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ON REAL J -HOMOMORPHISMS

Dedicated to Professor A. Komatu on his 70th birthday

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1. In the present work we consider a Real analogue of J -homomorphisms in the sense of [3]. We use here the notation in [4], §§1 and 9 and [9], §2 for the equivariant homotopy groups which are discussed by Bredon [5] and Levine [10]. Moreover we shall use notations and terminologies of [4], §1 without any references.

Let us denote by $GL(n, \mathbf{C})$ (resp. $GL(\infty, \mathbf{C})$) the general linear group of degree n (resp. the infinite general linear group) over the complex numbers with involutions induced by complex conjugation. Let X be a finite pointed τ -complex. Then, by following the construction of usual J -homomorphisms (cf. [13], p. 314, [2]) we can define homomorphisms

$$(1.1) \quad \begin{aligned} J_{R,n}: [\Sigma^{p,q}X, GL(n, \mathbf{C})]^\tau &\rightarrow [\Sigma^{p+n, q+n}X, \Sigma^{n,n}]^\tau \\ \text{and} \quad J_R: [\Sigma^{p,q}X, GL(\infty, \mathbf{C})]^\tau &\rightarrow \pi_s^{0,0}(\Sigma^{p,q}X) \end{aligned}$$

for $p \geq 0$ and $q \geq 1$ where let $\pi_s^{0,0}(\Sigma^{p,q}X) = \lim_{n \rightarrow \infty} [\Sigma^{p+n, q+n}X, \Sigma^{n,n}]^\tau$. We now give definitions of $J_{R,n}$ and J_R below. Let $\Omega_d^{n,n} \Sigma^{n,n}$ denote the subspace of $\Omega^{n,n} \Sigma^{n,n}$ consisting of maps of degree d in the usual sense. Let γ be the τ -map of $\Sigma^{n,n}$ induced by the correspondence of $R^{n,n}$ such that $(x_1, \dots, x_{2n}) \mapsto (x_1, \dots, x_{2n-1}, -x_{2n})$. By adding γ to the elements of $\Omega_1^{n,n} \Sigma^{n,n}$ with respect to the loop addition along fixed coordinates of $\Sigma^{n,n}$ we have a τ -map $t: \Omega_1^{n,n} \Sigma^{n,n} \rightarrow \Omega_0^{n,n} \Sigma^{n,n}$. Then we obtain $J_{R,n}$ by assigning to a base-point-preserving τ -map $f: \Sigma^{p,q}X \rightarrow GL(n, \mathbf{C})$ the adjoint of the composite

$$\Sigma^{p,q}X \xrightarrow{f} GL(n, \mathbf{C}) \xrightarrow{i} \Omega_1^{n,n} \Sigma^{n,n} \xrightarrow{t} \Omega_0^{n,n} \Sigma^{n,n}$$

where i is the canonical inclusion map.

As is easily seen the diagram

$$\begin{array}{ccc} [\Sigma^{p,q}X, GL(n+1, \mathbf{C})]^\tau & \xrightarrow{J_{R,n+1}} & [\Sigma^{p+n+1, q+n+1}X, \Sigma^{n+1, n+1}]^\tau \\ \uparrow j_* & & \uparrow \Sigma_*^{1,1} \\ [\Sigma^{p,q}X, GL(n, \mathbf{C})]^\tau & \xrightarrow{J_{R,n}} & [\Sigma^{p+n, q+n}X, \Sigma^{n,n}]^\tau \end{array}$$

is commutative under the identification $\Sigma^{r,s} \wedge \Sigma^{p,q} = \Sigma^{r+p, s+q}$ where j_* is the

homomorphism induced by a canonical inclusion map $j: GL(n, \mathbf{C}) \subset GL(n+1, \mathbf{C})$ and $\Sigma_*^{1,1}$ is the suspension homomorphism ([4], (7.2)). Therefore, by taking the direct limits we get a homomorphism

$$J_{R,\infty}: \lim_{n \rightarrow \infty} [\Sigma^{p,q} X, GL(n, \mathbf{C})]^\tau \rightarrow \pi_s^{0,0}(\Sigma^{p,q} X).$$

Also, as X is compact we have a canonical isomorphism $\mu: \lim_{n \rightarrow \infty} [\Sigma^{p,q} X, GL(n, \mathbf{C})]^\tau \rightarrow [\Sigma^{p,q} X, GL(\infty, \mathbf{C})]^\tau$. So we define J_R to be the composite $J_{R,\infty} \mu^{-1}$.

Taking $X = S^{0,1}$ in (1.1) J_R becomes the homomorphism from $\pi_{p,q}(GL(\infty, \mathbf{C}))$ to $\pi_{p,q}^s$. The aim of this paper is to prove the following theorem for the homomorphism

$$(1.2) \quad J_R: \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})) \rightarrow \pi_{2p-2k, 2p+2k-1}^s$$

for $p \geq k \geq 0$ and $p+k \geq 1$.

Theorem. *The image $J_R(\pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})))$ of the homomorphism (1.2) is a cyclic group of the following order:*

$$\begin{aligned} m(2p) & \quad \text{if either } p, k \text{ are even or odd} \\ \frac{1}{2} m(2p) & \quad \text{if } p \text{ is even and } k \text{ is odd} \\ m(2p) \text{ or } 2m(2p) & \quad \text{if } p \text{ is odd and } k \text{ is even} \end{aligned}$$

where $m(t)$ is the numerical function as in [1], II, p. 139.

2. Let X be a compact pointed τ -space throughout this section.

Let KR denote the Real K -functor [3]. Then a similar proof to the complex case gives rise to a canonical isomorphism

$$(2.1) \quad [X, GL(\infty, \mathbf{C})]^\tau \cong \widetilde{KR}(\Sigma^{0,1} X)$$

(cf. [8], Chap. I, Theorem 7.6) and so we may consider J_R of (1.1) the homomorphism from $\widetilde{KR}^{-1}(\Sigma^{p,q} X)$ to $\pi_s^{0,0}(\Sigma^{p,q} X)$ through this isomorphism. In particular, there exist isomorphisms

$$(2.2) \quad \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})) \cong \widetilde{KR}(\Sigma^{2p-2k, 2p+2k}) \cong \widetilde{KO}(S^{4k}) \cong Z$$

by (2.1) and the Real Thom isomorphism theorem [3]. Similarly we have isomorphisms

$$(2.3) \quad \pi_{4p-1}(GL(\infty, \mathbf{C})) \cong \tilde{K}(S^{4p}) \cong \tilde{K}(S^{4k}) \cong Z$$

in the complex K -theory.

Let $\psi: \pi_{p,q}(X) \rightarrow \pi_{p+q}(X)$ and $\psi: \pi_{p,q}^s(X) \rightarrow \pi_{p+q}^s(X)$ denote the forgetful homomorphisms [4,5]. Then, from the above discussion we have the following commutative diagram:

$$(2.4) \quad \begin{array}{ccc} \widetilde{KO}(S^{4k}) & \xrightarrow{c} & \widetilde{K}(S^{4k}) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})) & \xrightarrow{\psi} & \pi_{4p-1}(GL(\infty, \mathbf{C})) \\ \downarrow J_R & & \downarrow J_U \\ \pi_{2p-2k, 2p+2k-1}^s & \xrightarrow{\psi} & \pi_{4p-1}^s \end{array}$$

where c is the natural complexification homomorphism and J_U is the complex stable J -homomorphism.

In the following we identify $\Sigma^{r,s} \wedge \Sigma^{p,q}$ with $\Sigma^{r+p,s+q}$. Regarding $\Sigma^{1,0}$ as the one-point compactification of $R^{1,0}$ with ∞ as base-point, the quotient $\Sigma^{1,0}/\{0, \infty\}$ is homeomorphic to $S^1 \vee S^1$ as τ -spaces where $S^1 \vee S^1$ has the involution T interchanging factors. For a base-point-preserving map $f: S^{p+q} \rightarrow X$, define a τ -map $\tilde{f}: \Sigma^{p,q} \rightarrow X$ by the composition

$$\begin{aligned} \Sigma^{p,q} &= \Sigma^{p-1,0} \wedge \Sigma^{1,0} \wedge \Sigma^{0,q} \xrightarrow{1 \wedge \pi \wedge 1} \Sigma^{p-1,0} \wedge (\Sigma^{1,0}/\{0, \infty\}) \wedge \Sigma^{0,q} \\ &\approx (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \vee (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \xrightarrow{f \vee \tau f \tau'} X \end{aligned}$$

for $p, q \geq 1$ where π is the natural projection, τ is the involution of X and τ' is the involution of $(\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \vee (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q})$ induced by that of $\Sigma^{p-1,q+1} = \Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}$ and T . Then the correspondence $f \mapsto \tilde{f}$ determines a homomorphism

$$(2.5) \quad \alpha: \pi_{p+q}(X) \rightarrow \pi_{p,q}(X)$$

for $p, q \geq 1$ (cf. [5], p. 286, [4], (10.5)).

Let $J_{U,n}: \pi_{4p-1}(GL(n, \mathbf{C})) \rightarrow \pi_{4p-1+2n}(S^{2n})$ be the complex J -homomorphism. Let $\alpha_n: \pi_{4p-1}(GL(n, \mathbf{C})) \rightarrow \pi_{2p-2k, 2p+2k-1}(GL(n, \mathbf{C}))$ and $\alpha_n: \pi_{4p-1+2n}(S^{2n}) \rightarrow \pi_{2p-2k+n, 2p+2k-1+2n}(\Sigma^{n,n})$ denote the homomorphisms of (2.5) for $X = GL(n, \mathbf{C})$ and $X = \Sigma^{n,n}$ respectively. Then we have the commutative diagram:

$$\begin{array}{ccc} \pi_{4p-1}(GL(n, \mathbf{C})) & \xrightarrow{\alpha_n} & \pi_{2p-2k, 2p+2k-1}(GL(n, \mathbf{C})) \\ \downarrow J_{U,n} & & \downarrow J_{R,n} \\ \pi_{4p-1+2n}(S^{2n}) & \xrightarrow{\alpha_n} & \pi_{2p-2k+n, 2p+2k-1+2n}(\Sigma^{n,n}). \end{array}$$

The commutativity is proved as follows. For a τ -map $g: \Sigma^{2p-2k, 2p+2k-1} \rightarrow GL(n, \mathbf{C})$

we denote by adg the adjoint of the composition: $\Sigma^{2p-2k, 2p+2k-1} \xrightarrow{g} GL(n, \mathbf{C}) \subset \Omega_1^{n,n}$

$\Sigma^{n,n} \xrightarrow{\tilde{t}} \Omega_0^{n,n} \Sigma^{n,n}$. Then $J_{R,n}$ is given by the assignment $g \mapsto \text{adg}$ as in §1. In the above, forgetting the Z_2 -action we get the homomorphism $J_{U,n}$. Hence we also use the same notation for maps in the complex case. Let us define a map $\lambda: S^n \wedge S^{2p-2k} \wedge S^n \wedge S^{2p+2k-1} \rightarrow S^n \wedge S^n \wedge S^{2p-2k} \wedge S^{2p+2k-1}$ by $\lambda(u_1 \wedge v_1 \wedge u_2 \wedge v_2) = u_1 \wedge u_2 \wedge v_1 \wedge v_2$ ($u_1, u_2 \in S^n, v_1 \in S^{2p-2k}, v_2 \in S^{2p+2k-1}$). And we define a map

$f' : S^{4p-1+2n} \rightarrow S^{2n}$ by $f' = (\text{adf})\lambda$ for a map $f : S^{4p-1} \rightarrow GL(n, \mathbf{C})$. Then $f' \simeq \text{adf}$ since the degree of λ is 1, and so $\tilde{f}' \simeq \widetilde{\text{adf}}$. Besides we see easily that $\tilde{f}' = \text{adf}$. Therefore $\widetilde{\text{adf}} \simeq_{\tau} \text{adf}$ which implies $\alpha_n J_{U,n}([f]) = J_{R,n} \alpha_n([f])$ where $[f]$ denotes the homotopy class of f .

Here, by taking the direct limits we get the commutative diagram

$$(2.6) \quad \begin{array}{ccc} \pi_{4p-1}(GL(\infty, \mathbf{C})) & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})) \\ \downarrow J_U & \alpha & \downarrow J_R \\ \pi_{4p-1}^s & \longrightarrow & \pi_{2p-2k, 2p+2k-1}^s \end{array}$$

where each α is defined as the direct limit of α_n . As in proof of the commutativity of the above diagram, we can show that the lower homomorphism α is well-defined.

By the definition of α it follows that the realification homomorphism $r : \tilde{K}^{-1}(S^{4p-1}) \rightarrow \widetilde{KR}^{-1}(\Sigma^{2p-2k, 2p+2k-1})$ [12] coincides with $\alpha : \pi_{4p-1}(GL(\infty, \mathbf{C})) \rightarrow \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C}))$ through the natural isomorphisms. Because, $\psi\alpha = 1 + *$, $\psi = c$, $cr = 1 + *$ and c is injective where $*$ is the operation on $K(X)$ defined in [12], §2. Thus, by (2.2), (2.3) and (2.6) we get the commutative diagram

$$(2.7) \quad \begin{array}{ccc} \tilde{K}(S^{4k}) & \xrightarrow{r} & \widetilde{KO}(S^{4k}) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{4p-1}(GL(\infty, \mathbf{C})) & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C})) \\ \downarrow J_U & \alpha & \downarrow J_R \\ \pi_{4p-1}^s & \longrightarrow & \pi_{2p-2k, 2p+2k-1}^s \end{array}$$

where r is the realification homomorphism.

Let $GL(\infty, \mathbf{R})$ denote the infinite general linear group over the real numbers and J_o denote the real stable J -homomorphism in stable dimensions $4p-1$. Let us put

$$g_\Lambda = J_\Lambda(1), \quad \Lambda = O, U \text{ or } R,$$

identifying $\pi_{4p-1}(GL(\infty, \mathbf{R}))$, $\pi_{4p-1}(GL(\infty, \mathbf{C}))$ and $\pi_{2p-2k, 2p+2k-1}(GL(\infty, \mathbf{C}))$ with Z . Then, from (2.4), (2.7) and [12], (2.2) we see that

$$(2.8) \quad \alpha(g_U) = \begin{cases} 2g_R & \text{if } k \text{ is even} \\ g_R & \text{if } k \text{ is odd} \end{cases}$$

and
$$\psi(g_R) = \begin{cases} g_U & \text{if } k \text{ is even} \\ 2g_U & \text{if } k \text{ is odd} \end{cases}$$

Furthermore it is known that

$$(2.9) \quad g_U = \begin{cases} 2g_O & \text{if } p \text{ is even} \\ g_O & \text{if } p \text{ is odd} \end{cases}$$

and the order of g_o is equal to the number $m(2p)$ ([1], II, Theorem (2.7) and [11]) which is divisible by 8 ([1], II, p.139).

Let $o(p,k)$ denote the order of the image of (1.2). Then, by (2.8) and (2.9), we obtain

Lemma. For $p > k$,

$$o(p,k) = \begin{cases} dm(2p) & \text{if either } k, p \text{ are even or odd} \\ 2dm(2p) & \text{if } k \text{ is even and } p \text{ is odd} \\ \frac{1}{2}dm(2p) & \text{if } k \text{ is odd and } p \text{ is even} \end{cases}$$

where $d = \frac{1}{2}$ or 1.

We shall give a proof of Theorem in §§3–5.

3. Proof for $p > k$, k odd and p even. By [5], Fig. we have an exact sequence

$$\pi_{2p-2k-1, 2p+2k}^s \xrightarrow{\psi} \pi_{4p-1}^s \xrightarrow{\alpha} \pi_{2p-2k, 2p+2k-1}^s$$

(cf. [4], (10.5)). Therefore, if we suppose that $o(p,k) = \frac{1}{4}m(2p)$ then $\alpha(\frac{1}{2}m(2p)g_o) = \frac{1}{4}m(2p)g_R = 0$ by (2.8), (2.9) and so there exists an equivariant map

$$f: \Sigma^{2p-2k-1+n, 2p+2k+n} \rightarrow \Sigma^{n,n} \quad \text{for } n \text{ sufficiently large}$$

such that the image of the homotopy class of f by ψ is $\frac{1}{2}m(2p)g_o$.

Since k is odd,

$$\widetilde{KR}(\Sigma^{2p-2k-1+n, 2p+2k+n}) \cong \widetilde{KO}(S^{4k+1}) = 0$$

and $\widetilde{KR}(\Sigma^{2p-2k-1+n, 2p+2k+n+1}) \cong \widetilde{KO}(S^{4k+2}) = 0.$

Therefore we have the commutative diagram

$$\begin{array}{ccccccc} 0 \leftarrow \widetilde{KR}(\Sigma^{n,n}) & \leftarrow & \widetilde{KR}(\Sigma^{n,n} \cup C\Sigma^{2p-2k-1+n, 2p+2k+n}) & \leftarrow & \widetilde{KR}(\Sigma^{2p-2k-1+n, 2p+2k+n+1}) & \leftarrow & 0 \\ & & c \downarrow \cong & & c \downarrow & & = 0 \\ 0 \leftarrow \widetilde{K}(S^{2n}) & \leftarrow & \widetilde{K}(S^{2n} \cup_{f'} CS^{4p-1+2n}) & \leftarrow & \widetilde{K}(S^{4p+2n}) & \leftarrow & 0 \end{array}$$

where f' is a representative of $\frac{1}{2}m(2p)g_o$, CA is the cone of A and c is the natural complexification homomorphism ([12], §2). This diagram implies that $e_c(f') = 0$, which contradicts to the fact that $e_c(f') = \frac{1}{2}$ ([1], IV, §7). Hence we see by Lemma that $o(p,k) = \frac{1}{2}m(2p)$.

4. Proof for $p > k$ and p, k even or odd. Using the notation of Landweber for the stable homotopy groups [9], by [5], Fig. and (12) we have the following commutative diagram in which the columns and the rows are exact sequences:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 \lambda_{2p-2k-1, 2p+2k}^s & \xrightarrow{\psi^*} & \pi_{4p-1}^s & \xrightarrow{\alpha^*} & \lambda_{2p-2k, 2p+2k-1}^s \\
 & & \downarrow & \parallel & \downarrow \\
 \pi_{2p-2k-1, 2p+2k}^s & \xrightarrow{\psi} & \pi_{4p-1}^s & \xrightarrow{\alpha} & \pi_{2p-2k, 2p+2k-1}^s
 \end{array}$$

for $k \geq 0$. ($\lambda_{p,q}^s$ and $\pi_{p,q}^s$ are Bredon's $\pi_{p+q,p}^*$ and $\pi_{p+q,p}$ respectively.) If we assume that $o(p,k) = \frac{1}{2}m(2p)$, then $\alpha(\frac{1}{2}m(2p)g_O) = \frac{1}{2}m(2p)g_R = 0$ by (2.8), (2.9) and therefore there is an equivariant map

$$\tilde{f}: \Sigma^{2p-2k-1+n, 2p+2k+n} / \Sigma^{0, 2p+2k+n} \rightarrow \Sigma^{n,n} \text{ for } n \text{ sufficiently large}$$

such that the image of the homotopy class of \tilde{f} by ψ^* is $\frac{1}{2}m(2p)g_O$.

Consider the diagram

$$\begin{array}{ccc}
 \Sigma^{2p-2k-1+n, 2p+2k+n} / \Sigma^{0, 2p+2k+n} & & \tilde{f} \\
 \uparrow \pi & \searrow f & \\
 \Sigma^{2p-2k-1+n, 2p+2k+n} & \xrightarrow{\quad} & \Sigma^{n,n}
 \end{array}$$

where $f = \tilde{f}\pi$ and π is the map collapsing $\Sigma^{0, 2p+2k+n}$ to a point.

Putting

$$\begin{aligned}
 A &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n, 2p+2k+n} / \Sigma^{0, 2p+2k+n}), \\
 B &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n, 2p+2k+n}), \\
 C &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n, 2p+2k+n+1})
 \end{aligned}$$

and taking

$$n \equiv 0 \pmod{8},$$

we have by [9], Lemma 4.1

$$A \cong KO^{-2p-2k-n-1}(P^{2p-2k-2+n})$$

where P^m is the real projective m -space and we have by [6] and [9], Theorem 3.1

$$A \cong \begin{cases} 0 & \text{if } p=2q, k=2l \text{ and } q+l \text{ is odd} \\ & \text{or } p=2q+1, k=2l+1 \text{ and } q+l \text{ is even} \\ Z_2 \oplus Z_2 & \text{if } p=2q, k=2l \text{ and } q+l \text{ is even} \\ & \text{or } p=2q+1, k=2l+1 \text{ and } q+l \text{ is odd,} \end{cases}$$

$$B \cong Z, C \cong Z_2 \quad \text{if } p, k \text{ are even}$$

and

$$B \cong Z, C = 0 \quad \text{if } p, k \text{ are odd .}$$

In any case A, C are torsion groups and B is a free abelian group. Hence $f^* = \pi^* \tilde{f}^* : \widetilde{KO}_{Z_2}(\Sigma^{n,n}) \rightarrow B$ is a zero map since $\pi^* : A \rightarrow B$ is so. And therefore we have the commutative diagram

$$(4.1) \quad \begin{array}{ccccc} 0 \leftarrow \widetilde{KO}_{Z_2}(\Sigma^{n,n}) & \leftarrow & \widetilde{KO}_{Z_2}(\Sigma^{n,n} \cup_f C\Sigma^{2p-2k-1+n, 2p+2k+n}) & \leftarrow & C \\ & \rho \downarrow & \rho \downarrow & & \rho \downarrow \\ 0 \leftarrow \widetilde{KO}(S^{2n}) & \leftarrow & \widetilde{KO}(S^{2n} \cup_{f'} CS^{4p-1+2n}) & \leftarrow & \widetilde{KO}(S^{4p+2n}) \leftarrow 0 \\ & & & & \cong Z \end{array}$$

where f' is a representative of $\frac{1}{2}m(2p)g_0$ and ρ is the forgetful homomorphism.

From [9], Theorem 3.1 and Proposition 3.4 we see that $\widetilde{KO}_{Z_2}(\Sigma^{8m,8m})$ is a free $RO(Z_2)$ -module with a single generator u for which the Adams operation ψ^k satisfy

$$(4.2) \quad \psi^k(u) = \begin{cases} k^{8m}u + \frac{1}{2}k^{8m}(H-1)u & \text{if } k \text{ is even} \\ k^{8m}u + \frac{1}{2}(k^{8m} - k^{4m})(H-1)u & \text{if } k \text{ is odd} \end{cases}$$

for $m > 0$ where H is a canonical, non-trivial, 1-dimensional representation of Z_2 . Since $\rho(u)$ becomes a generator of $\widetilde{KO}(S^{16m})$, (4.1) and (4.2) imply that $e'_R(f') = 0$. On the other hand $e'_R(f') = \frac{1}{2} ([1], IV, \S 7)$. This contradiction and Lemma show that $o(p, k) = m(2p)$.

5. Proof for $p = k$. Considering the following diagram

$$\begin{array}{ccc} \pi_{0,4p-1}(GL(\infty, \mathbf{C})) & \xrightarrow{\varphi} & \pi_{4p-1}(GL(\infty, \mathbf{R})) \\ \downarrow J_R & & \downarrow J_O \\ \pi_{0,4p-1}^s & \xrightarrow{\varphi} & \pi_{4p-1}^s \end{array}$$

where φ is the fixed-point homomorphism [4,5] we see that this diagram is commutative and therefore $o(p, p)$ is divisible by $m(2p)$.

Let us denote by $\Omega_d^n S^n$ the space of base-point-preserving maps of S^n into itself of degree d , by $GL(n, \mathbf{R})$ the general linear group of degree n over the real numbers and by $GL(n, \mathbf{R})_0$ its identity component. Then the real J -homomorphism $J_{0,n} : \pi_{4p-1}(GL(n, \mathbf{R})) \rightarrow \pi_{4p-1+n}(S^n)$ is induced by the composition

$$GL(n, \mathbf{R})_0 \xrightarrow{i'} \Omega_1^n S^n \xrightarrow{t'} \Omega_0^n S^n$$

where i' is the inclusion map and t' is a similar one to t in §1 ([2], §1). Particularly, if $n \geq 4p + 1$ then we may consider $J_{0,n}$ the stable real J -homomorphism $J_0: \pi_{4p-1}(GL(\infty, \mathbf{R})) \rightarrow \pi_{4p-1}^s$.

For a map $f: S^{4p-1} \rightarrow \Omega_1^n S^n$, define a map $f': S^{4p-1} \rightarrow \Omega_1^{n,n} \Sigma^{n,n}$ by $f'(x) = f(x) \wedge f(x)$ ($x \in S^{4p-1}$). Here we regard $S^n \wedge S^n$ as a space with involution switching factors and then $S^n \wedge S^n \approx \Sigma^{n,n}$ as τ -spaces. The assignment $f \mapsto f'$ determines a homomorphism $\omega': \pi_{4p-1}(\Omega_1^n S^n) \rightarrow \pi_{0,p-1}(\Omega_1^{n,n} \Sigma^{n,n})$. And so we define a homomorphism

$$\omega: \pi_{4p-1}(\Omega_1^n S^n) \rightarrow \pi_{0,4p-1}^s$$

by the composition

$$\begin{aligned} \pi_{4p-1}(\Omega_1^n S^n) &\xrightarrow{\omega'} \pi_{0,4p-1}(\Omega_1^{n,n} \Sigma^{n,n}) \\ &\xrightarrow{t'_*} \pi_{0,4p-1}(\Omega_0^{n,n} \Sigma^{n,n}) \rightarrow \pi_{0,4p-1}^s \end{aligned}$$

where the unlabelled arrow is the obvious homomorphism. Then we can easily check that the diagram with the natural isomorphism $\pi_{4p-1}(GL(n, \mathbf{R})) \cong \pi_{4p-1}(GL(\infty, \mathbf{R}))$

$$\begin{array}{ccc} \pi_{0,4p-1}(GL(\infty, \mathbf{C})) & \xrightarrow{\varphi} & \pi_{4p-1}(GL(\infty, \mathbf{R})) \cong \pi_{4p-1}(GL(n, \mathbf{R})) \\ \downarrow J_R & & \downarrow i'_* \\ \pi_{0,4p-1}^s & \xleftarrow{\omega} & \pi_{4p-1}(\Omega_1^n S^n) \end{array}$$

is commutative for $n \geq 4p + 1$. From the commutativity of this diagram and the fact that J_0 factors into the following three homomorphism:

$$\begin{aligned} \pi_{4p-1}(GL(n, \mathbf{R})) &\xrightarrow{i'_*} \pi_{4p-1}(\Omega_1^n S^n) \xrightarrow{t'_*} \pi_{4p-1}(\Omega_0^n S^n) \\ &\cong \pi_{4p-1+n}(S^n) \end{aligned}$$

for $n \geq 4p + 1$ ([12], §1), it follows that $m(2p)$ is divisible by $o(p, p)$. This completes the proof of Theorem.

6. Finally we observe examples for the case k even and p odd.

By [5], (8) and [7], Table 1 we obtain

$$\lambda_{2,1}^s \cong Z_{12} \text{ and } \lambda_{6,5}^s \cong Z_{504}$$

using the Landweber's notation and so, making use of the exact sequence of [9], p.129, we have

$$\pi_{2,1}^s \cong Z_{24} \text{ and } \pi_{6,5}^s \cong Z_{504}.$$

Since $m(2p) = 24$ and $m(2p) = 504$ if $p = 1$ and $p = 3$ respectively, we get by Lemma

and the above isomorphisms $o(p, k) = m(2p)$ for $(p, k) = (1, 0), (3, 0)$. We therefore conjecture that $o(p, k) = m(2p)$ for k even and p odd generally.

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