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MIXED TORELLI PROBLEM FOR TODOROV SURFACES

Dedicated to Professor H. Hironaka on the occasion of his sixtieth birthday

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Introduction

There is an approach to the Torelli problem by using degeneracy loci. Namikawa and Friedman succeeded to prove the generic Torelli theorem for curves [21] and the Torelli theorem for algebraic $K3$ surfaces [14] respectively in this direction.

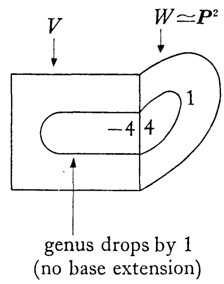
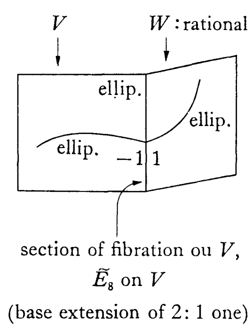
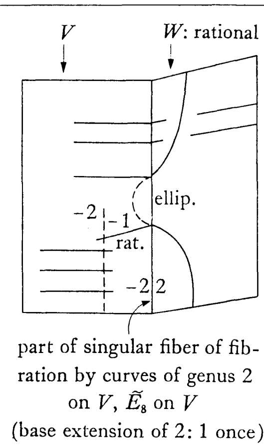
In case of Todorov surfaces X, since the period map sending X to the Hodge structure on $H^2(X)$ has positive dimensional fibers ([29], [30], [31], [32], [33]) it is necessary to consider the mixed period map which sends X to the mixed Hodge structure on $H^2(X-C)$, where C is the unique canonical curve of X ([34], [23]). On the other hand, we can observe that Todorov surfaces are connected by "tame" degenerations and smooth deformations. It is the purpose of the present paper to try to solve mixed Torelli problem for Todorov surfaces by using the "tame" degenerations. At present we have formulated the problem inductively and obtained some results but we have not yet arrived at the final destination.

We give examples of "tame" degenerations of double covers of surfaces as Table 0 on the next page. Degenerations of type (I_1) in Table 0 are observed for Todorov surfaces and surfaces with $c_1^2=2p_g-3$, type (I_2) are observed for Kunev surfaces, and (II_1) are observed for surfaces on the Noether line ([36], [37]). Recently these phenomena are observed more widely ([18], [4], [2], [3]). So our present trial can be seen as a miniature of a more ambitious attempt, namely, to attack (mixed) Torelli problem for surfaces of general type via degeneracy loci.

§1 is a Hodge theoretic preliminary. We recall, after [28], the constructions of (filtered) cohomological mixed Hodge complexes whose hypercohomologies yield the terms in a mixed version of the Clemens-Schmid sequence. We distinguish filtrations corresponding to the openness of the varieties in question and to their singularity and see their relationships. We prove partial results on the exactness of the mixed Clemens-Schmid sequence.

§2 contains an observation that the moduli spaces of Todorov surfaces are

Table 0

degeneration of branch locus	central fiber of semi-stable degeneration of pairs: (X_0, Y_0), $X_0=V+W$	change of (p_g, q, c_1^2) of V	local monodromy on $H^2(X_\infty)$
(I_1) passing an isolated branch point A_1		$(0, 0, -1)$	I
(I_2) passing D_4		$(0, +1, -1)$	I
(II_1) having ordinary quadruple point		$(-1, 0, -1)$	II

connected by "tame" degenerations, i.e., type (I_1) in Table 0. We use the results in [20].

In §3, we recall the moduli spaces of Todorov surfaces contructed in [20] and the formulation of a mixed period map in [34]. We give a candidate of a global monodromy.

In §4, we prove the splitting of the local monodromy over $\boldsymbol{Z}$ by using the result in §1 and extend the mixed period map over the "tame" degenerations.

§5 contains a useful result in the induction step of our framework. We also prove partially the infinitesimal mixed Torelli theorem for the extended mixed period map.

1. Mixed version of Clemens-Schmid sequence

(1.1) Let

(1.1.1) $$f\colon (\mathcal{X}, \mathcal{Y}) \to \Delta$$

be a semi-stable degeneration of pairs, i.e., $\mathcal{X}$ is a submanifold of $\boldsymbol{P}^N \times \Delta$, the restriction of the projection $f\colon \mathcal{X} \to \Delta$ is a flat morphism over a disc Δ whose fiber $X_t := f^{-1}(t)$ over $t \in \Delta$ is smooth for $t \neq 0$ and X_0 is a reduced divisor with simple normal crossings, $\mathcal{Y}$ is a reduced divisor of $\mathcal{X}$ flat with respect to f, and $X_0 + \mathcal{Y}$ has simple normal crossings. Any projective 1-parameter degeneration of pairs can be reduced to this case after a finite base extension (cf. [17, II], [23, I.9]).

We use the following notation:

(1.1.2)
$$\begin{aligned}
&X_t := f^{-1}(t) \quad (t \in \Delta), && Y_t := \mathcal{Y} \cap X_t,\\
&\mathcal{X}^* := \mathcal{X} - X_0, && \mathcal{Y}^* := \mathcal{Y} - Y_0,\\
&\mathring{\mathcal{X}} := \mathcal{X} - \mathcal{Y}, && \mathring{\mathcal{X}}^* := \mathring{\mathcal{X}} \cap \mathcal{X}^*,\\
&\Delta^* := \Delta - \{0\}, && \tilde{\Delta}^* \to \Delta^* \quad \text{universal cover } u \mapsto \exp(2\pi\sqrt{-1}u),\\
&X_\infty := \mathcal{X}^* \times_{\Delta^*} \tilde{\Delta}^*, && Y_\infty := \mathcal{Y}^* \times_{\Delta^*} \tilde{\Delta}^*,\\
&\mathring{X}_\infty := X_\infty - Y_\infty .
\end{aligned}$$

For a variety Z with simple normal crossings, we use the following notation:

$$\begin{aligned}
&Z^{(i)}\colon \text{locus of points in } Z \text{ of multiplicity} \geqslant i .\\
&a\colon \tilde{Z}^{(i)} \to Z^{(i)} \subset Z \quad \text{the normalization.}
\end{aligned}$$

We consider the diagram

(1.1.3)
$$\begin{array}{ccc}
\mathring{\mathcal{X}}^* & \xrightarrow{\ \mathring{j}\ } & \mathring{\mathcal{X}}\\
\big\downarrow \mathring{\imath} & & \big\downarrow \imath\\
\mathcal{X}^* & \xrightarrow{\ j\ } & \mathcal{X}
\end{array}$$
$$\mathring{f} := f|_{\mathring{\mathcal{X}}}\colon \mathring{\mathcal{X}} \to \Delta$$

Since (1.1.1) is locally C^∞-trivial over Δ^*, $R^n \mathring{f}_* \boldsymbol{Q}_{\mathring{\mathcal{X}}^*}$ is a local system and the Gysin filtration G induced from the canonical filtration τ (see [10, II. (1.4.6)]) of $R\mathring{\iota}_* \boldsymbol{Q}_{\mathring{\mathcal{X}}^*}$ (i.e., the Leray filtration for $\mathring{f}$: $\mathring{\mathcal{X}}^* \xrightarrow{\mathring{\iota}} \mathcal{X}^* \xrightarrow{f} \Delta^*$) consists of local subsystems. We denote by

$$(1.1.4) \qquad (\mathcal{V}, G, \nabla), \quad \mathcal{V} := R^n \mathring{f}_* \boldsymbol{Q}_{\mathring{\mathcal{X}}^*} \otimes_{\boldsymbol{Q}} \mathcal{O}_{\Delta^*}$$

the associated filtered vector bundle with the Gauss-Manin connection.

Since $\mathring{f}^{-1}\mathcal{O}_{\Delta^*} \to \Omega^{\boldsymbol{\cdot}}_{\mathring{\mathcal{X}}^*/\Delta^*}$ is a resolution and $\mathring{\iota}$ in (1.1.3) is Stein, $R\mathring{\iota}_* \mathring{f}^{-1}\mathcal{O}_{\Delta^*}$ is represented by $\mathring{\iota}^m_* \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*/\Delta^*}$ hence by $\Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*)$ [9,II. (3.3.1), (3.14.i)], which together with the canonical filtration τ and with the weight filration $W(\mathcal{Y}^*)$ are filtered quasi-isomorphic [10,II. (3.1.8)]. Therefore,

$$(1.1.5) \qquad (\mathcal{V}, G) \simeq (R^n f_* \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*), W(\mathcal{Y}^*)).$$

By the same reasoning, an exact sequence

$$(1.1.6) \qquad 0 \to R\mathring{\iota}_* \mathring{f}^{-1}\Omega^1_{\Delta^*}[-1] \to R\mathring{\iota}_* \mathring{f}^{-1}\Omega^{\boldsymbol{\cdot}}_{\Delta^*} \to R\mathring{\iota}_* \mathring{f}^{-1}\mathcal{O}_{\Delta^*} \to 0$$

is represented by

$$(1.1.7) \qquad \begin{aligned} &0 \to f^{-1}\Omega^1_{\Delta^*} \otimes_{f^{-1}\mathcal{O}_{\Delta^*}} \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*)\,[-1] \to \\ &\qquad \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*}(\log \mathcal{Y}^*) \to \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}^*/\Delta^*}(\log \mathcal{Y}^*) \to 0, \end{aligned}$$

hence we see that the Gauss-Manin connection ∇ of $\mathcal{V}$ is induced as the connecting homomorphism of the hypercohomology sequence of (1.1.7) [16].

The following lemma can be found in [9,II. (5.2), (7.11)], [27, (2.16)] and [28, (5.3)].

Lemma (1.1.8). $\tilde{\mathcal{V}} := R^n f_* \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))$ *is the canonical extension of* $(\mathcal{V}, G, \nabla)$, *i.e., the following hold:*

(i) $\tilde{\mathcal{V}}$ *is a vector bundle on* Δ *with* $\tilde{\mathcal{V}}|_{\Delta^*} = \mathcal{V}$.

(ii) G *on* $\mathcal{V}$ *extends uniquely to a filtration of* $\tilde{\mathcal{V}}$ *by subbundles, also denoted by* G.

(iii) ∇ *extends to a connection of* $\tilde{\mathcal{V}}$ *with logarithmic pole at* $0 \in \Delta$ *with* $\mathrm{Res}_0(\nabla)$ *nilpotent.*

Idea of Proof. (i) and (ii) follow from a fundamental observaiton: For $X_\infty \xrightarrow{k} \mathcal{X} \xleftarrow{i} X_0$ and $u = \log(t/2\pi\sqrt{-1})$,

$$(1.1.9) \qquad \begin{aligned} \Omega^{\boldsymbol{\cdot}}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \otimes \mathcal{O}_{X_0} &\underset{QIS}{\overset{\psi_t}{\leftarrow}} i^{-1}\Omega^{\boldsymbol{\cdot}}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))\,[u] \\ &\underset{QIS}{\to} i^{-1} k_* \Omega^{\boldsymbol{\cdot}}_{X_\infty}(\log Y_\infty), \end{aligned}$$

where QIS means quasi-isomorphic and

$$\psi_t(\sum \omega_j u^j) := (\text{image of } \omega_0)\,. \tag{1.1.10}$$

(iii) follows from an exact sequence

$$\begin{aligned} 0 \to f^{-1}\Omega^1_\Delta(\log 0)\otimes_{f^{-1}\mathcal{O}_\Delta} \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\,[-1] \to \\ \Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0)) \to \Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0)) \to 0\,, \end{aligned} \tag{1.1.11}$$

which is an extension of (1.1.7), and a direct computation of the residue. For details, see the above references. ∎

(1.2) We recall the construction of the mixed version of the Steenbrink complex $A^{\cdot}$ in [28, §5] (see also [22, §14], [12]). In the situation and the notation in (1.1), we consider a diagram

$$\begin{array}{ccccc} \mathring{X}_\infty & \xrightarrow{\mathring{k}} & \mathring{\mathcal{X}} & & \\ \downarrow & & \downarrow{\scriptstyle \ell} & & \\ X_\infty & \xrightarrow{k} & \mathcal{X} & \xleftarrow{i} & X_0 \end{array} \tag{1.2.1}$$

$$k' := \ell\mathring{k} : \mathring{X}_\infty \to \mathring{\mathcal{X}}$$

By the Eilenberg-Zilber theorem [26, p.232], we see

$$k'_*\Delta^{\cdot}(\mathring{X}_\infty) \underset{QIS}{\simeq} s^{\cdot}(\ell_*\Delta^{\cdot}(\mathring{\mathcal{X}})\otimes_{\boldsymbol{Q}} k_*\Delta^{\cdot}(X_\infty))$$

[28, (5.20)], where $\Delta^{\cdot}(Z)$ is the complex of sheaves of germs of singular $\boldsymbol{Q}$-cochains on a topological space Z. Since $\Delta^{\cdot}(Z)$ is a fine resolution of $\boldsymbol{Q}_Z$, we see by the above result, that

$$\begin{aligned} &I^{\cdot}(\mathring{\mathcal{X}}) := i^{-1}\ell_*\Delta^{\cdot}(\mathring{\mathcal{X}})\,, \quad I^{\cdot}(X_\infty) := i^{-1}k_*\Delta^{\cdot}(X_\infty) \quad \text{and} \\ &I^{\cdot}(\mathring{X}_\infty) := s^{\cdot}(I^{\cdot}(\mathring{\mathcal{X}})\otimes_{\boldsymbol{Q}} I^{\cdot}(X_\infty)) \end{aligned} \tag{1.2.2}$$

are representatives of $i^{-1}R\ell_*\boldsymbol{Q}_{\mathring{\mathcal{X}}}$, $i^{-1}Rk_*\boldsymbol{Q}_{X_\infty}$ and $i^{-1}Rk'_*\boldsymbol{Q}_{\mathring{X}_\infty}$ respectively.

$I^{\cdot}(\mathring{X}_\infty)$ is of course a candidate of the $\boldsymbol{Q}$-structure but the monodromy logarithm $\log T$ can not be lifted on this complex. In order to rescue this situation, we need a rather complicated construction of $A^{\cdot}_{\boldsymbol{Q}}$ in the following way.

The automorphism $(x, u)\mapsto(x, u-1)$ on $X_\infty := \mathcal{X}^*\times_{\Delta^*}\tilde{\Delta}^*$ induces an automorphism T of $I^{\cdot}(X_\infty)$. Define

$$\begin{aligned} &B^{\cdot}(X_\infty) := \bigcup_{m\geq 0} \operatorname{Ker}(T-1)^{m+1} \subset I^{\cdot}(X_\infty)\,, \\ &B^{\cdot} := B^{\cdot}(\mathring{X}_\infty) := I^{\cdot}(\mathring{\mathcal{X}})\otimes_{\boldsymbol{Q}} B^{\cdot}(X_\infty) \subset I^{\cdot}(\mathring{X}_\infty)\,. \end{aligned} \tag{1.2.3}$$

Then these inclusions are quasi-isomorphisms [28, (5.9)] (more precisely, see

[22, §14]), and

$$(1.2.4) \qquad \delta := \log T\colon B^\bullet(X_\infty) \to B^\bullet(X_\infty)$$

is well-defined by construction.

Let

$$\rho(B)^\bullet := \rho(B^\bullet(\overset{\circ}{X}_\infty), 1\otimes\delta)^\bullet \simeq s^\bullet(I^\bullet(\overset{\circ}{\mathfrak{X}})\otimes\rho(B^\bullet(X_\infty), \delta))$$

be the mapping cone, i.e.,

$$\rho(B)^p := B^p \oplus B^{p-1}, \quad d(x, y) := (dx, (1\otimes\delta)\, x - dy)$$

We define a morphism of complexes

$$(1.2.5) \qquad \theta\colon \rho(B)^\bullet \to \rho(B)^\bullet[1] \quad by \quad \theta(x, y) := (0, x)\,.$$

Let

$$\tau_q'(K^\bullet \otimes L^\bullet) := (\tau_q K^\bullet)\otimes L^\bullet\,, \quad \tau_q''(K^\bullet \otimes L^\bullet) := K^\bullet \otimes (\tau_q L^\bullet)\,,$$

be the partial canonical filtratiosn for a tensor product of complexes $K^\bullet$ and $L^\bullet$, where τ is the canonical filtration.

A double complex $A_Q^{\bullet\bullet} = A_Q^{\bullet\bullet}(\overset{\circ}{X}_\infty)$ is defined as

$$(1.2.6) \qquad \begin{aligned} &A_Q^{\bullet q} := \begin{cases} (\rho(B)^\bullet/\tau_q'')\,[q+1] & \text{if } p \geqslant -1 \text{ and } q \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases} \\ &d'\colon A_Q^{pq} \to A_Q^{p+1,q} \quad \text{is induced from } (-1)^{q+1}\, d_{\rho(B)}\,, \quad \text{and} \\ &d''\colon A_Q^{pq} \to A_Q^{p,q+1} \quad \text{is induced from } \theta\,. \end{aligned}$$

The ***Q**-structure of the mixed version of the Steenbrink complex* is the associated single complex:

$$(1.2.7) \qquad A_Q^\bullet := s^\bullet(A_Q^{\bullet\bullet})\,, \quad d := (-1)^q\, d' + d'' = -d_{\rho(B)} + \theta \quad \text{on} \quad A_Q^{pq}$$

It can be seen that the map $B^\bullet \hookrightarrow A_Q^\bullet$ defined by $B^p \ni x \mapsto (0, x) \in A_Q^{p,0}$ is a quasi-isomorphism [28, (5.13)].

Let $\tilde{\delta}$ and ν be endomorphisms of the complex $A_Q^\bullet$ defined by

$$(1.2.8) \qquad \begin{aligned} &\tilde{\delta}\colon A_Q^{pq} \to A_Q^{pq} && \tilde{\delta}(x, y) := ((1\otimes\delta)\, x, (1\otimes\delta)\, y)\,, \quad \text{and} \\ &\nu\colon A_Q^{pq} \to A_Q^{p-1,q+1} && \text{projection.} \end{aligned}$$

These are homotopic [28, (5.14)]. In fact, it is easy to verify that the map given by

$$h\colon A_Q^{p+1,q-1} \to A_Q^{p,q-1} \quad h(x, y) := (y, 0)$$

satisfies $\nu - \tilde{\delta} = dh + hd$. Moreover the endomorphisms $1\otimes\delta$ of $B^\bullet$ and $\tilde{\delta}$ of $A_Q^\bullet$

are compatible with $B^{\bullet} \hookrightarrow A_{\boldsymbol{Q}}^{\bullet}$. Hence ν on $A_{\boldsymbol{Q}}^{\bullet}$ induces $\log T$ on the hyper-cohomology, which is the significance of the complex $A_{\boldsymbol{Q}}^{\bullet}$.

Let $W(X_0)$ be the partial weight filtration of the complex $\Omega_{\mathcal{X}}^{\bullet}(\log(\mathcal{Y}+X_0))$, i.e., $W_q(X_0)\,\Omega_{\mathcal{X}}^{p}(\log(\mathcal{Y}+X_0)) := \Omega_{\mathcal{X}}^{q}(\log(\mathcal{Y}+\overset{\circ}{X_0})) \wedge \Omega_{\mathcal{X}}^{p-q}(\log \mathcal{Y})$. We define a double complex $A_{\boldsymbol{C}}^{\bullet\bullet} = A_{\boldsymbol{C}}^{\bullet\bullet}(X_\infty)$ by

(1.2.9)
$$A_{\boldsymbol{C}}^{\bullet q} := \begin{cases} (\Omega_{\mathcal{X}}^{\bullet}(\log(\mathcal{Y}+X_0))/W_q(X_0))\,[q+1] & \text{if } p, q \geqslant 0\,, \\ 0 & \text{otherwise,} \end{cases}$$
$$d' : A_{\boldsymbol{C}}^{pq} \to A_{\boldsymbol{C}}^{p+1,q} \quad \text{is induced from } (-1)^{q+1}(\text{exterior differential}), \text{ and}$$
$$d'' : A_{\boldsymbol{C}}^{pq} \to A_{\boldsymbol{C}}^{p,q+1} \quad \text{is induced from } \theta \wedge,$$

where

(1.2.10) $\theta := f^* d \log t / 2\pi\sqrt{-1}\,, \quad t$: a parameter of the disc Δ.

The ***C**-structure of the mixed version of the Steenbrink complex* is the associated single complex:

(1.2.11)
$$A_{\boldsymbol{C}}^{\bullet} := s^{\bullet}(A_{\boldsymbol{C}}^{\bullet\bullet})\,, \quad d := (-1)^q d' + d'' = -(\text{exterior differential}) + \theta \wedge \quad \text{on } A_{\boldsymbol{C}}^{pq}$$

Let ν be the endomorphism of the complex $A_{\boldsymbol{C}}^{\bullet}$ defined by

(1.2.12) $\nu : A_{\boldsymbol{C}}^{pq} \to A_{\boldsymbol{C}}^{p-1,q+1}$ projection.

In order to see the relation between the $\boldsymbol{Q}$-structure and the $\boldsymbol{C}$-structure, we set

(1.2.13)
$$\tilde{B}^{\bullet}(X_\infty) := i^{-1}\,\Omega_{\mathcal{X}}^{\bullet}(\log X_0)\,[u]\,, \quad \text{where } u = \log t/2\pi\sqrt{-1}\,, \quad \text{and}$$
$$\tilde{B}^{\bullet} := \tilde{B}^{\bullet}(\overset{\circ}{X}_\infty) := \Omega_{\mathcal{X}}^{\bullet}(\log \mathcal{Y}) \otimes_{\boldsymbol{C}} \tilde{B}^{\bullet}(X_\infty)$$

and construct a complex

(1.2.14) $\tilde{A}_{\boldsymbol{C}}^{\bullet}$ from $\tilde{B}^{\bullet}$

in the same way as the construction of $A_{\boldsymbol{Q}}^{\bullet}$ from $B^{\bullet}$. We define an endomorphism

(1.2.15) δ of $\tilde{B}^{\bullet}(X_\infty)$ by $\delta(\sum \omega_j u^j/j!) := -\sum \omega_j u^{j-1}/(j-1)!\,,$

and denote the induced ones by

(1.2.16)
$$1 \otimes \delta \text{ on } \tilde{B}^{\bullet}\,,$$
$$\tilde{\delta} \text{ on } \tilde{A}_{\boldsymbol{C}}^{\bullet}\,, \qquad \tilde{\delta}(x, y) := ((1\otimes\delta)\,x, (1\otimes\delta)\,y)\,.$$

We denote also by

(1.2.17) ν on $\tilde{A}_C^\bullet$: the one induced from the projection $\tilde{A}_C^{pq} \to \tilde{A}_C^{p-1,q+1}$.

Then we have compatible quasi-isomorphisms:

$$
\begin{array}{lccl}
 & \tilde{\delta} & \text{on} & A_Q^\bullet \otimes \boldsymbol{C} \\
 & & & \uparrow\wr\, QIS \qquad [28,(5.13)] \\
 & \delta & \text{on} & B^\bullet \otimes \boldsymbol{C} \\
 & & & |\wr\, QIS \qquad [28,(5.18)] \\
 & 1\otimes\delta & \text{on} & \tilde{B}^\bullet \\
 & & & \downarrow\wr\, QIS \qquad [28,(5.13)] \\
(1.2.18) & \tilde{\delta} & \text{on} & \tilde{A}_C^\bullet \\
 & & & \| \qquad [28,(5.14)] \\
 & \nu & \text{on} & \tilde{A}_C^\bullet \\
 & & & \psi_t\downarrow\wr\, QIS \qquad [28,(5.18)] \\
 & \nu & \text{on} & A_C^\bullet \\
 & & & (-1)^\bullet\theta\wedge\uparrow\wr\, QIS \qquad [28,(5.5)],\ [27,(4.16)] \\
 & & & \Omega^\bullet_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\otimes\mathcal{O}_{X_0},
\end{array}
$$

where ψ_t above is induced from a morphism of dobule complexes defined by

(1.2.19) $\psi_t\colon \tilde{A}_C^{pq} \to A_C^{pq}$, $\psi_t(\sum x_j u^j/j!, \sum y_j u^j/j!) := x_0 + du\wedge y_0$.

Taking hypercohomology, (1.2.18) induces a compatible isomorphism (cf. [27, (4.22)]):

$$
\begin{array}{lccl}
 & \log T & \text{on} & H^n(X_\infty, \boldsymbol{C}) \\
(1.2.20) & & & \psi_t\downarrow\wr \\
 & -2\pi\sqrt{-1}\,\mathrm{Res}_0(\nabla) & \text{on} & \tilde{\mathcal{V}}(0) = H^n(X_0, \Omega^\bullet_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\otimes\mathcal{O}_{X_0}),
\end{array}
$$

where ∇ is the Gauss-Manin connection in (1.1.8). In this sense, we hereafter denote

$$N := \log T = -2\pi\sqrt{-1}\,\mathrm{Res}_0(\nabla). \tag{1.2.21}$$

Remark **(1.2.22)**. [27,(4.24)] explains how the isomorphism ψ_t in (1.2.20) depends on the choice of the parameter t of Δ (cf. also [22,(14.18)]). This can be also explained in the following way.

Let $\{e_1, \cdots, e_r\}$ be a multi-valued flat frame of $\mathcal{V}$ in (1.1.5). Modifying

$$\tilde{e}_j := \exp(-u\log T)\, e_j$$

we get an invariant frame $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ which extends over Δ and induces a basis of the central fiber $\tilde{\mathcal{V}}(0)$ of the canonical extension [9, II. §5], also denoted by the same symbols. Let M_∇ and M_T be the matrices such that

$$(\nabla\tilde{e}_1, \cdots, \nabla\tilde{e}_r) = (\tilde{e}_1, \cdots, \tilde{e}_r)\, M_\nabla\,, \quad \text{and}$$
$$(Te_1, \cdots, Te_r) = (e_1, \cdots, e_r)\, M_T$$

Then

$$\psi_t(e_j) = \tilde{e}_j \quad \text{for all } j\,,$$

and *under this identification we have* (cf. [9,II.(1.17), (5.6)])

$$\log M_T = -2\pi\sqrt{-1}\,\mathrm{Res}_0(M_\nabla)\,. \quad \blacksquare$$

We define filtrations of $A^\bullet$ by

(1.2.23)
$$\begin{aligned}
G_iA^{\bullet q} &:= \mathrm{image}\begin{cases}(\tau'_i\,\rho(B)^\bullet)\,[q+1] \to A_Q^{\bullet q}\,,\\ W_i(\mathcal{Y})\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))\,[q+1] \to A_C^{\bullet q}\,,\end{cases}\\
L_jA^{\bullet q} &:= \mathrm{image}\begin{cases}(\tau''_{j+2q+1}\,\rho(B)^\bullet)\,[q+1] \to A_Q^{\bullet q}\,,\\ W_{j+2q+1}(X_0)\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))\,[q+1] \to A_C^{\bullet q}\,,\end{cases}\\
W_kA^{\bullet q} &:= \mathrm{image}\begin{cases}(\tau_{k+2q+1}\,\rho(B)^\bullet)\,[q+1] \to A_Q^{\bullet q}\,,\\ W_{k+2q+1}(\mathcal{Y}+X_0)\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))\,[q+1] \to A_C^{\bullet q}\,,\end{cases}\\
F^pA_C^{\bullet\bullet} &:= \bigoplus_{p'\geq p} A_C^{p'\bullet}\,.
\end{aligned}$$

The *convolution* (or *amalgamation*) $F'*F''$ of two filtrations F' and F'' is defined by

$$(F'*F'')_k := \sum_{i+j=k} F'_i \cap F''_j \quad [28,(1.4)]\,.$$

Lemma (1.2.24). **(i)** $(A^\bullet, G*L) \to (A^\bullet, W)$ *is a filtered quasi-isomorphism.*
(ii) *G on $A^\bullet$ satisfies*

$$\nu G_i \subset G_i\,, \quad \mathrm{gr}_i^G A^\bullet \underset{QIS}{\xrightarrow{\sim}} A^\bullet(\widetilde{Y}_\infty^{(i)})\,[-i]\,,$$

and induces the Gysin filtration on the hypercohomology, where $\widetilde{Y}_\infty^{(i)}$ is the normalization of the i-ple locus of Y_∞.

(iii) $(A_Q^\bullet, L)\otimes \boldsymbol{C} \underset{FQIS}{\cong} (A_C^\bullet, L)$, *where FQIS means filtered quasi-isomorphic, and L on $A_Q^\bullet$ induces the N-filtration on the hypercohomology, i.e., $NL_j \subset L_{j-2}$ and N^j: $\mathrm{gr}_j^L \xrightarrow{\sim} \mathrm{gr}_{-j}^L$ on $H^n(X_0, A_Q^\bullet) = H^n(\mathring{X}_\infty, \boldsymbol{Q})$.*

Proof. Set $\Omega := \Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))$. Then

$$\begin{aligned}
&(G*L)_k\, A_C^{\bullet q}\\
&= \Big(\sum_{i+j=k}((W_i(\mathcal{Y}) + W_q(X_0)) \cap (W_{j+2q+1}(X_0) + W_q(X_0)))/W_q(X_0)\Big)\,\Omega\,[1]\\
&= \Big(\sum_{i+j=k}(W_i(\mathcal{Y}) \cap W_q(X_0) + W_{j+2q+1}(X_0))/W_q(X_0)\Big)\,\Omega\,[1]
\end{aligned}$$

$$= ((W_{k+2q+1}(\mathcal{Y}+X_0)+W_q(X_0))/W_q(X_0))\,\Omega[1] = W_k A_C^{\cdot q}\,.$$

Similarly we have $(G*L)_k A_Q^{\cdot} \subset W_k A_Q^{\cdot}$. These together with [10, II.(3.18)] yield a commuatitive diagram:

$$\begin{array}{ccccc} (A_Q^{\cdot}, G*L)\otimes C & \underset{FQIS}{\overset{\sim}{\leftarrow}} & (A_C^{\cdot}, \tau'*\tau''[-2q-1]) & \underset{FQIS}{\overset{\sim}{\rightarrow}} & (A_C^{\cdot}, G*L) \\ \downarrow & & & & \| \\ (A_Q^{\cdot}, W)\otimes C & \underset{FQIS}{\overset{\sim}{\leftarrow}} & (A_C^{\cdot}, \tau[-2q-1]) & \underset{FQIS}{\overset{\sim}{\rightarrow}} & (A_C^{\cdot}, W) \end{array}$$

From this we get the assertion for the $\boldsymbol{Q}$-structure. This proves (i).

The first assertion of (ii) is immediate by definition. As for the second,

$$\begin{aligned} \mathrm{gr}_i^G A_Q^{\cdot q} &\simeq (((\tau_i'+\tau_q'')/(\tau_{i-1}'+\tau_q''))\,\rho(B)^{\cdot})\,[q+1] \\ &\simeq ((\tau_i'/(\tau_{i-1}'+\tau_i'\cap\tau_q''))\,\rho(B)^{\cdot})\,[q+1] \\ &\simeq (\mathrm{gr}_i^{\tau'}\,\rho(B)^{\cdot}/\tau_q''\,\rho(B)^{\cdot})\,[q+1] \\ &\underset{QIS}{\overset{\sim}{\rightarrow}} (a_*\boldsymbol{Q}_{\tilde{\mathcal{Y}}^{(i)}}[-i]\otimes(\rho(B^{\cdot}(X_\infty))^{\cdot}/\tau_q''))\,[q+1] \\ &= a_*\boldsymbol{Q}_{\tilde{\mathcal{Y}}^{(i)}}[-i]\otimes A_Q^{\cdot q}(X_\infty) \underset{QIS}{\overset{\sim}{\rightarrow}} A_Q^{\cdot q}(\tilde{Y}_\infty^{(i)})\,[-i]\,. \end{aligned}$$

Similarly we have the second assertion for the $\boldsymbol{C}$-structure. The last assertion follows from these. This proves (ii).

The first assertion of (iii) is easy by construction. We prove the second assertion.

$$\nu L_j A_Q^{pq} = \nu\tau_{j+2q+1}'' A_Q^{pq} = \tau_{j+2q+1}'' A_Q^{p-1,q+1} = L_{j-2} A_Q^{p-1,q+1}\,.$$

Hence $NL_j \subset L_{j-2}$ on $H^n(X_\infty, \boldsymbol{Q})$. Next we observe that

$$(A_Q^{\cdot}, G) \underset{QIS}{\overset{\sim}{\rightarrow}} (R\ell_{t*}\boldsymbol{Q}_{\overset{\circ}{X}_t}, \tau)\,, \qquad \text{where } \ell_t\colon \overset{\circ}{X}_t \hookrightarrow X_t \quad (t\in\Delta^*)$$

and that the latter is a part of the functorial cohomological mixed Hodge complex for X_∞ (see [10, III(8.1)]) hence the spectial sequence of $(R\Gamma R\ell_{t*}\boldsymbol{Q}_{\overset{\circ}{X}_t}, \tau)$ degenerates in $E_2 = E_\infty$. We also observe that under

$$\mathrm{gr}_i^G A^{\cdot\cdot}(\overset{\circ}{X}_\infty) \underset{QIS}{\overset{\sim}{\rightarrow}} A^{\cdot-i,\cdot}(\tilde{Y}_\infty^{(i)})$$

$L_j\,\mathrm{gr}_i^G A^{\cdot\cdot}(\overset{\circ}{X}_\infty)$ corresponds to $W_j A^{\cdot-i,\cdot}(\tilde{Y}_\infty^{(i)})$ and the d_1 of the above spectral sequence are morphisms of mixed Hodge structures (actually, Gysin maps), so $L=W$ on E_1 is strict for d_1 [10, II.(2.3.5.iii)]. It follows that taking cohomology and gr^L commute [10, II.(1.1.11.ii)]. By [27, (5.9)] (see (A.1) below),

$$N^j\colon \mathrm{gr}_j^L E_1^{pq} \overset{\sim}{\rightarrow} \mathrm{gr}_{-j}^L E_1^{pq}\,.$$

Hence

$$N^j\colon \mathrm{gr}_j^L E_2^{-i,n+i} \overset{\sim}{\rightarrow} \mathrm{gr}_{-j}^L E_2^{-i,n+i}\,.$$

That is

$$N^j\colon \mathrm{gr}_j^L\,\mathrm{gr}_i^G\,H^n(\overset{\circ}{X}_\infty, \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{gr}_{-j}^L\,\mathrm{gr}_i^G\,H^n(\overset{\circ}{X}_\infty, \boldsymbol{Q})\,.$$

This implies

$$N^j\colon \mathrm{gr}_j^L\,H^n(\overset{\circ}{X}_\infty, \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{gr}_{-j}^L\,H^n(\overset{\circ}{X}_\infty, \boldsymbol{Q})\,. \quad \blacksquare$$

[11] generalized the notion of cohomological mixed Hodge complex (CMHC, for short) in [10, III.(8.1)] to:

DEFINITION (**1.2.25**). *$(M, G)=((M^{\cdot}_{\boldsymbol{Q}}, G, W), (M^{\cdot}_{\boldsymbol{C}}, G, W, F), \alpha)$ is a G-filtered CMHC on a topological space Z if it satisfies the following conditions*:

(**i**) *M is a $\boldsymbol{Q}$-CMHC on Z. $\alpha\colon (M^{\cdot}_{\boldsymbol{Q}}, G, W)\otimes\boldsymbol{C}\simeq(M^{\cdot}_{\boldsymbol{C}}, G, W)$ is a bifiltered quasi-isomorphism.*

(**ii**) *$\mathrm{gr}_i^G M$ is a $\boldsymbol{Q}$-CMHC on Z for each i.*

(**iii**) Dec *W and gr^G commute on $\widetilde{M}^{\cdot}:=R\Gamma M^{\cdot}_{\boldsymbol{Q}}$.*

(**iv**) *The spectral sequence of $(\widetilde{M}^{\cdot}, G)$ degenerates in $E_2=E_\infty$.*

Recall that the Hodge filtration F on $\mathcal{V}$ in (1.1.5) is the one induced from the stupid filtration

$$F^p\Omega^{\cdot}_{\mathcal{X}^*}(\log \mathcal{Y}^*):=\sum_{p'\geqslant p}\Omega^{p'}_{\mathcal{X}^*}(\log \mathcal{Y}^*)$$

The following lemma can be found in [28, §5, (6.9), (3.13), Appledix].

Lemma (1.2.26). *$((A^{\cdot}_{\boldsymbol{Q}}, G, W), (A^{\cdot}_{\boldsymbol{C}}, G, W, F), \alpha)$ is a G-filtered CMHC on X_0, whose hypercohomology yields a limit of the variation of mixed Hodge structure arising from $f\colon \mathcal{X}^*\to\Delta^*$, that is, the following hold*:

(**i**) *W on $A^{\cdot}_{\boldsymbol{Q}}$ induces the G-relative N-filtration on the hypercohomology, i.e., $NW_k\subset W_{k-2}$ and $N^j\colon \mathrm{gr}^W_{i+k}\,\mathrm{gr}^G_i\xrightarrow{\sim}\mathrm{gr}^W_{i-k}\,\mathrm{gr}^G_i$ on $H^n(X_0, A^{\cdot}_{\boldsymbol{Q}})=H^n(\overset{\circ}{X}_\infty, \boldsymbol{Q})$.*

(**ii**) *F on $\mathcal{V}$ extends to a filtration of $\widetilde{\mathcal{V}}$ in* (1.1.8) *such that $F^p\mathrm{gr}_i^G\widetilde{\mathcal{V}}$* is *locally free and $F^p\,\widetilde{\mathcal{V}}(0)=F^p\,H^n(X_0, A^{\cdot}_{\boldsymbol{C}})$ for each i and p.*

Proof. By (1.2.24.i) and [38, II.(A.1)], we have

$$\begin{aligned}\mathrm{gr}_k^W A^{\cdot}_{\boldsymbol{C}} &\simeq \bigoplus_{i+j=k}\mathrm{gr}_i^G\,\mathrm{gr}_j^L\,A^{\cdot}_{\boldsymbol{C}}\\ &\overset{\mathrm{Res}}{\xrightarrow{\sim}} \bigoplus_{i+j=k}\ \bigoplus_{q\geqslant\max\{0,-k\}} a_*\Omega^{\cdot}_{\widetilde{\mathcal{Y}}^{(i)}\cap\widetilde{X}_0^{(j+2q+1)}}[-k-2q]\,,\end{aligned}$$

where $\widetilde{\mathcal{Y}}^{(i)}\cap\widetilde{X}_0^{(j')}$ is the normalization of (i-ple in $\mathcal{Y}$, j'-ple in X_0)-locus of $\mathcal{Y}+X_0$ and $a\colon \widetilde{\mathcal{Y}}^{(i)}\cap\widetilde{X}_0^{(j')}\to X_0$ is the projection (cf. [28, (5.22)]). The above isomorphism is compatible with F and we have a similar decomposition for the $\boldsymbol{Q}$-structure. (1.2.25.i) follows. By (1.2.24.i), [38, II(A.1)] and [28, (1.5)], we have

$$\mathrm{gr}_k^W A^{\cdot} \simeq \bigoplus_{i+j=k}\mathrm{gr}_i^G\,\mathrm{gr}_j^L\,A^{\cdot} \simeq \bigoplus_i \mathrm{gr}_k^W\,\mathrm{gr}_i^G\,A^{\cdot}\,.$$

This is compatible with F. (1.2.25.ii) follows. (1.2.25.iii) also follows by [28, (6.8)]. (1.2.25.iv) is already shown in the proof of (1.2.24.iii). This proves the first half of the assertion.

The proof of (i) in the second assertion is analogous to that of (1.2.24.iii) and we omit it.

As for (ii), set $\Omega(t):=\Omega^{\cdot}_{\mathcal{X}/\Delta}(\log(\mathcal{Y}+X_0))\otimes\mathcal{O}_{X_t}$ $(t\in\Delta)$. We first note that, for $\theta:=f^*d\log t/2\pi\sqrt{-1}$,

$$\theta\wedge:(\Omega(0),F)\underset{FQIS}{\xrightarrow{\sim}}(A^{\cdot}_{C},F)\quad(\text{cf. [27, (4.16)]}).$$

This implies $F^p\widetilde{\mathcal{V}}(0)=F^pH^n(X_0,A^{\cdot}_{C})$. As we have seen in the proof of (1.2.24.iii), F on the E_1 of the spectral sequence of $(R\Gamma R\ell_{t*}\boldsymbol{Q}_{\overset{\circ}{\mathcal{X}}_t},\tau)$ is strict for d_1. Hence gr_F commutes with taking cohomology, and we can compute as

$$\begin{aligned}\mathrm{gr}^p_F\,\mathrm{gr}^G_i\,\widetilde{\mathcal{V}}(t)&=\mathrm{gr}^p_F\,\mathrm{gr}^G_i\,H^n(R\Gamma\Omega(t))\\&=\mathrm{gr}^p_F\,E_2^{-i,n+i}(R\Gamma\Omega(t),G)=\mathrm{gr}^p_F\,E_1^{-i+n,i}(R\Gamma\Omega(t),\mathrm{Dec}\,G)\\&=\mathrm{gr}^p_F\,H^n(R\Gamma\,\mathrm{gr}^{\mathrm{Dec}\,G}_{i-n}\Omega(t))=H^n(R\Gamma\,\mathrm{gr}^p_F\,\mathrm{gr}^{\mathrm{Dec}\,G}_{i-n}\Omega(t))\\&=H^n(X_t,\mathrm{gr}^p_F\,\mathrm{gr}^{\mathrm{Dec}\,G}_{i-n}\Omega(t)).\end{aligned}$$

From this, we see that $\dim\mathrm{gr}^p_F\,\mathrm{gr}^G_i\,\widetilde{\mathcal{V}}(t)$ is upper semi-continuous in $t\in\Delta$. On the other hand, $\dim\mathrm{gr}^G_i\,\widetilde{\mathcal{V}}(t)$ is constant. Hence $\mathrm{gr}^p_F\,\mathrm{gr}^G_i\,\widetilde{\mathcal{V}}$ is locally free by the continuity theorem. ∎

(1.3) In the situation of (1.1), we recall a construction of a CMHC $K^{\cdot}$ whose hypercohomology gives the functorial mixed Hodge structure on the cohomology of $\overset{\circ}{X}_0$ (cf. [10, III.(8.1.12)]).

Let $K^{\cdot\cdot}_{Q}$ be a double complex defined by

$$K^{pq}_{Q}:=\begin{cases}I^p(\overset{\circ}{\mathcal{X}})\otimes_Q a_*\boldsymbol{Q}_{\widetilde{X}_0^{(q+1)}} & \text{if } p,q\geqslant 0,\\ 0 & \text{otherwise,}\end{cases}$$

$$d':K^{pq}_{Q}\to K^{p+1,q}_{Q}\quad\text{is } (-1)^{q+1}d_{\Gamma(\overset{\circ}{\mathcal{X}})},\quad\text{and}$$

$$d'':K^{pq}_{Q}\to K^{p,q+1}_{Q}\quad\text{is the Mayer-Vietoris map } 1\otimes(\textstyle\sum_i(-1)^i\delta_i^*).$$

where $a\colon\widetilde{X}_0^{(q+1)}\to X_0$ is the projection and $I^{\cdot}(\overset{\circ}{\mathcal{X}})$ is the complex in (1.2.2). The $\boldsymbol{Q}$-structure is defined as the associated single complex

(1.3.1) $\quad K^{\cdot}_{Q},\quad d:=(-1)^q d'+d''=-d_{\Gamma^{\cdot}(\overset{\circ}{\mathcal{X}})}+1\otimes(\sum_i(-1)^i\delta_i^*)\quad$ on K^{pq}_{Q}

Let $K^{\cdot\cdot}_{C}$ be a double complex defined by

$$K^{pq}_{C}:=\begin{cases}a_*\Omega^p_{\widetilde{X}_0^{(q+1)}}(\log(\mathcal{Y}\cap\widetilde{X}_0^{(q+1)})) & \text{if } p,q\geqslant 0,\\ 0 & \text{otherwise,}\end{cases}$$

$$d' : K_C^{pq} \to K_C^{p+1,q} \quad \text{is } (-1)^{q+1}(\text{exterior differential}), \quad \text{and}$$
$$d'' : K_C^{pq} \to K_C^{p,q+1} \quad \text{is the Mayer-Vietoris map } \sum_i (-1)^i \delta_i^* .$$

The $\boldsymbol{C}$-structure is defined as the associated single complex

(1.3.2) $K_C^\bullet$, $d := (-1)^q d' + d'' = -(\text{exterior differential}) + \sum_i (-1)^i \delta_i^*$ on K_C^{pq}

We define filtrations of $K_Q^\bullet$ and $K_C^\bullet$ by

(1.3.3)
$$\begin{aligned} G_i K^\bullet &:= \begin{cases} \tau_i' K_Q^\bullet & \text{over } \boldsymbol{Q}, \\ W_i(\mathcal{Y}) K_C^\bullet & \text{over } \boldsymbol{C}, \end{cases} \\ L_j K^{\bullet\bullet} &:= \bigoplus_{q \geq -j} K^{\bullet q} \quad \text{over } \boldsymbol{Q} \text{ as well as over } \boldsymbol{C}, \\ W_k K^{\bullet q} &:= \begin{cases} \tau_{k+q}' K_Q^{\bullet q}, \\ W_{k+q}(\mathcal{Y}) K_C^{\bullet q}, \end{cases} \\ F^p K_C^{\bullet\bullet} &:= \bigoplus_{p' \geq p} K_C^{p'\bullet} . \end{aligned}$$

Lemma (1.3.4). **(i)** *$(K^\bullet, G*L) \to (K^\bullet, W)$ is a filtered quasi-isomorphism.*

(ii) $(K_Q^\bullet, G) \otimes \boldsymbol{C} \underset{FQIS}{\xrightarrow{\sim}} (K_C^\bullet, G)$ *and G on $K^\bullet$ satisfies*

$$\nu G_i \subset G_i , \quad \mathrm{gr}_i^G K^\bullet \underset{QIS}{\xrightarrow{\sim}} a_* K^\bullet(\widetilde{\mathcal{Y}}^{(i)} \cap X_0) [-i] ,$$

hence induces the Gysin filtration on the hypercohomology.

(iii) $(K_Q^\bullet, L) \otimes \boldsymbol{C} \underset{FQIS}{\xrightarrow{\sim}} (K_C^\bullet, L)$ *and L on $K_Q^\bullet$ satisfies*

$$\mathrm{gr}_j^L K_Q^\bullet \underset{QIS}{\xrightarrow{\sim}} a_* \boldsymbol{Q}_{\tilde{\mathring{X}}_0^{(-j+1)}} [j] ,$$

hence induces the Mayer-Vietoris filtration on the hypercohomology.

(iv) *$K := ((K_Q^\bullet, W), (K_C^\bullet, W, F), \alpha)$ is a CMHC over $\boldsymbol{Q}$ on X_0, whose hypercohomology yields the functorial mixed Hodge structure on $H^\bullet(\mathring{X}_0, \boldsymbol{Q})$.*

(v) *If the spectral sequence of $R\Gamma K^\bullet$ by the filtration G (resp. L) degenerates in $E_2 = E_\infty$, then K with G (resp. L) is a G-filtered (resp. L-filtered) CMHC over $\boldsymbol{Q}$,*

(vi) *$K_C^\bullet = \mathrm{Ker}\{\nu : A_C^\bullet \to A_C^\bullet\}$ and the filtrations G, L, W and F on both terms coincide respectively.*

Proof. (i): $(G*L)_k K^{\bullet q} = \sum_{i+j=k} (G_i \cap L_j) K^{\bullet q} = G_{k+q} K^{\bullet q} = W_k K^{\bullet q}$.

The first assertion of (ii) follows immediately by definition. As for the second,

$$\begin{aligned} \mathrm{gr}_i^G K_C^{\bullet q} &= \mathrm{gr}_i^{W(\mathcal{Y})} a_* \Omega^\bullet_{\tilde{X}_0^{(q+1)}} (\log (\mathcal{Y} \cap \tilde{X}_0^{(q+1)})) \\ &\simeq a_* \Omega^\bullet_{\widetilde{\mathcal{Y}}^{(i)} \cap \tilde{X}_0^{(q+1)}} [-i] = a_* K_C^{\bullet q}(\widetilde{\mathcal{Y}}^{(i)} \cap X_0^{(q+1)}) [-i] . \end{aligned}$$

Similarly, we get the assertion for the $\boldsymbol{Q}$-structure. The third assertion follows from these.

The first assertion of (iii) follows immediately by definition.

$$\begin{aligned}\mathrm{gr}^L_j K^\bullet_C &= K^{\bullet-j}_C[j]\\ &= a_*\Omega^\bullet_{\tilde{X}_0^{(-j+1)}}(\log(\mathcal{Y}\cap\tilde{X}_0^{(-j+1)}))[j] \underset{QIS}{\xleftarrow{\sim}} a_* C_{\tilde{\mathring{X}}_0^{(-j+1)}}[j]\,.\end{aligned}$$

Similarly, we get the assertion for the $\boldsymbol{Q}$-structure. The third assertion follows from these.

(iv) is found in [10, III.(8.1.12)]. (v) is easy to verify by using (i) and [28, (6.8)]. (vi) is immediate by construction. ∎

We now recall a construction of a $\boldsymbol{Q}$-CMHC $C^\bullet$ whose hypercohomology gives the functorial mixed Hodge structure on the cohomology of $(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)$ (cf. [15, IV.5]).

We are working on a diagram:

(1.3.5)
$$\begin{array}{ccccc} \mathring{\mathcal{X}}^* & \xrightarrow{\mathring{j}} & \mathring{\mathcal{X}} & & \\ \downarrow & & \downarrow \ell & & \\ \mathcal{X}^* & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & X_0 \end{array}$$

As in (1.2.2), the complexes

(1.3.6)
$$\begin{aligned}&I^\bullet(\mathring{\mathcal{X}}) := i^{-1}\ell_*\Delta^\bullet(\mathring{\mathcal{X}}),\quad I^\bullet(\mathcal{X}^*) := i^{-1}j_*\Delta^\bullet(\mathcal{X}^*)\quad\text{and}\\ &I^\bullet(\mathring{\mathcal{X}}^*) := s^\bullet(I^\bullet(\mathring{\mathcal{X}})\otimes_Q I^\bullet(\mathcal{X}^*))\,.\end{aligned}$$

are representatives of $i^{-1}R\ell_*\boldsymbol{Q}_{\mathring{\mathcal{X}}}$, $i^{-1}Rj_*\boldsymbol{Q}_{\mathcal{X}^*}$ and $i^{-1}R(\ell\mathring{j})_*\boldsymbol{Q}_{\mathring{\mathcal{X}}^*}$ respectively. The complexes $C^\bullet_Q$ and $C^\bullet_C$ and their filtrations are defined as

(1.3.7)
$$\begin{aligned}
C^\bullet &:= \begin{cases} (I^\bullet(\mathring{\mathcal{X}}^*)/I^\bullet(\mathring{\mathcal{X}}))[1] & \text{over } \boldsymbol{Q}\,,\\ (\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))/\Omega^\bullet_{\mathcal{X}}(\log\mathcal{Y}))[1] & \text{over } \boldsymbol{C}\,,\end{cases}\\
G_i C^\bullet &:= \text{image}\begin{cases} (\tau'_i I^\bullet(\mathring{\mathcal{X}}^*))[1] \to C^\bullet_Q\,,\\ W_i(\mathcal{Y})\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^\bullet_C\,,\end{cases}\\
L_j C^\bullet &:= \text{image}\begin{cases} (\tau''_{j+1} I^\bullet(\mathring{\mathcal{X}}^*))[1] \to C^\bullet_Q\,,\\ W_{j+1}(X_0)\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^\bullet_C\,,\end{cases}\\
W_k C^\bullet &:= \text{image}\begin{cases} (\tau_{k+1} I^\bullet(\mathring{\mathcal{X}}^*))[1] \to C^\bullet_Q\,,\\ W_{k+1}(\mathcal{Y}+X_0)\,\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1] \to C^\bullet_C\,,\end{cases}\\
F^p C^\bullet_C &:= \text{image of } F^p(\Omega^\bullet_{\mathcal{X}}(\log(\mathcal{Y}+X_0))[1]) \to C^\bullet_C\,.
\end{aligned}$$

Lemma (1.3.8). (i) *$(C^\bullet, G*L)\to(C^\bullet, W)$ is a filtered quasi-isomorphism.*
(ii) *$(C^\bullet_Q, G)\otimes\boldsymbol{C} \underset{FQIS}{\simeq} (C^\bullet_C, G)$ and G on $C^\bullet$ satisfies*

$$\nu G_i \subset G_i, \quad \mathrm{gr}_i^G C^\cdot \underset{QIS}{\overset{\sim}{\to}} C^\cdot(\widetilde{\mathcal{Y}}^{(i)}, \widetilde{\mathcal{Y}}^{(i)} \cap \mathcal{X}^*)[-i],$$

hence induces the Gysin filtration on the hypercohomology.

(iii) $(C_Q^\cdot, L) \otimes \boldsymbol{C} \underset{FQIS}{\overset{\sim}{\to}} (C_C^\cdot, L)$ *and* L *on* $C_Q^\cdot$ *satisfies*

$$\mathrm{gr}_j^L C_Q^\cdot \underset{QIS}{\overset{\sim}{\to}} a_* \boldsymbol{Q}_{\overset{\circ}{\tilde{X}}{}_0^{(j+1)}} [-j],$$

hence induces the Mayer-Vietoris filtration on the hypercohomology.

(iv) $C := ((C_Q^\cdot, W), (C_C^\cdot, W, F), \alpha)$ *is a CMHC over* $\boldsymbol{Q}$ *on* $\overset{\circ}{X}_0$, *whose hypercohomology yields the functorial mixed Hodge structure on* $H^\cdot(\overset{\circ}{\mathcal{X}}, \overset{\circ}{\mathcal{X}}{}^*; \boldsymbol{Q})$ [2].

(v) *If the spectral sequence of* $R\Gamma C^\cdot$ *by the filtration* G (*resp.* L) *degenerates in* $E_2 = E_\infty$, *then* C *with* G (*resp.* L) *is a* G*-filtered* (*resp.* L*-filtered*) *CMHC over* $\boldsymbol{Q}$.

(vi) $C_C^\cdot = \mathrm{Coker}\{\nu : A_C^\cdot \to A_C^\cdot\}$ *and the filtrations* G, L, W *and* F *on both terms coincide respectively.*

Proof. (i), (ii) and (iii) are proved analogously as (1.2.24.i), (1.2.24.ii) and (1.3.4.iii) respectively hence we omit it. (iv) is found in [15, IV.5]. In fact, by the Künneth formula and the residue formula,

$$\mathrm{gr}_k^W C_Q^\cdot \underset{QIS}{\simeq} \bigoplus_{i+j=k} a_* \boldsymbol{Q}_{\widetilde{\mathcal{Y}}^{(i)} \cap \tilde{X}_0^{(j)}} [-k].$$

$$\mathrm{gr}_k^W C_C^\cdot \underset{QIS}{\simeq} \bigoplus_{i+j=k} a_* \boldsymbol{C}_{\widetilde{\mathcal{Y}}^{(i)} \cap \tilde{X}_0^{(j)}} [-k].$$

These show that $\mathrm{gr}^W C^\cdot$ is a CHC hence $C^\cdot$ is a CMHC. (v) is easy to verify by using (i) and [28, (6.8)]. (vi) is immediate by construction. ∎

(1.4) In the situation of (1.1), we shall construct a mixed version of the Clemens-Schmid sequence after [38, §7].

Let

(1.4.1) $$\nu : A^\cdot \to A^\cdot$$

be the mixed version of the Steenbrink complex and the lifting of the monodromy logarithm in (1.2). From (1.4.1), we have an exact sequence

(1.4.2)
$$\begin{array}{ccccc} 0 \to \mathrm{Ker}(\nu) \to A^\cdot & & \xrightarrow{\nu} & & A^\cdot \to \mathrm{Coker}(\nu) \to 0 \\ & \searrow & & \nearrow & \\ & & \mathrm{Im}(\nu) & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

Taking the hypercohomology, we have two long sequences (for n odd or even)

(1.4.3) $$\to H^n(\overset{\circ}{\mathcal{X}}, \overset{\circ}{\mathcal{X}}{}^*) \to H^n(\overset{\circ}{X}_0) \to H^n(\overset{\circ}{X}_\infty) \xrightarrow{N} H^n(\overset{\circ}{X}_\infty) \to H^{n+2}(\overset{\circ}{\mathcal{X}}, \overset{\circ}{\mathcal{X}}{}^*) \to$$

over $\boldsymbol{Q}$ by (1.2.25), (1.3.4.iv) and (1.3.8.iv). This is a mixed version of the Clemens-Schmid sequence.

The following is the Poincaré duality (for the prooi, see [26]).

Lemma (1.4.4). $H^n(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*; \boldsymbol{Z}) \simeq H_{2d+2-n}(X_0, Y_0; \boldsymbol{Z})$ *where* $d+1=\dim \mathcal{X}$.

Proposition (1.4.5). *In the situation of* (1.1), *we have for the cohomology with coefficients in* $\boldsymbol{Q}$, *the folowing*:

(i) (1.4.3) *is a sequence of mixed Hodge structures over* $\boldsymbol{Q}$.

(ii) *The filtrations* G *as well as* L *on each term of* (1.4.3) *are compatible respectively.*

(iii) *If* $\mathcal{Y}$ *is smooth (possibly reducible), then* (1.4.3) *is a sequence of* G-*filtered mixed Hodge structures over* $\boldsymbol{Q}$.

(iv) *If* $\mathcal{Y}$ *is smooth (possibly reducible), the Gysin map* $H^0(Y_\infty) \to H^2(X_\infty)$ *is injective and* $H_{2d-1}(X_0)=0$, *where* $d=\dim X_0$, *then the following parts of* (1.4.3) *are exact*:

$$H^1(\mathring{X}_0) \to H^1(\mathring{X}_\infty) \xrightarrow{N} H^1(\mathring{X}_\infty) \to H^3(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*).$$

$$H^0(\mathring{X}_\infty) \to H^2(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \to H^2(\mathring{X}_0) \to H^2(\mathring{X}_\infty) \xrightarrow{N} H^2(X_\infty).$$

Proof. (i), (ii) and (iii) follow from (1.2.25), (1.3.4) and (1.3.8).

In order to prove (iv), we first note that taking gr^G on each term of (1.4.2) yield the following commutative diagram consisting of the Clemens-Schmid sequences as horizontal lines and the Thom-Gysin sequences as vertical lines:

$$\text{(1.4.6)}\quad \begin{array}{ccccccccc}
H^{n-1}(\mathcal{Y}, \mathcal{Y}^*) & \to & H^{n-1}(Y_0) & \to & H^{n-1}(Y_\infty) & \to & H^{n-1}(Y_\infty) & \to & H^{n+1}(\mathcal{Y}, \mathcal{Y}^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^n(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) & \to & H^n(\mathring{X}_0) & \to & H^n(\mathring{X}_\infty) & \to & H^n(\mathring{X}_\infty) & \to & H^{n+2}(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^n(\mathcal{X}, \mathcal{X}^*) & \to & H^n(X_0) & \to & H^n(X_\infty) & \to & H^n(X_\infty) & \to & H^{n+2}(\mathcal{X}, \mathcal{X}^*) \\
\uparrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow \\
H^{n-2}(\mathcal{Y}, \mathcal{Y}^*) & \to & H^{n-2}(Y_0) & \to & H^{n-2}(Y_\infty) & \to & H^{n-2}(Y_\infty) & \to & H^n(\mathcal{Y}, \mathcal{Y}^*)
\end{array}$$

We shall prove the exactness of the second sequence in (iv) by chasing the diagram (1.4.6). As for the first sequence, the proof is similar and easier and we omit it.

At the first term $H^0(\mathring{X}_\infty)$, the exactness follws from

$$H^0(X_\infty) \xrightarrow{\sim} H^0(\mathring{X}_\infty), \quad H^2(\mathcal{X}, \mathcal{X}^*) \xrightarrow{\sim} H^2(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)$$

by (1.4.4) and from the exactness at $H^0(X_\infty)$ in the usual Clemens-Schmid sequence [8]. In the same way, the exatness at $H^2(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)$ follows from

$H^2(\mathscr{X}, \mathscr{X}^*) \xrightarrow{\sim} H^2(\overset{\circ}{\mathscr{X}}, \overset{\circ}{\mathscr{X}}{}^*)$, the assumption of the injectivity of $H^0(Y_\infty) \to H^2(X_\infty)$, $H^0(Y_0) \xrightarrow{\sim} H^0(Y_\infty)$ and from the exactness at $H^2(\mathscr{X}, \mathscr{X}^*)$ in the usual Clemens-Schmid sequence. Similarly the exactness at $H^2(\overset{\circ}{X}_0)$ follows from the injectivity of $H^1(Y_0) \to H^1(Y_\infty)$ in the usual Clemens-Schmid sequence, $H^0(Y_0) \xrightarrow{\sim} H^0(Y_\infty)$ and from the exactness at $H^2(X_0)$ in the usual Clemens-Schmid sequence. As for the exactness at the first $H^2(X_\infty)$, notice that $H^2(\mathscr{Y}, \mathscr{Y}^*) \to H^4(\mathscr{X}, \mathscr{X}^*)$ is injective because $\{H^3(\mathscr{X}, \mathscr{X}^*) \to H^3(\overset{\circ}{\mathscr{X}}, \overset{\circ}{\mathscr{X}}{}^*)\}$ is isomorphic to $\{H_{2d-1}(X_0) \to H_{2d-1}(X_0, Y_0)\}$ by (1.4.4) and the latter is an isomorphism. Now the desired exactness follows similarly from the exactness of the usual Clemens-Schmid sequence, $H^3(\mathscr{X}, \mathscr{X}^*) \simeq H_{2d-1}(X_0) = 0$ by (1.4.4) and the assumption, and from the above remark. ∎

It is not yet known in general whether (1.4.3) is exact or not. Proposition (1.4.5.iv) is only a partial result but it is sufficient enough for our later use in the present paper.

PROBLEM **(1.4.7)**. *Prove the exactness of* (1.4.3).

Appendix to § 1

(A.1) As [12, II.(3.18)] has pointed out, there is a part which is not clear in the proof of [27, (5.9)], i.e., "This implies that $\zeta \in P^{q-r}(\widetilde{Y}^{(r+1)}, \boldsymbol{Q})(-r)$." [ibid, p.254, ↑11]. We explain the point more precisely. In the notation there, we have

$$\begin{array}{ccccccccc} E_1^{r-1,q-r} & = & H^{q-r}(\widetilde{Y}^{(r)}) \oplus & H^{q-r-2}(\widetilde{Y}^{(r+2)}) & \oplus & H^{q-r-4}(\widetilde{Y}^{(r+4)}) & \oplus \cdots \\ (-1)^{(r-1)} d_1 \downarrow & & \theta\searrow & \swarrow -\gamma & \theta\searrow & \swarrow -\gamma & \cdots \\ E_1^{r,q-r} & = & H^{q-r}(\widetilde{Y}^{(r+1)}) & \oplus & H^{q-r-2}(\widetilde{Y}^{(r+3)}) & \oplus & \cdots \end{array}$$

where the θ and the γ are the Mayer-Vietoris maps and the Gysin maps respectively and we omit the coefficients of the cohomologies as well as the Tate twists. Let

$$\xi = (\xi_i)_{i \geqslant 0} \in Z(E_1^{-r,q+r}) \subset \bigoplus_{i \geqslant 0} H^{q-r-2i}(\widetilde{Y}^{(r+2i)})$$

be a primitive element such that $\nu^r \xi \in B(E_1^{r,q-r})$ as in the situation in question. Then, by the above diagram, there exists

$$\eta = (\eta_i)_{i \geqslant -1} \in E_1^{r-1,q-r} = \bigoplus_{i \geqslant -1} H^{q-r-2-2i}(\widetilde{Y}^{(r+2+2i)}) \quad \text{such that}$$
$$\nu^r \xi = (-1)^{r-1} d_1 \eta = (\theta\eta_{i-1} - \gamma\eta_i)_{i \geqslant 0}\,.$$

In particular, $\xi_0 = \nu^r \xi_0 = \theta\eta_{-1} - \gamma\eta_0 \in P^{q-r}(\widetilde{Y}^{(r+1)})$, but it is not known whether $\theta\eta_{-1}$ is primitive or not, hence we can not conclude $\theta\eta_{-1} = 0$ (ξ is assumed as $\nu^r \xi = \theta\eta_{-1}$ there!) by the argument using the polarization on $P^{q-r}(\widetilde{Y}^{(r+1)})$.

However, we can rescue the claim (A.1.1) below (cf. [ibid, p.254, ↑0]) along the line of the original proof by using the polarization Q on the whole

$$(E_1^{-r,q+r})_{prim} \overset{\mathfrak{v}^r}{\xrightarrow{\sim}} (E_1^{r,q-r})_{prim} = \bigoplus_{i\geq 0} P^{q-r-2i}(\tilde{Y}^{(r+1+2i)}) .$$

[ibid, (5.9)] now follows from (A.1.1).

Claim (A.1.1). $\xi = 0$.

Proof. We keep the notation in [ibid, §5]. Identifying by $\mathfrak{v}^r$ above, we have

$$\xi_i = \theta\eta_{i-1} - \gamma\eta_i \quad (i\geq 0) .$$

Since θ and γ are adjoint, we can compute as

$$\begin{aligned} Q(\xi, \xi) &:= \sum_{i\geq 0} Q(\xi_i, \xi_i) \\ &= \varepsilon \sum_{i\geq 0} \int_{\tilde{Y}^{(r+1+2i)}} L_0^{n-q} \wedge C\,\xi_i \wedge \xi_i \\ &= \varepsilon \sum_{i\geq 0} \int_{\tilde{Y}^{(r+1+2i)}} L_0^{n-q} \wedge C(\theta\eta_{i-1} - \gamma\eta_i) \wedge \xi_i \\ &= \varepsilon \sum_{i\geq 0} \left(\int_{\tilde{Y}^{(r+2i)}} L_0^{n-q} \wedge C\eta_{i-1} \wedge \gamma\xi_i - \int_{\tilde{Y}^{(r+2+2i)}} L_0^{n-q} \wedge C\eta_i \wedge \theta\xi_i \right) \\ &= \varepsilon \sum_{i\geq 1} \int_{\tilde{Y}^{(r+2i)}} L_0^{n-q} \wedge C\eta_{i-1} \wedge (\gamma\xi_i - \theta\xi_{i-1}) = 0 , \end{aligned}$$

where $\varepsilon = (-1)^{(q-r)(q-r-1)/2}$. We used $(\theta\xi_{i-1} - \gamma\xi_i)_{i\geq 1} = (-1)^{r+1} d_1\xi = 0$ in the last equality. Hence, by the positive definiteness of Q, we get the assertion. ■

(A.2) In [8], the Clemens-Schmid sequences are constructed by combining "Wang sequences" and the local cohomology sequences. The mixed versions can be also contsructed in this manner.

Lemma (A.2.1). *In the situation and the notation in* (1.1)–(1.3), *we have a commutative diagram*

$$\begin{array}{ccccccc} 0 \to & A_Q^\bullet(\mathring{X}_\infty)[-1] & \to & \rho(A_Q^\bullet(\mathring{X}_\infty), \nu) & \to & A_Q^\bullet(\mathring{X}_\infty) \to & 0 \\ & QIS \uparrow \wr & & QIS \uparrow \wr & & QIS \uparrow \wr & \\ 0 \to & B^\bullet(\mathring{X}_\infty)[-1] & \to & \rho(B^\bullet(\mathring{X}_\infty), \delta) & \to & B^\bullet(\mathring{X}_\infty) \to & 0 \\ & & & QIS \uparrow \wr & & \| & \\ 0 & & \to & I^\bullet(\mathring{\mathfrak{X}}^*) & \to & B^\bullet(\mathring{X}_\infty) \overset{\delta}{\to} & B^\bullet(X_\infty) \to 0 \end{array}$$

whose horizontal lines are exact. The hypercohomology of each horizontal sequence yield a "Wang sequence" in the category of mixed Hodge structures.

The proof is standard and easy by the construction hence we omit it.

In the notation in (1.1.3), (1.2.2) and (1.3.6), we set

$$\begin{aligned} &\bar{I}^\cdot(\mathring{\mathcal{X}}^*) := i^{-1}(\mathring{lj})_* \Delta^\cdot(\mathring{\mathcal{X}}^*)\,, \\ \text{(A.2.2)} \qquad &\bar{I}^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) := \operatorname{Ker}\{I^\cdot(\mathring{\mathcal{X}}) \to \bar{I}^\cdot(\mathring{\mathcal{X}}^*)\} \quad \text{and} \\ &I^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) := \operatorname{Coker}\{I^\cdot(\mathring{\mathcal{X}}) \to I^\cdot(\mathring{\mathcal{X}}^*)\}\,[-1]\,. \end{aligned}$$

Lemma (A.2.3). *In the situation and the notation in* (1.1)–(1.3) *and* (A.2.2), *we have a commutative diagram*

$$\begin{array}{ccccccccc} & 0 & \to & K_Q^\cdot & \to & \rho(A_Q^\cdot(\mathring{X}_\infty), \nu) & \to & C_Q^\cdot[-1] & \to 0 \\ & & & QIS \uparrow\wr & & QIS\uparrow\wr & & QIS\uparrow\wr & \\ & 0 & \to & I^\cdot(\mathring{\mathcal{X}}) & \to & I^\cdot(\mathring{\mathcal{X}}^*) & \to & I^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*)[1] & \to 0 \\ & & & \| & & QIS\uparrow\wr & & & \\ 0 \to & \bar{I}^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) & \to & I^\cdot(\mathring{\mathcal{X}}) & \to & \bar{I}^\cdot(\mathring{\mathcal{X}}^*) & \to & 0 & \end{array}$$

where the top horizontal sequence is QIS exact and the other horizontal sequences are exact. In particular,

$$\bar{I}^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \underset{QIS}{\overset{\sim}{\to}} I^\cdot(\mathring{\mathcal{X}}, \mathring{\mathcal{X}}^*) \underset{QIS}{\overset{\sim}{\to}} C_Q^\cdot[-2]\,.$$

The hypercohomology of each horizontal sequence yields a local cohomology sequence in the category of mixed Hodge structures.

Proof. We shall show that the top horizontal sequence is QIS exact in the middle term. The other assertions are standard and easy to see by the construction and we omit their proofs.

Let $(x, y) \in A^n \oplus A^{n-1}$ such that $d(x, y) = (d_A x, \nu x - d_A y) = 0$ and that there exists $\bar{\eta} \in C^{n-1}$ with $\bar{y} = -d_C \bar{\eta}$ in $C^\cdot$. Then, since $\overline{y + d_A \eta} = 0$, there exists $\xi \in A^{n-1}$ satisfying $y + d_A \eta = \nu\xi$. Hence

$$\begin{aligned} &(x, y) - d(\xi, \eta) = (x - d_A \xi, y + d_A \eta - \nu\xi) = (x - d_A \xi, 0) \quad \text{and} \\ &\nu(x - d_A \xi) = \nu x - d_A \nu\xi = \nu x - d_A(y + d_A \eta) = \nu x - d_A y = 0\,. \end{aligned}$$

Therefore $x - d_A \xi \in K^n$. ∎

Combining the mixed versions of the "Wang sequences" in (A.2.1) and the local cohomology sequences in (A.2.3) as in [8], we get mixed versions of the Clemens-Schmid sequences.

2. General degenerations of Todorov surfaces

In this section, we recall and modify the results in [20] (cf. also [30]) for our later use.

(2.1) We recall first some facts from coding theory. Let $\boldsymbol{F}_2 := \boldsymbol{Z}/2\boldsymbol{Z}$. A *binary linear code* $(V \subset \boldsymbol{F}_2^I)$ on a finite set I is a vector subspace V of the $\boldsymbol{F}_2$-vector space $\boldsymbol{F}_2^I$ of all maps from I to $\boldsymbol{F}_2$. The *distance* of $\varphi \in \boldsymbol{F}_2^I$ is $\#\{i \in I \mid \varphi(i)=1\}$. A binary code $(V \subset \boldsymbol{F}_2^I)$ is *equidistant* if all non-zero elements of V have the same distance; this common distance is called the *distance of the code*. Let $(V \subset \boldsymbol{F}_2^I)$ be a binary linear code. The *linear subcode* associated o to a subset $J \subset I$ is defined as $(\{\varphi \in V \mid \varphi(i)=0 \text{ if } i \notin J\} \subset \boldsymbol{F}_2^J)$.

In the case that the set I itself has a structure of $\boldsymbol{F}_2$-vector space of dimension 4, we define a binary linear code

$$\mathcal{D} := (\{\text{affine linear function on } I\} \subset \boldsymbol{F}_2^I)\,.$$

Assigning a pair of integers $(k, \alpha)(C) := (\# J, \dim V)$ to a linear subcode $C = (V \subset \boldsymbol{F}_2^J)$ of $\mathcal{D}$, we get

Lemma (2.1.1). *There is an order preserving bijection*

$$\{\textit{linear subcode of } \mathcal{D}\}/(\textit{isom. as abstract codes})$$
$$\updownarrow$$
$$\{(k, \alpha) \in \boldsymbol{Z}^2 \mid 0 \leqslant \alpha \leqslant 5,\ 2^4 - 2^{4-\alpha} \leqslant k \leqslant \alpha + 11\}\,,$$

where we endow these sets orders defined respectively by

$$C' \leqslant C \Leftrightarrow C' \textit{ is isomorphic to a linear subcode of } C\,,$$
$$(k', \alpha') \leqslant (k, \alpha) \Leftrightarrow \alpha' \leqslant \alpha \textit{ and } \alpha - \alpha' \leqslant k - k'\,.$$

The proof is found in [20, (1.2)]. The assertion about the orders are implicit there, but a careful reading of that proof leads us to this assertion.

(2.2) We recall here the definition of Todorov surfaces and $K3$ surfaces of Todorov type and their relationships.

Definition **(2.2.1)**. *A canonical model $\bar{X}$ of a smooth minimal surface X is called a Todorov surface if $\chi(\mathcal{O}_{\bar{X}})=2$ and $\bar{X}$ has an involution σ such that $\bar{X}/\sigma$ is $K3$ surface only with rational double points. A pair of integers $(\ell, \alpha) := (c_1^2(X), \log_2 \#(2\text{-torsion of } \operatorname{Pic}(X)))$ is called the type of $\bar{X}$.*

[20, § 5] shows that the values of (ℓ, α) are as in the table (2.3.3) below.

Let (Y, E) be a pair of a smooth minimal K3 surface Y and a disjoint union $E = \sum_{i \in I} E_i$ of (-2)-curves on Y. By using the cup product pairing on $H^2(Y, \boldsymbol{Z})$ and the reduction modulo 2, we have a homomorphism of modules:

$$\delta: (\text{primitive span of } \sum_i \boldsymbol{Z}[E_i] \text{ in } H^2(Y, \boldsymbol{Z})) \to \operatorname{Hom}(\sum_i \boldsymbol{Z}[E_i], \boldsymbol{F}_2) \simeq \boldsymbol{F}_2^I\,.$$

$(\operatorname{Im}\delta \subset \boldsymbol{F}_2^I)$ is called the *binary linear code of* (Y, E).

DEFINITION (**2.2.2**). *Let (l, α) be one of the* 11 *values for Todorov surfaces in the table* (2.3.3) *below. A* $K3$ *surface of Todorov type (l, α) is a triple $(\bar{Y}, L, E)$ consisting of a* $K3$ *surface $\bar{Y}$ only with rational double points, an ample line bundle L on $\bar{Y}$ and a disjoint union $E=\sum_{i\in I} E_i$ of (-2)-curves contained in the exceptional locus of the minimal resolution μ: $Y \to \bar{Y}$, such that $\mu^*L\otimes\mathcal{O}_Y(E)$ is* 2-*divisible in* $\mathrm{Pic}(Y)$ *and that $L\cdot L=2l$ and* $\dim \mathrm{Im}\, \delta=\alpha$ *for the associated code. E is called the distinguished (-2)-curves.*

Let $\bar{X}$ be a Todorov surface of type (l, α) and consider the following diagram:

$$\begin{array}{ccccc} \bar{C} & \to & \bar{X} & \leftarrow & \hat{X} \\ \downarrow & & \downarrow \pi & & \downarrow \hat{\pi} \\ \bar{B} & \to & \bar{Y} & \overset{\mu}{\leftarrow} & Y \end{array} \tag{2.2.3}$$

where $\bar{Y}:=\bar{X}/\sigma$, $\bar{C}$ is the canonical curve of $\bar{X}$, $\bar{B}:=\pi(\bar{C})$, μ is the minimal resolution, and $\hat{X}:=\bar{X}\times_{\bar{Y}} Y$.

Lemma (2.2.4). *In the above notation, let $\mu^*\bar{B}+E$ be the branch locus of the double cover $\hat{\pi}$. Then there is a bijection:*

$$\{\bar{X} \mid \textit{Todorov surface of type } (l, \alpha)\}/\textit{isom.}$$
$$\updownarrow$$
$$\{(\bar{Y}, \bar{B}, E) \mid (\bar{Y}, \mathcal{O}_{\bar{Y}}(\bar{B}), E) \textit{ is a K3 surfaces of Todorov type } (l, \alpha) \textit{ and } \bar{B} \textit{ satisfies Condition (2.2.5) below}\}/\textit{isom.}$$

Condition (2.2.5). *On the smooth minimal model Y, $B:=\mu^*\bar{B}$ is reduced and has at most simple singularities and $B\cap E=\emptyset$.*

The proof of (2.2.4) is found in [20, §4, §5]. We call a data $(\bar{Y}, \bar{B}, E)$ in (2.2.4) a *Todorov triple.*

For a $K3$ surface $(\bar{Y}, L, E)$ of Todorov type (l, α), it is known that $\# I= l+8$, where $E=\sum_{i\in I} E_i$ (see [20, (5.2.ii)]).

(**2.3**) Finally, we summarize the main result in [20] about the moduli spaces of Todorov surfaces together with an observation of their general degenerations.

DEFINITION (**2.3.1**). *A numerical* $K3$ *surface is a smooth minimal surface with $p_g=1$, $q=0$ and $c_1^2=0$ (cf.* [35]).

Note that a numerical $K3$ surface has an elliptic fibration.

Proposition (2.3.2). *The values of type (l, α) of Todorov surfaces are as in the table* (2.3.3) *below. For each of these values of (l, α), there exists the moduli* *space of Todorov surfaces of type* (ℓ, α) *which is irreducible. The general degenerations of Todorov surfaces are those of type* (I_1) *in Table* 0 *in Introductions and except the case* $(2, 1)\to(0, 1)$, *they go down one step in the direction* $\downarrow$ *or* $\to$ *freely under the controle of the associated binary linear code. In case of* $(2, 1)\to(0, 1)$, *they go down two steps.*

$$
(2.3.3) \qquad (\ell, \alpha) = \begin{array}{lllcll}
(8, 5) & & & & & \\
(7, 4) & & & & & \\
(6, 3) & & & & & \\
(5, 2) & (4, 2) & & & & \\
(4, 1) & (3, 1) & (2, 1) & \vdots & & (0, 1) \\
(3, 0) & (2, 0) & (1, 0) & \vdots & (0, 0) & (-1, 0)
\end{array}
$$

The left hand side of the vertical dots in the table (2.3.3) *correspond to Todorov surfaces.*

(0, 1) *corresponds to numerical* $K3$ *surfaces with two double fibers.*

(0, 0) *corresponds to numerical* $K3$ *surfaces with one double fiber.*

$(-1, 0)$ *corresponds to* $K3$ *surfaces blown up one point.*

Proof. The first half of the proposition is proved in [20] by using the coding theory, a suitable version of Nikulin's embedding theorem, and the Torelli theorem and the surjectivity of the period map for $K3$ surfaces of Todorov type. We prove here the assertion about the degenerations which is implicit there.

There are sixteen (-2)-curves $E=\sum_i E_i$ on a smooth minimal Kummer surface $Y=\mathrm{Km}(A)$ which correspond to the 2-torsion points of the abelian surface A. They form a 4-dimensional $\boldsymbol{F}_2$-vector space and it is known that the binary linear code of (Y, E) is $\mathcal{D}$ in (2.1). This is the key point of the relationship of the abstract coding theory and the geometry from which it is deduced that the binary code associated to any $K3$ surface of Todorov type is isomorphic to a linear subcode of $\mathcal{D}$ (see [20, (2.1)]).

Let $(\bar{Y}, L, E)$ be a general $K3$ surface of Todorov type (ℓ, α), i.e., the smooth minimal model Y of $\bar{Y}$ has the Picard number $k+1=\ell+9$. Let $C=(V\subset \boldsymbol{F}_2^I)$ be the associated binary linear subcode. In case $(\ell, \alpha)\neq(8, 5)$, $(2, 1)$ or $(1, 0)$, L is very ample on $\bar{Y}$ and $\bar{Y}$ has oly $k=\ell+8$ ordinary double points which correspond to E [20, (7.7)]. By (2.1.1), if $(\ell-1, \alpha')$, $\alpha'=\alpha$ or $\alpha-1$, appears in the table (2.3.3), there is a distinguished (-2)-curve, say E_1, such that the linear subcode of C associated to the subset $I-\{1\}\subset I$ has invariants $(\ell+8-1, \alpha')$. Take a general member $\bar{B}_1\in|L|$ and a general member $\bar{B}_0\in|L|$ subjected that B_0 passes through the ordinary double point on $\bar{Y}$ corresponding to E_1. Let Δ be a small disc in the parameter space of the the pencil generated by $\bar{B}_0$ and $\bar{B}_1$ whose center $0\in\Delta$ corresponds to $\bar{B}_0$. Denote by $\bar{\mathcal{B}}\subset\bar{Y}\times\Delta$ the total space of the family $\{\bar{B}_t\}_{t\in\Delta}$ and by $\mathcal{B}\subset Y\times\Delta$ the proper transform of $\bar{\mathcal{B}}$, which is the total space of the family $\{B_t\}_{t\in\Delta}$ on Y. We can perform a semi-stable reduction of the family of pairs of the double cover of Y branched along B_t+E ($t\in\Delta$) and their ramification lici in the following way: (i) Set $\mathcal{E}_i:=E_i\times\Delta$ ($i\in I$). Let α: $\mathcal{Y}\to Y\times\Delta$ be the blowing-up along $\mathcal{B}\cap\mathcal{E}_1$. Denote by $W_{\mathcal{Y}}$ the exceptional divisor. (ii) Take the double cover β: $\hat{\mathcal{X}}\to\mathcal{Y}$ branched along the proper transform $\alpha^{-1}(\mathcal{B}+\sum\mathcal{E}_i)$. (iii) Since the $(\alpha\beta)^{-1}\mathcal{E}_i$ are the total space of families of (-1)-curves on the fibers of $\hat{\mathcal{X}}\to\Delta$, we can contract them to obtain γ: $\hat{\mathcal{X}}\to\mathcal{X}$. Set $\mathcal{B}_{\mathcal{X}}:=\gamma(\alpha\beta)^{-1}\mathcal{B}$ and $W_{\mathcal{X}}:=\gamma\beta^{-1}W_{\mathcal{Y}}$. Figure 1 below is the central fiber on each step. We obtain a family of pairs

(2.3.4) $$f\colon(\mathcal{X},\mathcal{B}_{\mathcal{X}})\to\Delta\,.$$

It is easy to see that this is a semi-stable degeneration of pairs of smooth minimal models of Todorov surfaces of type (l,α) and their smooth canonical curves whose central fiber X_0 is as the stage (a) in Figure 1 consisting of $\boldsymbol{P}^2$ and a smooth minimal model of a Todorov surface of type $(l-1,\alpha')$ (for details, cf. [36, (1.3)]).

In case $(l,\alpha)=(8,5)$, $\bar{Y}$ is a Kummer surface which can be represented as a quartic surface with 16 ordinary dobule points in $\boldsymbol{P}^3$ by $|2\Theta|$, where Θ is the theta divisor of the associated abelian surface, $L=\mathcal{O}_{\bar{Y}}(2)$ and E corresponds to the above 16 ordinary double points. Hence we can go on in the same way as before.

In case $(l,\alpha)=(2,1)$, it can be seen that the linear system $|L|$ gives a finite double cover $\bar{Y}\to\bar{Y}\subset\boldsymbol{P}^3$ over a quadric cone $\bar{Z}$ whose branch locus is a union of two smooth quadric sections Q_i ($i=1,2$) meeting transversally (cf. [7], [20, (5.4)]).

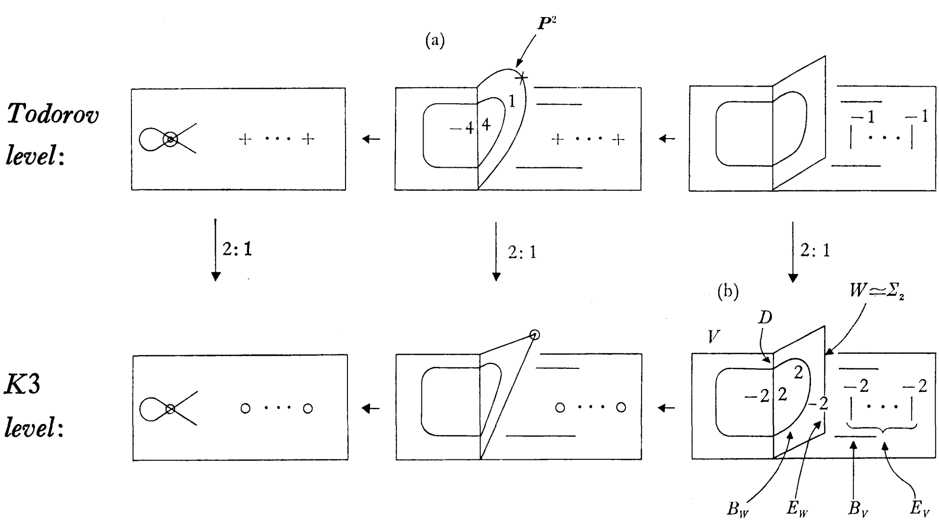


Figure 1

The 8+2 ordinary double points on $\bar{Y}$ come from $Q_1 \cap Q_2$ and from the vertex of $\bar{Z}$ counted once and twice respectively. Hence we can find a desired degenerating branch locus $\bar{B}_0 \in |L|$ as a pull-back of a suitable hyperplane section $\bar{H}_0$ of $\bar{Z} \subset \boldsymbol{P}^3$. The remaining steps of the construction are the same as before and we get a family of pairs like (2.3.4). We remark here that the central fiber X_0 of the resulting semi-stable degeneration of pairs consists of two $\boldsymbol{P}^2$ and the main component whose type drops as $(2, 1) \to (0, 1)$ in the table (2.3.3) if and only if the hyperplane section $\bar{H}_0$ contains the vertex of $\bar{Z}$.

In case $(\ell, \alpha)=(1, 0)$, the linear system $|L|$ give a finite double cover $\bar{Y} \to \boldsymbol{P}^2$ branched along a union of two smooth cubics C_i $(i=1, 2)$ meeting transversally (cf. [6], [20, (5.4)]), and we can go on as in the previous case (for details, see [35], [36]). ∎

3. Moduli and mixed period map

In this section, we shall formulate a mixed period map for smooth pairs of Todorov surfaces and their canonical curves. For that purpose, (2.2.4) allows us to use Todorov triples instead of Todorov surfaces.

(3.1) Let $(\bar{Y}_r, L_r, E_r)$ be a reference $K3$ surface of Todorov type (ℓ, α), $\bar{B}_r \in |L_r|$ a reference smooth curve, and $(\bar{Y}_r, \bar{B}_r, E_r)$ a reference Todorov triple (see (2.2)). Let μ: $Y_r \to \bar{Y}_r$ be the minimal resolution and $B_r := \mu^* \bar{B}_r$. We denote by

(3.1.1)
$$[\Lambda] = \Lambda(\bar{Y}_r, \bar{B}_r, E_r)$$

the Thom-Gysin exact sequence

$$\begin{array}{ccccccc} H^2(Y_r, \boldsymbol{Z}) & \to & H^2(\overset{\circ}{Y}_r, \boldsymbol{Z}) & \to & H^1(B_r, \boldsymbol{Z}) & \to & 0 \\ \| & & \| & & \| & & \\ \Lambda_Y & & \tilde{\Lambda} & & \Lambda_3 & & \end{array}$$

together with the cup product pairings on Λ_Y and on Λ_3 and with the fundamental classes

$$b := [B_i],\ e := [E_{r,i}] \in \Lambda_Y \quad (i \in I),$$

where $E_r = \sum_{i \in I} E_{r,i}$, $\overset{\circ}{Y}_r := Y_r - (B_r + E_r)$ and $\{e_i \mid i \in I\}$ is considered as an unordered set. We also denote by

$$[\Lambda_Y] = \Lambda(\bar{Y}_r, L_r, E_r)$$

the partial data consisting of Λ_Y, the cup product pairing on it and the fundamental classes b, $\{e_i \mid i \in I\}$, and by

$$[\Lambda_3] = \Lambda(\bar{B}_r)$$

the data Λ_3 equiped with the cup product pairing on it.

(3.2) Let $(\Lambda, G, \overset{\circ}{F}_r)$ be the reference mixed Hodge structure defined by the complex structure on $\overset{\circ}{Y}_r$, where

$$\begin{aligned}&\Lambda := \tilde{\Lambda}/\text{torsion},\\ \textbf{(3.2.1)}\quad &G := (G_1 = 0 \subset G_2 = \operatorname{Im}\{\Lambda_Y \to \Lambda\} \subset G_3 = \Lambda) \quad \text{weight filtration},\\ &F_r := (F_r^0 = \Lambda \otimes \boldsymbol{C} \supset F_r^1 \supset F_r^2 \supset F_r^3 = 0) \quad \text{Hodge filtration.}\end{aligned}$$

Set

$$\textbf{(3.2.2)}\qquad \Lambda_2 := \{\lambda \in \Lambda_Y \mid \lambda \cdot b = \lambda \cdot e_i = 0 \ (i \in I)\}\ .$$

Then

$$\begin{aligned}&\operatorname{gr}_2^G \Lambda = G_2 \Lambda \hookrightarrow \Lambda_2 \quad \text{with finite cokernel},\\ &\operatorname{gr}_3^G \Lambda \xrightarrow{\sim} \Lambda_3\ .\end{aligned}$$

Denote

$$\textbf{(3.2.3)}\qquad f^p := \dim F_r^p\ , \quad f_i^p := \dim \operatorname{gr}_i^G F_r^p\ .$$

Since Y_r is a smooth minimal K3 surface and B_r is isomorphic to the canonical curve C_r of the Todorov surface of type (ℓ, α) corresponding to $(\bar{Y}_r, \bar{\boldsymbol{B}}_r, E_r)$, we can compute as

$$\textbf{(3.2.4)}\qquad \begin{aligned}&f_2^2 = 1\ , \quad f_2^1 = \operatorname{rank} \Lambda_2 - 1 = 12 - \ell\ ,\\ &f_3^2 = \operatorname{genus} B_r = \operatorname{genus} C_r = (2(C_r)^2 + 2)/2 = \ell + 1\ ,\\ &f_3^1 = 2 f_3^2 = 2(\ell + 1)\ ,\\ &f^2 = f_2^2 + f_3^2 = \ell + 2\ , \quad f^1 = f_2^1 + f_3^1 = \ell + 14\ .\end{aligned}$$

Let

$$\textbf{(3.2.5)}\qquad \begin{aligned}&\mathcal{F}_i := \operatorname{Flag}(\Lambda_i \otimes \boldsymbol{C}; f_i^1, f_i^2)\ ,\\ &\mathcal{F} := \{F \in \operatorname{Flag}(\Lambda_i \otimes \boldsymbol{C}; f^1, f^2) \mid \operatorname{gr}_i^G F \in \mathcal{F}_i \text{ for all } i\}\ .\end{aligned}$$

and let

$$\textbf{(3.2.6)}\qquad \operatorname{gr}: \mathcal{F} \to \mathcal{F}_2 \times \mathcal{F}_3\ , \quad F \mapsto (\operatorname{gr}_2^G F, \operatorname{gr}_3^G F)\ .$$

The classifying spaces D_i and D of Hodge filtrations on Λ_i and on Λ are defined respectively by

D_2: the one of the two connected components of

$\{F \in \mathcal{F}_2 \mid F^2 \cdot F^2 = 0,\ \omega \cdot \bar{\omega} > 0\ (0 \neq \omega \in F^2)\}$

(3.2.7) which contains the reference Hodge filtration $\operatorname{gr}_2^G F_r$,

$$D_3 := \{F \in \mathcal{F}_3 \mid F^2 \cdot F^2 = 0, \sqrt{-1}\, \omega \cdot \bar{\omega} > 0 \ (0 \neq \omega \in F^2)\}$$
$$D := \mathrm{gr}^{-1}(D_2 \times D_3) \subset \mathcal{F},$$

(cf. [24, Appendix. §6, II. §7], [34], [23, I.2]).

(3.3) Let $(\bar{Y}, L, E)$ be any K3 surface of Todorov type (l, α) and $\bar{B} \in |L|$ any smooth curve.

Definition **(3.3.1)**. *A $[\Lambda, D]$-marking of a Todorov triple $(\bar{Y}, \bar{B}, E)$ is an isomorphism of data*

$$\eta = (\eta_Y, \tilde{\eta}, \eta_3): \Lambda(\bar{Y}, \bar{B}, E) \xrightarrow{\sim} [\Lambda]$$

sending the Hodge filtration on $H^2(\mathring{Y}, \boldsymbol{C})$ into D.

A $[\Lambda_Y, D_2]$-marking of a K3 surface $(\bar{Y}, L, E)$ of Todorov type is an isomorphism of data

$$\eta_Y: \Lambda(\bar{Y}, L, E) \xrightarrow{\sim} [\Lambda_Y]$$

sending the Hodge filtration on $H^2(Y, \boldsymbol{C})$ into D_2.

A $[\Lambda_3]$-marking of a curve $(\bar{B})$ is an isometry

$$\eta_3: \Lambda(\bar{B}) \xrightarrow{\sim} [\Lambda_3].$$

Notice that a $[\Lambda_Y, D_2]$-marking introduced above coincides with a "special marking" in [20, §7].

We denote by $\mathrm{Aut}[\Lambda, D]$, $\mathrm{Aut}[\Lambda_Y, D_2]$ and $\mathrm{Aut}[\Lambda_3]$ the groups of automorphisms of the data $[\Lambda]$, $[\Lambda_Y]$ and $[\Lambda_3]$ respectively which preserve, in the first two cases, the components D and D_2 respectively.

Lemma (3.3.2) *The natural map*

$$\mathrm{Aut}[\Lambda, D] \to \mathrm{Aut}[\Lambda_Y, D_2] \times \mathrm{Aut}[\Lambda_3]$$

is surjective.

Proof. Since Λ_3 is $\boldsymbol{Z}$-free, there exists a $\boldsymbol{Z}$-submodule $\Lambda_3' \subset \Lambda$ such that $\Lambda = \mathrm{Im}\{\Lambda_Y \to \Lambda\} \oplus \Lambda_3'$. Notice also that an automorphism of the data $[\Lambda]$ preserves D if and only if its restriction on the data $[\Lambda_Y]$ preserves D_2. The lemma follows from these observations ∎

For a K3 surface $(\bar{Y}, L, E)$ of Todorov type, let $\mu: Y \to \bar{Y}$ be the minimal resolution. Let $W(\bar{Y})$ be the group of isometries of the lattice $H^2(Y, \boldsymbol{Z})$ generated by the reflections $x \mapsto x + (x \cdot d)d$ $(x \in H^2(Y, \boldsymbol{Z}))$ where d runs over the fundamental classes of all the exceptional (-2)-curves of μ. We denote by $W(\bar{Y}, E)$ the subgroup of $W(\bar{Y})$ consisting of those elements which preserve the unordered set $\{[E_i] \mid i \in I\}$ of the fundamental classes of the distinguished (-2)-curves

$E=\sum_{i\in I} E_i$. Notice that $w\in W(\bar{Y}, E)$ acts on the set of $[\Lambda_Y, D_2]$-markings by $\varphi_Y \mapsto \varphi_Y w^{-1}$. We call an element of the set

$$\{[\Lambda_Y, D_2]\text{-marking of } (\bar{Y}, L, E)\}/W(\bar{Y}, E)$$

a *marking* of the K3 surface $(\bar{Y}, L, E)$ of Todorov type or of a Todorov triple $(\bar{Y}, \bar{B}, E)$ $(\bar{B}\in |L|)$.

(3.4) [20, (7.5)] constructs the coarse moduli space of Todorov surfaces in the following way.

By the Torelli theorem and the surjectivity of the period map for K3 surfaces of Todorov type (l, α), the local universal families are glued together to make up a universal family

$$g: (\bar{\mathcal{Y}}, \mathcal{L}, \mathcal{E}, \bar{\varphi}_{\mathcal{Y}}) \to D_2$$

of marked K3 surfaces of Todorov type (l, α). Let $\mathcal{V}=\mathcal{V}_{(l,\alpha)}$ be the Zariski open subset of the projective bundle $\boldsymbol{P}(g_*\mathcal{L})$ over D_2 consisting of marked Todorov triples $(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y)$, i.e., $\bar{B}\in |L|$ satisfies Condition (2.2.5). Let

$$f: (\bar{\mathcal{Y}}, \bar{\mathcal{B}}, \mathcal{E}, \bar{\varphi}_{\mathcal{Y}}) \to \mathcal{V}$$

be the universal family of the marked Todorov triples of type (l, α). Then the action of $\gamma\in \mathrm{Aut}\,[\Lambda_Y, D_2]$ on D_2 lifts onto $\boldsymbol{P}(g_*\mathcal{L})$ by the Torelli theorem for K3 surfaces of Todorov type. In fact, if $\gamma(\bar{Y}, L, E, \bar{\varphi}_Y)=(\bar{Y}', L', E', \bar{\varphi}'_Y)$ and φ_Y (resp. φ'_Y) is a lifting of $\bar{\varphi}_Y$ (resp. $\bar{\varphi}'_Y$), there exist uniquely $w\in W(\bar{Y}, E)$ and an isomorphism $\tilde{\gamma}: (\bar{Y}, L, E) \simeq (\bar{Y}', L', E')$ such that $(\tilde{\gamma}^{-1})^* = (\varphi'_Y)^{-1}\gamma\varphi_Y w$: $\Lambda(\bar{Y}, L, E) \simeq \Lambda(\bar{Y}', L', E')$. Now define the action of $\gamma\in \mathrm{Aut}\,[\Lambda_Y, D_2]$ on $\boldsymbol{P}(g_*\mathcal{L})$ by

$$\gamma(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y) = (\bar{Y}', \tilde{\gamma}\bar{B}, E', \bar{\varphi}'_Y)\,.$$

This action on $\boldsymbol{P}(g_*\mathcal{L})$ is properly discontinuous since so is that on D_2. The quotients $\mathcal{V}/\mathrm{Aut}\,[\Lambda_Y, D_2]$ and $D_2/\mathrm{Aut}\,[\Lambda_Y, D_2]$ are the required coarse moduli spaces of Todorov surfces of type (l, α) and of K3 surfaces of Todorov type (l, α) respectively.

(3.5) We recall here a formulation of a mixed period map for Todorov surfaces with smooth canonical curves.

Let

$$\mathcal{U} = \mathcal{U}_{(l,\alpha)} \subset \mathcal{V}_{(l,\alpha)} \subset \boldsymbol{P}(g_*\mathcal{L}) \tag{3.5.1}$$

be the Zariski open subset consisting of those marked Todorov triples $(\bar{Y}, \bar{B}, E, \bar{\varphi}_Y)$ which satisfy the following Condition (3.5.2).

Condition (3.5.2). *On the minimal resolution μ: $Y\to\bar{Y}$, $B:=\mu^*\bar{B}$ is smoo-*

th and $B \cap E = \emptyset$.

We define a mixed period map

$$(3.5.3) \qquad \begin{aligned} &\Phi\colon \mathcal{U}/\mathrm{Aut}\,[\Lambda_Y, D_2] \to D/\mathrm{Aut}\,[\Lambda, D]\,, \\ &\Phi(\bar{Y}, \bar{B}, E) := \varphi(\text{Hodge filtration on } H^2(\mathring{Y}, \boldsymbol{C}))\,, \end{aligned}$$

where φ is any $[\Lambda, D]$-marking and $\mathring{Y} = Y - (B + E)$. We see that Aut $[\Lambda, D]$ acts on D properly discontinuously (cf. [34, II]) and that, with the aid of the univereal family over $\mathcal{U}$, Φ is holomorphic.

4. Extension of mixed period map

In this section, we shall prove that the local monodromy on $H^2(\mathring{Y}_\infty, \boldsymbol{Z})$, arround a "tame" degeneration of Todorov triples in (2.3.2) splits and we shall extend the mixed period map Φ in (3.5.3) to $\bar{\Phi}$ over these degenerations. We continue to work on the stage (b) of Figure 1 in (2.3). We use the notation in §2.

(4.1) We recall first a general result on the splitting of a nilpotent endomorphism on a vector space over a field. Let

V: a finite dimensional vector space over a field,

G: an increasing filtration of V,

N: a niplotent endomrophism of V which is compatible with G

The following lemma is found in [28, (2.11), (2.16)].

Lemma (4.1.1). *In the above notation, if* length $G \leqslant 2$, *i.e., for some* i, $G_i = 0$ *and* $G_{i+2} = V$, *then the following are equivalent to each other*:

(i) *$G*L$ yields the G-relative N-filtration, where L is the N-filtration.*

(ii) *The G-relative N-filtration exists.*

(iii) *G is strict for N^j for all non-negative integers j.*

(iv) *G has an N-stable splitting.*

We can show implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i). The assumption length $G \leqslant 2$ is necessary for the step (ii)$\Rightarrow$(iii). For details, see the above reference.

(4.1.1) is a remarkable fact but it is not sufficient for our use. We need an investigation of the local monodromy over $\boldsymbol{Z}$.

(4.2) Let

$$(4.2.1) \qquad \bar{\mathcal{U}} := \bar{\mathcal{U}}_{(l,\alpha)} \subset \boldsymbol{P}(g_* \mathcal{L})$$

be a partial compactification of $\mathcal{U}$ in (3.5.1) obtained by adding those triples $(\bar{Y}, \bar{B}, E)$ which satisfy the following Condition (4.2.2).

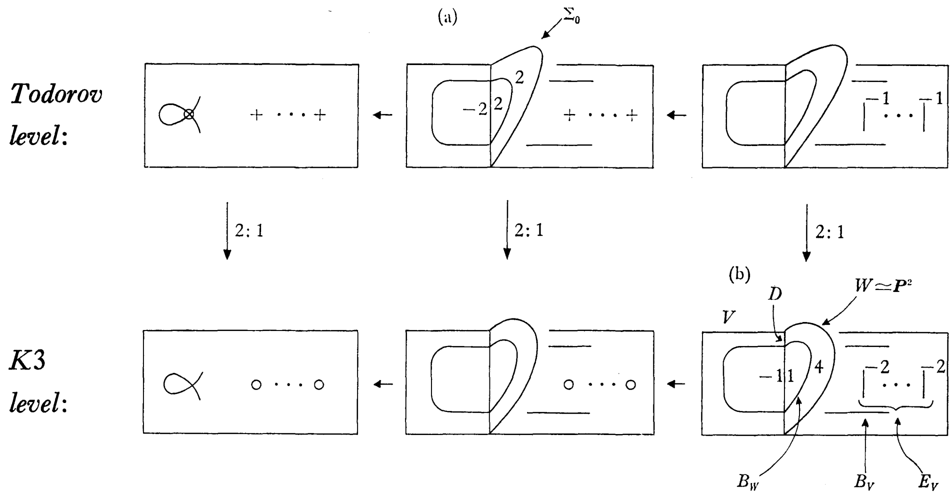


Figure 2

Condition (**4.2.2**). *The Picard number of the minimal resolution Y of $\bar{Y}$ is $l+9$ and $\bar{B}$ is an irreducible, reduced curve with one node. We devide the cases*:

(**i**) *$\bar{B}$ passes through one of the double points on $\bar{Y}$.*

(**ii**) *$\bar{B}$ is apart from all the double points on $\bar{Y}$*

Then, by the same arguments as (3.4), the quotient

$$\bar{\mathcal{U}}/\mathrm{Aut}\,[\Lambda_Y, D_2] \tag{4.2.3}$$

is the coarse moduli space of the triples in question. The central fiber $Y_0 = V \cup W$ of a semi-stable reduction of the degeneration of types (4.2.2.i) and (4.2.2.ii) are given in Figure 1 in (2.3) and Figure 2 above respectively.

(**4.3**) Let

$$f: (\mathcal{Y}, \mathcal{B}+\mathcal{E}) \to \Delta \tag{4.3.1}$$

be a semi-stable degeneration of type (4.2.2.i) on the stage (b) in Figure 1. By (A.2), we can consider the Thom-Gysin-Clemens-Schmid diagram (1.4.6) over $\boldsymbol{Z}$ with exact columns. In order to adjust that diagram for our use, we set

$$\begin{aligned}
&\tilde{G}_3(\mathring{Y}_\infty) := H^2(\mathring{Y}_\infty, \boldsymbol{Z}),\\
&\tilde{G}_2(\mathring{Y}_\infty) := \mathrm{Im}\{H^2(Y_\infty, \boldsymbol{Z}) \to \tilde{G}_3(\mathring{Y}_\infty)\},\\
&\tilde{G}_3(\mathring{Y}_0) := \mathrm{Coker}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \boldsymbol{Z}) \to H^2(\mathring{Y}_0, \boldsymbol{Z})\},\\
&\tilde{G}_2(\mathring{Y}_0) := \mathrm{Im}\{H^2(Y_0, \boldsymbol{Z}) \to \tilde{G}_3(\mathring{Y}_0)\},\\
&\tilde{G}_3(\mathring{V}) := \mathrm{Coker}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \boldsymbol{Z}) \to H^2(\mathring{Y}_0, \boldsymbol{Z}) \to H^2(\mathring{V}, \boldsymbol{Z})\},\\
&\tilde{G}_2(\mathring{V}) := \mathrm{Im}\{H^2(V, \boldsymbol{Z}) \to \tilde{G}_3(\mathring{V})\},
\end{aligned} \tag{4.3.3}$$

$$\widetilde{G}_3(\overset{\circ}{W}) := H^2(\overset{\circ}{W}, \boldsymbol{Z}),$$
$$\widetilde{G}_2(\overset{\circ}{W}) := \operatorname{Im}\{H^2(W, \boldsymbol{Z}) \to \widetilde{G}_3(\overset{\circ}{W})\} = \widetilde{G}_3(\overset{\circ}{W}),$$

where we use the notation as (1.1.2) applied for (4.3.1) as well as the notation on the stage (b) in Figure 1. Since $H^1(B_0, \boldsymbol{Z})$ (resp. $H^1(B_\infty, \boldsymbol{Z})$, $H^1(B_{\overset{\circ}{V}}, \boldsymbol{Z})$) is $\boldsymbol{Z}$-free, the Thom-Gysin exact sequence implies that the torsion of $\widetilde{G}_i(\overset{\circ}{Y}_0)$ (resp. $\widetilde{G}_i(\overset{\circ}{Y}_\infty)$, $\widetilde{G}_i(\overset{\circ}{V})$) for $i=2, 3$ coincide. We denote

$$\begin{aligned} G_i(\overset{\circ}{Y}_\infty) &:= \widetilde{G}_i(\overset{\circ}{Y}_\infty)/(\text{torsion}) \\ G_i(\overset{\circ}{Y}_0) &:= \widetilde{G}_i(\overset{\circ}{Y}_0)/(\text{torsion}) \\ G_i(\overset{\circ}{V}) &:= \widetilde{G}_i(\overset{\circ}{V})/(\text{rorsion}) \end{aligned} \tag{4.3.3}$$

In the notation (4.3.3), the Thom-Gysin-Clemens-Schmid diagram becomes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & H^1(B_0, \boldsymbol{Z}) & \to & H^1(B_\infty, \boldsymbol{Z}) & \overset{N_B}{\to} & H^1(B_\infty, \boldsymbol{Z}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & G_3(\overset{\circ}{Y}_0) & \to & G_3(\overset{\circ}{Y}_\infty) & \overset{N}{\to} & G_3(\overset{\circ}{Y}_\infty) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & G_2(\overset{\circ}{Y}_0) & \to & G_2(\overset{\circ}{Y}_\infty) & \overset{0}{\to} & G_2(\overset{\circ}{Y}_\infty) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array} \tag{4.3.4}$$

We notice that all the columns of the diagram (4.3.4) are exact by (A.2) and the construction. The top row is the case of curves and the exactness is well-known. The bottom row is exact by construction. Hence we see, by chasing the diagram, that the middle row is also exact.

Lemma (4.3.5). *For type (4.2.2.i), there exists a $\boldsymbol{Z}$-basis $\{e_1, \cdots, e_{m+2g}\}$ of $G_3(\overset{\circ}{Y}_\infty)$ satisfying the following conditions.*

(i) *$\{e_1, \cdots, e_m\}$ is a $\boldsymbol{Z}$-basis of $G_2(\overset{\circ}{Y}_\infty)$ and $\{e_{m+1}, \cdots, e_{m+2g}\}$ is a lifting of a symplectic $\boldsymbol{Z}$-basis of $H^2(B_\infty, \boldsymbol{Z})$.*

(ii) $$N(e_i) = \begin{cases} -2e_{m+1} & \text{if } i = m+g+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{e_1, \cdots, e_{m+2g}\}$ be a $\boldsymbol{Z}$-basis of $G_3(\overset{\circ}{Y}_\infty)$ satifying the condition (i). By the Picard-Lefschetz formula on $H^1(B_\infty, \boldsymbol{Z})$ (cf. [25, XV.3.4]) and the (4.1.1.iii), we may assume

$$N(e_i) = \begin{cases} -2e_{m+1}+x & \text{if } i = m+g+1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $x \in G_2(\mathring{Y}_\infty)$. Hence it is enough to show

Claim. Im N *in* $G_3(\mathring{Y}_\infty)$ *is not primitive.*

By this claim, x is 2-divisible and, replacing e_{m+1} by $e_{m+1}+x/2 \in G_3(\mathring{Y}_\infty)$, we get the desired basis.

We now prove the above claim. Since the restriction map $H^2(W, \boldsymbol{Z}) \to H^2(D, \boldsymbol{Z})$ is surjective and the fundamental class of B_W is sent to the 2-divisible element $[B_D]$ of $H^2(D, \boldsymbol{Z})$, where $B_D := B_W \cap D =: \{p, q\}$, the Mayer-Vietoris sequence implies an exact sequence

$$(4.3.6) \qquad 0 \to \tilde{G}_2(\mathring{Y}_0) \xrightarrow{r} \tilde{G}_2(\mathring{V}) \oplus \tilde{G}_2(\mathring{W}) \to H^2(D, \boldsymbol{Z})/\boldsymbol{Z}[B_D] \to 0 .$$

Since $\tilde{G}_2(\mathring{W})$ and $H^2(D, \boldsymbol{Z})/\boldsymbol{Z}[B_D]$ are isomorphic through the above map, (4.3.6) splits hence we have, in praticular,

$$(\text{torsion of Im } r) \oplus \tilde{G}_2(\mathring{W}) = (\text{torsion of } \tilde{G}_2(\mathring{V})) \oplus \tilde{G}_2(\mathring{W}) .$$

It is easy to compute, by the Thom-Gysin sequence and the Mayer-Vietoris sequence, the following:

$$H^1(\mathring{V}, \boldsymbol{Z}) = H^1(\mathring{W}, \boldsymbol{Z}) = 0 . \quad H^2(\mathring{W}, \boldsymbol{Z})\text{: 2-torsion.}$$
$$H^3(\mathring{Y}_0, \boldsymbol{Z}) = 0 .$$

By the Clemens-Schmid-Thom-Gysin diagram, we see

$$H^2(\mathcal{Y}, \mathcal{Y}^*; \boldsymbol{Z}) \simeq H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \boldsymbol{Z}) ,$$

so, by chasing the diagram, we have

$$\text{Im}\{H^2(\mathring{\mathcal{Y}}, \mathring{\mathcal{Y}}^*; \boldsymbol{Z}) \to H^2(\mathring{Y}_0, \boldsymbol{Z})\} \subset \text{Im}\{H^2(Y_0, \boldsymbol{Z}) \to H^2(\mathring{Y}_0, \boldsymbol{Z})\} .$$

Notice also that

$$\text{Im}\{H^0(B_V, \boldsymbol{Z}) \oplus H^0(B_W, \boldsymbol{Z}) \xrightarrow{\text{restriction}} H^0(B_D, \boldsymbol{Z})\} = \boldsymbol{Z}(p+q) ,$$
$$\text{Im}\{H^0(\mathring{D}, \boldsymbol{Z}) \xrightarrow{\text{residue}} H^0(B_D, \boldsymbol{Z})\} = \boldsymbol{Z}(p-q) .$$

Hence, by the above results, the Mayer-Vietoris-Thom-Gysin diagram is arranged as

$$(4.3.7) \qquad \begin{array}{ccc} 0 & & \\ \uparrow & & \\ \dfrac{\boldsymbol{Z}p+\boldsymbol{Z}q}{\boldsymbol{Z}(p+q)+\boldsymbol{Z}(p-q)} & 0 & 0 \\ \uparrow & \uparrow & \uparrow \end{array}$$

$$\begin{array}{ccccccccc}
0 \to & \dfrac{\boldsymbol{Z}p+\boldsymbol{Z}q}{\boldsymbol{Z}(p+q)} & \to & H^1(B_0, \boldsymbol{Z}) & \to & H^1(B_V, \boldsymbol{Z}) & \to & 0 & \\
& \uparrow & & \uparrow & & \uparrow & & & \\
0 \to & H^1(\mathring{D}, \boldsymbol{Z}) & \to & \widetilde{G}_3(\mathring{Y}_0) & \to & \widetilde{G}_3(\mathring{V}) \oplus \widetilde{G}_3(\mathring{W}) & \to & 0 & \\
& \uparrow & & \uparrow & & \uparrow & & & \\
& 0 & \to & \widetilde{G}_3(\mathring{Y}_0) & \to & \widetilde{G}_2(\mathring{V}) \oplus \widetilde{G}_2(\mathring{W}) & \to & \dfrac{H^2(D, \boldsymbol{Z})}{\boldsymbol{Z}[B_D]} & \to 0 \\
& & & \uparrow & & \uparrow & & & \\
& & & 0 & & 0 & & &
\end{array}$$

We see, from (4.3.7) together with the remarks after (4.3.2) and (4.3.6), that the image $H^1(\mathring{D}, \boldsymbol{Z})$ in $G_3(\mathring{Y}_0)$ is 2-divisible. Put

$$\hat{H}^1(\mathring{D}): \text{primitive span of image of } H^1(\mathring{D}, \boldsymbol{Z}) \text{ in } G_3(\mathring{Y}_0)\,.$$

Then we have

$$\text{(4.3.8)} \qquad \begin{array}{ccccccc}
0 \to & \dfrac{\boldsymbol{Z}p+\boldsymbol{Z}q}{\boldsymbol{Z}(p+q)} & \to & H^1(B_0, \boldsymbol{Z}) & \to & H^1(B_V, \boldsymbol{Z}) & \to 0 \\
& \uparrow\wr & & \uparrow\wr & & \uparrow\wr & \\
0 \to & \hat{H}^1(\mathring{D}) & \to & \mathrm{gr}_3^{G(\mathring{Y}_0)} & \to & \mathrm{gr}_3^{G(\mathring{V})} & \to 0
\end{array}$$

From (4.3.4), (1.4.5.ii) and the primitivity of $\hat{H}^1(\mathring{D})$, we have a commutative diagram

$$\text{(4.3.9)} \qquad \begin{array}{ccccccc}
0 \to & \mathrm{Im}\, N & \to & \mathrm{Ker}\, N & \to & \mathrm{gr}_2^L\, G_3(\mathring{Y}_\infty) & \to 0 \\
& \downarrow \alpha & & \uparrow\wr & & \downarrow & \\
0 \to & \hat{H}^1(\mathring{D}) & \to & G_3(\mathring{Y}_0) & \to & G_3(\mathring{V}) & \to 0
\end{array}$$

We see, from (4.3.4), that

$$\text{(4.3.10)} \qquad \begin{array}{l}
\mathrm{Ker}\, N/G_2(\mathring{Y}_\infty) \xrightarrow{\sim} \mathrm{Ker}\, N_B\,, \\
(\mathrm{Im}\, N + G_2(\mathring{Y}_\infty))/G_2(\mathring{Y}_\infty) \xrightarrow{\sim} \mathrm{Im}\, N_B
\end{array}$$

by the induced maps. Taking gr_3^G of (4.3.9), we have, by (4.3.10) and (4.3.8), a commutative exact diagram

$$\text{(4.3.11)} \qquad \begin{array}{ccccccc}
0 \to & \mathrm{Im}\, N_B & \to & \mathrm{Ker}\, N_B & \to & \mathrm{gr}_2^L\, H^1(B_\infty, \boldsymbol{Z}) & \to 0 \\
& \downarrow \alpha_B & & \uparrow\wr & & \downarrow & \\
0 \to & \dfrac{\boldsymbol{Z}p+\boldsymbol{Z}q}{\boldsymbol{Z}(p+q)} & \to & H^1(B_0, \boldsymbol{Z}) & \to & H^1(B_V, \boldsymbol{Z}) & \to 0
\end{array}$$

Since $\mathrm{Im}\, N_B$ in $\mathrm{Ker}\, N_B$ is 2-divisible and $H^1(B_V, \boldsymbol{Z})$ is $\boldsymbol{Z}$-free, α_B in (4.3.11) is not isomorphic hence not so is α in (4.3.9). This proves our claim. ∎

REMARK **(4.3.12).** *The same assertion as Lemma* (4.3.5) *holds also for the type* (4.2.2.ii) *on the stage* (*b*) *in Figure* 2 *in* (4.2). The proof is similar, but now terms $\tilde{G}_i(\mathring{V})\oplus\tilde{G}_i(\mathring{W})$ ($i=3, 2$) etc. are replaced by

$$\tilde{G}_3(\mathring{V}\sqcup\mathring{W}) := (H^2(\mathring{V}, \boldsymbol{Z})\oplus H^2(\mathring{W}, \boldsymbol{Z}))/\boldsymbol{Z}([D_{\mathring{V}}]-[D_{\mathring{W}}]),$$
$$\tilde{G}_2(\mathring{V}\sqcup\mathring{W}) := \operatorname{Im}\{H^2(V, \boldsymbol{Z})\oplus H^2(W, \boldsymbol{Z})\to\tilde{G}_3(\mathring{V}\sqcup\mathring{W})\} \quad \text{etc.},$$

where $[D_{\mathring{V}}]$ and $[D_{\mathring{W}}]$ are the fundamental classes of $\mathring{D}\subset\mathring{V}$ and $\mathring{D}\subset\mathring{W}$ respectively. The splitting of (4.3.6) in the present case is given by the image of $H^2(W, \boldsymbol{Z})$ in $\tilde{G}_2(\mathring{V}\sqcup\mathring{W})$. We omit the details.

(4.4) Let

(4.4.1) $$\{e_1, \cdots, e_{m+2g}\}$$

be a $\boldsymbol{Z}$-basis of Λ in (3.2.1) satisfying the condition (4.3.5.i) and let N be an endomorphism of Λ defined by

(4.4.2) $$N(e_i) = \begin{cases} -e_{m+1} & \text{if } i = m+g+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Since N splits, we can construct easily a partial compactification of the classifying space $D/\operatorname{Aut}[\Lambda, D]$ in (3.5.3) added only the boundary component of codimension 1 associated to N by the method of toroidal compactifications for locally symmetric Siegel spaces (cf. [1], [5]). As a set, this is defined by

(4.4.3) $$\begin{aligned}&\overline{D/\operatorname{Aut}[\Lambda, D]} := (D/\operatorname{Aut}[\Lambda, D])\sqcup(\exp(\boldsymbol{C}N)\,D/\exp(\boldsymbol{C}N))/\operatorname{Norm}_{\boldsymbol{Z}}(N)\,,\\ &\text{where}\quad \operatorname{Norm}_{\boldsymbol{Z}}(N) := \{\gamma\in\operatorname{Aut}[\Lambda, D]\,|\,\gamma^{-1}N\gamma = N\}\,.\end{aligned}$$

The analytic structure is defined through the following construction of $\overline{D/\operatorname{Aut}[\Lambda, D]}$.

Let $D_c\simeq\Delta^\lambda\times(U\times\Delta^{\mu-1})\times\Delta^\nu$ be a small open subset of D, where Δ is the unit disc, U is the upper half plane and the decomposition is the one into $D_2\times D_3\times$(extension data) (see (3.2.7)). Construct

(4.4.4) $$\begin{array}{ccc} D_c & \simeq & \Delta^\lambda\times(U\times\Delta^{\mu-1})\times\Delta^\nu \\ \downarrow\varepsilon & & \\ D_c/\exp(\boldsymbol{Z}N) & \simeq & \Delta^\lambda\times(\Delta^*\times\Delta^{\mu-1})\times\Delta^\nu \\ \cap & & \\ \overline{D_c/\exp(\boldsymbol{Z}N)} & \simeq & \Delta^\lambda\times\Delta^\mu\times\Delta^\nu \\ \downarrow & & \\ \overline{D_c/\exp(\boldsymbol{Z}N)}/\operatorname{Norm}_{\boldsymbol{Z}}(N) & & \end{array}$$

where $\mathcal{E}:=1\times(\exp 2\pi\sqrt{-1}(\ )\times 1)\times 1$. Patching up by

$$(4.4.5)\qquad \begin{array}{c} (D_c/\exp(\boldsymbol{Z}N))/\mathrm{Norm}_{\boldsymbol{Z}}(N) \hookrightarrow D/\mathrm{Aut}[\Lambda, D] \\ \cap \\ \overline{D_c/\exp(\boldsymbol{Z}N)}/\mathrm{Norm}_{\boldsymbol{Z}}(N)\,, \end{array}$$

we obtain $\overline{D/\mathrm{Aut}[\Lambda, D]}$. As in the case of locally symmetric Siegel spaces, this has a structure of V-manifold (=orbifold).

Proposition (4.4.6). *The mixed period map* Φ *in* (3.5.3). *extends holomorphically to*

$$\bar{\Phi}\colon \bar{\mathcal{U}}/\mathrm{Aut}[\Lambda_Y, D_2] \to \overline{D/\mathrm{Aut}[\Lambda, D]}$$

which sends a boundary point to its nilpotent orbit, where the source is (4.2.3) *and the target* (4.4.3).

Proof. By construction (4.2.1), the boundary $\bar{\mathcal{U}}-\mathcal{U}$ is a smooth divisor on $\bar{\mathcal{U}}$. Localizing the situation at a boundary point, we may assume

$$\mathcal{U} = \Delta^{11-\ell}\times(\Delta^{*}\times\Delta^{\ell})\subset\bar{\mathcal{U}} = \Delta^{11-\ell}\times\Delta^{\ell+1}$$

with local coordinates $t=(t_1, t')$, where $t_1=0$ is the boundary and t' the other coordinates. Take a point $\tau\in\mathcal{U}$ and fix an isomorphism of the data in (3.1.1)

$$\pi\colon \Lambda(\bar{Y}_\infty, \bar{B}_\infty, E_\infty) \xrightarrow{\sim} \Lambda(\bar{Y}_\tau, \bar{B}_\tau, E_\tau)\,.$$

By definition (or by (3.3.2)), for any $\boldsymbol{Z}$-basis $\{e_1(\infty), \cdots, e_{m+2g}(\infty)\}$ of $G_3(\mathring{Y}_\infty)$ and the monodromy logarithm N_∞ satisfying the condition (4.3.5) (see also (4.3.12)), there exists a $[\Lambda, D]$-making in (3.3.1)

$$\begin{aligned} &\eta = (\eta_Y, \bar{\eta}, \eta_3)\colon \Lambda(\bar{Y}_\tau, \bar{B}_\tau, E_\tau) \xrightarrow{\sim} [\Lambda] \quad \text{such that} \\ &\eta\pi e_i(\infty) = e_i \quad (m+1\leqslant i\leqslant m+2g)\,. \end{aligned}$$

Hence we have $N_\infty=(\eta\pi)^{-1}(2N)(\eta\pi)$ for the types (i) and (ii) in (4.2.2) on the stage (b). For each fixed t', let $F(\infty, t')$ be the limit Hodge filtration as $t_1\to 0$. We define

$$\bar{\Phi}(0, t')\colon = \exp(\boldsymbol{C}N)\,(\eta\pi F(\infty, t'))/\exp(\boldsymbol{C}N) \bmod \mathrm{Norm}_{\boldsymbol{Z}}(N)\,.$$

In order to see that $\bar{\Phi}$ is holomorphic, we observe its period matrix. We first examine the type (4.2.2.i). By (4.3.5.ii), gr_2^W and the extension data of the period matrix are invariant under the action of the local monodromy $T_\infty:=\exp N_\infty$. gr_3^W of the period matrix is the only part which is affected by T_∞. To see this part more precisely, let $\{e_1(t), \cdots, e_{m+2g}(t)\}$ be a horizontal frame of the local system $\{\Lambda(t):=H^2(\mathring{Y}_t, \boldsymbol{Z})/(\text{torsion})\}_{t\in\mathcal{U}}$ which coincides with $\eta^{-1}\{e_1, \cdots, e_{m+2g}\}$

at $t=\tau$. $e_{m+g+1}(t)$ is multi-valued. Let $\{\omega_1(t), \cdots, \omega_{g+1}(t)\}_{t\in\mathcal{U}}$ be a frame of the Hodge filter F^2 satisying $\omega_{1+i}(t)\equiv e_{m+g+i}(t) \bmod \sum_{j=1}^{m+g} Ce_j(t)$ $(1\leq i\leq g)$. Then the period matrix for F^2 is of the form

$$(\omega_1(t); \cdots, \omega_{g+1}(t))$$
$$= (e_1(t), \cdots; e_{m+1}(t), \cdots; e_{m+g+1}(t), \cdots, e_{m+2g}(t)) \begin{pmatrix} A(t) & B(t) \\ 0 & Z(t) \\ 0 & 1_g \end{pmatrix}$$

The (1, 1)–part $z_{11}(t)$ of $Z(t)$ is the only part which is multi-valued. By (4.3.5.ii), we can compute as

$$z_{11}(t) = 2(\log t_1)/2\pi\sqrt{-1}+s(t)\,,$$

where $s(t)$ extends holomorphically over $\bar{\mathcal{U}}$, which is equivalent to the existence of the limit Hodge filter $F^2(\infty, t')$. Hence, by (4.4.4) and (4.4.5), we have

$$\begin{array}{ccccc} \Phi: \mathcal{U} & \to & D_c/\exp(\boldsymbol{Z}N) & \to & D/\mathrm{Aut}[\Lambda, D] \\ \cap & & \cap & & \cap \\ \bar{\Phi}: \bar{\mathcal{U}} & \to & \overline{D_c/\exp(\boldsymbol{Z}N)} & \to & \overline{D/\mathrm{Aut}[\Lambda, D]} \end{array}$$

where the (1.1)-part of $Z(t)$ on the middle stage becomes

(4.4.7) $$\exp(2\pi\sqrt{-1}\, z_{11}(t)) = t_1^2 \exp(2\pi\sqrt{-1}\, s(t))\,.$$

This shows that $\bar{\Phi}$ is holomorphic for the type (4.2.2.i).

As for the type (4.2.2.ii), a similar argument works and instead of (4.4.7) we get

(4.4.8) $$\exp(2\pi\sqrt{-1}\, z_{11}(t)) = t_1 \exp(2\pi\sqrt{-1}\, s(t))$$

because of the (2:1) base extension in the semi-stable reduction of pairs in Figure 2 in (4.2). ∎

5. Inheritance of induction hypothesis and infinitesimal mixed Torelli theorem

(5.1) The following result is useful for our inductive approach of the mixed Torelli problem by using the degeneration of the type (4.2.2.i).

Proposition (5.1.1). *In the notation of* (4.3), *we see for the family* (4.3.1) *the following*:

(i) $G_3(\mathring{Y}_0)\cong\mathrm{Ker}\{N\colon G_3(\mathring{Y}_\infty)\to G_3(\mathring{Y}_\infty)\}$.

(ii) *The Gysin filtrations G are isomorphic under (i).*

(iii) *The Mayer-Vietoris filtration L and the N-filtration L are isomorphic* *under* (i).

(**iv**) *Before the shiftings* [2], $W_0 \supset (G*L)_0$ *with index* 2 *on* $G_3(\overset{\circ}{Y}_0)$ *and* $W = G*L$ *on* $G_3(\overset{\circ}{Y}_\infty)$.

(**v**) (i) *is an isomorphism of mixed Hodge structures with weight filtrations* $(G*L)$ [2] *on both terms.*

Proof. (i) and (ii) follow immediately from (4.3.4). Since $\widetilde{G}_3(\overset{\circ}{W}) = H^2(\overset{\circ}{W}, \boldsymbol{Z})$ is a 2-torsion and N satisfies (4.3.5.ii), (4.3.9) implies a commutative exact diagram

$$\text{(5.1.2)}\quad \begin{array}{ccccccccc} 0 & \to & \operatorname{Im} N & \to & \operatorname{Ker} N & \to & \operatorname{gr}_2^L G_3(\overset{\circ}{Y}_\infty) & \to & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\ 0 & \to & H^1(\overset{\circ}{D}, \boldsymbol{Z}) & \to & G_3(\overset{\circ}{Y}_0) & \to & G_3(\overset{\circ}{V}) \oplus \widetilde{G}_3(\overset{\circ}{W}) & \to & 0\,. \end{array}$$

This shows (iii).

As for the first part of (iv), we see in the same way as (1.3.1), (1.3.3) and (1.3.4.i) that a complex $K^{\cdot}_{\boldsymbol{Z}}$ over $\boldsymbol{Z}$ and its filtrations G, L and W are defined and that they satisfy $W = G*L$ on $K^{\cdot}_{\boldsymbol{Z}}$. The spectral sequence of hypercohomology of $(K^{\cdot}_{\boldsymbol{Z}}, W)$ degenerates in $E_2 = E_\infty$ [10, II]. We compute the E_2^2:

$$\begin{aligned} E_1^{-k,2+k} &= H^2(Y_0, \operatorname{gr}_k^W K^{\cdot}_{\boldsymbol{Z}}) = \bigoplus_{i+j=k} H^{2-i+j}((\widetilde{\mathcal{B}+\mathcal{E}})^{(i)} \cap \widetilde{Y}_0^{(-j+1)}, \boldsymbol{Z})\,. \\ E_2^{1,1} &= E_1^{1,1} = H^1(D, \boldsymbol{Z}) = 0 \\ E_2^{-1,3} &= E_1^{-1,3} = H^1(B_V, \boldsymbol{Z}) \\ E_2^{-2,4} &= E_1^{-2,4} = 0\,. \end{aligned}$$

Hence, before shifting [2], we have

$$W_{-1} = 0 \subset W_0 = \operatorname{Ker}\{H^2(\overset{\circ}{Y}_0, \boldsymbol{Z}) \overset{\alpha}{\to} H^1(B_V, \boldsymbol{Z})\} \subset W_1 = H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})\,,$$

where α is the composite of the Mayer-Vietoris map and the residue map. On the other hand, since

$$\begin{aligned} G_{-1} &= 0 \subset G_0 = \operatorname{Im}\{H^2(Y_0, \boldsymbol{Z}) \to H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})\} \subset G_1 = H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})\,, \\ L_{-2} &= 0 \subset L_{-1} = \operatorname{Im}\{H^1(\overset{\circ}{D}, \boldsymbol{Z}) \to H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})\} \subset L_0 = H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})\,, \end{aligned}$$

we have

$$(G*L)_{-1} = 0 \subset (G*L)_0 = G_0 + L_{-1} \subset (G*L)_1 = H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})$$

before the shiftings. By the part of the original Mayer-Vietoris-Thom-Gysin diagram like (4.3.7), we see

$$(G*L)_0 \subset W_0 \quad \text{with index 2 on } H^2(\overset{\circ}{Y}_0, \boldsymbol{Z})$$

hence so is that on $G_3(\overset{\circ}{Y}_0)$.

The second part of (iv) follows from (4.3.5) and the argument of the proof of the step (iv)$\Rightarrow$(i) in (4.1.1) which is valid also over $\boldsymbol{Z}$ (cf. [28, (2.11)]). (v) is a consequence of (i)–(iv) and (1.3.4). ∎

(5.2) We have the following partial result at present for the infinitesimal mixed period map.

Proposition (5.2.1). *The infinitesimal mixed Torelli theorem holds for the extension $\bar{\Phi}$ of the mixed period map in* (4.4.6) *at interior points $\in \mathcal{U}$ and at boundary points $\in \bar{\mathcal{U}} - \mathcal{U}$ of the type* (4.2.2.*i*) *in the tangential directions of the boundary.*

Proof. Let $(Y, B+E)$ be a pair of the smooth minimal model and its branch locus of $(\bar{Y}, \bar{B}+E) \in \mathcal{U}$. By taking the dual, we see that

$$H^1(T_Y(-\log(B+E))) \to \mathrm{Hom}(H^1(\Omega_Y^1(\log(B+E))), H^2(\mathcal{O}_Y))$$

is injective. This proves the first half of our assertion.

For a degeneration of pairs of the type (4.2.2.i), the locally trivial (=equisingular) small deformations of the pair on the stage (a) in Figure 1 in (2.3) within the limits of these pairs corresponds exactly to those on the stage (b), i.e., there are no problems of ordinary double points nor the Todorov involution. On the stage (a), they are determined by the deformations of the pair of the main component and the union of its branch locus and the double curve. The latter are determined by the deformations of $(V, (B+E+D)_V)$ on the stage (b). As before,

$$H^1(T_V(-\log(B+E+D)_V)) \to \mathrm{Hom}(H^1(\Omega_V^1(\log(B+E+D)_V)), H^2(\mathcal{O}_V))$$

is injective. This implies the second half of our assertion, because, for a boundary point $t_0 \in \bar{\mathcal{U}}$, $\bar{\Phi}(t_0)$ induces the Hodge filtration on $G_3(\mathring{V}) \otimes \boldsymbol{C} = H^2(\mathring{V}, \boldsymbol{C})/\boldsymbol{C}[\mathring{D}]$ by (5.1.2) and the difference of

$$H^2(\mathring{V}, \boldsymbol{C})/\boldsymbol{C}[\mathring{D}] \hookrightarrow H^2(\mathring{V}-\mathring{D}, \boldsymbol{C}) = H^2(V-(B+E+D)_V, \boldsymbol{C})$$

affects their gr_F^p only for $p=2$. ∎

PROBLEM **(5.2.2).** *Solve infinitesimal mixed Torelli problem for $\bar{\Phi}$.*
PROBLEM **(5.2.3).** *Solve local mixed Torelli problem for $\bar{\Phi}$.*
PROBLEM **(5.2.4).** *Solve generic mixed Torelli problem for Φ.*

REMARK **(5.3.5).** **(i)** In cases $(\ell, \alpha)=(1, 0)$ and $(2, 1)$ the generic mixed Torelli theorem is verified for geometric monodromy in an elementary way ([19], [23, II.2]).

(ii) The number of moduli of Todorov surfaces is 12 for every (ℓ, α). On the other hand, a hypersurface of $(\bar{\mathcal{U}}/\mathrm{Aut}[\Lambda_Y, D_2])_{(\ell-1,\alpha')}$, $\alpha'=\alpha$ or $\alpha-1$, is glued as the boundary locus of the degenerations of the type (4.2.2.i). Hence the induction step will not proceed naively.

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