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Osaka University
ON REAL J-HOMOMORPHISMS

Dedicated to Professor A. Komatu on his 70th birthday

HARUO MINAMI

(Received June 7, 1978)

1. In the present work we consider a Real analogue of J-homomorphisms in the sense of [3]. We use here the notation in [4], §§1 and 9 and [9], §2 for the equivariant homotopy groups which are discussed by Bredon [5] and Levine [10]. Moreover we shall use notations and terminologies of [4], §1 without any references.

Let us denote by \( GL(n, \mathbb{C}) \) (resp. \( GL(\infty, \mathbb{C}) \)) the general linear group of degree \( n \) (resp. the infinite general linear group) over the complex numbers with involutions induced by complex conjugation. Let \( X \) be a finite pointed \( \tau \)-complex. Then, by following the construction of usual J-homomorphisms (cf. [13], p. 314, [2]) we can define homomorphisms

\[
J_{R,n}: [\Sigma^{p,q}X, GL(n, \mathbb{C})]^\tau \to [\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau
\]

and

\[
J_{R}: [\Sigma^{p,q}X, GL(\infty, \mathbb{C})]^\tau \to \pi_{n}^{0,0}(\Sigma^{p,q}X)
\]

for \( p \geq 0 \) and \( q \geq 1 \) where let \( \pi_{n}^{0,0}(\Sigma^{p,q}X) = \lim_{n \to \infty} [\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau \). We now give definitions of \( J_{R,n} \) and \( J_{R} \) below. Let \( \Omega_{d}^{\ast,n}\Sigma^{n,n} \) denote the subspace of \( \Omega^{\ast,n}\Sigma^{n,n} \) consisting of maps of degree \( d \) in the usual sense. Let \( \gamma \) be the \( \tau \)-map of \(\Sigma^{n,n} \) induced by the correspondence of \( R^{n,n} \) such that \((x_{1}, \ldots, x_{2n}) \mapsto (x_{1}, \ldots, x_{2n-1}, -x_{2n})\). By adding \( \gamma \) to the elements of \( \Omega_{d}^{\ast,n}\Sigma^{n,n} \) with respect to the loop addition along fixed coordinates of \(\Sigma^{n,n} \) we have a \( \tau \)-map \( t: \Omega_{d}^{\ast,n}\Sigma^{n,n} \to \Omega_{d+1}^{\ast,n}\Sigma^{n,n} \). Then we obtain \( J_{R,n} \) by assigning to a base-point-preserving \( \tau \)-map \( f: \Sigma^{p,q}X \to GL(n, \mathbb{C}) \) the adjoint of the composite

\[
\Sigma^{p,q}X \xrightarrow{f} GL(n, \mathbb{C}) \xrightarrow{i} \Omega_{d}^{\ast,n}\Sigma^{n,n} \xrightarrow{t} \Omega_{d+1}^{\ast,n}\Sigma^{n,n}
\]

where \( i \) is the canonical inclusion map.

As is easily seen the diagram

\[
\begin{array}{ccc}
[\Sigma^{p,q}X, GL(n+1, \mathbb{C})]^\tau & \xrightarrow{J_{R,n+1}} & [\Sigma^{p+n+1,q+n+1}X, \Sigma^{n+1,n+1}]^\tau \\
\uparrow j_{*} & & \uparrow \Sigma_{1}^{n+1} \\
[\Sigma^{p,q}X, GL(n, \mathbb{C})]^\tau & \xrightarrow{J_{R,n}} & [\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau
\end{array}
\]

is commutative under the identification \( \Sigma^{r,s} \wedge \Sigma^{p,q} = \Sigma^{r+p+s,q} \) where \( j_{*} \) is the
homomorphism induced by a canonical inclusion map $j: GL(n, \mathbb{C}) \subset GL(n+1, \mathbb{C})$ and $\Sigma^{k+1}_q$ is the suspension homomorphism ([4], (7.2)). Therefore, by taking the direct limits we get a homomorphism

$$J_{R^+} : \lim_{\rightarrow} [\Sigma^{p,q}X, GL(n, \mathbb{C})^\tau] \to \pi^{0,0}_{q}(\Sigma^{p,q}X).$$

Also, as $X$ is compact we have a canonical isomorphism $\mu : \lim_{\rightarrow} [\Sigma^{p,q}X, GL(n, \mathbb{C})^\tau] \to [\Sigma^{p,q}X, GL(\infty, \mathbb{C})^\tau]$. So we define $J_R$ to be the composite $J_{R^+} \mu^{-1}$.

Taking $X = S^{q+1}$ in (1.1) $J_R$ becomes the homomorphism from $\pi_{p+q}(GL(\infty, \mathbb{C}))$ to $\pi_{p+q}$. The aim of this paper is to prove the following theorem for the homomorphism

$$J_R : \pi_{2p-2k,2q+2k-1}(GL(\infty, \mathbb{C})) \to \pi_{2p-2k,2q+2k-1}^t$$

for $p \geq k \geq 0$ and $p + k \geq 1$.

**Theorem.** The image $J_R(\pi_{2p-2k,2q+2k-1}(GL(\infty, \mathbb{C})))$ of the homomorphism (1.2) is a cyclic group of the following order:

- $m(2p)$ if either $p$, $k$ are even or odd
- $\frac{1}{2} m(2p)$ if $p$ is even and $k$ is odd
- $m(2p)$ or $2m(2p)$ if $p$ is odd and $k$ is even

where $m(t)$ is the numerical function as in [1], II, p. 139.

2. Let $X$ be a compact pointed $\tau$-space throughout this section.

Let $KR$ denote the Real $K$-functor [3]. Then a similar proof to the complex case gives rise to a canonical isomorphism

$$(2.1) \quad [X, GL(\infty, \mathbb{C})^\tau] \approx \widetilde{KR}(\Sigma^{0,1}X)$$

(cf. [8], Chap. I, Theorem 7.6) and so we may consider $J_x$ of (1.1) the homomorphism from $\widetilde{KR}(\Sigma^{p,q}X)$ to $\pi^{0,0}_{q}(\Sigma^{p,q}X)$ through this isomorphism. In particular, there exist isomorphisms

$$(2.2) \quad \pi_{2p-2k,2q+2k-1}(GL(\infty, \mathbb{C})) \approx \widetilde{KR}(\Sigma^{2p-2k,2q+2k}) \approx \widetilde{KO}(S^{4k}) \approx Z$$

by (2.1) and the Real Thom isomorphism theorem [3]. Similarly we have isomorphisms

$$(2.3) \quad \pi_{4p-1}(GL(\infty, \mathbb{C})) \approx \check{K}(S^{4p}) \approx \check{K}(S^{2p}) \approx Z$$

in the complex $K$-theory.

Let $\psi : \pi_{p+q}(X) \to \pi_{p+q}(X)$ and $\check{\psi} : \pi_{p+q}(X) \to \pi_{p+q}(X)$ denote the forgetful homomorphisms [4,5]. Then, from the above discussion we have the following commutative diagram:
\[ \frac{KO(S^{4k})}{\cong} \xrightarrow{c} \frac{K(S^{4k})}{\cong} \]

\[ \pi_{2p-2k,2p+2k-1}(GL(\infty, C)) \xrightarrow{\psi} \pi_{4p-1}(GL(\infty, C)) \]

\[ \pi_{2p,2p+2k-1} \xrightarrow{J_U} \pi_{4p-1} \]

where \( c \) is the natural complexification homomorphism and \( J_U \) is the complex stable \( J \)-homomorphism.

In the following we identify \( \Sigma^{r,q} \) with \( \Sigma^{r+p,q} \). Regarding \( \Sigma^{1,0} \) as the one-point compactification of \( R^{1,0} \) with \( \infty \) as base-point, the quotient \( \Sigma^{1,0}/\{0, \infty \} \) is homeomorphic to \( S^1 \), where \( S^1 \) has the involution \( T \) interchanging factors. For a base-point-preserving map \( f: S^p \to X \), define a \( \tau \)-map \( \tilde{f}: \Sigma^{p,q} \to X \) by the composition

\[ \Sigma^{p,q} = (\Sigma^{p-1,0} \cap \Sigma^{0,q}) \to \Sigma^{p-1,0} \cap (\Sigma^{1,0}/\{0, \infty \}) \]  

for \( p, q \geq 1 \) where \( \pi \) is the natural projection, \( \tau \) is the involution of \( X \) and \( \tau' \) is the involution of \( (\Sigma^{p-1,0} \cap \Sigma^{0,q}) \to (\Sigma^{p-1,0} \cap \Sigma^{0,q}) \) induced by that of \( \Sigma^{p-1,0} \to \Sigma^1 \cap \Sigma^{0,q} \) and \( T \). Then the correspondence \( f \to \tilde{f} \) determines a homomorphism

\[ \alpha: \pi_{p,q}(X) \to \pi_{p,q}(X) \]

for \( p, q \geq 1 \) (cf. [5], p. 286, [4], (10.5)).

Let \( J_{U,n}: \pi_{4p-1}(GL(n, C)) \to \pi_{4p-1+2n}(S^{2n}) \) be the complex \( J \)-homomorphism. Let \( \alpha_n: \pi_{4p-1}(GL(n, C)) \to \pi_{2p-2k,2p+2k-1}(GL(n, C)) \) and \( \alpha_n: \pi_{4p-1+2n}(S^{2n}) \to \pi_{2p-2k,n+2p+2k-1}(\Sigma^{n,n}) \) denote the homomorphisms of (2.5) for \( X = GL(n, C) \) and \( X = \Sigma^{n,n} \) respectively. Then we have the commutative diagram:

\[ \pi_{4p-1}(GL(n, C)) \xrightarrow{\alpha_n} \pi_{2p-2k,2p+2k-1}(GL(n, C)) \]

\[ \pi_{4p-1+2n}(S^{2n}) \xrightarrow{\alpha_n} \pi_{2p-2k,n+2p+2k-1}(\Sigma^{n,n}) \]

The commutativity is proved as follows. For a \( \tau \)-map \( g: \Sigma^{2p-2k,2p+2k-1} \to GL(n, C) \) we denote by \( \text{ad} g \) the adjoint of the composition: \( \Sigma^{2p-2k,2p+2k-1} \xrightarrow{g} GL(n, C) \subseteq \Omega^{2n}_{n} \Sigma^n \). Then \( J_{U,n} \) is given by the assignment \( g \mapsto \text{ad} g \) as in §1. In the above, forgetting the \( Z_2 \)-action we get the homomorphism \( J_{U,n} \). Hence we also use the same notation for maps in the complex case. Let us define a map \( \lambda: S^n \times S^{2p-2k} \times S^n \times S^{2p+2k-1} \to S^n \times S^n \times S^{2p-2k} \times S^{2p+2k-1} \) by \( \lambda(u_1, v_1, u_2, v_2) = u_1 \times u_2 \times v_1 \times v_2 \) (\( u_1, u_2 \in S^n, v_1, v_2 \in S^{2p-2k}, v_2 \in S^{2p+2k-1} \)). And we define a map
$f': S^{4p-1+2k} \to S^{2n}$ by $f' = (\text{ad}f)\lambda$ for a map $f: S^{4p-1} \to GL(n, \mathbb{C})$. Then $f' = \text{ad}f$ since the degree of $\lambda$ is 1, and so $f' = \text{ad}f$. Besides we see easily that $f' = \text{ad}f$. Therefore $\text{ad}f = \text{ad}f$ which implies $\alpha_\ast J_{U,\ast}([f]) = J_{R,\ast}\alpha_\ast([f])$ where $[f]$ denotes the homotopy class of $f$.

Here, by taking the direct limits we get the commutative diagram

$$
\begin{array}{ccc}
\pi_{4p-1}(GL(\infty, \mathbb{C})) & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbb{C})) \\
\downarrow J_U & & \downarrow J_R \\
\pi_{4p-1} & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}
\end{array}
$$

where each $\alpha$ is defined as the direct limit of $\alpha_\ast$. As in proof of the commutativity of the above diagram, we can show that the lower homomorphism $\alpha$ is well-defined.

By the definition of $\alpha$ it follows that the realification homomorphism $r: \hat{K}^{-1}(S^{4p-1}) \to \tilde{KR}^{-1}(S^{2p-2k, 2p+2k-1})$ coincides with $\alpha: \pi_{4p-1}(GL(\infty, \mathbb{C})) \to \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbb{C}))$ through the natural isomorphisms. Because, $\psi \alpha = 1 + \ast$, $\psi = e$, $\sigma = 1 + \ast$ and $\ast$ is the operation on $K(X)$ defined in [12], §2. Thus, by (2.2), (2.3) and (2.6) we get the commutative diagram

$$
\begin{array}{ccc}
\hat{K}(S^{4p-1}) & \xrightarrow{r} & \tilde{KO}(S^{4p-1}) \\
\downarrow \cong & & \downarrow \cong \\
\pi_{4p-1}(GL(\infty, \mathbb{C})) & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbb{C})) \\
\downarrow J_U & & \downarrow J_R \\
\pi_{4p-1} & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}
\end{array}
$$

where $r$ is the realification homomorphism.

Let $GL(\infty, \mathbb{R})$ denote the infinite general linear group over the real numbers and $J_\Lambda$ denote the real stable $J$-homomorphism in stable dimensions $4p-1$. Let us put

$$g_\Lambda = J_\Lambda(1), \quad \Lambda = O, \, U \, \text{or} \, R,$$

identifying $\pi_{4p-1}(GL(\infty, \mathbb{R})), \pi_{4p-1}(GL(\infty, \mathbb{C}))$ and $\pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbb{C}))$ with $Z$. Then, from (2.4), (2.7) and [12], (2.2) we see that

$$\alpha(g_\Lambda) = \begin{cases} 
2g_R & \text{if } k \text{ is even} \\
\frac{g_R}{g} & \text{if } k \text{ is odd}
\end{cases}$$

and

$$\psi(g_\Lambda) = \begin{cases} 
g_R & \text{if } k \text{ is even} \\
2g_R & \text{if } k \text{ is odd}
\end{cases}$$

Furthermore it is known that

$$g_\Upsilon = \begin{cases} 
2g_\Upsilon & \text{if } p \text{ is even} \\
g_\Upsilon & \text{if } p \text{ is odd}
\end{cases}$$
and the order of \( g_0 \) is equal to the number \( m(2p) \) ([1], II, Theorem (2.7) and [11]) which is divisible by 8 ([1], II, p.139).

Let \( o(p, k) \) denote the order of the image of (1.2). Then, by (2.8) and (2.9), we obtain

**Lemma.** For \( p > k \),

\[
o(p, k) = \begin{cases} 
    dm(2p) & \text{if either } k, p \text{ are even or odd} \\
    2dm(2p) & \text{if } k \text{ is even and } p \text{ is odd} \\
    \frac{1}{2} \text{dm}(2p) & \text{if } k \text{ is odd and } p \text{ is even}
\end{cases}
\]

where \( d = \frac{1}{2} \) or 1.

We shall give a proof of Theorem in §§3–5.

3. Proof for \( p > k \), \( k \) odd and \( p \) even. By [5], Fig. we have an exact sequence

\[
\pi^*_{2p-2k-1, 2p+2k+1} \xrightarrow{\psi} \pi^*_{4p-1} \xrightarrow{\alpha} \pi^*_{2p-2k, 2p+2k-1}
\]

(cf. [4], (10.5)). Therefore, if we suppose that \( o(p, k) = \frac{1}{4} m(2p) \) then \( \alpha(\frac{1}{2}m(2p)\ g_0) = \frac{1}{4} m(2p)g_0 = 0 \) by (2.8), (2.9) and so there exists an equivariant map

\[
f : \Sigma^{2p-2k-1+n, 2p+2k+n} \to \Sigma^{n, n}
\]

for \( n \) sufficiently large such that the image of the homotopy class of \( f \) by \( \psi \) is \( \frac{1}{2} m(2p)g_0 \).

Since \( k \) is odd,

\[
\widetilde{KR}(\Sigma^{2p-2k-1+n, 2p+2k+n}) \simeq \widetilde{KO}(S^{4k+1}) = 0
\]

and

\[
\widetilde{KR}(\Sigma^{2p-2k-1+n, 2p+2k+n+1}) \simeq \widetilde{KO}(S^{4k+2}) = 0
\]

Therefore we have the commutative diagram

\[
\begin{array}{ccc}
0 & \leftarrow & \widetilde{KR}(\Sigma^{n, n}) \\
\downarrow c & & \downarrow c \\
\widetilde{K}(S^{2n}) & \overset{f'}{\leftarrow} & \widetilde{K}(S^{2n} \cup CS^{4p-1+2n}) \\
& & c \downarrow \\
0 & \leftarrow & \widetilde{K}(S^{2n})
\end{array}
\]

where \( f' \) is a representative of \( \frac{1}{2} m(2p)g_0 \), \( CA \) is the cone of \( A \) and \( c \) is the natural complexification homomorphism ([12], §2). This diagram implies that \( e_c(f') = 0 \), which contradicts to the fact that \( e_c(f') = \frac{1}{2} \) ([1], IV, §7). Hence we see by Lemma that \( o(p, k) = \frac{1}{2} m(2p) \).
4. Proof for \( p > k \) and \( p, k \) even or odd. Using the notation of Landweber for the stable homotopy groups [9], by [5], Fig. and (12) we have the following commutative diagram in which the columns and the rows are exact sequences:

\[
\begin{array}{c}
0 \\
\downarrow \psi^* \\
\lambda_{4p-2k-1,2p+2k} \rightarrow \pi_{4p-1} \rightarrow \lambda_{4p-2k,2p+2k-1} \downarrow \\
\psi \\
\alpha \downarrow \alpha \\
\pi_{4p-2k-1,2p+2k} \rightarrow \pi_{4p-1} \rightarrow \pi_{4p-2k,2p+2k-1} \\
\end{array}
\]

for \( k \geq 0 \). (\( \lambda^*_{p,q} \) and \( \pi^*_{p,q} \) are Bredon's \( \pi^*_{p+q} \) and \( \pi^*_{p+q} \) respectively.) If we assume that \( o(p,k) = \frac{1}{2} m(2p) \), then \( \alpha(\frac{1}{2} m(2p)g_0) = \frac{1}{2} m(2p)g_0 = 0 \) by (2.8), (2.9) and therefore there is an equivariant map

\[ f: \Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n} \rightarrow \Sigma^n, \]

such that the image of the homotopy class of \( f \) by \( \psi^* \) is \( \frac{1}{2} m(2p)g_0 \).

Consider the diagram

\[
\begin{array}{c}
\Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n} \\
\uparrow \pi \\
\Sigma^{2p-2k-1+n,2p+2k+n} \rightarrow \Sigma^n, \]
\]

where \( f = f \pi \) and \( \pi \) is the map collapsing \( \Sigma^{0,2p+2k+n} \) to a point.

Putting

\[
A = \widetilde{KO}_{\mathbb{Z}_2}(\Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n}),
\]

\[
B = \widetilde{KO}_{\mathbb{Z}_2}(\Sigma^{2p-2k-1+n,2p+2k+n}),
\]

\[
C = \widetilde{KO}_{\mathbb{Z}_2}(\Sigma^{2p-2k-1+n,2p+2k+n+1})
\]

and taking

\[ n \equiv 0 \mod 8, \]

we have by [9], Lemma 4.1

\[ A \cong KO^{-2p-2k-n-1}(P^{2p-2k-2+n}) \]

where \( P^m \) is the real projective \( m \)-space and we have by [6] and [9], Theorem 3.1

\[ A \cong \begin{cases} 
0 & \text{if } p = 2q, k = 2l \text{ and } q+l \text{ is odd} \\
& \text{or } p = 2q+1, k = 2l+1 \text{ and } q+l \text{ is even} \\
Z_2 \oplus Z_2 & \text{if } p = 2q, k = 2l \text{ and } q+l \text{ is even} \\
& \text{or } p = 2q+1, k = 2l+1 \text{ and } q+l \text{ is odd} 
\end{cases} \]
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$B \simeq Z, C \simeq Z_2$ if $p, k$ are even

and

$B \simeq Z, C = 0$ if $p, k$ are odd.

In any case $A, C$ are torsion groups and $B$ is a free abelian group. Hence $f^* = \pi^* f^*: \widetilde{KO}_{Z_2}(\Sigma^n, \tau) \to B$ is a zero map since $\pi^*: A \to B$ is so. And therefore we have the commutative diagram

\[
\begin{array}{ccc}
0 & \leftarrow & \widetilde{KO}_{Z_2}(\Sigma^n, \tau) \\
\rho & \downarrow & \rho \\
0 & \leftarrow & \widetilde{KO}(S^{2n}) \\
\end{array}
\]

\[
(4.1)
\]

where $f'$ is a representative of $\frac{1}{2} m(2p) g_0$ and $\rho$ is the forgetful homomorphism.

From [9], Theorem 3.1 and Proposition 3.4 we see that $\widetilde{KO}_{Z_2}(\Sigma^{5m, 5m})$ is a free $RO(Z_2)$-module with a single generator $u$ for which the Adams operation $\psi^*$ satisfy

\[
\psi^*(u) = \begin{cases} 
\frac{1}{2} k^{8m}(H-1)u & \text{if } k \text{ is even} \\
\frac{1}{2} k^{8m} (k^{8m} - k^{4m}) (H-1)u & \text{if } k \text{ is odd}
\end{cases}
\]

for $m > 0$ where $H$ is a canonical, non-trivial, 1-dimensional representation of $Z_2$. Since $\rho(u)$ becomes a generator of $\widetilde{KO}(S^{10m})$, (4.1) and (4.2) imply that $e'(f') = 0$. On the other hand $e'(f') = \frac{1}{2}$ ([1], IV, §7). This contradiction and Lemma show that $o(p, k) = m(2p)$.

5. Proof for $p = k$. Considering the following diagram

\[
\begin{array}{ccc}
\pi_{0, 4p-1} (GL(\infty, C))^\varphi & \xrightarrow{\varphi} & \pi_{0, 4p-1} (GL(\infty, R)) \\
\downarrow J_K & \cong & \downarrow J_0 \\
\pi_{0, 4p-1} i' & \xrightarrow{\varphi} & \pi_{0, 4p-1} t'
\end{array}
\]

where $\varphi$ is the fixed-point homomorphism [4, 5] we see that this diagram is commutative and therefore $o(p, p)$ is divisible by $m(2p)$.

Let us denote by $\Omega_2 S^n$ the space of base-point-preserving maps of $S^n$ into itself of degree $\tilde{d}$, by $GL(n, R)$ the general linear group of degree $n$ over the real numbers and by $GL(n, R)_0$ its identity component. Then the real $J$-homomorphism $J_{0, n}: \pi_{4p-1} (GL(n, R)) \to \pi_{4p-1+n} (S^n)$ is induced by the composition

\[
GL(n, R)_0 i' \subseteq \Omega_2 \rightarrow \Omega_2 S^n
\]
where \( i' \) is the inclusion map and \( t' \) is a similar one to \( t \) in §1 ([2], §1). Particularly, if \( n \geq 4p+1 \) then we may consider \( J_{0,*} \) the stable real \( J \)-homomorphism \( J_0: \pi_{4p-1}(GL(\infty, R)) \rightarrow \pi_{4p-1}^s \).

For a map \( f: S^{4p-1} \rightarrow \Omega_1^n S^n \), define a map \( f': S^{4p-1} \rightarrow \Omega_1^n S^n \) by \( f'(x) = f(x) \wedge f(x) \) (\( x \in S^{4p-1} \)). Here we regard \( S^n \wedge S^n \) as a space with involution switching factors and then \( S^n \wedge S^n \approx S^n \times S^n \) as \( \tau \)-spaces. The assignment \( f \mapsto f' \) determines a homomorphism \( \omega': \pi_{4p-1}(\Omega_1^n S^n) \rightarrow \pi_{0, p-1}(\Omega_1^n S^n) \). And so we define a homomorphism

\[
\omega: \pi_{4p-1}(\Omega_1^n S^n) \rightarrow \pi_{0, 4p-1}^s \]

by the composition

\[
\pi_{0, 4p-1}(\Omega_1^n S^n) \xrightarrow{\omega'} \pi_{0, 4p-1}(\Omega_1^n S^n) \xrightarrow{t^*} \pi_{0, 4p-1}(\Omega_1^n S^n) \rightarrow \pi_{0, 4p-1}^s
\]

where the unlabelled arrow is the obvious homomorphism. Then we can easily check that the diagram with the natural isomorphism \( \pi_{4p-1}(GL(n, R)) \approx \pi_{4p-1}(GL(\infty, R)) \)

\[
\pi_{0, 4p-1}(GL(\infty, C)) \xrightarrow{i'} \pi_{4p-1}(GL(\infty, R)) \approx \pi_{4p-1}(GL(n, R)) \xrightarrow{i''} \]

is commutative for \( n \geq 4p+1 \). From the commutativity of this diagram and the fact that \( J_0 \) factors into the following three homomorphism:

\[
\pi_{4p-1}(GL(n, R)) \xrightarrow{i''} \pi_{4p-1}(\Omega_1^n S^n) \xrightarrow{t^*} \pi_{4p-1}(\Omega_1^n S^n) \approx \pi_{4p-1}(\Omega_1^n S^n)
\]

for \( n \geq 4p+1 \) ([12], §1), it follows that \( m(2p) \) is divisible by \( o(p, p) \). This completes the proof of Theorem.

6. Finally we observe examples for the case \( k \) even and \( p \) odd.

By [5], (8) and [7], Table 1 we obtain

\[
\lambda_{k, 1}^* \approx Z_{12} \text{ and } \lambda_{0, 5}^* \approx Z_{504}
\]

using the Landweber's notation and so, making use of the exact sequence of [9], p.129, we have

\[
\pi_{k, 1}^* \approx Z_{24} \text{ and } \pi_{0, 5}^* \approx Z_{504}.
\]

Since \( m(2p) = 24 \) and \( m(2p) = 504 \) if \( p = 1 \) and \( p = 3 \) respectively, we get by Lemma
and the above isomorphisms \( o(p,k) = m(2p) \) for \((p,k) = (1,0), (3,0)\). We therefore conjecture that \( o(p,k) = m(2p) \) for \( k \) even and \( p \) odd generally.

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References
