \leq \varepsilon_i$$

for $i \in \boldsymbol{N}$. Let $J_i = [\tau, t_i]$ and $J = [\tau, +\infty)$. We define a number

$$m(i) = \max\{ \|A(t,x)\| a_1 + \|F(t,x)\| : t \in J_i, \|x\| \leq a_1 \}$$

and denote the following sets.

$$C(i) = \{\ z \in C(J) : z \in C(i\text{-}1),\ \|z(t) - z(r)\| \le m(i)|t - r|$$
$$\text{for } t, r \in J_i, \text{ and } \|z(t) - x_1(t)\| \le \varepsilon_i \text{ for } t \ge t_i\ \}$$

for $i \in \boldsymbol{N}$, where

$$C(0) = \{\ z \in C(J) : \|z(t)\| \le a_1 \text{ for } t \in J\ \}.$$

Let $D(i)$ be the subset in $C(J_i)$ such that, for any $y \in D(i)$, there exists a $z \in C(i)$ satisfying $z(t) = y(t)$ for $t \in J_i$. Then $D(i)$ is convex and closed. By Ascoli-Arzela's theorem, it is a relatively compact subset in $C(J_i)$ for $i \in \boldsymbol{N}$.

Consider an initial value problem

$$(\mathrm{N}_y) \qquad x' = A(t, y(t))x + F(t, y(t)), \quad x(t_i) = y(t_i)$$

for $t \in J_i$, where $y \in D(i)$. There exists one and only one solution $x_y$ of $(\mathrm{N}_y)$, which belongs to $C(J_i)$, such that

$$(7.10) \qquad x_y(t) = X_y(t)X_y^{-1}(t_i)y(t_i) + \int_{t_i}^{t} X_y(t)X_y^{-1}(s)F(s, y(s))ds$$

for $t \in J_i$. It follows that, by (7.2), (7.5), (7.6), (7.7), and the definition of $D(i)$,

$$\|x_y(t)\| \le K\exp(K\delta)\|y(t_i)\| + K\exp(K\delta)\int_0^{+\infty} \sup_{\|y\| \le a_1}\|F(s, y)\|ds$$

$$\le K\exp(K\delta)\left( \frac{a_1(1 - CK\exp(K\delta) - a_0)}{K\exp(K\delta)} + \varepsilon_i + a_1 C \right)$$

$$< a_1$$

for $t \in J_i$. Since $x_y(t) = y(t_i) + \int_{t_i}^{t} [A(s,y(s))x_y(s) + F(s,y(s))]ds$

for $t \in J_i$, we obtain

$$\|x_y(t) - x_y(r)\| \leq \left| \int_r^t [\|A(s,y(s))\|a_1 + \|F(s,y(s))\|]ds \right|$$

(7.11)

$$\leq m(i)|t - r|$$

for $t$, $r \in J_i$. We shall show that $x_y$ belongs to $D(i-1)$. It suffices that $\|x_y(t) - x_1(t)\| \leq \varepsilon_{i-1}$ for $t \in I_i = [t_{i-1}, t_i]$. Since

$$x_y(t) = X_B(t)X_B^{-1}(t_i)y(t_i) + X_B(t)\int_{t_i}^{t} X_B^{-1}(s)[A(s,y(s)) - B(s)]x_y(s)ds$$

(7.12)

$$+ X_B(t)\int_{t_i}^{t} X_B^{-1}(s)F(s,y(s))ds$$

and $x_1(t) = X_B(t)X_B^{-1}(t_i)x_1(t_i)$ for $t \in I_i$, we have, by the definition of $D(i)$, (7.1), (7.7), (7.8), and (7.9),

$$\|x_y(t) - x_1(t)\| \leq K\varepsilon_i + Ka_1\int_{t_{i-1}}^{t_i} \|A(s,y(s)) - B(s)\|ds + K\int_{t_{i-1}}^{t_i} \|F(s, y(s))\|ds$$

(7.13) $$\leq 3K\varepsilon_i$$

$$\leq \varepsilon_{i-1}$$

for $t \in I_i$, and hence $x_y \in D(i)$. We can define an operator $V : D(i) \to D(i)$ by $V(y) = x_y$. It can be seen that

$$[V(y)](t) = X_y(t)X_y^{-1}(t_i)y(t_i) + X_y(t)\int_{t_i}^{t} X_y^{-1}(s)F(s,y(s))ds$$

for $t \in J_i$, then $V$ is continuous on $D(i)$. As $n \to +\infty$, if $y_n \to y_0$ in $D(i)$, then $X_{y_n}(t) \to X_{y_0}(t)$ uniformly for $t \in J_i$. By using Schauder's fixed point theorem, $V$ has at least one fixed point $x_f$ in $D(i)$ for any $i \in \boldsymbol{N}$. From the definition of $D(i)$, there exists a $z \in C(i)$ such that $z(t) = x_f(t)$ for $t \in J_i$. Let $x_i = z$. Then we obtain a sequence $\{x_i\}$, where $\|x_i(t)\| \leq a_1$ for $t \in J_i$, and $\{ x_i \}$ is equicontinuous on any compact interval in $[\tau,+\infty)$. In a similar way to the proof of Ascoli-Arzela's theorem, there exists a subsequence, which limit is a solution $x$ of (N) as well as satisfies

(7.14) $$\|x(t) - x_1(t)\| \to 0$$

as $t \to +\infty$.

Conversely, by Lemma 7.2.2, there exists a positive number $\eta < a_1$ such that, when $\tau \geq 0$ and $\|\xi\| \leq \eta$, the solution of (N)

$$\|y(t;\tau,\xi)\| \leq \frac{a_1(1 - CK\exp(K\delta) - a_0)}{K\exp(K\delta)},$$

where $a_0$ and $a_1$ are the same as in (7.5). We denote $y(\cdot)$ by the solution $y(\cdot;\tau,\xi)$ and $X_y$ the fundamental matrix for solutions of $x' = A(t,y(t))x$.

Consider the following sequence $\{ \varepsilon_i > 0 \}$ such that

$$\varepsilon_0 \leq \left( a_1 - \eta, \; \frac{a_0 a_1}{K\exp(K\delta)} \right) \quad \textit{and that } \varepsilon_i \leq \frac{\varepsilon_{i-1}}{3K\exp(K\delta)},$$

and $\{ t_i \geq \tau \}$ satisfying the foregoing (7.8) and (7.9). We define

$$m_1(i) = \max\{ \|B(t)\| a_1 : t \in J_i, \; \|x\| \leq a_1 \}$$

and the following sets $C_1(i)$, $D_1(i)$ ( for $i \in \mathbf{N}$ ) and $C_1(0)$.

$$C_1(i) = \{ z \in C(J) : z \in C_1(i-1), \; \|z(t) - z(r)\| \leq m_1(i)|t - r| \text{ for } t,r \in J_i, \text{ and } \|z(t) - y(t)\| \leq \varepsilon_i \text{ for } t \geq t_i \};$$

$$C_1(0) = \{ z \in C(J) : \|z(t)\| \leq a_1 \text{ for } t \in J \};$$

$$D_1(i) = \{ u \in C(J_i) : \text{there exists a } z \in C_1(i) \text{ such that } z(t) = u(t) \text{ for any } t \in J_i \}.$$

For any $i \in \mathbf{N}$ and $u \in D_1(i)$, there exists one and only one solution $x_u$ for the following initial value problem

$$x' = B(t)x, \quad x(t_i) = u(t_i),$$

where $t \in J_i$. Since $x_u(t) = X_B(t)X_B^{-1}(t_i)u(t_i)$, we have

$$\|x_u(t)\| \leq K\left( \frac{a_1(1 - CK\exp(K\delta) - a_0)}{K\exp(K\delta)} + \varepsilon_i \right) \leq a_1$$

for $t \in J_i$. This yields

$$\|x_u(t) - x_u(r)\| \leq \left| \int_r^t \|B(s)\| a_1 \, ds \right|$$

$$\leq m_1(i)|t - r|$$

for $t, r \in J_i$. Furthermore

$$\|x_u(t) - y(t)\| \leq \varepsilon_{i-1}$$

for $t \in I_i$. In fact, it is follows that

$$x_u(t) = X_y(t)X_y^{-1}(t_i)u(t_i) + X_y(t)\int_{t_i}^{t} X_y^{-1}(s)[B(s)-A(s,y(s))]x_u(s)ds$$

and

$$y(t) = X_y(t)X_y^{-1}(t_i)y(t_i) + X_y(t)\int_{t_i}^{t} X_y^{-1}(s)F(s,y(s))ds$$

for $t \in I_i$, we obtain

$$\|x_u(t) - y(t)\| \leq K\exp(K\delta)\|y(t_i) - u(t_i)\|$$

$$+ K\exp(K\delta)a_1\int_{t_{i-1}}^{t_i} \|B(s) - A(s,y(s))\|ds + K\exp(K\delta)\int_{t_{i-1}}^{t_i} \|F(s,y(s))\|ds$$

$$\leq \varepsilon_{i-1}$$

for $t \in I_i$. Hence $x_u$ belongs to $D_1(i)$. In the same argument used in (7.14), there exists a solution $x$ of (L) such that

$$\|x(t) - y(t)\| \to 0$$

as $t \to +\infty$. This completes the proof. **Q.E.D.**

Under the condition that (7.4) holds, the asymptotic equivalence between (L) and (N), solutions of which are uniformly bounded, respectively, is proved by Schauder's fixed point theorem.

**Theorem 7.3.2.** *Suppose that the same assumption in* Lemma 7.2.3 *holds. Then* (L) *and* (N) *are asymptotically equivalent.*

*Proof.* From Hypothesis 7.1, for any $\alpha > 0$, there exists a $\beta > 0$ such that, when $\tau \geq 0$ and $\|\xi\| \leq \alpha$, the solution of (N)

$$\|x_1(t;\tau,\xi)\| \leq \beta$$

for $t \geq \tau$. Let $x_1(\cdot) = x_1(\cdot;\tau,\xi)$. By (7.4) we can choose a $b > \beta$ such that

$$\int_0^{+\infty} \sup_{\|x\| \leq b} \|F(s,x)\| ds \leq bC.$$

Let $\{ \varepsilon_i > 0 \}$ be a sequence satisfying $\varepsilon_0 \leq b - \beta$ and $\varepsilon_i \leq \varepsilon_{i-1}/(3K)$ for $i \in \boldsymbol{N}$. From Hypothesis 7.1 and (7.4), there exists a divergent sequence $\{ t_i \geq \tau : t_0 = \tau,\ t_i > t_{i-1}$ for $i \in \boldsymbol{N} \}$ such that

$$b(1 + C)\exp(K\delta)\int_{t_{i-1}}^{+\infty} \sup_{\|x\| \leq b} \|A(s,x) - B(s)\| ds \leq \varepsilon_i$$

and that

$$\int_{t_{i-1}}^{+\infty} \sup_{\|x\| \leq b} \|F(s,x)\| ds \leq \varepsilon_i$$

for $i \in \boldsymbol{N}$. We define a number $n(i)$ and sets $E(i)$ (for $i \in \boldsymbol{N}$), $E(0)$ as follows:

$n(i) = \max\{ \|A(t,x)\|b + \|F(t,x)\| : t \in J_i = [\tau, t_i],\ \|x\| \le b \};$

$E(i) = \{ z \in C(J) : z \in E(i-1),\ \|z(t) - z(r)\| \le n(i)|t - r|$

$\quad \text{for } t,\ r \in J_i, \text{ and } \|z(t) - x_1(t)\| \le \varepsilon_i \text{ for } t \ge t_i \};$

$E(0) = \{ z \in C(J) : \|z(t)\| \le b \text{ for } t \in J = [\tau, +\infty) \}.$

Let $F(i)$ be the subset in $C(J_i)$ such that, for any $y \in F(i)$, there exists a $z \in E(i)$ such that $z(t) = y(t)$ for $t \in J_i$. $F(i)$ is a convex and compact subset in $C(J_i)$ for $i \in \mathbf{N}$.

Consider an initial value problem

$$(\mathrm{N}_y) \quad x' = A(t,y(t))x + F(t,y(t)), \qquad x(t_i) = y(t_i)$$

for $t \in J_i$, where $y \in F(i)$. The solution $x_y$ of the above problem satisfies (7.12) for $i \in \mathbf{N}$ and $t \in I_i = [t_{i-1}, t_i]$. We have

$$\|x_y(t)\| \le K(\|x_1(t)\| + \varepsilon_{i-1}) + K\int_t^{t_i} \|A(s,y(s)) - B(s)\|\,\|x_y(s)\|ds$$

$$+ K\int_t^{t_i} \|F(s,y(s))\|ds$$

$$\le bK + K\int_t^{t_i} \|A(s,y(s)) - B(s)\|\,\|x_y(s)\|ds + bCK,$$

so that, by using Gronwall's lemma, we obtain

$$\|x_y(t)\| \le bK(1 + C)\exp(K\delta)$$

for $t \in I_i$. Moreover $\|x_y(t)\| \le b$ for $t \in J_i$. In a similar way to (7.13), we have

$$\|x_y(t) - x_1(t)\| \leq K\varepsilon_i + bK(1 + C)\exp(K\delta)\int_{t_{i-1}}^{t_i}\|A(s,y(s)) - B(s)\|ds$$

$$+ K\int_{t_{i-1}}^{t_i}\|F(s,y(s))\|ds$$

$$\leq \varepsilon_{i-1}$$

for $i \in \boldsymbol{N}$ and $t \in I_i$, and hence

$$\|x_y(t)\| \leq \|x_1(t)\| + \varepsilon_{i-1} \leq \beta + \varepsilon_0 \leq b$$

for $t \in J_i$. In a similar way to (7.11), we get

$$\|x_y(t) - x_y(r)\| \leq n(i)|t - r|$$

for $t$, $r \in J_i$. Therefore $x_y$ belongs to $F(i)$.

In the same manner as (7.14), there exists a solution $x$ of (N) such that

$$\|x(t) - x_1(t)\| \to 0$$

as $t \to +\infty$.

From Lemma 7.2.3, for any $\alpha > 0$, there exists a $\beta_1 > 0$ such that, when $\tau \geq 0$ and $\|\xi\| \leq \alpha$, the solution $y$ of (N) satisfies $\|y(t)\| \leq \beta_1$ for $t \geq \tau$, where $y(\tau) = \xi$. In the same way as (7.14), there exists a solution $x$ of (L) such that

$$\|x(t) - y(t)\| \to 0$$

as $t \to +\infty$. This completes the proof. **Q.E.D.**

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